

Electrical Variational System

❖ Electrical variational system

$$\nabla \cdot \vec{D} = \sigma$$

Premultiply by allowable variation of φ , $\delta\varphi$

$$\int_V (\nabla \cdot \vec{D}) \delta\varphi dV = \int_V \sigma \delta\varphi dV$$

$\delta\varphi = 0$ or fixed conditions

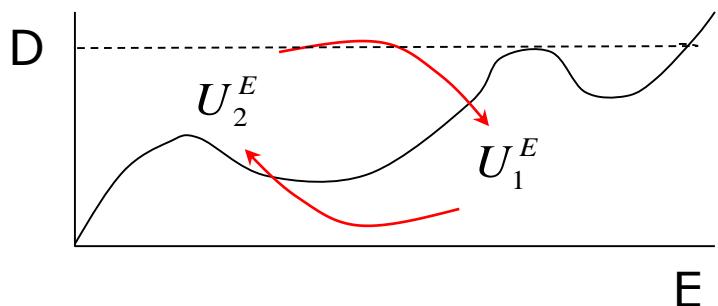
$\delta\varphi = \text{const.}$ along conductors

$$\delta E = -\nabla \delta\varphi$$

divergence theorem
 $\iiint \text{div} \vec{F} dv = \iint \vec{F} \cdot \vec{n} dA$

$$\int_V D \cdot (\delta\varphi \vec{D}) dV - \int_V \vec{D} \cdot \nabla \delta\varphi dV = \int_V \sigma \delta\varphi dV$$

$$\int_S \delta\varphi \vec{D} \cdot \vec{dS} \xrightarrow{0}$$



$$\int_V \vec{D} \cdot \delta \tilde{E} dV = \int_V \sigma \delta\varphi dV \xrightarrow{\sum q_i \delta\varphi_i (\text{discrete})}$$

$$\delta U_1^E = \int_V \vec{D} \cdot \delta \tilde{E} dV, \quad \delta W_1^E = \sum q_i \delta\varphi_i$$

$$U_1^E = \int_V \int_0^E \vec{D}(\tilde{E}) \cdot \delta \tilde{E} dV$$

$$\delta(U_1^E - W_1^E) = 0$$

Electrical Variational System

❖ Non-complementary principle

- Allow variation of free charge distribution
- Constant with E equal

$$\mathbf{D} \bullet \vec{\delta D} = \delta\sigma$$

- Pre-multiply by φ

$$\int_V \nabla \mathbf{D} \bullet \vec{\delta D} \varphi dV = \int_V \delta\sigma \varphi dV$$

$$\int_V \nabla \bullet (\delta \vec{D} \varphi) dV - \int_V \delta \vec{D} \bullet \nabla \varphi dV = \int_V \delta\sigma \varphi dV$$

$$\int_V \delta \vec{D} \bullet \vec{E} dV = \int_V \delta\sigma \varphi dV$$

$$\int_V \delta \vec{D} \bullet \vec{E} dV = \sum_i \delta q_i \varphi_i$$

$$\delta(U_2^E - W_2^E) = 0$$

$$U_2^E = \int_V \int_0^D \vec{E}(D) \bullet dD dV$$

linear relationship $\vec{D} = \epsilon \vec{E}$

$$\Rightarrow U_1^E = \frac{1}{2} \int_V \vec{E}^T \epsilon \vec{E} dV$$

$$U_2^E = \frac{1}{2} \int_V \vec{D}^T \{ \epsilon^{-1} \} \vec{D} dV$$

Electrical Variational System

❖ Non-complimentary principle (continued)

