Basics of Fluid Flow

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Introduction

- Fluids
  - No resistance to external shear forces
  - Regarded as continuum, i.e., continuous substance
- Fluid flow is caused by the action of externally applied forces.
  - Surface forces: pressure or shear forces
  - Body forces: gravity, forces induced by rotation
- Flow speed affects its properties
  - Creeping flow: Speed low enough; inertia ignored
  - Laminar flow: Inertia important, but each fluid particle follows a smooth trajectory
  - Turbulent flow: Speed increase leads to instability; random type of flow

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Introduction – Cont.

- Flow speed/Sound speed (Ma: Mach #)
  - Compressibility
    - Ma < 0.3: Incompressible flow
    - Ma > 0.3: Compressible flow
  - Thermal effects
    - Ma < 1.0: Subsonic flow
    - Ma > 1.0: Supersonic flow
    - Ma > 5.0: Hypersonic flow (change in chemical nature)
  - Viscosity
    - Viscous flow vs. Inviscid flow
  - Newton's law
    - Newtonian flow vs. Non-Newtonian flow
  - Other phenomena
    - Heat transfer, Bouyancy, Multi-phase, …

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Conservation Principles

- Control volume approach
  - In fluid flows, it is more convenient to deal with the flow within a certain spatial region (CV), rather than in a parcel of matter which quickly passes through the region of interest.
- Conservation of two extensive properties
  - Rate of change of the property amount in a given CM to externally determined effects
  - Mass $\frac{dm}{dt} = 0 \, .$ (1.1)
  - Momentum (Newton's 2nd law of motion)

$$\frac{d(m\mathbf{v})}{dt} = \sum \mathbf{f} \, , \quad (1.2)$$

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Conservation Principles – Cont.

- Transformation into CV form
  - Fundamental variables are intensive, i.e., independent of amount of matter considered. Ex. Density (mass/volume), Velocity (momentum/mass).
  - Corresponding extensive property

$$\Phi = \int_{\Omega_{CM}} \rho\phi \,\mathrm{d}\Omega \,, \qquad (1.3)$$

  - LHS of each conservation equation for a CV (HW#1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{CM}} \rho\phi\,\mathrm{d}\Omega = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{CV}} \rho\phi\,\mathrm{d}\Omega + \int_{S_{CV}} \rho\phi(\boldsymbol{v}-\boldsymbol{v}_b)\cdot\boldsymbol{n}\,\mathrm{d}S \,, \qquad (1.4)$$

  - For a fixed CV, $\boldsymbol{v}_b = \boldsymbol{0}$ and the first derivative on RHS becomes a local (partial) derivative.

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Mass Conservation

- Continuity equation
  - Set $\phi = 1$:

$$\frac{\partial}{\partial t}\int_{\Omega} \rho\,\mathrm{d}\Omega + \int_S \rho\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = 0 \,. \qquad (1.5)$$

  - Applying the Gauss' divergence theorem, and allowing CV to become infinitesimally small (HW #2)

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\boldsymbol{v}) = 0 \,. \qquad (1.6)$$

  - Cartesian form in tensor notation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} = 0 \,, \qquad (1.7)$$

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Momentum Conservation

- Momentum conservation equation
  - In Eq. (1.2) and (1.4), replace ϕ with v,

$$\frac{\partial}{\partial t}\int_{\Omega} \rho \boldsymbol{v}\, \mathrm{d}\Omega + \int_{S} \rho \boldsymbol{v}\boldsymbol{v}\cdot \boldsymbol{n}\, \mathrm{d}S = \sum \boldsymbol{f} . \tag{1.8}$$

  - RHS can be expressed in terms of intensive properties by considering
    - Surface forces (pressure, normal and shear stresses, surface tension, etc.)
    - Body forces (gravity, centrifugal and Coriolis forces, electromagnetic forces, etc.)
  - To close the system of equations, assume that the fluid is Newtonian (HW #3).

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Momentum Conservation – Cont.

