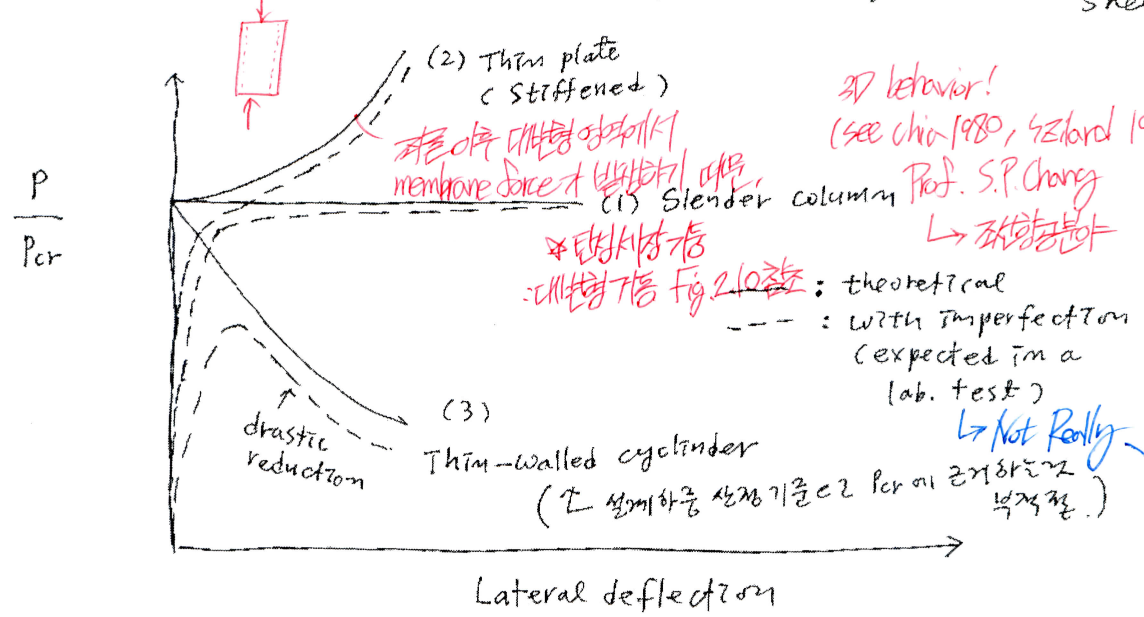
Postbuckling Behavior

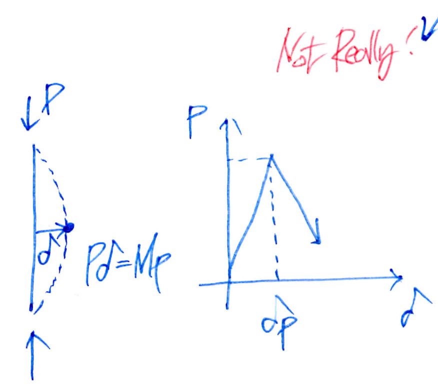
(1) Load-deflection relationship in the postbuckling range have an important bearing on the structural design significance of the "critical load".

(2) For the idealized "perfect" compression element (one that is perfectly elastic, devoid of imperfection), three different types of postbuckling behavior are typified by
(i) the column, (ii) the stiffened plate, and (iii) cylindrical shell.



: "Elastic" postbuckling curves for compressed members

↑ postbuckling curve 의 기울기에 주목할 것.
(후좌굴곡선)



제 1장 보충자료 (Stability theory)

1. Introduction

- "Instability" : a condition wherein "a compression member" loses the ability to resist increasing loads and exhibits instead a decrease in load-carrying capacity ; instability occurs at the maximum point on the load-deflection curve.

- Categories in instability problem

  (i) bifurcation of equilibrium :
  axially compressed columns, plates, and cylindrical shells.
  「critical load에서 deformation mode의 급작스런 변동을 수반」

  (ii) limit-load instability (without bifurcation) :
  shallow arches / spherical caps subjected to uniform external pressure
  ↳ 단일의 변형 모드가 limit load까지 이어짐.

2. Bifurcation buckling

2.1 Initially Perfect Systems

1) "real" imperfect member의 붕괴하중 예측에는 복잡함
2) 초기 imperfection을 반영하여 비선형 하중-변위 곡선의 전 영역을 고려해야 실제 부재의 failure load를 산정할 수 있음
3) simple model을 사용하여 elastic buckling 및 '분기' 이후의 postbuckling behavior 기본특성을 이해할 수 있음.

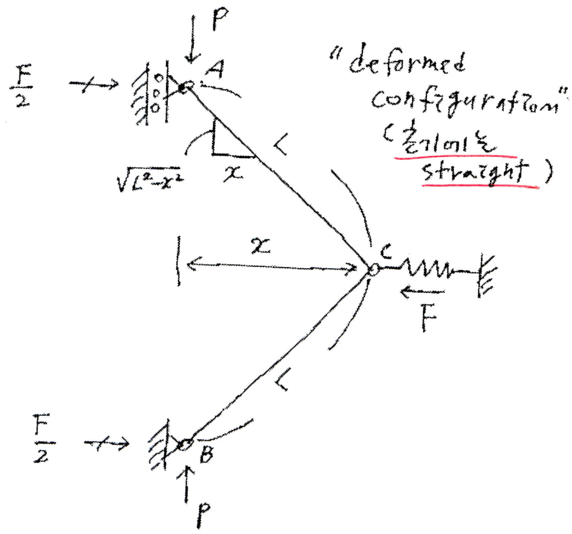
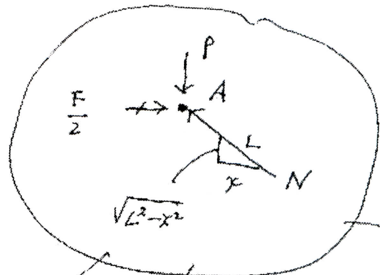


Fig.1 : Bifurcation-buckling model
(two rigid bars hinged and laterally supported by a nonlinear spring)
↑ classical model

비선형 스프링의 복원력 (횡지지력) lateral restraint

$$F = k_1 \varepsilon - k_2 \varepsilon^2 + k_3 \varepsilon^3 \quad \text{---- (1)}$$

where $\varepsilon = \dfrac{x}{L}$



$$\overset{+}{\rightarrow} \Sigma X_i = \frac{F}{2} - \left(\frac{x}{L}\right) N = 0 \;;\; F = 2\varepsilon N$$

$$+\!\downarrow \Sigma Y_i = P - \frac{\sqrt{L^2 - x^2}}{L} N = 0,$$

$$P - \sqrt{1-\varepsilon^2}\, N = 0 \;;\; N = \frac{P}{\sqrt{1-\varepsilon^2}}$$

$$F = 2\varepsilon \times \frac{P}{\sqrt{1-\varepsilon^2}}$$

따라서 deformed configuration 에서의 P − F 사이의 관계식은

$$P\varepsilon = \frac{F}{2} \times \sqrt{1-\varepsilon^2} = \frac{1}{2}(k_1 \varepsilon - k_2 \varepsilon^2 + k_3 \varepsilon^3)\underbrace{(1-\varepsilon^2)}_{\text{고차항}}{}^{1/2} \quad \text{---- (2)}$$

만일 ε 값이 매우 작다면, (2)식에서 고차항을 무시하면

$$P_{cr} \times \varepsilon = \frac{1}{2}(k_1 \varepsilon) \quad \text{or} \quad \underbrace{P_{cr} = \frac{1}{2} k_1}_{\text{critical load}} \quad \text{---- (3)}$$

← small deformation assumption → $\sin\alpha \doteq \alpha$ etc

부재의 후좌굴 거동특성은 후좌굴 강도곡선의 기울기에서, 즉 임계하중 (critical load) 근방을 고찰하여 파악 가능, 따라서 ε 값을 "small but finite" 한 것으로 가정하여 (2)식을 간략히 하면 (Koiter 1970),

→ 즉 geometric nonlinearity만 제거.

$$P\varepsilon \simeq \frac{F}{2} = \frac{1}{2}(k_1 \varepsilon - k_2 \varepsilon^2 + k_3 \varepsilon^3) \quad \text{--- (4)}$$

$$P = \frac{1}{2}(k_1 - k_2 \varepsilon + k_3 \varepsilon^2)$$

$$= \frac{1}{2} k_1 \left(1 - \frac{k_2}{k_1}\varepsilon + \frac{k_3}{k_1}\varepsilon^2\right)$$

$$P = P_{cr}(1 - a\varepsilon + b\varepsilon^2) \quad \text{---- (5)} \; *$$

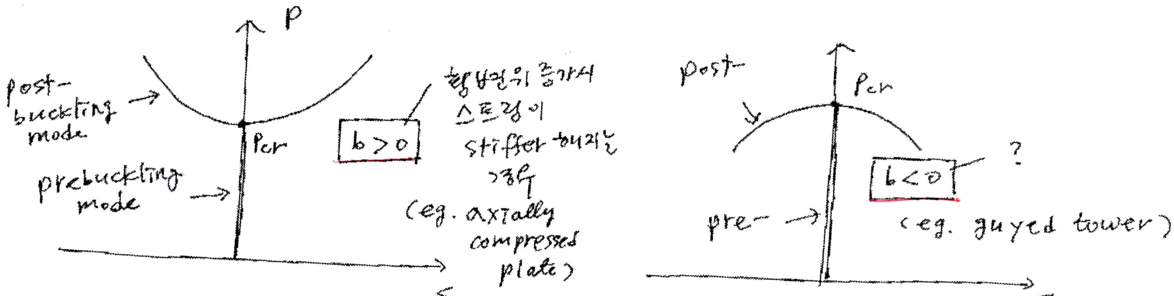
where $a = k_2 / k_1$ and $b = k_3 / k_1$

$P=P_{cr}(1-a\varepsilon+b\varepsilon^2)$ --- (5)

"좌굴 특성이 변형의 방향에 영향을 받지 않는 구조물의 경우 (5)식에서 $a=0$로 취하여 시뮬레이션 가능" ↖ 가령 압축력을 받는 기둥.

↗ "symmetric buckling"

$$P = P_{cr}(1+b\varepsilon^2) \quad \text{----} \quad (6)$$



(a) Stable postbuckling curve

(b) unstable postbuckling curve

Fig. 2 : Symmetric buckling of a bifurcation model (based on Eq. 6)

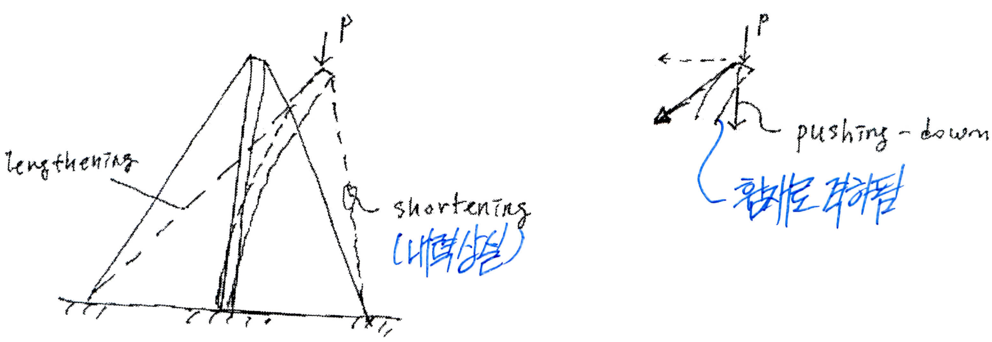


Fig. 3 : Guyed tower (unstable post-buckling example)

" 좌굴 특성이 변형의 방향에 영향을 받는 경우 "

