- LINEAR UNSTEADY TRANSONICS → ACOUSTICS
  - Nonlinear Unsteady Transonics
    Physical Aspects
    Mathematical Aspects

$M \simeq 0$	$M < 1$	$M \approx 1$	$M > 1$	$M >> 1$
Incompressible	Linearize	Non-linear	Linearize	Non-linear

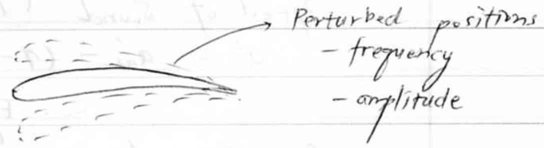
Steady
Irrotational
$\nabla^2 \phi = 0$
ϕ : Velocity Potential
Pressure

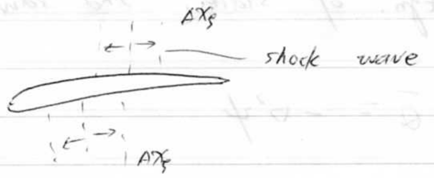
$M \equiv$ Mach number $= V_\infty / a_\infty$
$M^2 =$ Directed Energy / Thermal Energy

NON-LINEAR TRANSONICS

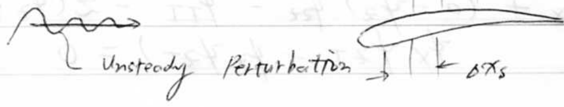
1. Introduction / context
2. Outline of Governing Equations
   Inviscid Fluids
3. Relationship between Mean Flow (Steady) and the unsteady perturbation

$M_\infty \approx 1$ (Transonic)





$M_\infty \simeq 1$ (Transonic)



- Non-linear Unsteady Transonics
  - Governing Equations
    - Assumptions
      - 2-D
      - Inviscid Fluid
      - Small Disturbance
  - Conservation of Linear Momentum
    - N-S Equations
    - Euler Equations
    - Bernoulli's Equations

$$\frac{1}{\rho}\frac{D\rho}{DT} = -\frac{1}{a^2}\frac{D}{DT}\left[\frac{\partial \psi}{\partial T} + \frac{Q^2 - U_\infty^2}{2}\right]$$

T : time

ρ : mass density

ψ : velocity potential

Q : speed

U : speed in the free stream

a : speed of sound

Bernoulli's

$$\frac{1}{\rho}\frac{D\rho}{DT} = -\frac{1}{a^2}\left[\frac{\partial^2 \psi}{\partial T^2} + \frac{\partial Q^2}{\partial T} + \bar{Q}\cdot\nabla\left(\frac{Q^2}{2}\right)\right]$$

Speed of Sound (1st law of thermodynamics, ...

$$a^2 = a_\infty^2 - (\gamma-1)\left[\psi_T + \frac{1}{2}\left(\psi_x^2 + \psi_z^2 - U_\infty^2\right)\right]$$

... Eqn. of state, 2nd law of Thermodynamics)

Conservation of Mass

$$\frac{1}{\rho}\frac{D\rho}{DT} = -\nabla\cdot\bar{Q} = -\nabla^2\psi$$

Combine the conservation principles,

$$(a^2-\psi_x^2)\psi_{xx} + (a^2-\psi_z^2)\psi_{zz} - \psi_{TT} - 2(\psi_z\psi_x\psi_{xz} + \psi_x\psi_{zT} + \psi_z\psi_{zT}) = 0$$

$\psi = \psi(x, z, t)$ subject to

Boundary Conditions + Initial Conditions

$$\Psi(X, Z, T) = U_\infty [X + \Phi(X, Z, T) + \cdots]$$

$\Phi(X, Z, T)$: Perturbation Velocity Potential

$$(1-M^2)\Phi_{XX} + \Phi_{ZZ} - 2\frac{M^2}{U_\infty}\Phi_{XT} - \frac{1}{a_\infty^2}\Phi_{TT} =$$
$$M^2\{[(\gamma+1)\Phi_X + \Phi_X^2]\Phi_{XX} + \frac{(\gamma-1)}{2}[\frac{2}{U_\infty}\Phi_T + \Phi_X^2 + \Phi_Z^2]$$
$$(\Phi_{XX} + \Phi_{ZZ}) + [(\gamma-1)\Phi_X + \Phi_Z^2]\phi_{ZZ}$$
$$+2[(1+\Phi_X)\Phi_{ZZ}\Phi_Z + \frac{1}{U_\infty}(\Phi_X\Phi_{XT} + \Phi_Z\Phi_{ZT})]\}$$

Neglect products of small terms, but retain "transonic" terms.

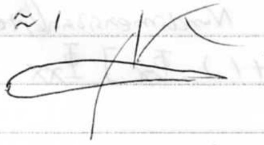
$$[(1-M^2) - \frac{M^2}{U_\infty}(\gamma-1)\Phi_T - M^2(\gamma+1)\Phi_X]\Phi_{XX}$$
$$+ \Phi_{ZZ} - 2\frac{M^2}{U_\infty}\Phi_{XT} - \frac{1}{a_\infty^2}\phi_{ZT} = 0$$

Non-dimensionalize Variables.

$$x = \frac{X}{c}, \quad z = \frac{Z}{c}, \quad t = T/(c/U_\infty), \quad \phi = \Phi/c$$

$$[1 - M^2 - M^2(\gamma-1)\phi_t - M^2(\gamma+1)\phi_x]\phi_{xx} + \phi_{zz}$$
$$- 2M^2\phi_{xt} - M^2\phi_{zt} = 0$$

$M_\infty \approx 1$



Energy gradient

Φ does not exist, $\bar{Q} \neq \nabla \cdot \phi$

Energy gradient are small

Φ does exist

$\bar{Q}$ does exist

· Non-linear Transonic Flows

Unsteady pressure distribution on aerodynamic surfaces

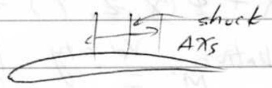
2-D

Unsteady

Isentropic, Irrotational

-- Crocco's law --- shock waves

weak, straight (Normal)



◦ Full Potential Equation

- Mass
- Linear Momentum
- Energy

⇒ Very Non-linear Equation

→ Finite Difference, Finite Volume, Finite Element, ...

◦ Full Potential

- Assumptions

... Flow = Uniform stream + Perturbation

$$\Phi(X,Z,T) = U_\infty [X + \bar{\Phi}(X,Z,T)]$$

Linearize the full potential, Nondimensionalize X, Z, T, Φ

$$[(1-M^2) - M^2(\gamma-1)\Phi_t - M^2(\gamma+1)\Phi_x]\Phi_{xx} + \Phi_{zz} - 2M^2\Phi_{xt} - M^2\Phi_{tt} = 0$$

We want the unsteady pressure distribution

Assume

$$\Phi(x,z,t) = \underbrace{\phi(x,z)}_{\text{steady}} + \underbrace{\hat{\varphi}(x,z,t)}_{\text{unsteady}}$$

$$\left.\begin{aligned} &[(1-M^2) - M^2(\gamma+1)\phi_x]\phi_{xx} + \phi_{zz} = 0 \quad \text{---(1)} \\ &[(1-M^2) - M^2(\gamma+1)\phi_x]\hat{\varphi}_{xx} - M^2(\gamma+1)\phi_{xx}\hat{\varphi}_x + \hat{\varphi}_{zz} \\ &\quad - M^2(\gamma-1)\phi_{xx}\hat{\varphi}_t - 2M^2\hat{\varphi}_{xt} - M^2\hat{\varphi}_{tt} = 0 \quad \text{---(2)} \end{aligned}\right\} \quad \begin{aligned} &\phi,\ \phi_x > \\ &\hat{\varphi},\ \hat{\varphi}_x,\ \hat{\varphi}_t \end{aligned}$$

