Solution of the Navier-Stokes Equations

Special Features

- The discretization principles described for a generic conservation equation apply to the momentum and continuity equations (Navier-Stokes Equations).
- How the terms in the momentum eqns which differ from those in the generic conservation eqn are treated.
- Unsteady, advection, and diffusive terms are same or similar.
- The momentum eqns contain a contribution from the pressure, which has no analog in the generic eqn. It may be regarded either as a source term (as a body force – non-conservatively) or as a surface force (conservatively).

Special Features – Cont.

- Due to the close connection of the pressure and the continuity eqn, it requires special attention.
- The fact that the principal variable is a vector allows more freedom in the choice of a grid.

SNUTT Seoul National University Towing Tank

Special Features – Cont.

- Discretization of Convective and Viscous Terms
  - Convective term in differential and integral forms

$$\frac{\partial(\rho u_i u_j)}{\partial x_j} \quad \text{and} \quad \int_S \rho u_i \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}S \; . \tag{7.1}$$

  - Viscous terms in differential and integral forms

$$\frac{\partial \tau_{ij}}{\partial x_j} \quad \text{and} \quad \int_S (\tau_{ij} \boldsymbol{i}_j) \cdot \boldsymbol{n} \, \mathrm{d}S \; , \tag{7.2}$$

where for a Newtonian fluid and incompressible flow

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \; . \tag{7.3}$$

SNUTT Seoul National University Towing Tank

Special Features – Cont.

- The part of the viscous term in the momentum eqns which corresponds to the diffusive term in the generic conservation eqn is

$$\frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) \quad \text{and} \quad \int_S \mu \, \mathrm{grad}\, u_i \cdot \boldsymbol{n} \, \mathrm{d}S \, . \quad (7.4)$$

Eq. (7.2)

- The extra terms which are non-zero when the viscosity is spatially variable in an incompressible flow

$$\frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_j}{\partial x_i}\right) \quad \text{and} \quad \int_S \left(\mu \frac{\partial u_j}{\partial x_i} \boldsymbol{i}_j\right) \cdot \boldsymbol{n} \, \mathrm{d}S \, , \quad (7.5)$$

Eq. (7.4)

This term vanishes for constant μ.

Special Features – Cont.

- Discretization of Pressure Terms and Body Forces
  - "Pressure": $p - \rho_0 \boldsymbol{g} \cdot \boldsymbol{r} + \mu \frac{2}{3} \boxed{\mathrm{div}\, \boldsymbol{v}}$ 0 in incompressible flows
  - In FVM, treated as a surface force

$$-\int_S p \boldsymbol{i}_i \cdot \boldsymbol{n} \, \mathrm{d}S \quad (7.6)$$

  - Or alternatively, a non-conservative approach

$$-\int_\Omega \mathrm{grad}\, p \cdot \boldsymbol{i}_i \, \mathrm{d}\Omega \, . \quad (7.7)$$

  - Other body forces are easy to treat in FDM. In FVM, these terms are integrated over the CV volume, i.e., the value at CV center is multiplied by cell volume.

Choice of Variable Arrangement

- Select the points in the domain at which the values of the unknown dependent variable are to be computed.

- Colocated Arrangement
  - Store all the variables at the same set of grid points and to use the same control volumes for all variables
  - # of coeffs that must be computed and stored is minimized, because many of the terms in each of the equations are essentially identical
  - Programming is simplified
  - When multigrid procedures are used, the same restriction and prolongation operators can be used for all variables.
  - Advantages in complicated solution domains, when the boundaries have slope discontinuities or the BCs are discontinuous

Choice of Variable Arrangement – Cont.

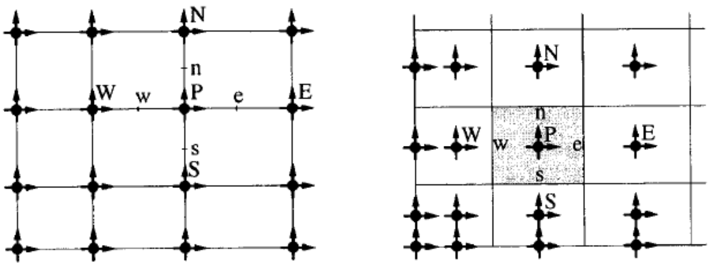

Fig. 7.1. Colocated arrangement of velocity components and pressure on a FD (left) and FV (right) grid

- The colocated arrangement was out of favor for incompressible flow computation due to the difficulties with p-v coupling and occurrence of oscillations in the pressure, because pressure is obtained from mass conservation
- Popularity began to rise again, when improved p-v coupling algorithms were developed

Choice of Variable Arrangement – Cont.