↳ $b=0$을 (5)식에 대입하여 시뮬레이션 가능

↗ Asymmetric buckling

$$P = P_{cr} \times (1-a\varepsilon) \quad \text{----} \quad (7)$$

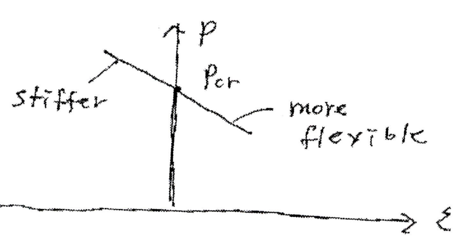
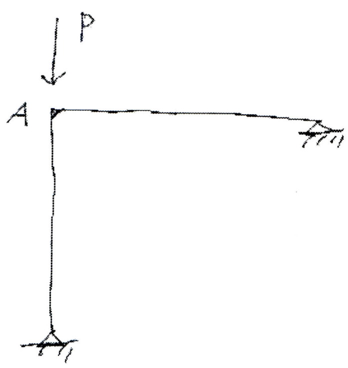
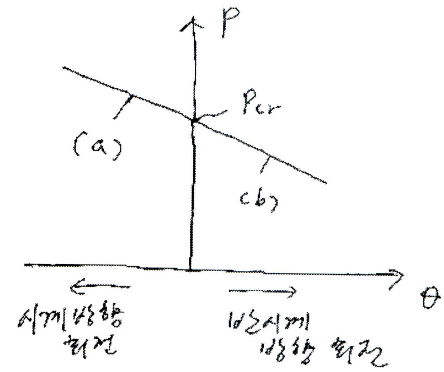
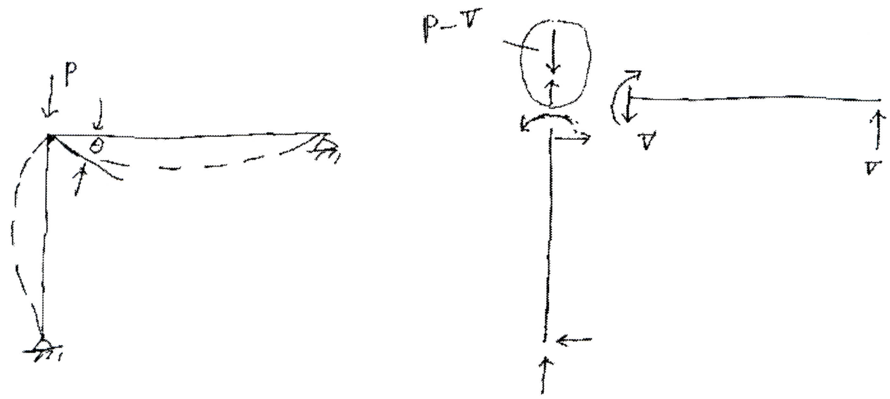


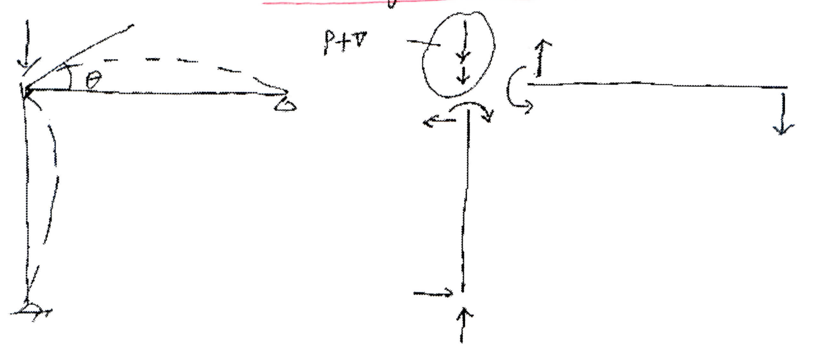
Fig 4 : Asymmetric buckling of a bifurcation model







(a) becoming stiffer (절점 A가 시계방향 회전 좌굴모드)



(b) becoming more flexible (절점 A가 반시계 방향 회전 좌굴모드)

Fig. 6 : Buckling of an L-shaped frame
(비대칭 분기좌굴의 예)

↑ 해석적으로 예견되던 거동을 실험적으로 입증 (Roorda 1965)

2.2 Initially Imperfect Systems

① The postbuckling curve of an "initially perfect" system does not by itself give sufficient information to allow one to determine when failure take place.

↳ all "real" structures ; initial imperfection & loading eccentricities

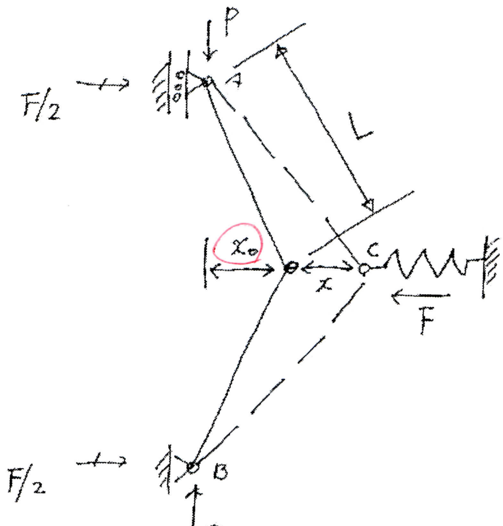
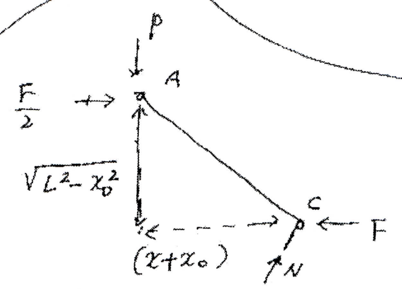


Fig. 6 : Model of an initially imperfect system



$$\sum M_{about\ c} = P(x+x_0) = \frac{F}{2}\sqrt{L^2-x_0^2}$$

$$P(\varepsilon+\varepsilon_0) = \frac{F}{2}\sqrt{1-\varepsilon_0^2}$$

$$\fallingdotseq \frac{F}{2}$$ ↖ 고차항 무시

$$P(\varepsilon+\varepsilon_0) = \frac{1}{2}(k_1\varepsilon - k_2\varepsilon^2 + k_3\varepsilon^3) \quad \text{--- (8)}$$

where $\varepsilon_0 = x_0/L$.

$$P = \frac{P_{cr}(\varepsilon - a\varepsilon^2 + b\varepsilon^3)}{\varepsilon+\varepsilon_0} \quad \text{---- (9)}$$

For "symm." behavior $a=0$ and

$$P = \frac{P_{cr}(\varepsilon + b\varepsilon^3)}{\varepsilon+\varepsilon_0} \quad \text{---- (10)}$$

??

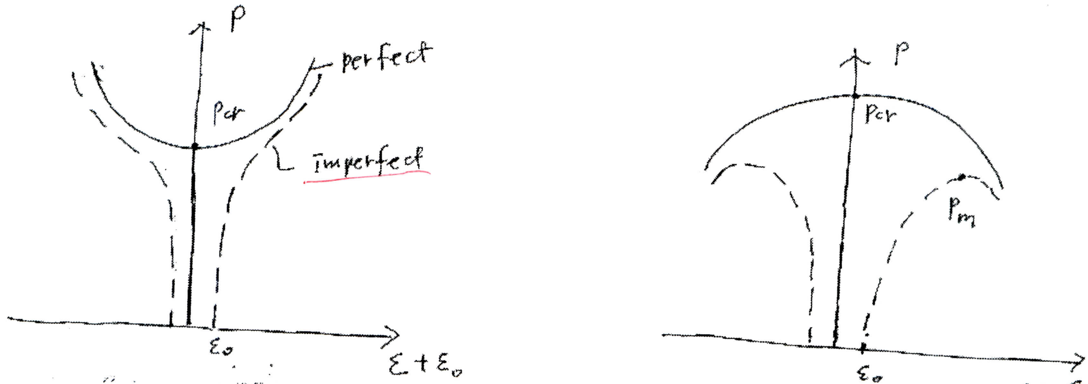
For "asymm." behavior $b=0$ and

$$P = \frac{P_{cr}(\varepsilon - a\varepsilon^2)}{\varepsilon+\varepsilon_0} \quad \text{---- (11)}$$

??

b>0 : stable post-buckling

b<0 : unstable post-buckling



(a) stable   (b) unstable

Fig. 7 : Postbuckling behavior of initially imperfect system

Notes : 1) Stable postbuckling system의 경우 '초기불완전' 값이 과도하지 않다면 불리한 영향을 크게 받지 않음 (P_{cr} 이상을 초과하는 내하력 유지)

(후좌굴 강도곡선의 기울기가 steep 할수록
P_{cr} 보다 P_y 값이 크면 클수록) postbuckling strength 증대

→ (axially compressed plates : steep slope
axially loaded columns : extremely small slope

2) 반면 unstable postbuckling system의 경우 초기불완전 값이 작더라도 심각한 강도저하가 수반된다 ; $P_m \ll P_{cr}$

← Imperfection-sensitive structures

☆ HW #

(1) Fig. 7(b)의 P_m이 다음과 같이 됨을 보여라. 즉 (10)식을 ε에 대하여 미분하여 구하면 될 것임.

$$\frac{P_m}{P_{cr}} = 1-3\left(-\frac{b}{4}\right)^{1/3} \varepsilon_0^{2/3} \quad \text{------ } (*)$$

(2) (*)식의 시사하는 바를 간략히 기술할 것.

In conclusion, the behavior of real imperfection members can be predicted from the shape of the post-buckling curve for perfect systems. Members with stable postbuckling curves will fail at loads equal to or above the critical load, whereas members with unstable postbuckling curves will fail at loads below the critical load.

2.3 Limit-load buckling ↔ a bifurcation of equilibrium model

↑ a second type of instability
(가령 shallow arch)

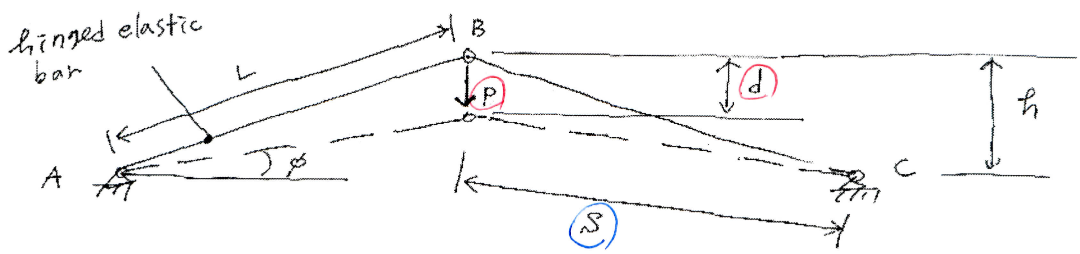
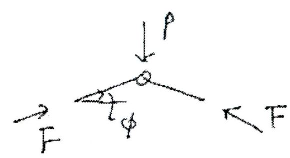


Fig. 8 : Limit-load buckling model

↳ P vs. d 관계

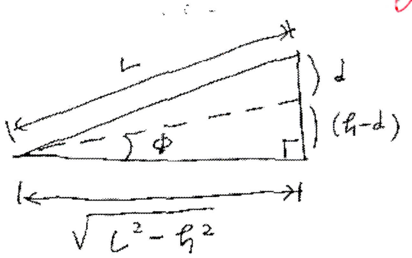


$2 \times F \sin\phi = P$, or

$$F = \frac{P}{2\sin\phi} = \frac{PS}{2(h-d)} \quad \text{---- (12)}$$

shortening

$$\Delta = \frac{F}{K\left(=\frac{AE}{L}\right)} = \frac{PS}{2K(h-d)} \quad \text{---- (13)}$$



$$S = \sqrt{\left(\sqrt{L^2-h^2}\right)^2 + (h-d)^2}$$
$$= \sqrt{L^2+d^2-2dh}$$

$\Delta = L - S$ (the axial shortening of each bar)

$$\Delta = L - S = \frac{PS}{2K(h-d)}$$

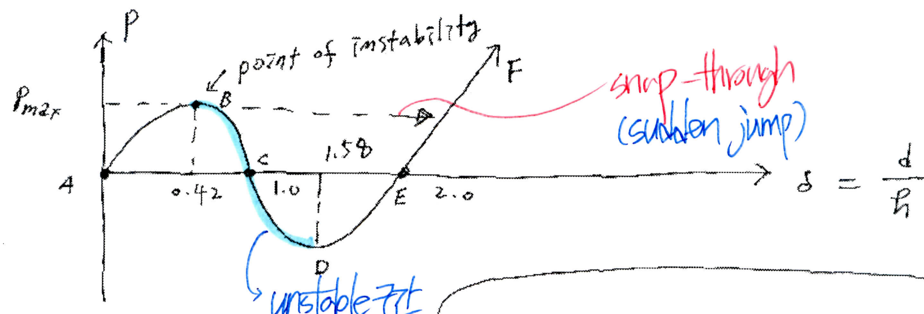
$$L - \sqrt{L^2 + d^2 - 2dh} = \frac{P\left(\sqrt{L^2+d^2-2dh}\right)}{2K(h-d)} \quad \cdots (14)$$

If $\frac{h}{L}$ is small (shallow arch), Eq. (14) reduces to

$$P = \frac{Kh^3}{L^2}\underbrace{(\delta^3 - 3\delta^2 + 2\delta)}_{\delta(\delta-1)(\delta-2)} \quad \cdots (15)$$