(1): Non-linear PDE, steady

(2): Linear, unsteady, variable coefficients dependent on ϕ, ϕ_x, ϕ_{xx}

- Parameters
  - τ : Airfoil Thickness Ratio $\sim t/c \ll 1$
  - $\dfrac{\Delta x_S}{\delta} = \dfrac{\text{Shock displacement}}{\text{Amplitude of oscillation}}$
- A family of solutions for τ

$$p = p(x, z, t ; \tau)$$

$$N\{\phi(x, z, \tau)\} = 0 \quad \text{--- (1)}$$

$$\frac{\partial}{\partial \tau}\left[N\{\phi(x, z, \tau)\}\right] = 0 \Rightarrow L[g(x, z, \tau)] = 0$$

$$\frac{\partial}{\partial \tau}[\text{Boundary Conditions}] = 0 \rightarrow g(x, z, \tau)|_B = 0$$

$$g \equiv \frac{\partial \phi}{\partial \tau}$$

$$\phi(x, z, \tau + \Delta\tau) = \phi(x, z, \tau) + \int g\, d\tau$$

"Method of Parametric Differentiation" (Landahl)

$$[(1-M^2) - M^2(\gamma+1)\phi_x]\phi_{xx} + \phi_{zz} = 0$$

$$\frac{\partial}{\partial \tau}[\qquad\qquad\qquad] = 0$$

$$-M^2(\gamma+1)\frac{\partial \phi_x}{\partial \tau} + [(1-M^2) - M^2(\gamma+1)\phi_x]\frac{\partial \phi_{xx}}{\partial \tau} + \frac{\partial \phi_{zz}}{\partial \tau} = 0$$

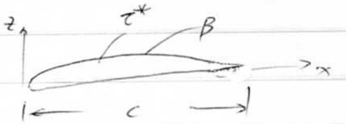
$$g = \frac{\partial \phi}{\partial \tau}$$

$$-M^2(\gamma+1)\frac{\partial g}{\partial x} + [(1-M^2) - M^2(\gamma+1)\underbrace{\phi_x}_{\sim \tau \text{ unknown}}]\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial z^2} = 0$$

For thin airfoil, boundary condition

$$B(x, z, t, \tau) = 0 \quad \text{--- instantaneous airfoil position}$$

$$B_t + B_x + \phi_z B_z = 0$$



$$\tau = \frac{\tau^*}{c}$$

$$\frac{\partial}{\partial \tau}[B_t + B_x + \phi_z B_z] = 0$$

$$B_{t\tau} + B_{x\tau} + g_z B_z + \phi_z B_{z\tau} = 0$$

- Method of Parametric Differentiation

Non-linear System $\longrightarrow$ Asymptotics $\tau = \tau_0$

$\downarrow$ | Define all coefficients at $\tau = \tau_0$

Parameter transformation

$\downarrow$

linear system $\longleftarrow$

$\downarrow$

Solution at $\tau = \tau_0 + \Delta\tau$ — Update coefficients $\tau_0 + \Delta\tau$

$\tau = \tau_0 + 2\Delta\tau$

$\tau = \tau_0 + 3\Delta\tau$

$\vdots$

$\downarrow$

Desired solution

PS 3 Question 1 --- due Friday (20)

2 --- Monday (30)

III. DYNAMIC AEROELASTICITY

∘ Introduction

Two principal phenomena :
- dynamic stability (flutter)
- response to dynamic loads as modified by aeroelastic effects

Flutter ··· self-excited vibration of a structure arising from the interaction of aerodynamics, elastic and inertial loads

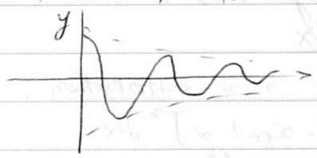
∘ Typical aircraft flutter problems
- flutter of wings
- " control surfaces
- " panel

- stability concept

If solution of dynamic systems may be written as

$$y(x,t) = \sum_{k=1}^{N} \bar{y}_k(x)\, e^{(\sigma_k + i\omega_k)t}$$

⇒ a) $\sigma_k < 0$, $\omega_k \neq 0$: convergent oscillations



"STABLE"

b) $\sigma_k = 0$; $\omega_k \neq 0$: simple harmonic motion
"STABILITY BOUNDARY"

c) $\sigma_k > 0$: $\omega_k \neq 0$ divergent oscillations
"UNSTABLE"

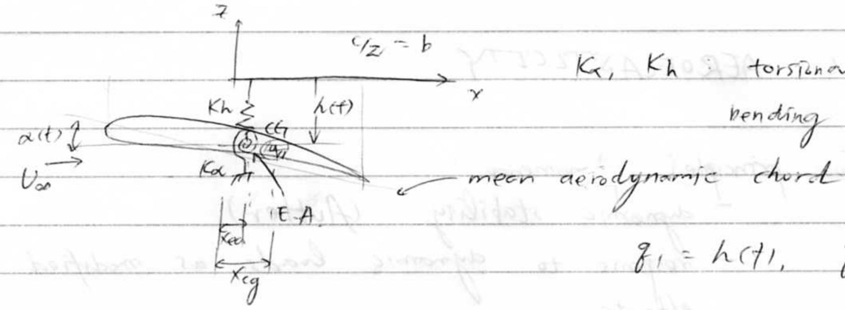
d) $\sigma_k < 0$. $\omega_k = 0$: continuous convergence → "STABLE"

e) $\sigma_k = 0$, $\omega_k = 0$: time independent solution
"STABILITY BOUNDARY"

f) $\sigma_k > 0$, $\omega_k = 0$: continuous divergence → "UNSTABLE"

∘ FLUTTER OF A WING

Let's first consider a typical section with 2 d.o.f.



K_α, K_h torsional and bending stiffness

$q_1 = h(t)$, $q_2 = \alpha(t)$

First step in Flutter analysis $\Rightarrow$ formulate equations of motion
The vertical displacement at any point along the mean aerodynamic chord from equilibrium position $z = 0$ will be taken as $z_a(x,t)$:

$$z_a(x,t) = \underset{\uparrow \, q_1}{-h} - (x - x_{ea})\underset{\uparrow \, q_2}{\alpha}$$

The equations of motion can be derived using Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$L = T - U$$

The total kinetic energy (T),

$$T = \frac{1}{2}\int_{-b}^{b} \rho \left(\frac{\partial z_a}{\partial t}\right)^2 dx$$

ρ ← chordwise mass distribution, $\rho = \rho(x)$

$$= \frac{1}{2}\int_{-b}^{b} \rho\left[\dot{h} + (x - x_{ea})\dot{\alpha}\right]^2 dx$$

$$= \frac{1}{2}\dot{h}^2 \underbrace{\int_{-b}^{b} \rho\, dx}_{\substack{m \\ \text{(Airfoil mass)}}} + \dot{h}\dot{\alpha}\underbrace{\int_{-b}^{b}(x - x_{ea})\rho\, dx}_{\substack{S_\alpha \\ \text{(static unbalance)}}} + \frac{1}{2}\dot{\alpha}^2 \underbrace{\int_{-b}^{b}(x - x_{ea})^2 \rho\, dx}_{\substack{I_\alpha \\ \text{(mass moment of inertia)}}}$$

Note: if $x_{ea} = x_{cg}$, then $S_\alpha = 0$ by definition of c.g.
Therefore,

$$T = \frac{1}{2} m\dot{h}^2 + \frac{1}{2} I_\alpha \dot{\alpha}^2 + S_\alpha \dot{h}\dot{\alpha}$$

The total potential energy (strain energy)

$$U = \frac{1}{2} K_h h^2 + \frac{1}{2} K_\alpha \alpha^2$$

Then, using Lagrange's equations with $L = T - U$

$q_1 = h, \quad q_2 = \alpha$

$$\Rightarrow \begin{cases} m\ddot{h} + S_\alpha \ddot{\alpha} + K_h h = Q_h \\ S_\alpha \ddot{\alpha} + I_\alpha \ddot{\alpha} + K_\alpha \alpha = Q_\alpha \end{cases}$$ --- Governing equation

where, Q_h, Q_α : are generalized forces associated with d.o.f.'s h, α respectively