$$U_1^E = D \bullet E - U_2^E$$

$$\delta(U_1^M - W_1^M) = 0$$

PMTPE

$$\delta(U_2^M - W_2^M) = 0$$

PMTCPE

$$\delta(U_1^E - W_1^E) = 0$$

Complementary

$$\delta(U_2^E - W_2^E) = 0$$

Non-Complementary

} Mechanical

$\delta u, \delta S$

$\delta T, \delta f$

$\delta \varphi, \delta E$

$\delta \sigma, \delta D$

Electrical Variational System

❖ Combined Electrical-Mechanical

$$U^{TOT} = U_1^M + U_2^E$$

$$\delta U^{TOT} = \delta U_1^M + \delta U_2^E = \delta W_1^M + \delta W_2^E = \delta W$$

$$\delta U_1^M = \int_V \vec{T} \bullet \delta \vec{S} dV_1$$

$\xrightarrow{\quad}$ $T(\bar{S}, \bar{D})$

$$\delta U_2^E = \int_V \vec{E} \bullet \delta \vec{D} dV$$

$\xrightarrow{\quad}$ $E(S, D)$

From thermodynamics

1st law of thermodynamics,

$$dU^{TOT} = dQ + dW$$

$$dU^{TOT} = \theta ds + \vec{T} \bullet d\vec{S} + \vec{E} \bullet d\vec{D}$$

$$dU = \frac{\partial U}{\partial s} ds + \frac{\partial U}{\partial \vec{S}} d\vec{S} + \frac{\partial U}{\partial \vec{D}} d\vec{D}$$

$$\theta = \left[\frac{\partial U}{\partial s} \right]_{\vec{S}, D}, \vec{T} = \left[\frac{\partial U}{\partial \vec{S}} \right]_{s, D}, \vec{E} = \left[\frac{\partial U}{\partial \vec{D}} \right]_{\vec{S}, s}$$

Electrical Variational System

❖ Combined Electrical-Mechanical (continued)

$$A = U - \theta s, dA = -sd\theta + \vec{T} \bullet d\vec{S} + \vec{E} \bullet d\vec{D} \quad : \text{Helmholtz Free Energy}$$

$$s = \left[\frac{\partial A}{\partial \theta} \right]_{\vec{S}, \vec{D}}, \vec{T} = \left[\frac{\partial A}{\partial s} \right]_{\theta, \vec{D}}, \vec{E} = \left[\frac{\partial A}{\partial \vec{D}} \right]_{\vec{S}, \theta}$$

$$G_1 = U - \theta s - \vec{T} \bullet \vec{S}, dG_1 = -sd\theta - \vec{S} \bullet d\vec{T} + \vec{E} \bullet d\vec{D} \quad : \text{Gibbs Free Energy}$$

$$\delta G_1 = \delta W \rightarrow -\delta U_2^M + \delta U_2^E + \delta W_2^M - \delta W_2^E = 0$$

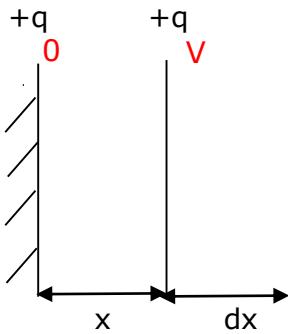
$$G_2 = U - \theta s - \vec{D} \bullet \vec{E}, dG_2 = -sd\theta + T \bullet d\vec{S} - \vec{D} \bullet d\vec{E}$$

$$\boxed{\delta U_1^M - \delta U_1^E - \delta W_1^M + \delta W_1^E = 0}$$

Displacement, Electrical Potential

Electrical Variational System

❖ Consider



$$\begin{aligned} q &= CV \\ &= \frac{\epsilon_0 A}{x} V \\ &= \epsilon_0 A E \\ E &= \frac{q}{\epsilon_0 A} \end{aligned}$$

$$\begin{aligned} U &= \frac{1}{2} \epsilon_0 \int_V E^2 dV \\ &= \frac{1}{2} \epsilon_0 \int_0^x \frac{q^2}{\epsilon_0^2 A^2} A dx \\ \frac{\partial U}{\partial x} &= \frac{1}{2} \frac{q^2}{\epsilon_0 A} \\ F &= - \left(\frac{\partial U}{\partial x} \right)^{q=const} = -\frac{1}{2} Eq \end{aligned}$$