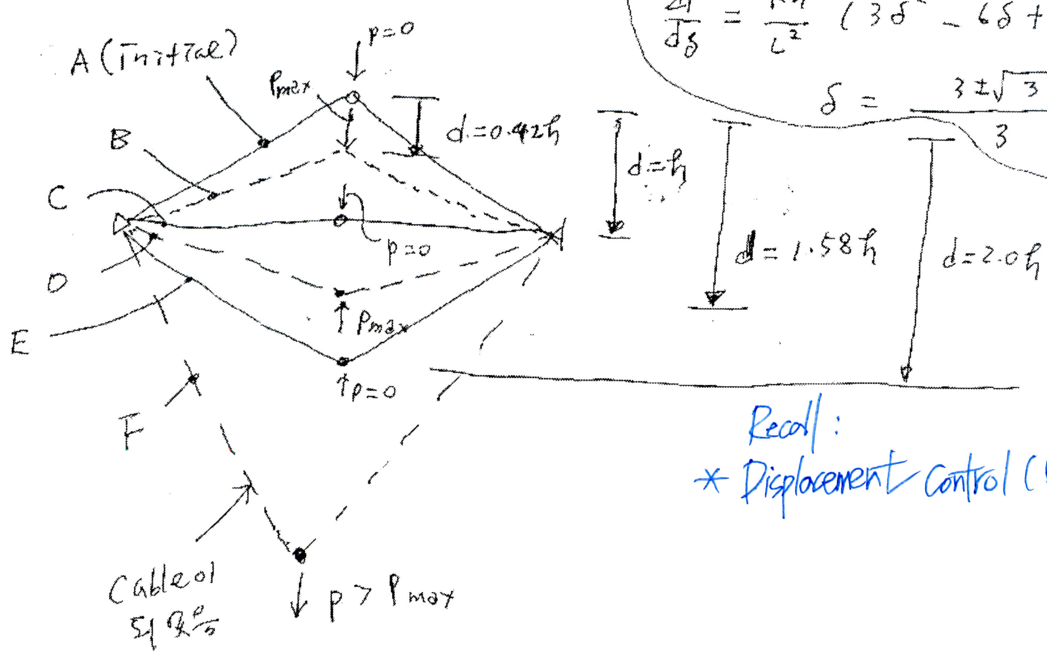
"Homework"
(유도과정)

where $\delta = \frac{d}{h}$

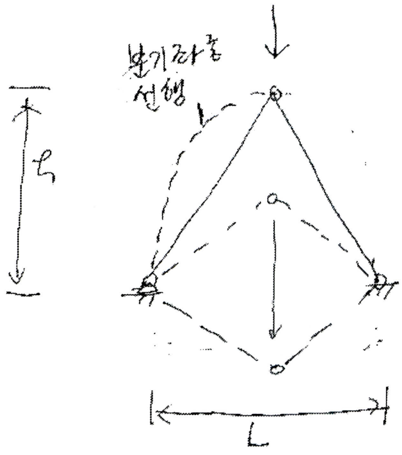


$$\frac{dP}{d\delta} = \frac{Kh^3}{L^2}(3\delta^2 - 6\delta + 2) = 0 \text{ only}$$

$$\delta = \frac{3 \pm \sqrt{3}}{3} = 0.42,\ 1.58$$

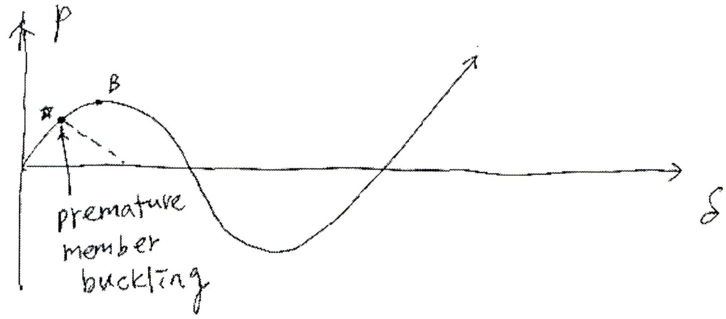


Recall:
* Displacement Control (변위제어) 사용

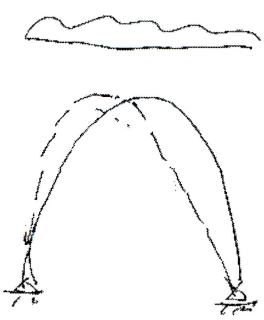


: "high" rise arch

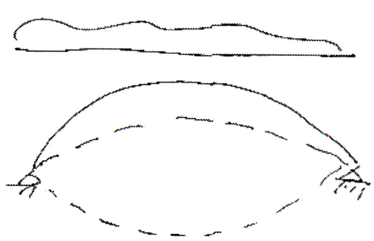
If the rise h of the model is large enough compared to L, the axial forces in the legs may reach their critical loads, causing the legs to buckle as hinged-hinged columns before the entire system reaches its limit load at point B. In that case buckling occurs as a result of a bifurcation of equilibrium at point ☆.



Note: high rise vs. shallow arches (and spherical caps) under uniform external pressure.



: Bifurcation buckling



: Limit-load buckling (symmetric)

1.1 see 교과서

1.2 Types of stability → 보충자료 참조

1.3 Methods of analysis in stability
- * Bifurcation approach
- * Energy "
- Dynamic " — beyond the scope of this book (불요) (간략한 설명, P.12 참조)

Bifurcation approach

1) Geometrically perfect system에 적용 가능

2) eigenvalue 해석을 통하여 2개 이상의 서로 다른 평형상태가 존재하는 가능점 (bifurcation point)을 찾아내는 것이 핵심.

3) procedure

i) 분기하중에서 구조시스템이 가질 수 있는 모든 평형상태 파악 (buckling shape 고려, 종종 自明)

ii) Deformed (Buckled) configuration에 대하여 평형조건식 적용

iii) 임계하중에서 구조물의 접선강성이 zero가 됨을 반영하여, 접선강성 행렬의 행렬값을 zero가 되도록 하여 system의 critical condition을 찾아냄 (the lowest eigenvalue is the critical load of the system)

(예제로서 가장 잘 설명됨)

☆ geometrically imperfect systems의 불안정 해석

하중을 가하는 즉시 변형이 발생함

→ The problem of a load-deflection rather than a bifurcation problem

→ 일종의 gradual buckling process로 치환하여 해석 가능

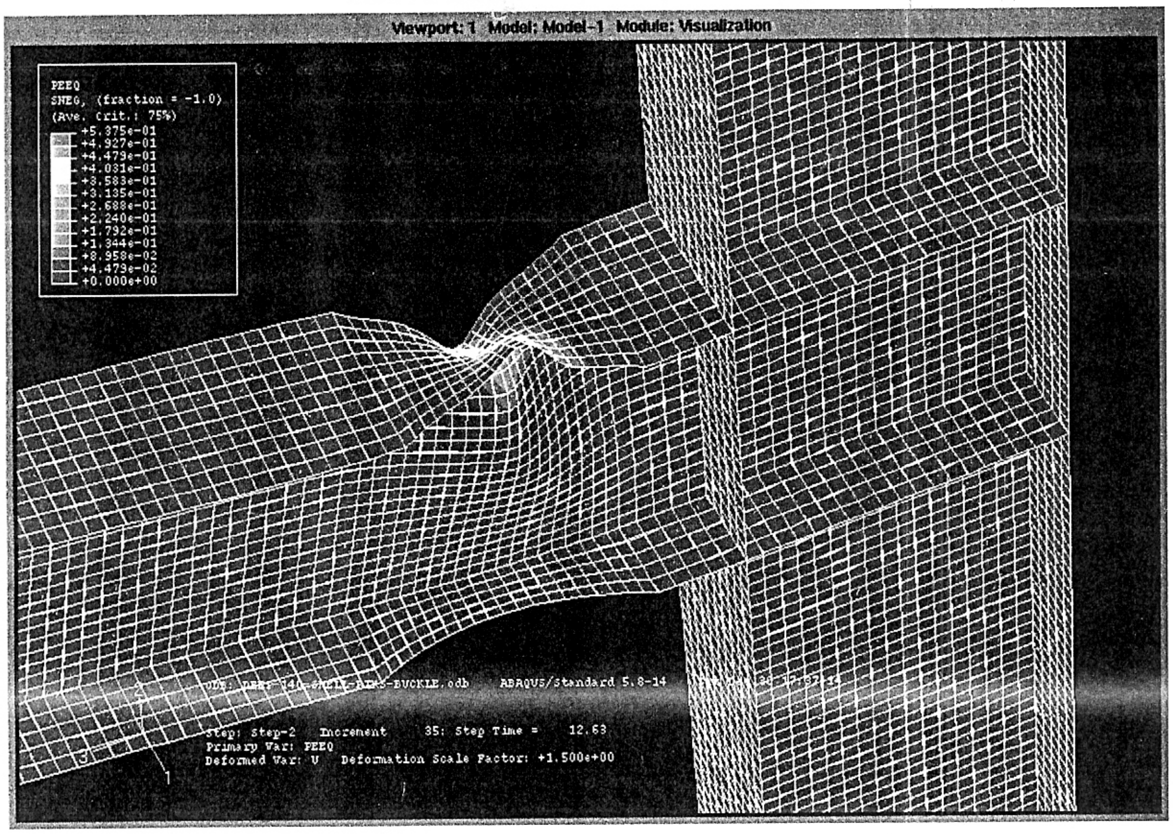
ex) Initial imperfection 도입 후 재료적/기하학적 비선형 해석

↳ C.H. Lee의 steel moment connection의 FEM cyclic analysis study

* "Real" plastic Hinge *

①-1

DEEP-140-BUCKLE-PEEQ-CASE4 (919x651x16 png)

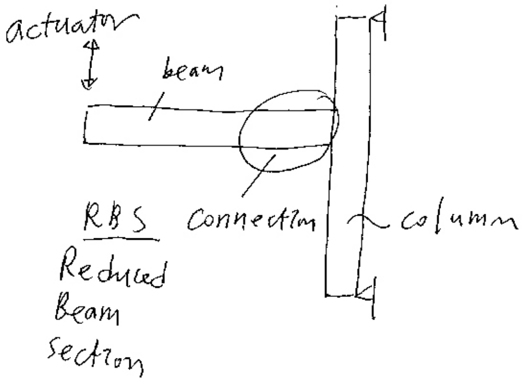


"Numerical Simulation of Deep Column Reduced Beam Section Connection"

- WLB (Web local buckling)
- FLB (Flange " ")
- LTB (Lateral Torsional buckling)

본 경우는 소성힌지가 형성된 이후에 발생한 FLB, WLB, LTB를 simulation 한 것임!,

(즉 compact section 이라해도 대변형 영역에서는 결국 local, global buckling이 발생함)



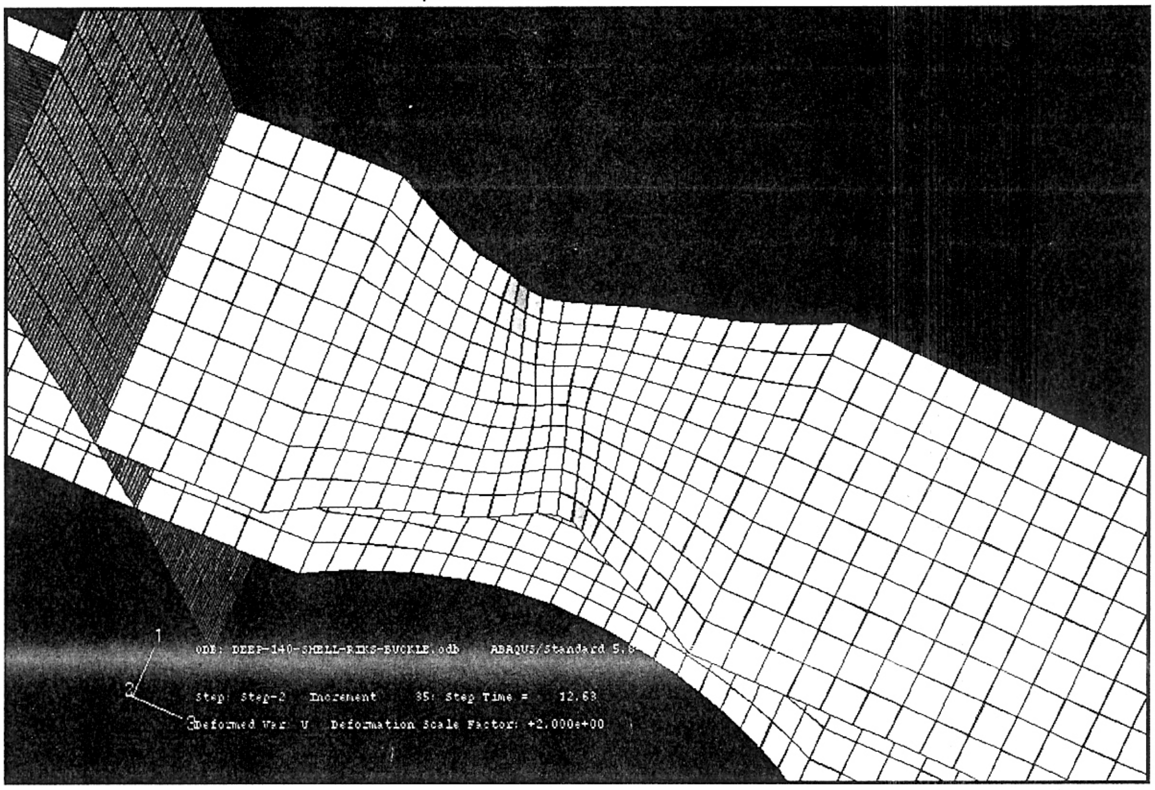
다음쪽 참조

: Initial imperfection 도입
plus geometric/material nonlinear postbuckling analysis
(RIKS algorithm)