$Q_i \; \delta q_i$

$(-L)\,\delta h$

$(M_{ea})\,\delta\alpha$

$Q_h = -L = -L(\alpha, h, \dot{\alpha}, \dot{h}, \ddot{\alpha}, \ddot{h}, \ldots)$

$Q_\alpha = M_{ea} = M_{ea}(\alpha, h, \dot{\alpha}, \dot{h}, \ddot{\alpha}, \ddot{h}, \ldots)$

Governing Equation

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} -L \\ M_{ea} \end{Bmatrix}$$

At first approximation, let's use quasi-steady aerodynamics

$$L = qS\,C_{L\alpha}(\alpha + \dot{h}/V_\infty)$$

$$M_{ac} = qSc\,C_{m\dot{\alpha}}\,\dot{\alpha}$$

$$\Rightarrow M_{ea} = \underbrace{(X_{ea} - X_{ac})}_{e} \cdot L + M_{ac}$$

$$= eqS\,C_{L\alpha}\left(\alpha + \frac{\dot{h}}{V_\infty}\right) + qSc\,C_{m\dot{\alpha}}\,\dot{\alpha}$$

Note : Three basic classifications of unsteadiness (linearized potential flow)

i) Quasi-steady aero : only circulatory terms due to the bound vorticity. Used for characteristic frequencies below 2 Hz e.g. conventional dynamic stability analysis

ii) Quasi-unsteady aerodynamics --- includes circulatory terms from both bound and wake vorticities.
satisfactory results for $2 < \omega < 10$ Hz.
Theodorsen is one that falls into here (without apparent mass terms)

iii) Unsteady Aerodynamics

quasi-unsteady + "apparent mass" terms (non-circulatory terms, inertial reactions : $\dot{\alpha}$, $\ddot{h}$)

For $\omega > 10\,Hz$, for conventional aircraft at subsonic speeds

The aeroelastic system of equations becomes:

$$\underbrace{\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix}}_{\text{mass matrix}} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \underbrace{\begin{bmatrix} qSC_{L\alpha}/U_\infty & 0 \\ -qSeC_{L\alpha}/U_\infty & -qSc\,C_{m\dot{\alpha}} \end{bmatrix}}_{\text{damping matrix (only from aerodynamics)}} \begin{Bmatrix} \dot{h} \\ \dot{\alpha} \end{Bmatrix}$$

$$+ \underbrace{\begin{bmatrix} K_h & qSC_{L\alpha} \\ 0 & K_\alpha - qSeC_{L\alpha} \end{bmatrix}}_{\text{stiffness}} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For stability, we can obtain characteristic equation of the system and analyze the roots.

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & qSC_{L\alpha} \\ 0 & K_\alpha - qSeC_{L\alpha} \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Much insight can be obtained by looking at the undamped system (Dowell, p. 83)

unsymmetric stiffness (typical of non-conservative problem)

Set $\alpha = \bar{\alpha} e^{ipt}$, $h = \bar{h} e^{ipt}$

$$\Rightarrow \underbrace{\begin{bmatrix} (mp^2 + K_h) & (S_\alpha p^2 + qSC_{L\alpha}) \\ S_\alpha p^2 & (I_\alpha p^2 + K_\alpha - qSeC_{L\alpha}) \end{bmatrix}}_{A} \begin{Bmatrix} \bar{h} \\ \bar{\alpha} \end{Bmatrix} e^{ipt} = 0$$

For nontrivial solutions, $\det A = 0$

→ characteristic eqn.

$$\underbrace{(mI_\alpha - S_\alpha^2)}_{A} p^4 + \underbrace{[K_h I_\alpha + (K_\alpha - qSeC_{L\alpha})m - qSC_{L\alpha}S_\alpha]}_{B} p^2 + \underbrace{K_h(K_\alpha - qSeC_{L\alpha})}_{C} = 0$$

- Undamped equations (p.86, 'Dowell')

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & qSC_{L\alpha} \\ 0 & K_\alpha - qSeC_{L\alpha} \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = 0$$

Set $h = \bar{h} e^{pt}$, $\alpha = \bar{\alpha} e^{pt}$ ($p \in \mathbb{C}$: complex)

Then,

$$\begin{bmatrix} (mp^2 + K_h) & (S_\alpha p^2 + qSC_{L\alpha}) \\ S_\alpha p^2 & (I_\alpha p^2 + K_\alpha - qSeC_{L\alpha}) \end{bmatrix} \begin{bmatrix} \bar{h} \\ \bar{\alpha} \end{bmatrix} = 0$$

For nontrivial solution, det $\Delta = 0$

Then, the characteristic equation of the form

$$Ap^4 + Bp^2 + C = 0$$

where

$$A = mI_\alpha - S_\alpha^2$$

$$B = K_h I_\alpha + (K_\alpha - qSeC_{L\alpha})m - qSC_{L\alpha}S_\alpha$$

$$C = K_h(K_\alpha - qSeC_{L\alpha})$$

The roots $p^2 = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

The signs of A, B, C determines the nature of the solution.

$A > 0$

$C > 0$, when $q < q_D$

Examine p as q increases

low $q \rightarrow p = \pm i\omega_1, \pm i\omega_2$ $(B^2 - 4AC > 0)$

higher $q \rightarrow p = \pm i\omega_1, \pm i\omega_2$ $(B^2 - 4AC = 0) \leftarrow (*)$

higher $q \rightarrow p = -\sigma_1 \pm i\omega_1, \sigma_2 \pm i\omega_2$ $(B^2 - 4AC < 0)$

↳ DYNAMIC INSTABILITY

(*) : STABILITY BOUNDARY

higher $q \rightarrow p = 0, 0 ; \pm i\omega_1$ $(C = 0)$

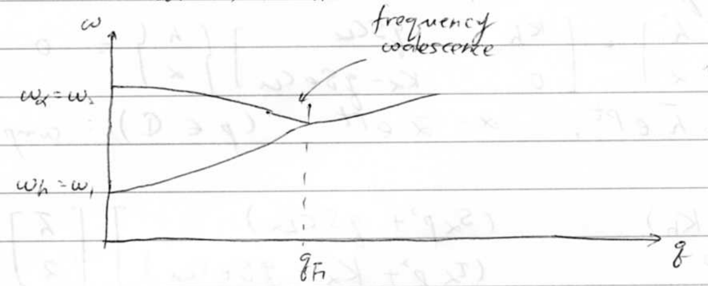
↑ q_D --- STABILITY BOUNDARY

In summary, Flutter condition $B^2 - 4AC = 0$

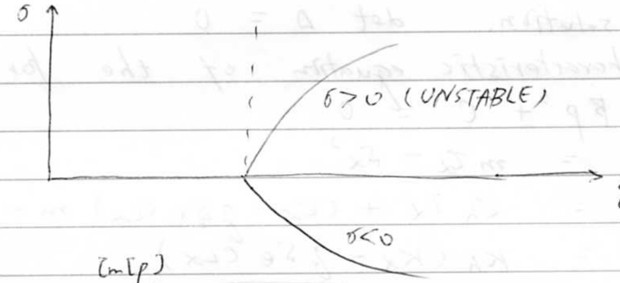
Divergence condition $C = 0$

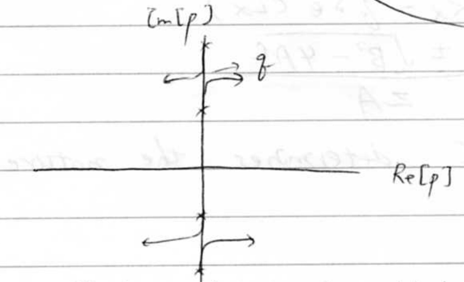
Graphically,

$$\omega_\alpha^2 = \frac{K_\alpha}{I_\alpha}, \quad \omega_h^2 = \frac{K_h}{m}$$



Ref : Bolotin "Nonconservative Problems of Theory of Elastic Stability", pp. 72-85





- Effects of Static Unbalance

In Dowell book, after Pines [1958] :