- For Newtonian fluids, the stress tensor T,

$$\mathsf{T} = -\left(p + \frac{2}{3}\mu\, \mathrm{div}\, \boldsymbol{v}\right)\mathsf{I} + 2\mu \mathsf{D} , \tag{1.9}$$

$$\mathsf{D} = \frac{1}{2}\left[\mathrm{grad}\, \boldsymbol{v} + (\mathrm{grad}\, \boldsymbol{v})^{T}\right] . \tag{1.10}$$

- or

For incompressible flow

$$T_{ij} = -\left(p + \frac{2}{3}\mu \frac{\partial u_j}{\partial x_j}\right)\delta_{ij} + 2\mu D_{ij} , \tag{1.11}$$

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) , \tag{1.12}$$

- Viscous part

$$\tau_{ij} = 2\mu D_{ij} - \frac{2}{3}\mu \delta_{ij}\, \mathrm{div}\, \boldsymbol{v} . \tag{1.13}$$

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Momentum Conservation – Cont.

- With *b* representing the body forces, in integral form

$$\frac{\partial}{\partial t}\int_\Omega \rho \boldsymbol{v}\,\mathrm{d}\Omega + \int_S \rho \boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_S \mathsf{T}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_\Omega \rho \boldsymbol{b}\,\mathrm{d}\Omega\,. \tag{1.14}$$

- By applying Gauss' divergence theorem, in coordinate free vector form

$$\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \mathrm{div}\,(\rho\boldsymbol{v}\boldsymbol{v}) = \mathrm{div}\,\mathsf{T} + \rho\boldsymbol{b}\,. \tag{1.15}$$

- i-th Cratesian component

$$\frac{\partial(\rho u_i)}{\partial t} + \mathrm{div}\,(\rho u_i\boldsymbol{v}) = \mathrm{div}\,\boldsymbol{t}_i + \rho b_i\,. \tag{1.16}$$

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Momentum Conservation – Cont.

- Integral form of Eq. (1.16)

$$\frac{\partial}{\partial t}\int_\Omega \rho u_i\,\mathrm{d}\Omega + \int_S \rho u_i\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_S \boldsymbol{t}_i\cdot\boldsymbol{n}\,\mathrm{d}S + \int_\Omega \rho b_i\,\mathrm{d}\Omega\,, \tag{1.17}$$

- where

$$\boldsymbol{t}_i = \mu\,\mathrm{grad}\,u_i + \mu\,(\mathrm{grad}\,\boldsymbol{v})^T\cdot\boldsymbol{i}_i - \left(p + \frac{2}{3}\mu\,\mathrm{div}\,\boldsymbol{v}\right)\boldsymbol{i}_i = \tau_{ij}\boldsymbol{i}_j - p\boldsymbol{i}_i\,. \tag{1.18}$$

- In Cartesian coordinates

$$\boldsymbol{t}_i = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\boldsymbol{i}_j - \left(p + \frac{2}{3}\mu\frac{\partial u_j}{\partial x_j}\right)\boldsymbol{i}_i\,. \tag{1.19}$$

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Momentum Conservation – Cont.

- The momentum equations are said to be in "strong conservation form" if all terms have the form of the divergence of a vector or tensor. The strong conservation form , when used with a FVM, automatically insures global momentum conservation.
- Eq. (1.16) is a strong conservation form. By employing the continuity equation and based on

$$\mathrm{div}\,(\rho \boldsymbol{v} u_i) = u_i \,\mathrm{div}\,(\rho \boldsymbol{v}) + \rho \boldsymbol{v} \cdot \mathrm{grad}\, u_i \ ;$$

- A non-conservation form is obtained

$$\rho \frac{\partial u_i}{\partial t} + \rho \boldsymbol{v} \cdot \mathrm{grad}\, u_i = \mathrm{div}\, \boldsymbol{t}_i + \rho b_i \ . \qquad (1.20)$$

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Momentum Conservation – Cont.

- The pressure gradient is regarded as a body force, i.e., non-conservative treatment of the pressure term, based on

$$\mathrm{div}\,(p\,\boldsymbol{i}_i) = \mathrm{grad}\, p \cdot \boldsymbol{i}_i \ .$$

- If Eq. (1.13) is substituted into Eq. (1.16), and gravity is the only body force, for Cartesian coordinates,

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (\rho u_j u_i)}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} + \rho g_i \ , \qquad (1.21)$$

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Conservation of Scalar Quantities

- Integral form of the equation describing conservation of a scalar quantity, ϕ,

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\phi\,\mathrm{d}\Omega + \int_{S}\rho\phi\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = \sum f_{\phi}\,, \quad (1.22)$$

  - Diffusive transport, by Fourier's law for heat diffusion or Fick's law for mass diffusion

$$f_{\phi}^{\mathrm{d}} = \int_{S}\Gamma\,\mathrm{grad}\,\phi\cdot\boldsymbol{n}\,\mathrm{d}S\,, \quad (1.23)$$

  - Integral form of the generic conservation equation

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\phi\,\mathrm{d}\Omega + \int_{S}\rho\phi\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{S}\Gamma\,\mathrm{grad}\,\phi\cdot\boldsymbol{n}\,\mathrm{d}S + \int_{\Omega}q_{\phi}\,\mathrm{d}\Omega\,, \quad (1.26)$$

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Cons. of Scalar Quantities – Cont.