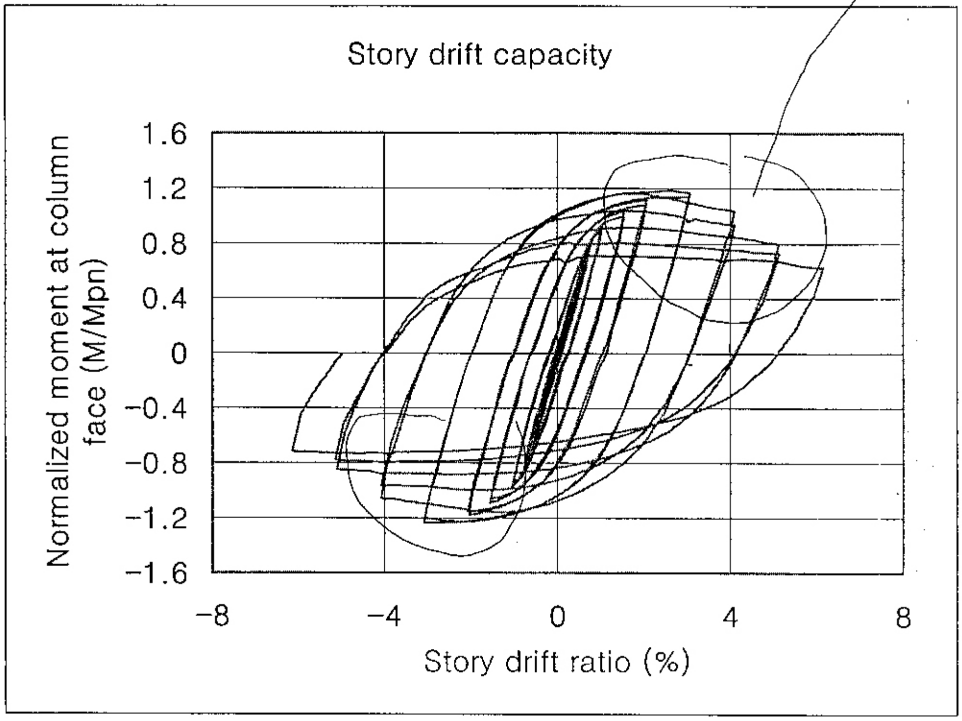
①-2

DEEP-140-BUCKLE-LTB1-CASE4 (919x651x256 png)

Viewport: 1 Model: Model-1 Module: Visualization

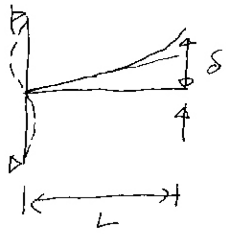


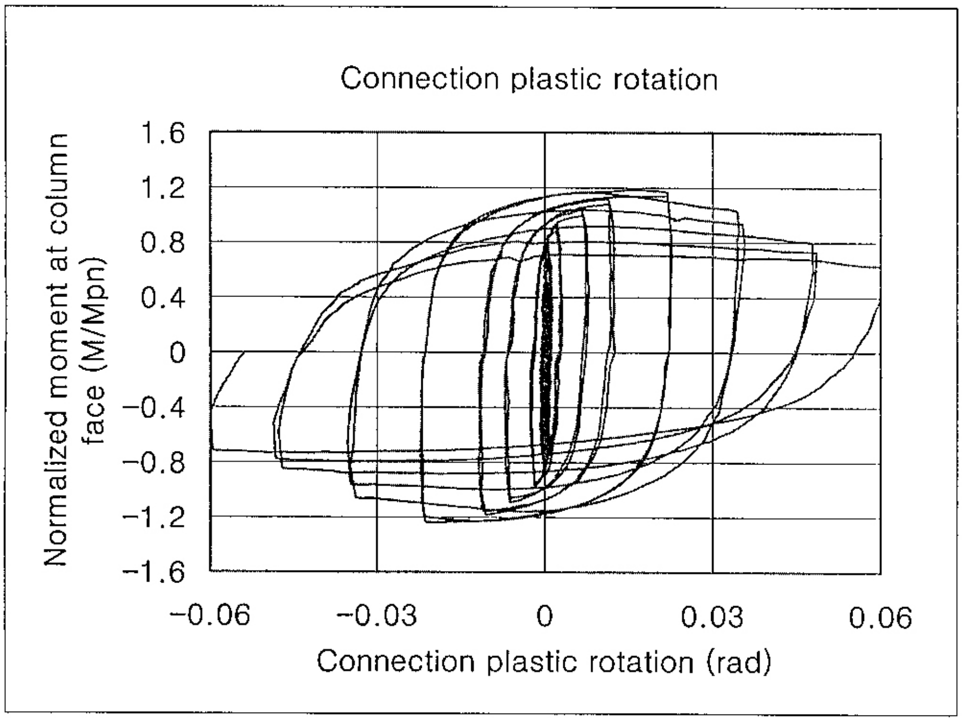
①-3



strength degradation are a result of WLB LTB, FLB. and

$\frac{\delta}{L}$ = equivalent story Drift Ratio





Energy approach (for elastic system under conservative forces)

① using "the principle of stationary value of total potential energy"

$\downarrow\uparrow$

Equilibrium condition

② procedure

i) Express the total potential energy as a function of a set of generalized displacements and the external applied forces

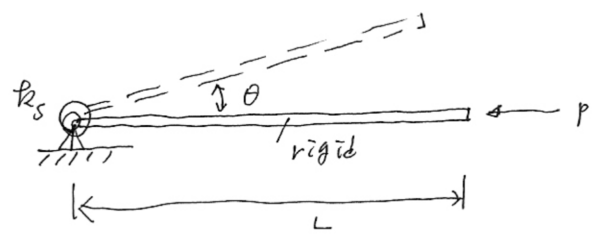
ii) By setting the first derivative of the total potential energy function w.r.t. each generalized displacement equal to zero, identify the equilibrium conditions of the system.

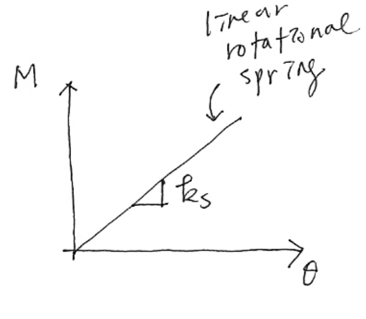
iii) Determine whether the equilibrium is stable or unstable by investigating higher order derivatives of the total potential energy function.

(예로서 가장 잘 설명됨)

1.4 Illustrative Examples — "small" deflection analysis

1.4.1 Rigid bar supported by a rotational spring

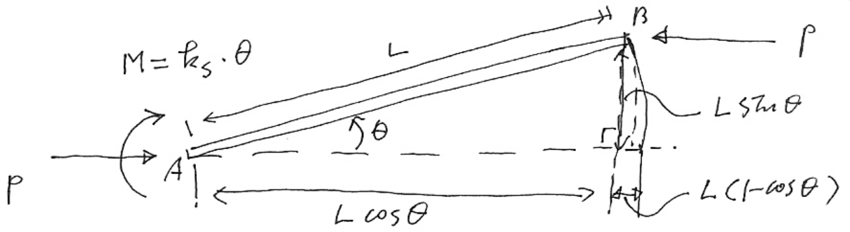




∴ 1dof system (1가지 좌굴모드만 있음)

Bifurcation approach solution

「만족되어야 하는 평형조건식
$\Sigma X = 0$ —①
$\Sigma Y = 0$ —②
$\Sigma M = 0$ —③」



: Deformed (or buckled) configuration

free body diagram이 완벽하게 그려져야」

따라서, 모멘트 평형 조건식,

$$\overset{+\circlearrowleft}{\Sigma M_{about\ A}} = k_s \cdot \theta - P \times (L \sin\theta)$$

$$\doteq k_s \cdot \theta - PL \cdot \theta = 0 \qquad (\text{for small } \theta,\ \sin\theta \doteq \theta)$$

$$(k_s - PL) \times \theta = 0$$ ← 가장 간단한 eigenvalue problem으로 볼 수 있음

For a nontrivial solution,

$k_s - PL = 0$ 이 되어야 $\left(\theta = \frac{0}{0} \leftarrow \text{不定型} \right)$

$$\therefore P = P_{cr} = \frac{K_s}{L}$$

Note: ① $\underset{(\text{유효강성})}{k_{eff}} = k_s - PL = \underset{\text{original stiffness}}{k_s} - \underset{\text{geometric stiffness}}{k_G}$ → zero 일때 좌굴 발생

② $P = P_{cr} = \frac{K_s}{L}$ 에서는 original horizontal position 및 slightly deflected ~~shape~~ position 모두에 대한 평형상태가 가능할 수 있다.

Energy approach solution (앞쪽 22p 참조)

Deformed 상태에서의 total potential energy는

$$\Pi = U + V$$

$$\left(U = \frac{1}{2} k_s \theta^2 \; ; \quad V = \underbrace{-PL(1-\cos\theta)}_{\text{외력 P의 "위치에너지" 감소분}} \right.$$

$$\therefore \Pi = \frac{1}{2} k_s \theta^2 - PL(1-\cos\theta) \quad \text{--- (1.4.5)}$$

1 dof - discrete system 이므로 functional이 아니라 θ의 function이 되었음

For equilibrium,

$$\frac{d\Pi}{d\theta} = 0 \quad (\text{필요조건}) \quad \leftarrow \text{first variation에 상응}$$

$$\text{or } \frac{d\Pi}{d\theta} = k_s \theta - PL(0 + \sin\theta) = 0$$

For small θ, $\sin\theta \doteq \theta$

$$\text{"} k_s \theta - PL\theta = 0 \text{"}$$

모멘트 평형조건식을 적용한 경우와 동일!

$$\therefore P_{cr} = \frac{K_s}{L} \quad \text{--- (1.4.8) 식}$$

Note: Energy approach의 경우, 임계하중 P_{cr} 뿐만 아니라 평형상태의 특성 (안정한 평형 or 불안정한 평형)도 파악할 수 있음.