$S_\alpha \leq 0 \rightarrow$ may avoid flutter

if $S_\alpha = 0$, $\frac{q_F}{q_D} = 1 - \frac{\omega_h^2}{\omega_\alpha^2}$

If $q_D < 0$ ($e < 0$) and $\omega_h / \omega_\alpha < 1.0$

$\Rightarrow q_F < 0$ (no flutter)

$q_D > 0$ and $\omega_h / \omega_\alpha > 1 \Rightarrow$ no flutter

- Inclusion of Damping :

For better accuracy,

$$m\ddot{q} + c\dot{q} + kq = 0$$

where

$$C = \begin{bmatrix} qSC_{L\alpha}/U_\infty & 0 \\ -qSeC_{L\alpha}/U_\infty & -qSc\,C_{m\dot{\alpha}} \end{bmatrix}$$

The characteristic equation is now of the form

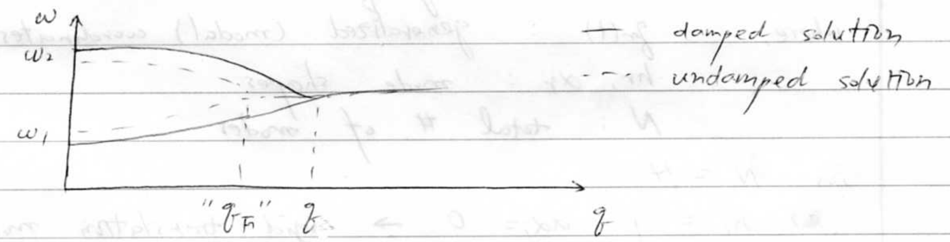
$$A_4 p^4 + A_3 p^3 + A_2 p^2 + A_1 p + A_0 = 0$$

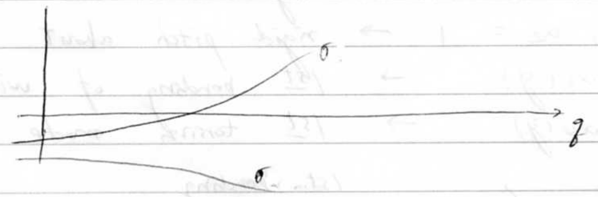
and we can examine p as q increases:

low $q \rightarrow p = -\sigma_1 \pm i\omega_1,\ -\sigma_2 \pm i\omega_2$ ← damped natural frequency

higher $q \rightarrow p = -\sigma_1 \pm i\omega_1,\ \pm i\omega_2$

higher $q \rightarrow p = -\sigma_1 \pm i\omega_1,\ +\sigma_2 \pm i\omega_2$ ← DYNAMIC INSTABILITY

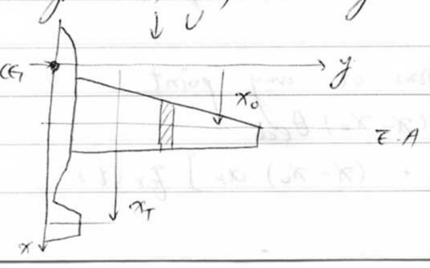




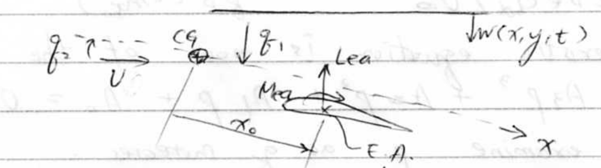
In summary,

- static instability --- $|K| = 0$
- dynamic instability
  - a) frequency coalescence (unsymmetric K)
  - b) negative damping ($C_{ij} < 0$)
  - c) unsymmetric damping (gyroscopic)

• Straight Aircraft Wing



Consider disturbance from equilibrium;



Using modal methods, the displacement (w_{ea}) and rotation (θ_{ea}) of elastic axis can be expressed as:

$$w_{ea} = \sum_{r=1}^{N} h_r(y)\, q_r(t)$$

$$\theta_{ea} = \sum_{r=1}^{N} \alpha_r(y)\, q_r(t)$$

where, $q_r(t)$: generalized (modal) coordinates

h_r, α_r : mode shapes

N : total # of modes

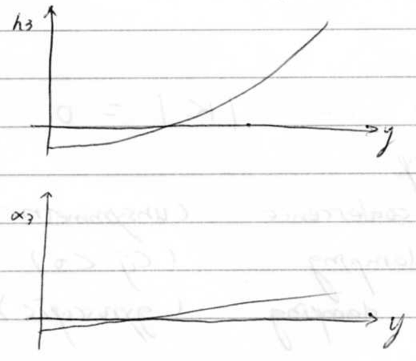
For $N = 4$

a) $h_1 = 1,\ \alpha_1 = 0 \rightarrow$ rigid translation mode ($\omega_1 = 0$)

b) $h_2 = x_0,\ \alpha_2 = 1 \rightarrow$ rigid pitch about CG ($\omega_2 = 0$)

c) $h_3(y),\ \alpha_3(y) \rightarrow$ 1st bending of wing ($\omega_3 \neq 0$)

d) $h_4(y),\ \alpha_4(y) \rightarrow$ 1st torsion mode ($\omega_4 \neq 0$)



1st bending
(bending - dominated)
mode

Modes can be assumed, or else calculated from mass-spring representation.

The displacement and rotations at any point:

$$w(x,y,t) = w_{ea} + (x - x_0)\,\theta_{ea}$$

$$= \sum_{r=1}^{N} \left[h_r + (x - x_0)\,\alpha_r \right] q_r(t)$$

$$\theta(x,y,t) = \theta_{ea} = \sum_{r=1}^{N} \alpha_r q_r(t)$$

The kinetic energy (T) is:

$$T = \frac{1}{2} \iint_{\frac{1}{2}\text{ aircraft}} m (\dot{w})^2 \, dx \, dy$$

$$= \frac{1}{2} \iint m \sum_r [h_r + (x - x_0)\alpha_r] \dot{q}_r \sum_s [h_s + (x - x_0)\alpha_s] \dot{q}_s \, dx\,dy$$

$$= \frac{1}{2} \sum_r \sum_s m_{rs} \dot{q}_r \dot{q}_s$$

where

$$m_{rs} = \int_0^L [M h_r h_s + I_\alpha \alpha_r \alpha_s + S_\alpha (h_r \alpha_s + h_s \alpha_r)] \, dy$$

and $M = \int_{LE}^{TE} m \, dx$ — mass / unit span

$S_\alpha = \int_{LE}^{TE} (x - x_0) m \, dx \rightarrow$ static unbalance / unit span

$I_\alpha = \int_{LE}^{TE} (x - x_0)^2 m \, dx \rightarrow$ moment of inertia about EA / unit span

The potential (strain) energy (U)

$$U = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 w_{ea}}{\partial y^2}\right)^2 dy + \frac{1}{2} \int_0^L GJ \left(\frac{\partial \theta_{ea}}{\partial y}\right)^2 dy$$

$$= \frac{1}{2} \int_0^L EI \sum_r h_r'' q_r \sum_s h_s'' q_s \, dy +$$

$$\frac{1}{2} \int_0^L GJ \sum_r \alpha_r' q_r \sum_s \alpha_s' q_s \, dy$$

$$= \frac{1}{2} \sum_r \sum_s K_{rs} q_r q_s$$

where,

$$K_{rs} = \int_0^L EI h_r'' h_s'' \, dy + \int_0^L GJ \alpha_r' \alpha_s' \, dy$$

[Note] $K_{rs} = 0$ for rigid body modes 1, 2 since $h_1'' = h_2'' = 0$ and $\alpha_1' = \alpha_2' = 0$

Finally, the work done by airloads

$$\delta W = -\int_0^L L_{ea} \, \delta w_{ea} \, dy + \int_0^L M_{ea} \, \delta\theta_{ea} \, dy \underbrace{- L_{HT} \, \delta w_{HT} + M_{HT} \, \delta\theta_{HT}}$$

↳ horizontal tail contribution (rigid fuselage assumption)

$$\delta W = -\int_0^L L_{ea} \sum_r h_r \, \delta q_r \, dy + \int_0^L M_{ea} \sum_r \alpha_r \, \delta q_r \, dy - L_{HT} \sum_r h_{r(HT)} \, \delta q_r + M_{HT} \sum \alpha_{r(HT)} \, \delta q_r$$

$$= \sum_{r=1}^{N} Q_r \, \delta q_r$$

where $Q_r = \int_0^l (-h_r L_{ea} + \alpha_r M_{ea})\, dy +$
$- h_{r(HT)} L_{HT} + \alpha_{r(HT)} M_{HT}$

[Note] $r = 1 \rightarrow \quad Q_1 = -\int_0^l L_{ea} dy - L_{HT} = -L_{TOTAL}/2$
$r = 2 \rightarrow \quad Q_2 = M_{TOTAL}/2 \quad (CG)$

- Straight wing (Cont.)