Electrical Variational System

❖ Consider constant voltage

$$\begin{aligned} U &= \frac{1}{2} \epsilon_0 \int_{Vol} E^2 dVol \\ &= \frac{1}{2} \epsilon_0 \int \left(\frac{V}{x} \right)^2 dVol \\ &= \frac{1}{2} \epsilon_0 \int_0^x \left(\frac{V}{x} \right)^2 d\tilde{x} \cdot A \end{aligned}$$

$$\begin{aligned} U &= \frac{1}{2} \epsilon_0 A \frac{V^2}{x} = \frac{1}{2} C V^2 \\ \frac{\partial U}{\partial x} \Big|^{V=const} &= -\frac{1}{2} \epsilon_0 A \frac{V^2}{x^2} \\ F &= -\left(\frac{\partial U}{\partial x} \right)^{V=const} = \frac{1}{2} \epsilon_0 A \frac{V^2}{x^2} \end{aligned}$$

What charge is required for V to be constant?

$$V = Ex = \frac{q}{\epsilon_0 A} x$$

$$dV = 0 = \frac{q}{\epsilon_0 A} dx + \frac{x}{\epsilon_0 A} dq$$

$$dq = -\frac{q}{x} dx$$

Electrical Variational System

❖ Total Energy

$$d\tilde{U} = dU - Vdq$$

$$dU = \frac{\partial \tilde{U}}{\partial x} dx$$

$$dU = \left(-\frac{1}{2} \frac{\epsilon_0 A}{x^2} V^2\right) dx$$

$$-Vdq = \frac{q}{x} U dx = \frac{\epsilon_0 A}{x^2} V^2 dx$$

$$dU = \frac{1}{2} \frac{\epsilon_0 A}{x^2} V^2 , F = -\frac{dU}{dx} = \frac{1}{2} \frac{\epsilon_0 A}{x^2} V^2$$

$$\begin{aligned} U &= U - \sum q\varphi \\ &= U - \vec{E} \cdot \vec{D} \quad : \text{electrical enthalpy} \end{aligned}$$

The force

$$F = -\frac{\partial U}{\partial x} \Bigg|^\varphi = \frac{\partial U}{\partial x} \Bigg|^\varphi = -\frac{\partial U}{\partial x} \Bigg|^\varphi$$

Electrical Variational System

❖ Total Energy (continued)

More formally

$$\begin{aligned} dU &= \left(\frac{\partial U}{\partial x} \right)^* dq + \left(\frac{\partial U}{\partial x} \right)^q dx \\ &= \varphi dq + F dx \quad : \text{total energy} \end{aligned}$$

$$\delta U = EdD - Tds \quad : \text{energy per volume}$$

$$U = U - \varphi q$$

$$\begin{aligned} \delta U &= \delta U - \delta \varphi q - \varphi \delta q \\ &= -q \delta \varphi - F \delta x \end{aligned}$$

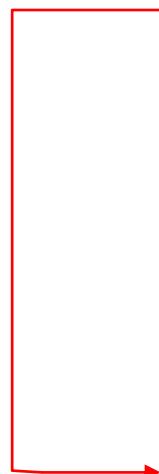
$$dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial q} dq$$

$$T = \left(\frac{\partial U}{\partial S} \right)^D, E = \left(\frac{\partial U}{\partial D} \right)^S$$

Electrical Variational System

❖ Total Energy (continued)