"Academc interest"

원래의 flat position ($\theta = 0$)의 평형상태의 안정, 불안정 조건을 판별하여 보자 (theory of extremum 보충 자료 참조)

$$\underbrace{\Pi(\theta)}_{\text{근방}} - \underbrace{\Pi(0)}_{\text{flat position}} = \left.\frac{d\Pi}{d\theta}\right|_{\theta=0} \times \theta + \frac{1}{2!} \left.\frac{d^2\Pi}{d\theta^2}\right|_{\theta=0} \times \theta^2 + O(\theta^3)$$

note:
$$\left.\begin{aligned} \frac{d\Pi}{d\theta} &= k_s \theta - PL\sin\theta \doteq k_s\theta - PL\theta \\ \frac{d^2\Pi}{d\theta^2} &= k_s - PL\cos\theta \doteq k_s - PL \end{aligned}\right) \text{ for "small" } \theta$$

or

$$\underbrace{\pi(\theta) - \pi(0)}_{\text{증분}} = \underbrace{\cancel{(k_s\theta - PL\theta)\Big|_{\theta=0} \times \theta}}_{\text{zero}} + \underbrace{\frac{1}{2!}(k_s - PL) \times \theta^2}_{\text{“dominant term”}} + O(\theta^3)$$

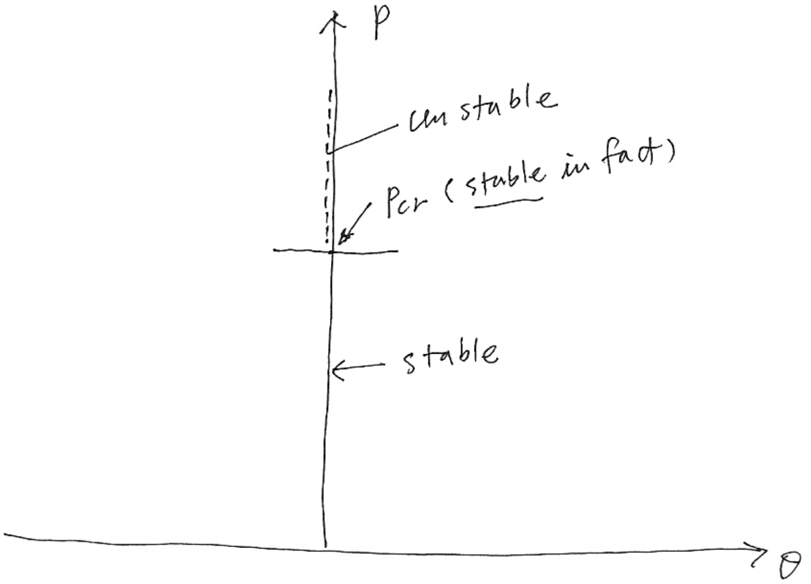
따라서

- $P < \frac{k_s}{L}$ 이면, $\pi(\theta) - \pi(0) > 0$; $\pi(0)$는 local min. (stable equilibrium)
- $P > \frac{k_s}{L}$ 이면, $\pi(\theta) - \pi(0) < 0$; $\pi(0)$는 local max. (unstable equilibrium)
- ※ $P = P_{cr}$ 이면, $\left(= \frac{k_s}{L}\right)$ "2차도함수"도 zero가 되어 안정/불안정에 대한 결론을 내릴 수 없음 (small deflection 해석이므로)

  ↓

  「대변위해석을 통하여 $P = P_{cr}$ 점이 stable 함을 보일 수 있음」

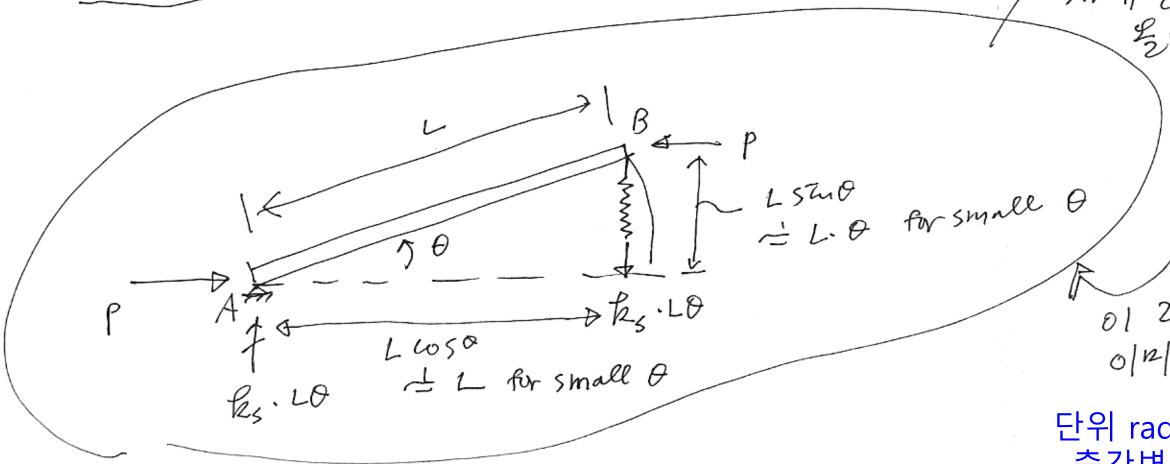
  ↳ 1.5절 참고



*골조의 sidesway Buckling 강도 산정에 활용가능)

1.4.2 Rigid bar supported by a translational spring
(예시 1 dof model)

Bifurcation approach



우선 자유물체도를 올바르게 그려야 ($\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma M = 0$)

이 2가지가 이미 반영되어 있음.

단위 radian의 층간변형각 유발횡력!

$$\overset{+\curvearrowright}{\Sigma M_{about\ A}} = (k_s \cdot L\theta)L - PL\theta = 0$$

$$(k_s L - P)\times\theta = 0$$

For a nontrivial solution, $P = P_{cr} = k_s L$ --- (1.4.11)

Energy approach

$$\Pi = U + V$$

$$= \frac{1}{2}k_s(L\theta)^2 - P(L - L\cos\theta)$$

$\sin\theta \doteq \theta$ for small θ

$$\frac{d\pi}{d\theta} = k_s(L\theta)\cdot L - PL(\sin\theta) = 0 \text{ 에서}$$

$$(k_s L - P)\times\theta = 0 \rightarrow \therefore P_{cr} = k_s L$$

note: flat position ($\theta = 0$)에서의 평형상태의 안정/불안정 검토

$$\frac{d\pi}{d\theta} = k_s L^2\theta - PL\sin\theta = k_s L^2\theta - PL\cdot\theta$$) for small θ

$$\frac{d^2\pi}{d\theta^2} = k_s L^2 - PL\cos\theta = k_s L^2 - PL$$

Dominant term

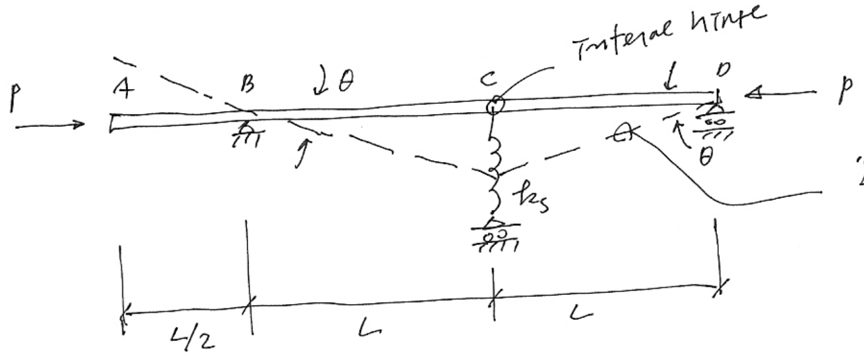
$$\underbrace{\pi(\theta)}_{\text{임의}} - \underbrace{\pi(0)}_{\text{flat position}} = \underbrace{(k_s L^2\theta - PL\theta)\Big|_{\theta=0}}_{\text{zero}} \times\theta + \frac{L}{2!}(k_s L - P)\Big|_{\theta=0}\times\theta^2 + O(\theta^3)$$

↳ $P < k_s L$ 이면, $\pi(\theta) - \pi(0) > 0$ local min ↳ stable

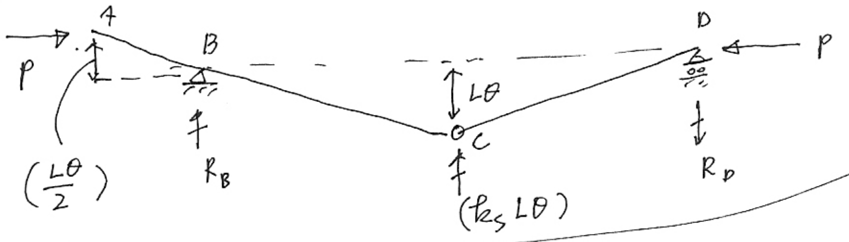
$P > k_s L$ 이면 → unstable

1.4.3 Two-bar system

↳ two-bar system 이지만 역시 1 dof 계열에 속함



"유일한" 가능 좌굴 모드



deformed shape에 대한

먼저 자유물체도를 완벽히 그려야 (정정구조물임)

↓

R_B, R_D를 우선 P로써 표시해야

overall equilibrium condition을 적용한 것에 지나지 않음

$\Sigma X = 0$, o.k.

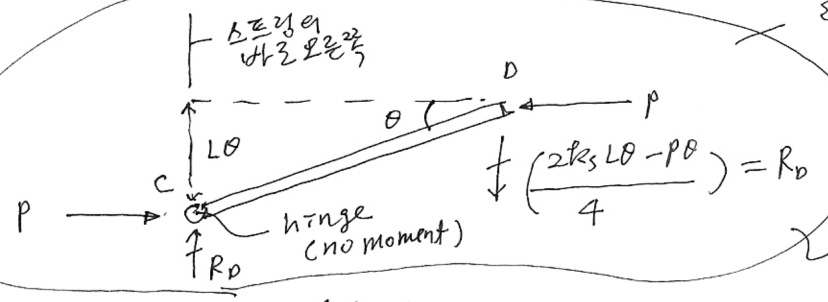
$\overset{+\uparrow}{\Sigma} Y_C = R_B + (k_s L\theta) - R_D = 0$

$\overset{+\circlearrowright}{\Sigma} M_{about\ B} = P\left(\frac{L\theta}{2}\right) - (k_s L\theta)(L) + R_D(2.0L) = 0$

$\therefore R_D = \dfrac{(k_s L\theta)(\cancel{L}) - P(\frac{\theta}{2})\cancel{L}}{2.0\cancel{L}} = \dfrac{2k_s L\theta - P\theta}{4}$

정리하면 $R_B = R_D - (k_s L\theta) = \boxed{}$

「Bifurcation approach」



$\Sigma X_C = 0$
$\Sigma Y_C = 0$ 이 만족되고 있음

HW#
「부재 AC를 택하여 P_{cr}을 산정하여 볼 것」

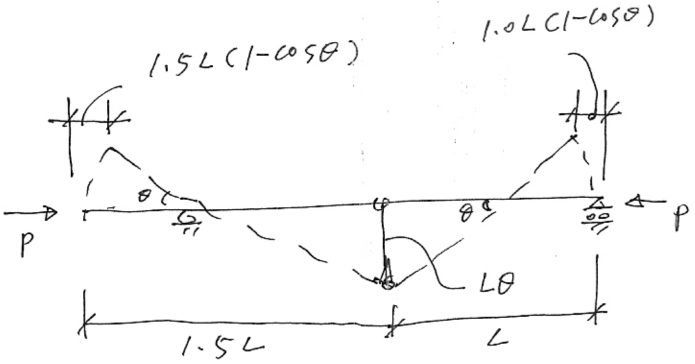
∴ 부재 CD의 자유물체도

$$\overset{+\circlearrowright}{\Sigma} M_{about\ C} = R_D \times L - PL\theta = \frac{2k_s L^2\theta - PL\theta}{4} - PL\theta = 0$$

$$\frac{1}{2}k_s L\theta - \frac{5}{4}P\theta = \left(\frac{1}{2}k_s L - \frac{5}{4}P\right)\times\theta = 0$$

$$\therefore P = P_{cr} = \frac{1}{2}k_s L \times \frac{4}{5} = \frac{2}{5}k_s L$$

「Energy approach」



$$\Pi = U + V$$

$$= \frac{1}{2} k_s (L\theta)^2 - P(2.5L)(1-\cos\theta)$$

For equilibrium,

$$\frac{d\Pi}{d\theta} = k_s(L^2\theta) - 2.5PL(\sin\theta)$$

$$= (k_s L^2 - 2.5PL) \times \theta = 0 \quad \text{(for small } \theta\text{)}$$

$$\therefore P = P_{cr} = \frac{k_s L}{2.5} = \underline{\underline{\frac{2}{5} k_s L}}$$

* $\theta = 0$ position 에서의 평형조건의 안정/불안정 판정

↳ $$\frac{d^2\Pi}{d\theta^2} = k_s L^2 - 2.5PL\cos\theta$$

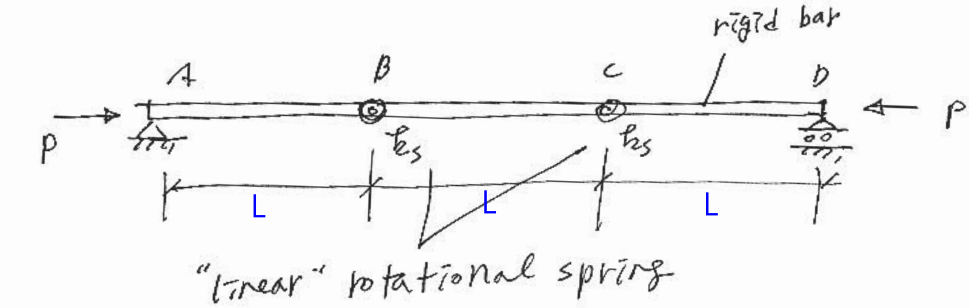
$$\doteq k_s L^2 - 2.5PL \quad \text{(for small } \theta\text{)}$$

For $P < P_{cr} \rightarrow \frac{d^2\Pi}{d\theta^2} > 0 \rightarrow \theta = 0$은 local min. point (stable)

For $P > P_{cr} \rightarrow \frac{d^2\Pi}{d\theta^2} < 0 \rightarrow \theta = 0$은 local max. point (unstable)

1.4.4 Three-bar system
(보충자료 참고)
↳ 이미 upload

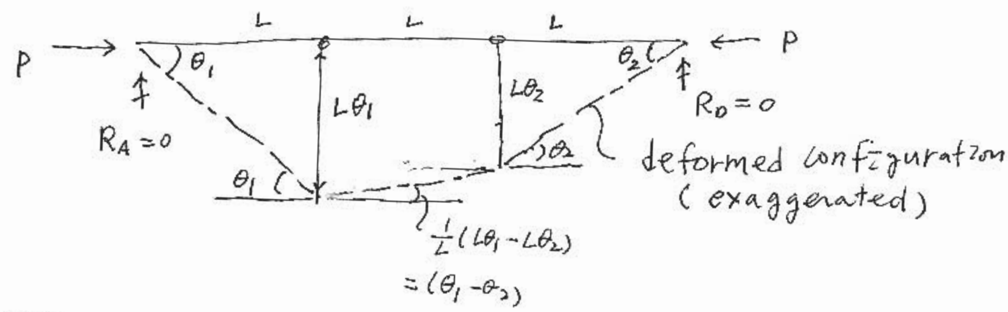
1.4.4 Three-Bar System



Bifurcation approach ← slightly deformed configuration에 대한 평형조건식을 우선 세워야 함.