Place T, U, Q_r into the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial U}{\partial q_r} = Q_r$$

yields eqn. of motion

$$[m_{rs}]\{\ddot{q}_r\} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & K_{33} & K_{34} \\ 0 & 0 & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \{Q_r\}$$

zeros associated with rigid body modes

[Note] If we used normal modes

$$w(x, y, t) = \sum_r \phi_r(x, y)\, q_r$$

free-free normal modes

The eqn. would have been uncoupled.

$$[m_{rs}] \longrightarrow \begin{bmatrix} \ddots & & \\ & m_{rr} & \\ & & \ddots \end{bmatrix}$$

$$[K_{rs}] \longrightarrow \begin{bmatrix} \ddots & & \\ & m_{rr}\omega_r^2 & \\ & & \ddots \end{bmatrix}$$

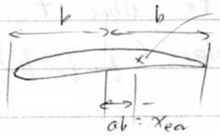
[Note] Free-free normal modes vs. Uncoupled Modes

Free-free normal modes	Uncoupled Modes
↓	↓
for entire structures	for individual components then couple together by Rayleigh-Ritz method
$M_r \ddot{q}_r + M_r \omega_r^2 q_r = Q_r$	$\Sigma m_{rs} \ddot{q}_s + \Sigma K_{rs} q_s = 0$
$\vdots$	$\vdots$
(more accurate)	(more versatile)

Now, let's introduce the aero by considering 2-D incompressible strip theory

$$L_{ea} = \pi\rho b^2[\ddot{w}_{ea} + U\dot{\theta}_{ea} - ba\ddot{\theta}_{ea}] + 2\pi\rho U b\, C(k)\cdot [\dot{w}_{ea} + U\theta_{ea} + b(\tfrac{1}{2} - a)\dot{\theta}_{ea}]$$

$C(k) = F(k) + iG(k)$



$a \equiv \frac{x_{ea}}{b}$

$$M_{ea} = \pi\rho b^3[a\ddot{w}_{ea} - U(\tfrac{1}{2} - a)\dot{\theta}_{ea} - b(\tfrac{1}{8} + a^2)\ddot{\theta}_{ea}] + 2\pi\rho U b^2(\tfrac{1}{2} + a)C(k)\cdot[\dot{w}_{ea} + U\theta_{ea} + b(\tfrac{1}{2} - a)\dot{\theta}_{ea}]$$

(see BA p. 104, p. 120
BAH p. 272)

However, as before:

$$w_{ea} = \sum_s h_s q_s$$

$$\theta_{ea} = \sum_s \alpha_s q_s$$

and placing this into L_{ea}, M_{ea} yields:

$$Q_r = \int_0^L (-h_r L_{ea} + \alpha_r M_{ea})\,dy + \cdots \text{ (H.T. terms)}$$

$$Q_r = Q_r(q_s, \dot{q}_s, \ddot{q}_s)$$

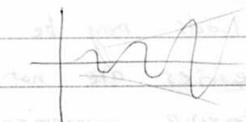
$\Rightarrow$ coupled set of homogeneous differential equations

For stability analysis, assume

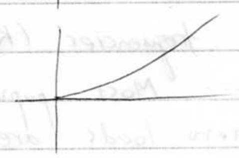
$$q_r(t) = \bar{q}_r e^{pt}$$

where, $p = \sigma + i\omega$, and for

a) $+\sigma$, $\omega \neq 0$ "flutter"



b) $+\sigma$, $\omega = 0$ "Divergence"



- Solutions of the Aeroelastic Equations of Motion

Two groups (Dowell, pp. 100~106)

a) Time domain

b) Frequency domain

- Time domain --- Fundamentally, a step by step solution for the time history.

  Direct integration methods ---

  a) equilibrium satisfied at discrete time t

  b) assumed variation of variables $(q, \dot{q}, \ddot{q})$ within the interval Δt

Example of methods

a) Central Difference

b) Newmark

c) Houbolt, etc.

Ref: Bathe, "Finite Element Procedures," Chap. 9

When selecting a method, three main issues to be aware:

a) efficient scheme

b) numerical stability

- conditionally stable --- dependent on Δt
- unconditionally stable

c) numerical accuracy

- amplitude decay
- period elongation

Advantage: straight forward method

Disadvantage: aero loads may be a problem

- theories are not well-developed
- intensive numerical calculations for small # of frequencies (K)

- Frequency domain --- Most popular approach

  main issue: aero loads are well developed for simple harmonic motion

  Disadvantage: two seporte solutions for stability and response to external loads

Consider Simple Harmonic Motion

$$q_r = \bar{q}_r e^{i\omega t}$$

and corresponding lift and moment,

$$L_{ea} = \bar{L}_{ea} e^{i\omega t}$$

$$M_{ea} = \bar{M}_{ea} e^{i\omega t}$$

where,

$$\bar{L}_{ea} = \pi\rho b^3 \omega^2 [\, l_h(k, M_\infty)\, \bar{w}_{ea}/b + l_\alpha(k, M)\, \bar{\theta} \,]$$

$$\bar{M}_{ea} = \pi\rho b^4 \omega^2 [\, m_h(k, M_\infty)\, \bar{w}_{ea}/b + m_\alpha(k, M)\, \bar{\theta} \,]$$

and where $l_\alpha,\ l_h,\ m_\alpha,\ m_h$: dimensionless complex functions of k, M_∞

Dowell p. 116

B.A. pp. 103 ~ 114

The governing equations become:

$$-\omega^2 \underbrace{[M]}_{\text{mass matrix}} \{\bar{q}\} + \underbrace{[K]}_{\text{stiffness matrix}} \{\bar{q}\} + \omega^2 \underbrace{[A(k, M_\infty)]}_{\substack{\text{aerodynamic operator} \\ \text{(aero mass matrix)}}} \{\bar{q}\} = 0$$

It's presumed that the following parameters

$$\underbrace{m,\ S_\alpha,\ I_\alpha}_{\text{inertia terms}} \qquad \underbrace{\omega_h,\ \omega_\alpha}_{\text{stiffness}}, \qquad \underbrace{b}_{c/2}$$

are known. The unknown quantities are

$$\bar{q},\ \omega,\ \underset{\text{altitude}}{\rho},\ M_\infty,\ k$$

i) k-method (V-g method)

Consider a system with just the right amount of structural damping, so the motion is simple harmonic.

$$-\omega^2 [M]\{\bar{q}\} + (1 + i\underbrace{g}_{\substack{\text{structural damping} \\ \text{coefficient}}})[K]\{\bar{q}\} + \omega^2 [A]\{\bar{q}\} = 0$$

[Note] structural damping --- restoring force in phase with velocity, but proportional to displacement

$$F_D = -g\, \dot{q}/|\dot{q}| \cdot |q|$$

viscous damping $F_c = -c\dot{q}$

Rewrite the eqn.