- Coordinate-free vector form,

$$\frac{\partial(\rho\phi)}{\partial t} + \mathrm{div}\,(\rho\phi\boldsymbol{v}) = \mathrm{div}\,(\Gamma\,\mathrm{grad}\,\phi) + q_{\phi}\,. \quad (1.27)$$

- In Cartesian coordinates and tensor notation

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho u_j\phi)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\Gamma\frac{\partial\phi}{\partial x_j}\right) + q_{\phi}\,. \quad (1.28)$$

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Simplified Mathematical Models

- Incompressible flow
  - Density is assumed constant – liquids or gases if Ma < 0.3
  - If flow is isothermal, the viscosity is also constant.

$$\operatorname{div} \boldsymbol{v} = 0 \, , \tag{1.32}$$

$$\frac{\partial u_i}{\partial t} + \operatorname{div}(u_i \boldsymbol{v}) = \operatorname{div}(\nu \operatorname{grad} u_i) - \frac{1}{\rho} \operatorname{div}(p \boldsymbol{i}_i) + b_i \, , \tag{1.33}$$

  - Kinematic viscosity $\nu = \mu/\rho$

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Simplified Models – Cont.

- Inviscid flow
  - Viscous effects are neglected altogether (far from solid surfaces)

$$\frac{\partial(\rho u_i)}{\partial t} + \operatorname{div}(\rho u_i \boldsymbol{v}) = -\operatorname{div}(p \boldsymbol{i}_i) + \rho b_i \, . \tag{1.34}$$

- Potential flow
  - Flow is assumed inviscid + Velocity field is irrotational.
  - Define velocity potential $\boldsymbol{v} = -\operatorname{grad} \Phi$.
  - Governing equation

$$\operatorname{div}(\operatorname{grad} \Phi) = 0 \, . \tag{1.36}$$

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Simplified Models – Cont.

- Stokes flow
  - When the flow velocity is very small, the fluid is very viscous, or the geometric dimensions are very small, i.e., Re is very small, the convective (inertial) terms play a minor role and can be neglected.
  - Momentum equation becomes (for steady non-convection flows)

$$\operatorname{div}(\mu \operatorname{grad} u_i) - \frac{1}{\rho}\operatorname{div}(p\,\boldsymbol{i}_i) + b_i = 0\,. \qquad (1.37)$$

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Simplified Models – Cont.

- Boussinesq approximation
  - In flows with heat transfer, the fluid properties are normally functions of temperature. The variations may be small and yet be the cause of the fluid motion.
  - If the density variation is not large, one may treat the density as constant in the unsteady and convection terms, and treat it as variable only in the gravitational term.

$$(\rho - \rho_0)g_i = -\rho_0 g_i \beta (T - T_0)\,, \qquad (1.38)$$

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Simplified Models – Cont.

- Boundary layer approximation
  - When the flow has a predominant direction and the variation of the geometry is gradual
  - Thin shear layer or boundary layer flows
    - Diffusive transport of momentum *in the principal flow direction* is much smaller than convection and can be neglected
    - The velocity component in the main flow direction is much larger than the components in other directions
    - The pressure gradient across the flow is much smaller than in the principal flow direction

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Simplified Models – Cont.

- Boundary layer approximation – Cont.
  - 2D BL equations

$$\frac{\partial(\rho u_1)}{\partial t} + \frac{\partial(\rho u_1 u_1)}{\partial x_1} + \frac{\partial(\rho u_2 u_1)}{\partial x_2} = \mu \frac{\partial^2 u_1}{\partial x_2^2} - \frac{\partial p}{\partial x_1} \,, \qquad (1.39)$$

and the momentum equation normal to the principal flow direction $\partial p / \partial x_2 = 0$.

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Mathematical Classification of Flows

- Hyperbolic flows
- Parabolic flows
- Elliptic flows
- Mixed flow types

  (HW)

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