① 2dof system ; 절점 A, D의 회전각으로서 모든 kinematics 표시 가능

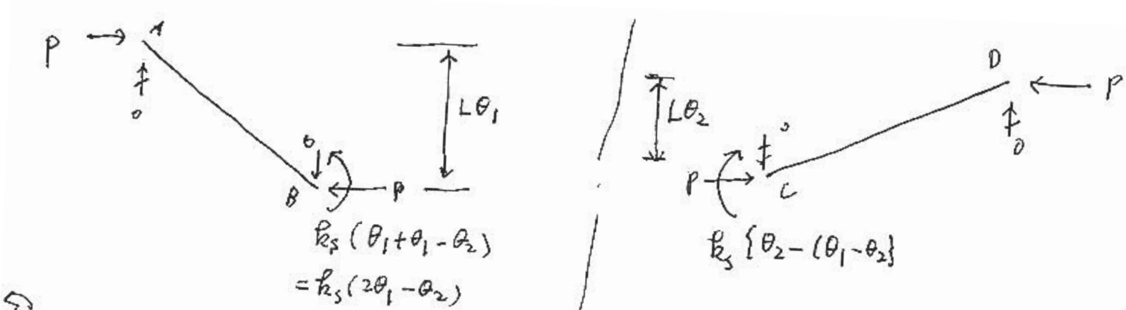
→ 2개의 평형조건식 필요 (미지수는 θ_1, θ_2 만 존재해야)



② 구조물 전체에 대한 평형조건식을 ($\Sigma M = 0$, $\Sigma Y = 0$) 적용하여,

$$R_A = 0 \;;\; R_D = 0$$

③ 절점 B 및 C를 'cut'하여 얻어진 2개의 free-body



$\Sigma M_B = 0$;

$$k_s(2\theta_1 - \theta_2) - PL\theta_1 = 0 \quad \text{--- } (*)$$

$\Sigma M_C = 0$;

$$k_s(2\theta_2 - \theta_1) - PL\theta_2 = 0 \quad \text{--- } (**)$$

In matrix form,

$$\begin{bmatrix} 2k_s - PL & -k_s \\ -k_s & 2k_s - PL \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution (by Cramer's rule),

$$\det\begin{bmatrix} 2k_s - pL & -k_s \\ -k_s & 2k_s - pL \end{bmatrix} = 0$$

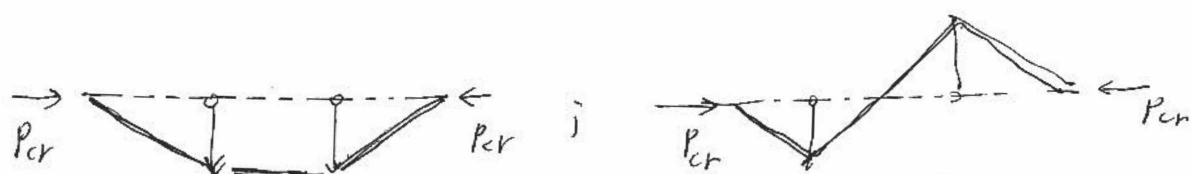
$$(2k_s - pL)^2 - k_s^2 = 0 \; ; \quad 2k_s - pL = \pm k_s$$

특성방정식

$$\therefore \left.\begin{matrix} p = \dfrac{k_s}{L} \\ p = \dfrac{3k_s}{L} \end{matrix}\right\} \text{ eigenvalues}$$

$\therefore$ eigenvectors

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



$$P_{cr} = \frac{k_s}{L} \qquad ; \qquad P_{cr} = \frac{3k_s}{L}$$

(The lowest value is the critical load of the system)

Effect of bracing

Brace

2차 mode를 강제할 것임.
(좌굴강도 3배로 증대)

Energy Approach

먼저 total potential energy $\Pi = U + V$ 를 구하면,

$$U = \frac{1}{2}k_s(2\theta_1 - \theta_2)^2 + \frac{1}{2}k_s(2\theta_2 - \theta_1)^2$$

$$V = -pL\left[(1-\cos\theta_1) + \{1-\cos(\theta_1-\theta_2)\} + (1-\cos\theta_2)\right]$$

$$\Pi = U + V = \frac{1}{2}k_s(2\theta_1-\theta_2)^2 + \frac{1}{2}k_s(2\theta_2-\theta_1)^2$$
$$-pL\{3-\cos\theta_1 - \cos\theta_2 - \cos(\theta_1-\theta_2)\}$$

For equilibrium, Π must be in stationary. In mathematical terms, this requires

$$\text{필요조건} \rightarrow \left(\begin{aligned} \frac{\partial \Pi}{\partial \theta_1} &= 2k_s(\cancel{2\theta_1} - \cancel{\theta_2}) - k_s(\cancel{2\theta_2} - \cancel{\theta_1}) - PL[\underbrace{\sin\theta_1 + \sin(\theta_1 - \theta_2)}_{2\theta_1 - \theta_2}] = 0 \\ \frac{\partial \Pi}{\partial \theta_2} &= -k_s(2\theta_1 - \theta_2) + 2k_s(2\theta_2 - \theta_1) - PL[\underbrace{\sin\theta_2 - \sin(\theta_1 - \theta_2)}_{2\theta_2 - \theta_1}] = 0 \end{aligned} \right\} \cdots (*)$$

Upon simplification and using small angle approximation,

$$\{ \sin\theta_1 \simeq \theta_1 \ ; \ \sin(\theta_1 - \theta_2) \simeq \theta_1 - \theta_2$$

$$\begin{bmatrix} 5k_s - 2PL & -4k_s + PL \\ -4k_s + PL & 5k_s - 2PL \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For nontrivial solution,

$$\det \left| \quad \right| = 0,$$

$$P_1 = \frac{k_s}{L} \ ; \ P_2 = \frac{3k_s}{L}$$

$$\therefore P_{cr} = P_1 = \frac{k_s}{L}$$

Academic interest.

(*) To study the nature of equilibrium for the system in its undeflected position ($\theta_1 = \theta_2 = 0$), we need to investigate higher order derivatives of Π.

$$(**) \cdots \left\{ \begin{aligned} \frac{\partial^2 \Pi}{\partial \theta_1^2} &= 5k_s - PL[\cos\theta_1 + \cos(\theta_1 - \theta_2)] \simeq 5k_s - 2PL \ \text{(for small angle approx.)} \\ \frac{\partial^2 \Pi}{\partial \theta_2^2} &= \qquad \simeq 5k_s - 2PL \ (\quad " \quad) \\ \frac{\partial^2 \Pi}{\partial \theta_1 \partial \theta_2} &= \qquad \simeq -4k_s + PL \ (\quad " \quad) \end{aligned} \right.$$

" 이상은 교과서의 전개와 동일 "

Note : 다변수 함수의 Taylor 급수전개 ↙ "formalism"

$$f(\underline{P}+\underline{H}) = f(\underline{P}) + \frac{(\underline{H}\cdot\underline{\nabla})f(\underline{P})}{1!} + \frac{(\underline{H}\cdot\underline{\nabla})^2 f(\underline{P})}{2!} + \cdots$$

가령, 2변수 함수의 경우,

$$\underline{\nabla} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \leftarrow \text{편미분 연산자}$$

$$\underline{P} = (a, b), \quad \underline{H}\cdot\underline{\nabla} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

$$\underline{H} = (x, y), \quad (\underline{H}\cdot\underline{\nabla})^2 = x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial}{\partial x\partial y}\bigg|_{\substack{x=a\\x=b}} + y^2\frac{\partial^2}{\partial y^2}$$

$$\therefore f(\underline{P}+\underline{H}) = f(a+x, b+y)$$

$$= f(a,b) + \left\{x\frac{\partial f(a,b)}{\partial x} + y\frac{\partial f(a,b)}{\partial y}\right\}$$

$$+ \frac{1}{2!}\left[x^2\frac{\partial^2 f}{\partial x^2}\bigg|_{\substack{x=a\\y=b}} + 2xy\frac{\partial^2 f(a,b)}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2}\bigg|_{\substack{x=a\\y=b}}\right] + \cdots$$

Back to the problem

$\underline{P} = (0, 0)$, $\underline{H} = (\theta_1, \theta_2)$ 에 상당

$$\pi(\theta_1,\theta_2) = \pi(0,0) + \left\{\frac{\partial\pi(0,0)}{\partial\theta_1}\times\theta_1 + \frac{\partial\pi(0,0)}{\partial\theta_2}\times\theta_2\right\}$$

$$+ \underbrace{\frac{1}{2!}\left\{\frac{\partial^2\pi(0,0)}{\partial\theta_1^2}\times\theta_1^2 + 2\theta_1\theta_2\cdot\frac{\partial^2\pi(0,0)}{\partial\theta_1\partial\theta_2} + \frac{\partial^2\pi(0,0)}{\partial\theta_2^2}\right\}}_{O(\theta^2)} + O(\theta^3)$$

$$\underbrace{\pi(\theta_1,\theta_2)}_{\text{varied path}} - \underbrace{\pi(0,0)}_{\text{extremum path}} = \left\{\frac{\partial\pi(0,0)}{\partial\theta_1}\times\theta_1 + \frac{\partial\pi(0,0)}{\partial\theta_2}\times\theta_2\right\} \rightarrow \text{zero 임, ③쪽 (*)식 참고}$$

$$+ \underbrace{O(\theta^2)}_{\text{dominant term}} + O(\theta^3)$$

따라서 $\pi(0,0)$이 stable equilibrium state 이 되기 위해서는 π 값이 local minimum이 되어야 하므로 제 2차 도함수로 구성된 항이 <u>(0,0)의 근방에서</u> '+' 값을 가져야 한다, 즉 $O(\theta^2) > 0$ 이어야 한다.