$$[M-A]\{\bar{q}\} = \underbrace{(1+ig)/\omega^2}_{\Lambda} [K]\{\bar{q}\}$$

$Re[\Lambda] \rightarrow \frac{1}{\omega^2}$

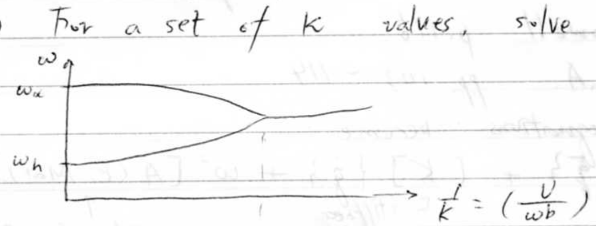
$Im[\Lambda] \rightarrow \frac{g}{\omega^2} \rightarrow g$

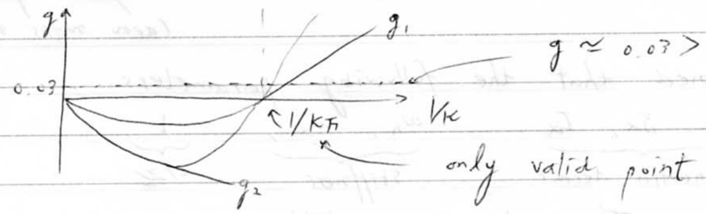
Solution process

(a) Given m, b, $\frac{\omega_h}{\omega_\alpha}$, S_α, I_α

(b) Assume ρ (fix altitude), $M_\infty = U/a$

(c) For a set of k values, solve a eigenproblem for Λ





(d) For $g_1 = 0 \Rightarrow \omega_1 = \omega_F$

$k_F = b\omega_F / U_F$

(e) Matching problem

$U_F \rightarrow M_F = M_\infty$

ii) p-method --- time dependent motion

$$q = \bar{q}e^{pt}, \quad p = \sigma + i\omega$$

and equations:

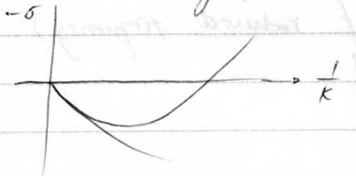
$$p^2[M]\{\bar{q}\} + [K]\{\bar{q}\} = [A(p,M)]\{\bar{q}\}$$

Now the aerodynamic becomes more approximate

a) quasi-steady aerodynamics

b) indicial lift function

c) flow eigenfunction



← all points are valid ones!

* k-method (V-g method)

only valid for simple harmonic motion ... $k \sim \omega$

$$q_r = \bar{q}_r e^{i\omega t}$$

· p-method

$$q = \bar{q} e^{pt}, \quad p = \sigma + i\omega$$

$$[M]\ddot{q} + [K]q = [A(p, m)]$$

· p-k method

The solution is assumed arbitrary (as in p-method). However, the aerodynamics is assumed to be:

$$A(p, M) \simeq A(k, M) \qquad (k\text{-part})$$

Then, the equation becomes:

$$\{p^2[M] + [K] - [A(k,M)]\}\{\bar{q}\} = 0$$

Solution process

a) specify k_i, M_i

b) solve for $p_0 = \sigma_0 + i\omega_0$ → k_0

c) check for double matching

$$k_0 = k_e\ ?$$

$$M_F = M_i\ ?$$

[Note] p-k method usually requires just a handful of interaction to converge

It's more expensive than V-g method

· Padé approximant solution

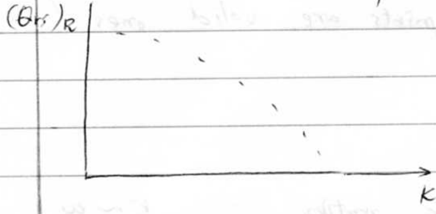
The generalized forces Q_r are computed for harmonic motion

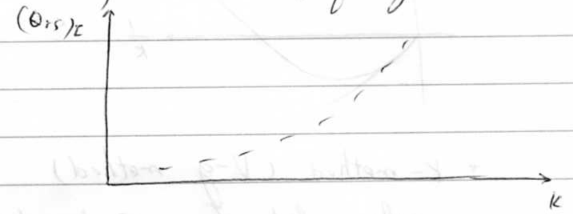
$$Q_r = \frac{1}{2}\rho U^2 Q_{rs} \bar{q}_s e^{i\omega t}$$

$$(\pi\rho\omega^2 A_{rs} \bar{q}_s e^{i\omega t})$$

where $Q_{rs} = (Q_{rs})_{REAL} + i\,(Q_{rs})_{IMAG}$

(complex functions of reduced frequency)





One can fit above by Padé Approximant in Laplace Transform domain p of form.

$$Q_r = \frac{1}{2}\rho U^2 \left[\underset{\text{mass}}{A_2 \left(\frac{b}{U}\right)^2 p^2} + \underset{\text{damp.}}{A_1\left(\frac{b}{U}\right)p} + \underset{\text{stiff}}{A_0} + \underset{\text{lag}}{\frac{A_3 \frac{b}{U} p}{\frac{b}{U}p + \beta_1}}\right] q_s$$

For harmonic motion : $p = i\omega$

$$\Rightarrow Q_r = \frac{1}{2}\rho U^2 \Big[\underbrace{\left(-A_2 k^2 + A_0 + k^2 A_3/(k^2+\beta_1^2)\right)}_{(Q_{rs})_R} + i\underbrace{\left(A_1 k - \frac{\beta_1 k A_3}{k^2+\beta_1^2}\right)}_{(Q_{rs})_I} \Big] q_s$$

and then evaluate coefficients $A_2, A_1, A_0, A_3, \beta_1$ to fit Q_{rs} over certain range of k, $0 \le k \le 2$

[Note] For better fit, use more lag terms.

$$Q_r = \frac{1}{2}\rho U^2 \left[A_2\left(\frac{b}{U}\right)^2 p^2 + \cdots + A_0 + \sum_{m=3}^{N} \frac{A_m\left(\frac{b}{U}\right)p}{\left(\frac{b}{U}\right)p + \beta_{m-2}} \right] q_s$$

Next, introduce new augmented state variables y_s, defined as

$$y_s = \frac{-\left(\frac{b}{U}\right)p}{\left(\frac{b}{U}\right)p + \beta_s} q_s = \frac{p}{p + \left(\frac{U}{b}\right)\beta_s} q_s$$

$$p\,y_s + \left(\frac{U}{b}\right)\beta_s y_s = p\,q_s$$

Returning to time domain

$$Q_r = \frac{1}{2}\rho U^2 \left[A_2\left(\frac{b}{U}\right)^2 \ddot{q}_s + A_1\left(\frac{b}{U}\right)\dot{q}_s + A_0 q_s + A_3 y_s \right]$$

$$\dot{y}_s + \left(\frac{U}{b}\right)\beta_s = \dot{q}_s$$

and the governing equations

$$\left\{ \; M\ddot{q} + C\dot{q} + Kq = \frac{1}{2}\rho U^2 \left[A_2\left(\frac{b}{U}\right)^2 \ddot{q} + \cdots + A_3 y_s \right] \right.$$

$\dot{y} + \left[\diagdown \frac{U\beta_s}{b} \diagdown \right] y = \dot{q}$

or,

$$\begin{bmatrix} M^* & 0 & 0 \\ 0 & M^* & 0 \\ 0 & 0 & I \end{bmatrix} \underbrace{\begin{Bmatrix} \ddot{q} \\ \dot{q} \\ \dot{y} \end{Bmatrix}}_{\dot{X}} + \begin{bmatrix} 0 & -M^* & 0 \\ K^* & C^* & G \\ 0 & -I & H \end{bmatrix} \underbrace{\begin{Bmatrix} q \\ \dot{q} \\ y \end{Bmatrix}}_{X}$$

where, $M^* = M - \frac{1}{2}\rho b^2 A_2$

$C^* = C - \frac{1}{2}\rho b A_1$

$K^* = K - \frac{1}{2}\rho U^2 A_0$

$G = \frac{1}{2}\rho U^2 A_3$

$H = \left[\diagdown \frac{U\beta}{b} \diagdown \right]$

and then,

$$\begin{Bmatrix} \dot{q} \\ \ddot{q} \\ \dot{y} \end{Bmatrix} = [A] \begin{Bmatrix} q \\ \dot{q} \\ y \end{Bmatrix} \rightarrow \dot{X} = AX$$