Gibbs free energy	: $G = U - T_{ij}S_{ij} - E_m D_m - \theta\sigma$	temperature enthalpy
Elastic free energy	: $G_1 = U - T_{ij}S_{ij} - \theta\sigma$	
Electric free energy	: $G_2 = U - E_m D_m - \theta\sigma$	
Helmholtz free energy	: $A = U - \theta\sigma$	
enthalpy	: $H = U - T \cdot S - E_m D_m$	
Elastic enthalpy	: $H_1 = U - T \cdot S$	
Electric enthalpy	: $H_2 = U - E_m D_m$	



$$dG = -\sigma d\theta - S_{ij} dT_{ij} - D_m dE_m$$

$$dG_1 = -\sigma d\theta - S_{ij} dT_{ij} + E_m dD_m$$

$$dG_2 = -\sigma d\theta + T_{ij} dS_{ij} - D_m dE_m$$

$$dA = -\sigma d\theta + T_{ij} dS_{ij} + E_m dD_m$$

$$dH = \theta d\sigma - S_{ij} dT_{ij} - D_m dE_m$$

$$dH_1 = \theta d\sigma - S_{ij} dT_{ij} + E_m dD_m$$

$$dH_2 = \theta d\sigma + T_{ij} dS_{ij} - D_m dE_m$$

$$dG = \cancel{dU} - T dS - \cancel{S dT} - E dD - \cancel{D_m dE} - \theta d\sigma - \sigma d\theta$$

$$\cancel{\theta d\sigma} + \cancel{E dD} + \cancel{T dS}$$

Electrical Variational System

❖ Total Energy (continued)

$$dG(\theta, T_{ij}, E_m) = \frac{\partial G}{\partial \theta} d\theta + \frac{\partial G}{\partial T_{ij}} dT_{ij} + \frac{\partial \sigma}{\partial E_m} dE_m$$

$$\sigma = - \left(\frac{\partial G}{\partial \theta} \right)^{T,E}$$

$$S_{ij} = - \left(\frac{\partial G}{\partial T_{ij}} \right)^{E,\theta}$$

$$D_m = - \left(\frac{\partial G}{\partial E_m} \right)^{T,\theta}$$

$$S_{ij} = s_{ijkl}^E T_{kl} + f_1(T, E)$$

$$D_m = \varepsilon_{mn}^T E_m + f_2(T, E)$$

Ref. 1."Dynamics and Mechanics of Electrical Systems" by Crandall - Chap. 6

2. "Principle and Applications of Ferroelectrics and Related Material" by M.E. Lines & A.M. Glass - Chap. 3

Electrical Variational System

❖ Total Energy (continued)

$$dG = -S_{ij}dT_{ij} - D_m dE_m$$

$$dG = \frac{\partial G}{\partial T_{ij}}dT_{ij} + \frac{\partial G}{\partial E_m}dE_m$$

$$S_{ij} = -\left(\frac{\partial G}{\partial T_{ij}}\right)^E, D_m = -\left(\frac{\partial G}{\partial E_m}\right)^T$$

Expand S_{ij}

$$dS_{ij} = \left(\frac{\partial S_{ij}}{\partial T_{kl}}\right)^E dT_{kl} + \left(\frac{\partial S_{ij}}{\partial E_m}\right)^T dE_m$$

$$S_{ijkl}^E = \left(\frac{\partial S_{ij}}{\partial T_{kl}}\right)^E = -\left(\frac{\partial^2 G}{\partial T_{ij} \partial T_{kl}}\right)^{E,\theta}$$

$$d_{ijk}^T = \left(\frac{\partial S_{ij}}{\partial E_k}\right)^T = -\left(\frac{\partial^2 G}{\partial T_{ij} \partial E_k}\right)^{T_1,\theta} : \text{Piezo free Strain}$$

$$\alpha_{ij}^{T_1 E} = \left(\frac{\partial S_{ij}}{\partial \theta}\right)^{T_1 E} = -\left(\frac{\partial^2 G}{\partial T_{ij} \partial \theta}\right)^{T_1 E} : \text{Coefficient of Thermal Expansion}$$

Linearize $dD_m \rightarrow D_m$

$$dS_{ij} \rightarrow S_{ij}$$

$$dT_{kl} \rightarrow T_{kl}$$

$$dE_m \rightarrow E_m$$