(3)쪽의 (**)을 그냥 대입해서는 θ_1, θ_2의 존재로 인해 의미있는 조건을 도출할 수 없음

5

$$\boxed{\frac{\partial^2 \pi(0,0)}{\partial \theta_1^2} \times \theta_1^2 + \frac{\partial^2 \pi(0,0)}{\partial \theta_2^2} \times \theta_2^2 + \frac{\partial^2 \pi(0,0)}{\partial \theta_1 \partial \theta_2}(2\theta_1\theta_2) > 0} \quad \cdots (***)$$

(i) $\theta_2 = 0$ & $\theta_1 \neq 0$ 인때 (perturb θ_1 only)

$$\frac{\partial^2 \pi(0,0)}{\partial \theta_1^2} > 0$$

(3)쪽의 (**)식 사용

$$(5k_s - 2PL) > 0 \quad --- (a)$$

(ii) $\theta_2 \neq 0$ and $\theta_1 = 0$ 인때 (perturb θ_2 only)

$$\frac{\partial^2 \pi(0,0)}{\partial \theta_2^2} > 0, \text{ 따라서}$$

$$(5k_s - 2PL) > 0 \quad ---- (b)$$

(iii) $\theta_1 \neq 0$ & $\theta_2 \neq 0$ 인때 (perturb θ_1 and θ_2 simultaneously)

(***) 식에서

산술평균

$$\left[\frac{\frac{\partial^2 \pi(0,0)}{\partial \theta_1^2}\theta_1^2 + \frac{\partial^2 \pi(0,0)}{\partial \theta_2^2}\theta_2^2}{2}\right] + \frac{\partial^2 \pi(0,0)}{\partial \theta_1 \partial \theta_2}(\theta_1\theta_2)$$

$$\geq \sqrt{\frac{\partial^2 \pi(0,0)}{\partial \theta_1^2}\theta_1^2 \times \frac{\partial^2 \pi(0,0)}{\partial \theta_2^2}\theta_2^2} + \frac{\partial^2 \pi(0,0)}{\partial \theta_1 \partial \theta_2}(\theta_1\theta_2) > 0$$

기하평균

$$\sqrt{\quad '' \quad} > -\frac{\partial^2 \pi(0,0)}{\partial \theta_1 \partial \theta_2}(\theta_1\theta_2)$$

$$\left\{\frac{\partial^2 \pi(0,0)}{\partial \theta_1^2}\right\} \times \left\{\frac{\partial^2 \pi(0,0)}{\partial \theta_2^2}\right\} \cancel{(\theta_1^2 \cdot \theta_2^2)} > \left[\frac{\partial^2 \pi(0,0)}{\partial \theta_1 \partial \theta_2}\right]^2 \cancel{(\theta_1^2\theta_2^2)}$$

$$\text{or} \quad \left[\frac{\partial^2 \pi(0,0)}{\partial \theta_1^2}\right]\left[\frac{\partial^2 \pi(0,0)}{\partial \theta_2^2}\right] > \left[\frac{\partial^2 \pi(0,0)}{\partial \theta_1 \partial \theta_2}\right]^2, \text{ 따라서}$$

$$\text{or} \quad (5k_s - 2PL)^2 - (4k_s - PL)^2 > 0$$

$$(9k_s - 3PL)(k_s - PL) > 0 \quad ---- (c)$$

$(5k_s - 2pL) > 0$ --- (a)

$(\frac{5}{?}k_s - 2pL) > 0$ -- (b)

$(k_s - pL)(3k_s - pL) > 0$ -- (c)

이 3개의 조건을 모두 만족시키면 undeflected position은 $(\theta_1 = \theta_2 = 0)$ stable equilibrium 상태에 있다.

$p < \frac{k_s}{L}$ 인때 : 위의 (a), (b), (c) 3개 조건 모두 만족 → stable equilibrium 상태에 있다.

$p > \frac{k_s}{L}$ 인때 : (C)의 조건에 위배되므로 $\Pi(0,0)$은 local maximum 이 되어 unstable equilibrium state에 있게 됨.

$O(\theta^2)$ 항이 "-"가 되므로

* For more rigorious and general mathematical treatment, refer to "Elastic stability of structures", pp. 8~14 by G. J. Simitses (1976)

1장 보충자료 : 1.5절 ~ 1.6절

1.5 Illustrative examples – large deflection analysis

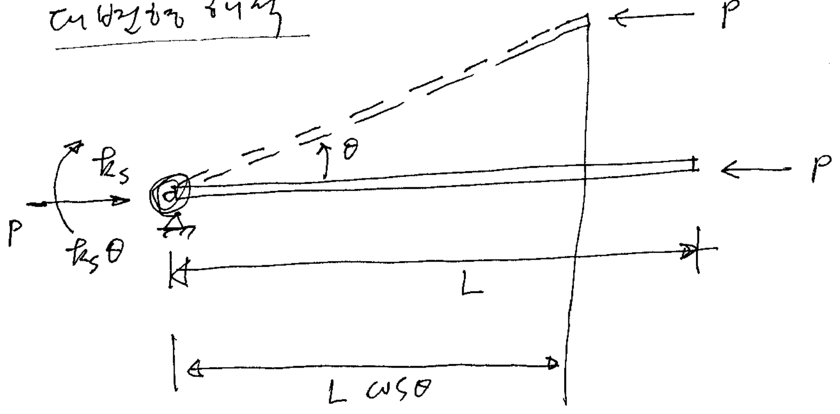
(1) 전절의 관심사 : P_{cr} 산정

↳ small deflection 해석으로 충분

(2) small deflection 해석에 의할 경우 system의 후좌굴거동 (postbuckling behavior)을 파악할 수 없는 단점이 있음

↳ large displacement 해석으로

1.5.1 Rigid bar supported by a rotational spring의 대변형 해석



: Fig. 1.16

Note : Fig. 1.11 참고 (미소변형 해석 결과를 정리한 것)

↳ ① stable or unstable when $P = P_{cr}$?

② no information about postbuckling behavior

Energy approach에 의한 대변형 해석 (Fig. 1.16 참조)

← 임계하중 뿐만 아니라 후좌굴 평형경로의 안정/불안정 판별이 가능하므로 (에너지법의 장점)

$$U = \frac{1}{2} k_s \theta^2$$

$$V = -PL(1-\cos\theta)$$

$$\Pi = U + V = \frac{1}{2} k_s \theta^2 - PL(1-\cos\theta)$$

For equilibrium, $\frac{d\pi}{d\theta} = k_s\theta - PL\sin\theta = 0$ ← $\theta = 0$이면 항상 평형조건이 만족됨 (trivial equilibrium path)

fundamental

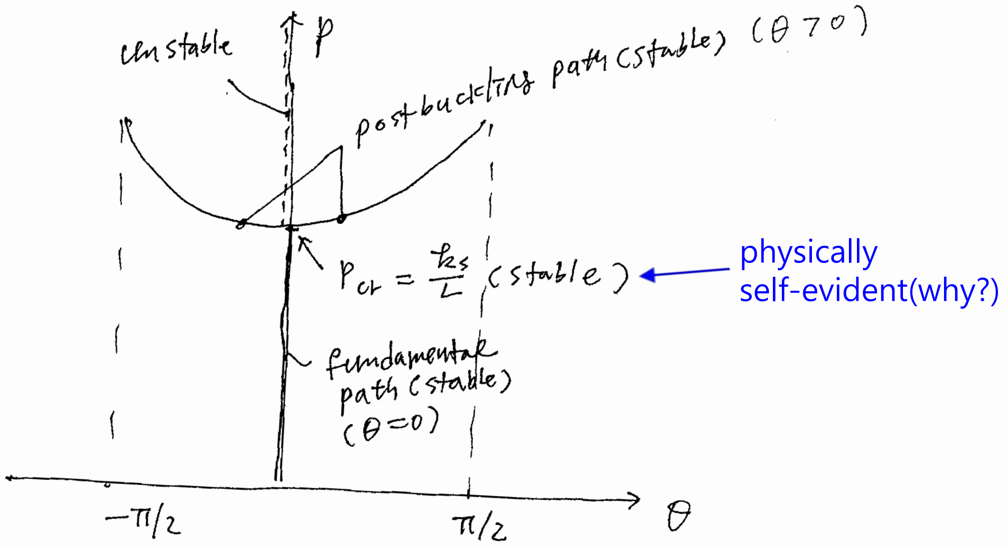
The postbuckling path ($\theta > 0$) is

$$\frac{d\pi}{d\theta} = K_s\theta - PL\sin\theta = 0 \text{ 에서}$$

$$P = \frac{K_s\theta}{L\sin\theta} \quad \text{---- (1.5.5)}$$

∘ 후좌굴 상태의 하중-변위 관계식

∘ even function for the range $\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$



Study of the stability of the equilibrium paths

$$\frac{d\pi}{d\theta} = k_s\theta - PL\sin\theta \quad \text{---- (1.5.4)}$$

$$\frac{d^2\pi}{d\theta^2} = k_s - PL\cos\theta \quad \text{---- (1.5.6)}$$

① For the fundamental path (or $\theta = 0$)

$$\left.\frac{d^2\pi}{d\theta^2}\right|_{\theta=0} = k_s - PL\cos 0 = k_s - PL \quad \text{---- (1.5.7)}$$

$$\pi(\theta) - \pi(0) = 0 + \frac{1}{2!}(k_s - PL)\times\theta^2$$

$P < \frac{k_s}{L}$ 이면 ? (stable)

$P > \frac{k_s}{L}$ 이면 ? (unstable)

(2강 참고)

② For the postbuckling path,

$$P = \frac{k_s \theta}{L \sin\theta}$$ 관계 만족 $(\theta > 0)$

↳ (1.5.4) 식의 조건 만족 (극값을 갖기 위한 필요조건)

따라서 (1.5.6) 식은

$$\frac{d^2\pi}{d\theta^2} = k_s - \left(\frac{k_s \theta}{L \sin\theta}\right) L \cos\theta$$

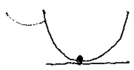
$$= k_s \left(1 - \frac{\theta}{\tan\theta}\right)$$

$$= k_s \left(1 - \frac{\theta}{\theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \cdots}\right)$$

< 1.0

항상 "+" 임; thus, the postbuckling path is always stable.

π function



(아래로 볼록한 곡률임, 따라서 local minimum) (앞쪽의 그림 참조)

③ At the critical point, $P = P_{cr} = \dfrac{K_s}{L}$

(1.5.7) 식에 $P = P_{cr} = \dfrac{k_s}{L}$ 를 대입하면 $\dfrac{d^2\pi}{d\theta^2} = 0$ 가 되어 안정/불안정의 판별을 위해서는 더욱 고차로 항수의 부호를 검토해야 할 것임:

dominant term의 부호를 고찰해야

$$\pi(\theta) - \pi(0) = \left.\frac{d\pi}{d\theta}\right|_{\theta=0} \times \theta + \frac{1}{2}\left.\frac{d^2\pi}{d\theta^2}\right|_{\theta=0} \times \theta^2 + \frac{1}{6}\left.\frac{d^3\pi}{d\theta^3}\right|_{\theta=0} \times \theta^3 + \frac{1}{24}\left.\frac{d^4\pi}{d\theta^4}\right|_{\theta=0} \times \theta^4 + O(\theta^5)$$

zero, (1.5.6) 식

zero (1.5.4) 식

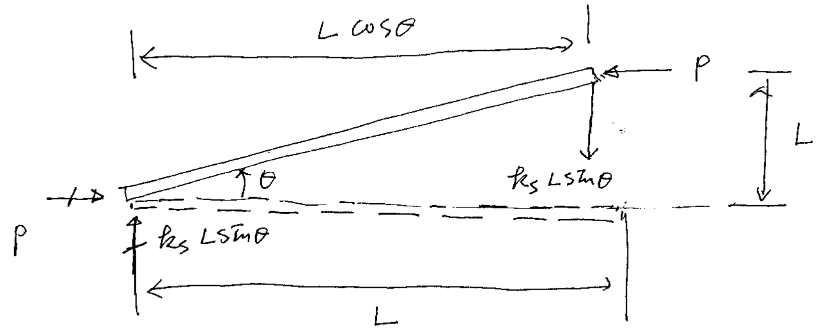
$PL\sin\theta\big|_{\theta=0} = \text{zero}$

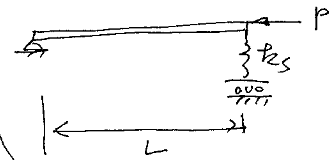
$PL\cos\theta\big|_{\theta=0} = PL > 0$

따라서 $\pi(0)$은 극소값 (따라서 stable!)