Ref : Abel, NASA TP-1367, 1979

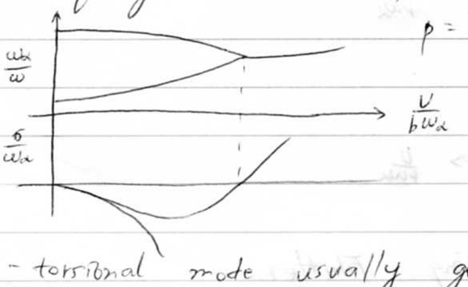
Tiffany and Adams, NASA TR-2776, July 1988

Karpel, AIAA J., Nov. 1991 p. 2007

- Types of Flutter

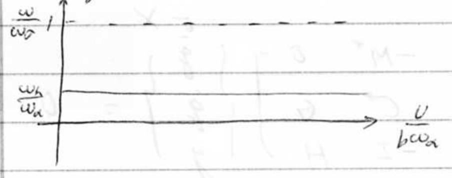
1) Frequency-coalescence flutter (2 d.o.f. flutter)

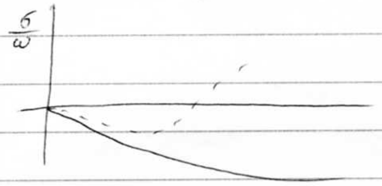
$p = \sigma + i\omega$



- torsional mode usually goes unstable
- flutter mode contains significant contributions of both bending and torsion
- out-of-phase forces are not qualitatively important (main effects come from non-symmetric $[K^*]$)

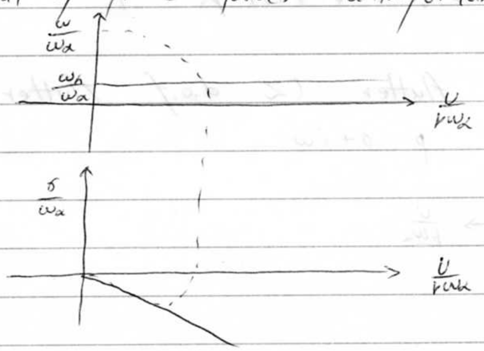
- Single d.o.f. flutter





- frequency of mode almost independent of reduced velocity
- results from negative damping
- out-of-phase part of aerodynamic operator is very important
- typical of systems with large mass ratio at large reduced velocity, e.g. - turbomachinery, bridges.

- divergence
- flutter at zero frequency
- single d.o.f. flutter
- out-of-phase forces unimportant



- Parameter Effects on Wing Flutter

When one nondimensionalize the flutter determinant (2-D), 5 parameters appear:

$$\mu = \frac{m}{\pi (2b)^2} \quad : \text{mass ratio}$$

$x_\alpha \equiv \frac{S_\alpha}{m b} = \frac{\text{distance CG is aft of E.A.}}{b}$

$r_\alpha \equiv \sqrt{\frac{I_\alpha}{m b^2}}$: $\frac{\text{radius of gyration about E.A.}}{b}$

$a = \frac{e}{b}$: $\frac{\text{distance E.A. is aft of midchord}}{b}$

$\frac{\omega_h}{\omega_\alpha}$: uncoupled bending-to-torsion frequency ratio

[Note] $\omega_\alpha t$ --- nondimensional time

M --- Mach No. (compressible effects)

$K_\alpha = \frac{\omega_\alpha b}{U}$ ... $\frac{1}{\text{reduced velocity}}$

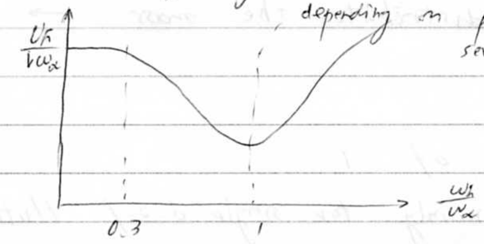
Then, for bending-torsion flutter,

$$\frac{U_F}{b\omega_\alpha} = f(\mu, x_\alpha, \frac{\omega_h}{\omega_\alpha}, a, r_\alpha, M)$$

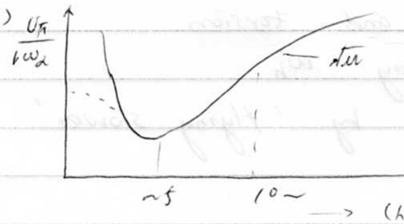
and the main trends are (Dowell pp 120~123),

a) $x_\alpha < 0$ (CG. ahead of E.A)

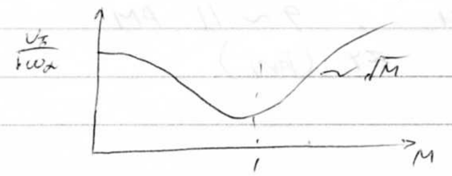
--- frequently no flutter occurs

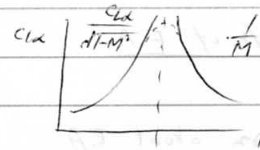
b) 

depending on parameters, this dip can be quite severe and approach to zero

c) 

→ (helicopter rotor blades)

d) 



$\frac{1}{M}$ "piston theory"

An approximate formula was obtained empirically by Theodorsen and Garrick for small ω_h/ω_α, large μ:

$$\frac{U_F}{b\omega_\alpha}\frac{1}{\sqrt{\mu}} \simeq \sqrt{\frac{r_\alpha^2}{2(\underbrace{\tfrac{1}{2}+a+x_\alpha}_{\text{distance between A.C and C.G.}})}}$$

AC EA CG (a, x_α)

(B.A.H. Eq 9-22)

Recall divergence

$$q_D = \frac{K_\alpha}{ec\, c_{l\alpha}} = \frac{1}{2}\rho U_D^2$$

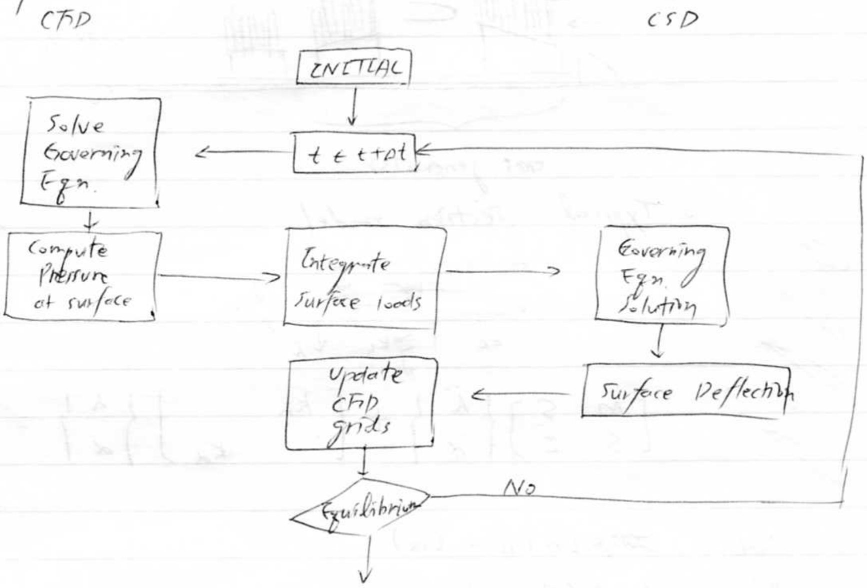
$$U_D = \sqrt{\frac{2K_\alpha}{\rho e c\, c_{l\alpha}}} = \cdots$$

$$\frac{U_D}{\omega_\alpha b}\frac{1}{\sqrt{\mu}} = \sqrt{\frac{r_\alpha^2}{2(\frac{1}{2}+a)}}$$

- Flutter Prevention
  - add mass or redistribute the mass $\rightarrow x_\alpha < 0$ ("mass balance")
  - increase ω_α
  - move $\frac{\omega_h}{\omega_\alpha}$ away of 1
  - add damping, mainly for single d.o.f. flutter
  - use composites:
    - couple bending and torsion
    - shift ω_α away ω_h
  - limit flight envelope by 'flying slower'

Announcement
- Controls in AE lectures
  May 8 and 11, 9 ~ 11 AM
- Apr 22 (Wed.), 24 (Fri.)