- Staggered Arrangements
  - No need for all variables to share the same grid
  - Several terms that require interpolation with the colocated arrangement can be calculated without interpolation
  - Since the pressure nodes lie at CV face centers and the velocity derivatives needed for the diffusive terms are readily computed at the CV faces, both the pressure and diffusion terms are natuarally approximated by CD approximation without interpolation
  - Strong coupling between the velocities and the pressure: biggest advantage

Choice of Variable Arrangement – Cont.

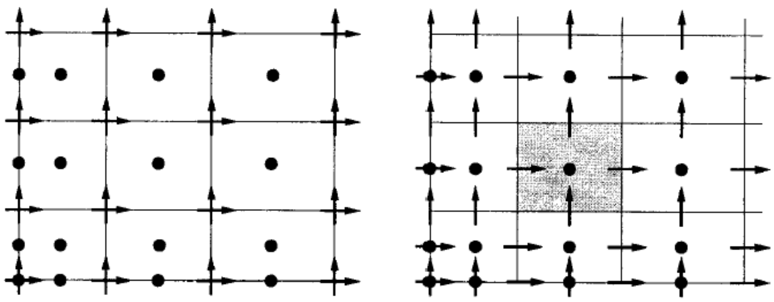

Fig. 7.2. Fully staggered arrangement of velocity components and pressure (right) and a partially staggered arrangement (left) on a FV grid

SNUTT Seoul National University Towing Tank

Calculation of the Pressure

- Soln of the NS eqns is complicated by the lack of an independent eqn for the pressure, whose gradient contributes to each of the 3 momentum eqns
- Treat this difficulty by constructing the pressure field so as to guarantee satisfaction of the continuity eqn
- Note: absolute pressure is of no significance; only the gradient of the pressure (pressure difference) affects the flow
- In compressible flows, the continuity eqn can be used to determine the density and the pressure is calculated from an eqn of state

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Pressure Eqn and its Solution
  - Momentum eqns determine the respective velocity components
  - Continuity eqn has to determine the pressure
  - How? Combining the momentum and continuity eqns
  - Take the divergence of the momentum eqn (1.15)

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}\boldsymbol{v}) = \operatorname{div} \mathsf{T} + \rho \boldsymbol{b} \,. \qquad (1.15)$$

$$\operatorname{div}(\operatorname{grad} p) = -\operatorname{div}\left[\operatorname{div}(\rho \boldsymbol{v}\boldsymbol{v} - \mathsf{S}) - \rho \boldsymbol{b} + \frac{\partial(\rho \boldsymbol{v})}{\partial t}\right] . \qquad (7.14)$$

  - In Cartesian coordinates,

$$\underbrace{\frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial x_i}\right)}_{\text{Laplace operator}} = -\frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_j}(\rho u_i u_j - \tau_{ij})\right] + \frac{\partial(\rho b_i)}{\partial x_i} + \frac{\partial^2 \rho}{\partial t^2} \,. \qquad (7.15)$$

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Pressure Eqn and its Solution – Cont.
  - *Note: RHS of the pressure eqn is the sum of derivatives of terms of the momentum eqns*
  - Note: Laplacian operator is the product of the divergence from the continuity and gradient from the momentum eqns → It is essential that the consistency of these operators be maintained, i.e., the outer and inner derivatives may be descretized using different schemes – they have to be those used in the momentum and continuity eqns

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- A Simple Explicit Time Advance Scheme
  - How the numerical Poisson eqn for the pressure is constructed
  - The role it plays in enforcing continuity
  - Semi-discretized (discrete in space but not time) momentum eqns (because spatial derivative approximation is not important)

$$\frac{\partial(\rho u_i)}{\partial t} = -\frac{\delta(\rho u_i u_j)}{\delta x_j} - \frac{\delta p}{\delta x_i} + \frac{\delta \tau_{ij}}{\delta x_j} = H_i - \frac{\delta p}{\delta x_i} , \qquad (7.17)$$

$\delta/\delta x$: discretized spatial derivative

H_i : shorthand for the advective and viscous terms

  - With the explicit Euler method for time advancement

$$(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left(H_i^n - \frac{\delta p^n}{\delta x_i} \right) . \qquad (7.18)$$

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- A Simple Explicit Time Advance Scheme – Cont.
  - Take the numerical divergence of Eq. (7.18)

$$\boxed{\frac{\delta(\rho u_i)^{n+1}}{\delta x_i}} - \boxed{\frac{\delta(\rho u_i)^n}{\delta x_i}} = \Delta t \left[\frac{\delta}{\delta x_i} \left(H_i^n - \frac{\delta p^n}{\delta x_i} \right) \right] . \quad (7.20)$$

zero

  - 1st term: divergence of the new velocity field that we want to be zero
  - 2nd term: assumed to be zero if continuity was enforced at *n*
  - Resulting in the discrete Poisson eqn for the pressure p^n

$$\frac{\delta}{\delta x_i} \left(\frac{\delta p^n}{\delta x_i} \right) = \frac{\delta H_i^n}{\delta x_i} . \quad (7.21)$$