Note : p. 28의 요약, 정리 등

1.5.2 Rigid bar supported by a translational spring 의 대변형 해석





$P_{cr} = k_s L$
(1.4.2절 참조)

Energy approach

$$\Pi = \frac{1}{2} k_s (L\sin\theta)^2 - PL(1-\cos\theta) \quad \text{--- (1.5.14)}$$

For equilibrium,

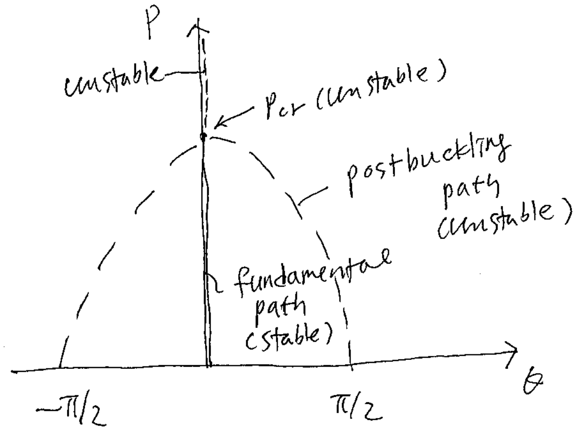
$$\frac{d\Pi}{d\theta} = (k_s L^2 \cos\theta - PL)\sin\theta = 0 \quad \text{--- (1.5.15)}$$

① $\theta = 0$ 이면 (1.5.15)식은 항상 만족 → fundamental equilibrium path

② postbuckling path (or $\theta > 0$)

(1.5.15)식 에서

$$(k_s L^2 \cos\theta - PL)\cancel{\sin\theta} = 0 \;;\; P = k_s L\cos\theta \quad \text{--- (1.5.16)}$$



= Fig. 1.19

Study of the stability of the equilibrium paths

↳ $$\frac{d^2\pi}{d\theta^2} = k_s L^2 (\cos^2\theta - \sin^2\theta) - PL\cos\theta \quad \cdots (1.5.17)$$

① For the fundamental path ($\theta = 0$)

$$\left.\frac{d^2\pi}{d\theta^2}\right|_{\theta=0} = k_s L^2 - PL = L(k_s L - P)$$

$$\pi(\theta) - \pi(0) = 0 + \underbrace{\frac{L}{2!}(k_s L - P)}_{} \times \theta^2 + O(\theta^3)$$

[$P < k_s L$ 이면 → stable
  $P > k_s L$ 이면 → unstable

② For the postbuckling path ($\theta > 0$)

$P = k_s L\cos\theta$ ← from (1.8.15)

↳ (1.5.17) 식에 대입

$$\frac{d^2\pi}{d\theta^2} = k_s L^2(\cos^2\theta - \sin^2\theta) - (k_s L\cos\theta)L\cos\theta \quad (1.5.18)$$

$$= \underline{-k_s L^2 \sin^2\theta} < 0$$

위로 볼록한 곡률 (local max. → unstable)

↳ 앞 쪽의 그림 참조

③ At the critical point ($P = P_{cr} = k_s L$),

↳ $\theta = 0$

$$\frac{d^2\pi}{d\theta^2} = k_s L^2 - (k_s L)L = 0 \quad \leftarrow \text{Eq. (1.5.18)}$$

"따라서 고차 도함수 검토 필요"

Fig. 1.19 참조 (앞 쪽)

Note: (1.5.17)을 1회 추가 미분하면 모든 항에 $\sin\theta$가 존재

↳ 따라서 $\left.\frac{d^3\pi}{d\theta^3}\right|_{\theta=0}$ = zero가 됨

unstable!

The first non-zero term in the series,

$$\frac{1}{24}\left.\frac{d^4\pi}{d\theta^4}\right|_{\theta=0} \times \theta^4 = \frac{1}{24}\left.(-4k_s L^2 + PL)\right|_{P=k_s L} \times \theta^4$$

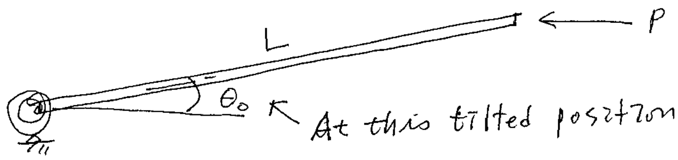
$$= -\frac{1}{8}k_s L^2 \times \theta^4 < 0$$ → 따라서 $\theta=0$은 local max.

HW # ... 이 과정 확인요

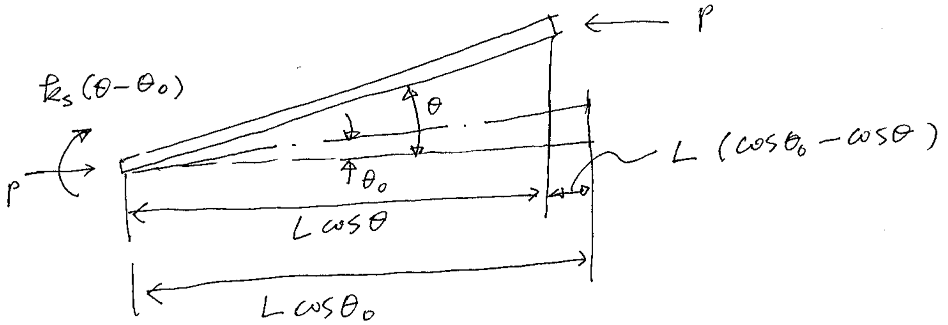
1.6 Imperfect systems

In the sense that the bars are slightly tilted ;
the bar will deflect as soon as the loaded is applied.
(엄밀히는 좌굴 문제는 아니지만 긴밀히 연계된 문제임)

1.6.1 Rigid bar supported by a rotational spring



At this tilted position, the spring is unstretched.



$$\Pi = \frac{1}{2}k_s(\theta-\theta_0)^2 - P\times L(\cos\theta_0-\cos\theta)$$

For equilibrium,

$$\frac{d\Pi}{d\theta} = k_s(\theta-\theta_0) - PL\sin\theta = 0 \quad \text{---- (1.6.4)}$$

or $$P = \frac{k_s(\theta-\theta_0)}{L\sin\theta} \quad \text{--- (1.6.5)}$$

"교과서 Fig. 1.21 참조" (p.33)

☆ Study of the stability of the equilibrium paths of the imperfect system,

(1.6.4)식 미분 → $$\frac{d^2\Pi}{d\theta^2} = k_s - PL\cos\theta \quad \left(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2} \text{ 범위}\right)$$

$$\begin{cases} P < \dfrac{k_s}{L\cos\theta} & \left(\text{stable} ;\ \dfrac{d^2\Pi}{d\theta^2} > 0\right) \\[2ex] P > \dfrac{k_s}{L\cos\theta} & \left(\text{unstable} ;\ \dfrac{d^2\Pi}{d\theta^2} < 0\right) \end{cases}$$

The equilibrium paths are stable if

$$P < \frac{k_s}{L\cos\theta} \quad --- (*)$$

or $\dfrac{k_s(\theta-\theta_0)}{L\sin\theta} < \dfrac{k_s}{L\cos\theta}$

(1.6.5)식

"수정요"

or $\cos\theta < \dfrac{\sin\theta}{\theta-\theta_0} \quad --- (*)'$

$$(\text{좌변}) = \cos\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - + \cdots\right)$$

$$(\text{우변}) = \frac{\sin\theta}{\theta-\theta_0} \cancel{\geq} \frac{\sin\theta}{\theta} = \frac{1}{\theta}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \cdots\right)$$

$$= \left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - + \cdots\right) > \cos\theta$$

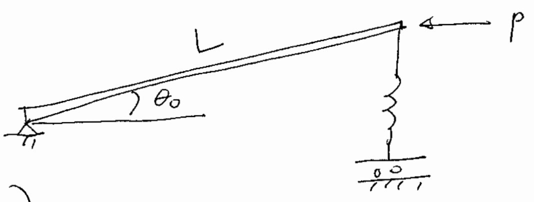
따라서 (*)식은 θ와 무관하게 항상 만족

(좌변)과 (우변)의 항별 비교를 해보면 (좌변) < (우변) 임을 알 수 있고, (*)식의 조건은 항상 만족이 된다.

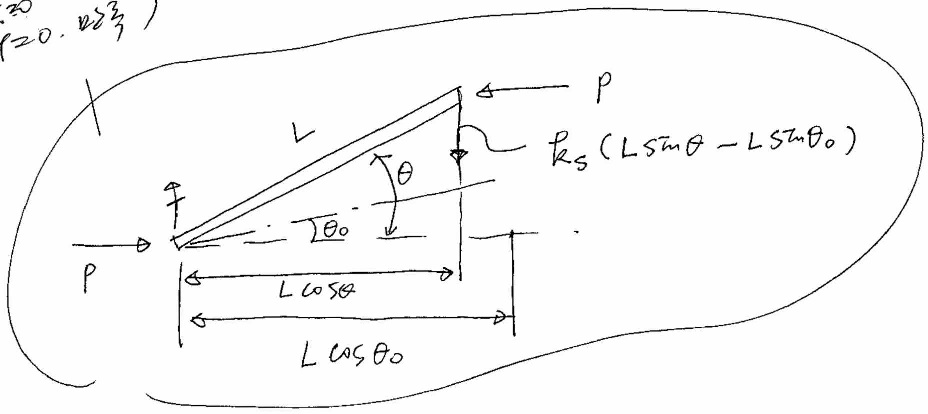
따라서 본 Imperfect system의 평형경로는 항상 stable 하다.

Fig. 1.21로도 自明

1.6.2 Rigid bar supported by a translational spring



($\Sigma X=0$, $\Sigma Y=0$ 만족)



Energy approach

$$\Pi = \frac{1}{2} k_s \{L(\sin\theta - \sin\theta_0)\}^2 - PL(\cos\theta_0 - \cos\theta)$$

For equilibrium,

$$\frac{d\pi}{d\theta} = k_s L^2 (\sin\theta - \sin\theta_0)\cos\theta - PL\sin\theta = 0 \quad \text{---} (1.6.12)$$

or $$P = K_s L\cos\theta \left(1 - \frac{\text{"}\sin\theta_0\text{"}}{\sin\theta}\right) \quad \text{---} (1.6.13)$$

← 양쪽의 free body에
$\Sigma M = 0$을 적용해서도
얻을 수 있는 식임!

→ Fig. 1.23 참조
(p.35의 설명도 참조)

☆ Fig. 1.23의 Pmax 값은,

$$\frac{dP}{d\theta} = k_s L\left(-\sin\theta + \frac{\sin\theta_0}{\sin^2\theta}\right) = 0 \text{ 에서}$$

$$\sin\theta_0 = \sin^3\theta \longrightarrow (1.6.13) \text{식에 대입}$$

$$\therefore P_{max} = k_s L\cos\theta \frac{(1-\sin^2\theta)}{\cos^2\theta}$$

$$= \underline{\underline{k_s L\cos^3\theta}}$$

→ the locus of the max. loads.

Note: 주어진 상수 θ_0

$\sin^3\theta_{max} = \sin\theta_0$

$\sin\theta_{max} = \sqrt[3]{\sin\theta_0}$

$\therefore \theta_{max} = \sin^{-1}\left(\sqrt[3]{\sin\theta_0}\right)$

☆ The stability of equilibrium paths

(1.6.12)식 미분, $$\frac{d^2\pi}{d\theta^2} = \boxed{}$$

← P 대신 (1.6.13)식 대입

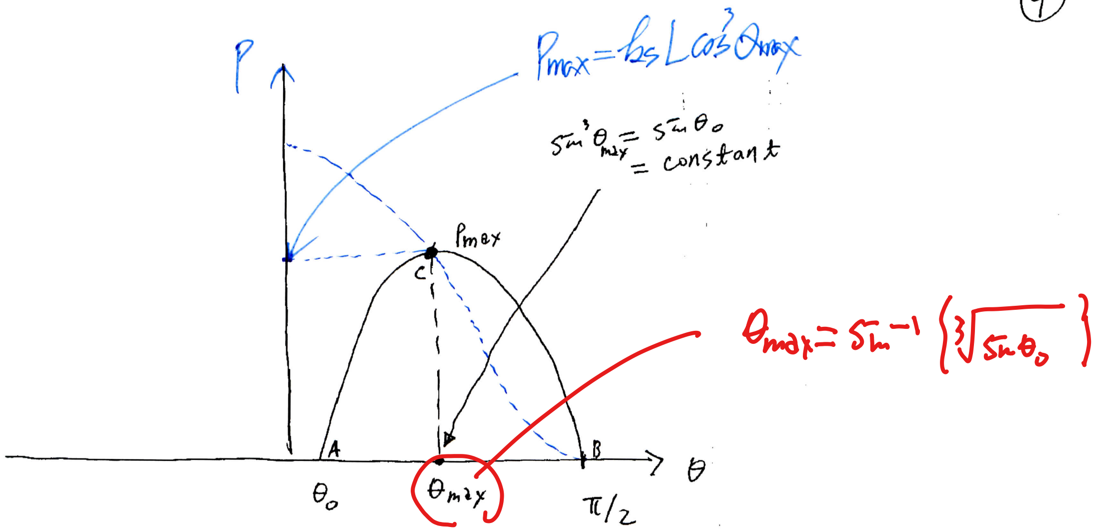
$$= k_s L^2 \left(\frac{\sin\theta_0 - \sin^3\theta}{\sin\theta}\right)$$

따라서 $\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$ 범위에서 $\underline{\sin\theta_0 > \sin^3\theta}$ 이면 $\frac{d^2\pi}{d\theta^2} > 0$ 이므로

equilibrium path는 stable 하게 될 것임. why?

→ (rising equilibrium path는 stable
falling " " unstable)

(Pmax 점이 $\sin\theta_0 = \sin^3\theta$를 만족하는 θ 값으로 주어지므로, 다음쪽 참조)



"AC branch (rising branch) 에서는 $\sin^3\theta < \sin\theta_0$ 일 것 임 → stable

$\theta_0 \le \theta \le \theta_{max}$

"CB branch (falling branch) 에서는 $\sin^3\theta > \sin\theta_0$ 일 것 임 → unstable

$\theta \ge \theta_{max}$

사실은 graphically evident!