• Tightly (or closely) - coupled Analysis
  - most popular
  - interaction between CFD and CSD codes occurs at every time step



  - guaranteed convergence and stability

• Loosely - coupled Analysis

  CFD and CSD are solved alternatively with occasional interaction only.

  Ref : Smith, Huttsell, et al. AIAA Paper 96-1513
  SDM conf. Apr. '96

  - Difficulties in convergence

• Intimately - coupled (Unified) Analysis

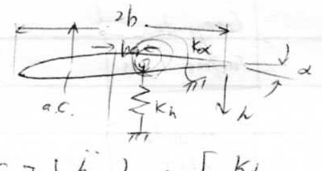
  governing equations are re-formulated and solved together

○ Introduction to Turbomachinery

Ref: AGARD manual in Aeroelasticity of Axial Flow Turbomachine, Vol. II, Chap. 19, F. Carta

Gas generator

○ Typical section model



$$\begin{bmatrix} M & S \\ S & I \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & \\ & K_\alpha \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \underbrace{\begin{Bmatrix} -L_M \\ M_M \end{Bmatrix}}_{\text{motion}} + \underbrace{\begin{Bmatrix} -L_G \\ M_G \end{Bmatrix}}_{\text{Gust}}$$

$$L_M = 2\pi\rho U b(\dot{h} + U\alpha)$$

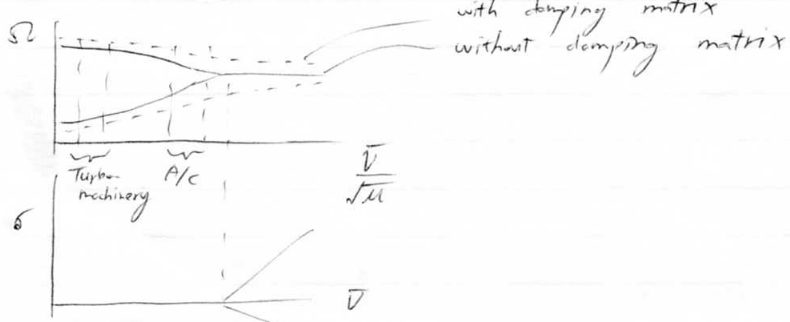
$$M_M = L \cdot b\left(a + \frac{1}{2}\right)$$

$$\begin{bmatrix} M & S \\ S & I \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + 2\pi\rho U b \begin{bmatrix} 1 & 0 \\ -b(a+\frac{1}{2}) & 0 \end{bmatrix} \begin{Bmatrix} \dot{h} \\ \dot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & 2\pi\rho U^2 b \\ 0 & K_\alpha - 2\pi\rho U^2 b(a \end{bmatrix} \cdot \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\begin{Bmatrix} h/b \\ \alpha \end{Bmatrix} = \underline{q}\, e^{j\omega t}$

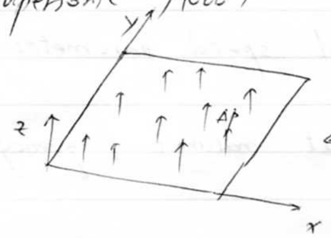
$$\left[-\Omega^2 \begin{bmatrix} 1 & x_a \\ x_a & r^2 \end{bmatrix} + j\Omega \frac{2\bar{V}}{\mu} \begin{bmatrix} 1 & 0 \\ -(a+\frac{1}{2}) & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2\frac{\bar{V}^2}{\mu} \\ 0 & \frac{K_\alpha}{b^2 K_h} - \frac{2\bar{V}^2}{\mu}(a+\frac{1}{2}) \end{bmatrix} \right]$$

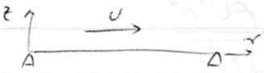
$$\Omega = \frac{\omega}{\sqrt{K_h/\mu}}$$



• Panel Flutter

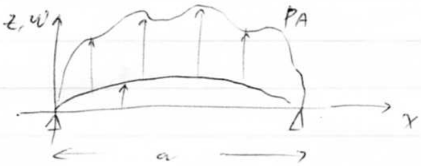
self-excited oscillation of the external skin of a flight vehicle when exposed to the air flow on that side (supersonic flow)





← amplitude limited by inplane forces

For simplicity, consider a 2-D simply supported panel in supersonic flow:



For a linear panel flutter analysis, the eqn. of motion

$$D\frac{\partial^4 w}{\partial x^4} + m\ddot{w} = p_A \quad \left[-N_x\frac{\partial^2 w}{\partial x^2}\right] \text{ nonlinear term}$$

$$N_x = \frac{Eh}{2a}\int_0^a\left(\frac{\partial w}{\partial x}\right)^2 dx$$

where, $D = \dfrac{Eh^3}{12(1-\nu^2)}$ (isotropic : plate stiffness)

m : mass/unit , h : thickness

p_A = aerodynamic pressure

for $M > 1.6$, $$p_A \cong -\frac{\rho U^2}{\sqrt{M^2-1}}\left\{\frac{\partial w}{\partial x} + \frac{M^2-2}{M^2-1}\frac{1}{U}\frac{\partial w}{\partial t}\right\}$$

Putting all together, the governing equation becomes :

$$D w^{IV} + \frac{\rho U^2}{\sqrt{M^2-1}} w' + \frac{\rho U}{\sqrt{M^2-1}}\frac{M^2-2}{M^2-1}\dot{w} + m\ddot{w} = 0$$

subject to : $w(0,t) = w(a,t) = 0$, $w''(0,t) = w''(a,t) = 0$ } simply-supported B.C.

Using Galerkin's method,

$$w(x,t) = \sum_{j=1}^{2} \sin j\frac{\pi x}{a}\, q_j(t)$$

satisfies all B.C.'s

and setting $q_j(t) = \bar{q}_j e^{pt}$, we get

$$\begin{bmatrix} (p^2 + a_\infty p + \omega_1^2) & -\dfrac{8\omega_1^2}{3\pi^4}\lambda \\ \dfrac{8\omega_1^2}{3\pi^4}\lambda & (p^2 + a_\infty p + 16\omega_1^2) \end{bmatrix} = 0$$

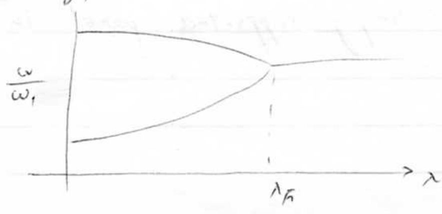
~ antisymmetric [K]

where, a_∞ : speed of sound

$\lambda \equiv \dfrac{\rho U^2 a^3}{D\sqrt{M^2-1}}$: critical speed parameter

$\omega_1 = \pi^2\sqrt{\dfrac{D}{\pi a^4}}$: lowest natural frequency

A typical result



[Note] $\lambda_F = \dfrac{\rho U_F^2 a^3}{D\sqrt{M^2-1}}$

If λ_F const. $\longrightarrow$ $E \nearrow \rightarrow D \nearrow \rightarrow \, q_F \nearrow$

$h \nearrow \rightarrow D \nearrow \rightarrow q_F \nearrow$

$a \downarrow \rightarrow q_F \nearrow$

$\frac{a}{b} \nearrow \rightarrow \lambda_F \nearrow$

· Computational Aeroelasticity

With the advance of computational resources and algorithms, there have been a great development in two areas.

CFD ··· Computational Fluid Dynamics

CSD ··· " Structural "

⇒ CAE ··· Computational Aero Elasticity

Difficulties arise from the nature of the methods.

CFD ··· finite difference discretization procedure based on Eulerian (spatial) description

CSD ··· finite element method based on Lagrangian (material) description

Define the nature of the coupling when combining the two numerical schemes.

There are broad classes :