  - If p^n satisfies this discrete Poisson eqn, the velocity field at *n+1* will be divergence free

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- A Simple Explicit Time Advance Scheme – Cont.
  - If the pressure gradient term had been treated implicitly, we would have p^{n+1} in place of p^n
  - Algorithm for time-advancing the NS eqns

- Start with a velocity field u_i^n at time t_n which is assumed divergence free. (As noted, if it is not divergence free this can be corrected.)
- Compute the combination, H_i^n, of the advective and viscous terms and its divergence (both need to be retained for later use).
- Solve the Poisson equation for the pressure p^n.
- Compute the velocity field at the new time step. It will be divergence free.
- The stage is now set for the next time step.

  - We have shown how solving the Poisson eqn for the pressure can assure that the velocity field satisfies the continuity eqn

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- A Simple Implicit Time Advance Scheme
  - Backward or implicit Euler method

Because of $\frac{\delta}{\delta x_i}\left(\frac{\delta u_i}{\delta x_j}\right)$

$$(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left(-\frac{\delta(\rho u_i u_j)^{n+1}}{\delta x_j} - \frac{\delta p^{n+1}}{\delta x_i} + \cancel{\frac{\delta \tau_{ij}^{n+1}}{\delta x_j}} \right) . \quad (7.22)$$

  - The corresponding Poisson eqn for the pressure

$$\frac{\delta}{\delta x_i}\left(\frac{\delta p^{n+1}}{\delta x_i}\right) = \frac{\delta}{\delta x_i}\left(\frac{-\delta(\rho u_i u_j)^{n+1}}{\delta x_j}\right) . \quad (7.23)$$

  - RHS term cannot be computed until the computation of the velocity field at *n+1* is complete and vice versa → the Poisson eqn and the momentum eqns have to be solved simultaneously

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- A Simple Implicit Time Advance Scheme – Cont.
  - Eq. (7.22) is a large system of non-linear eqns which must be solved for the velocity field
    - Newton-Raphson iteration or a secant method with the converged results from the preceding time step
    - Linearize the eqns about the result at the preceding time step

$$u_i^{n+1} = u_i^n + \Delta u_i \ , \quad (7.24)$$

For small Δt, $\Delta u_i \sim \Delta t \frac{\partial u_i}{\partial t}$

    - Then

Therefore $\Delta u_i \Delta u_j$: 2nd order in Δt

$$u_i^{n+1}u_j^{n+1} = u_i^n u_j^n + u_i^n \Delta u_j + u_j^n \Delta u_i + \cancel{\Delta u_i \Delta u_j} \ . \quad (7.25)$$

    - Rewrite Eq. (7.22)

$$\rho \Delta u_i = \Delta t \left(-\frac{\delta(\rho u_i u_j)^n}{\delta x_j} - \frac{\delta(\rho u_i^n \Delta u_j)}{\delta x_j} - \frac{\delta(\rho \Delta u_i u_j^n)}{\delta x_j} - \frac{\delta p^n}{\delta x_i} - \frac{\delta \Delta p}{\delta x_i} + \frac{\delta \tau_{ij}^n}{\delta x_j} + \frac{\delta \Delta \tau_{ij}}{\delta x_j} \right) . \quad (7.26)$$

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- A Simple Implicit Time Advance Scheme – Cont.
  - Still a large system of eqns → A reasonable strategy is to use the local lineaization based on Eq. (7.24) and update the eqns by the ADI method using the old pressure gradient

- Call the velocity field computed by updating the momentum equations with the old pressure gradient u_i^*. It does not satisfy the continuity equation.
- Solve a Poisson equation for the pressure correction:

$$\frac{\delta}{\delta x_i}\left(\frac{\delta \Delta p}{\delta x_i}\right) = \frac{1}{\Delta t}\frac{\delta(\rho u_i^*)}{\delta x_i} . \tag{7.27}$$

in Eq (7.20), $u_i^{n+1} \rightarrow u_i^*$ unsatisfied

→ remaining $\frac{\delta}{\delta x_i}\left(\frac{\delta \Delta p}{\delta x_i}\right)$

→ $\frac{\delta(\rho u_i^*)}{\delta x_i} - \frac{\delta(\rho u_i)^n}{\delta x_i}$

$= \Delta t \left\{ \frac{\delta}{\delta x_i}\left[\underbrace{\left(H_i^n - \frac{\delta p^n}{\delta x_i}\right)}_{=0} + \cancel{\left(\Delta H_i^n - \frac{\delta \Delta p^n}{\Delta x_i}\right)}\right]\right\}$ HOT

- Update the velocity:

$$u_i^{n+1} = u_i^* - \frac{\Delta t}{\rho}\frac{\delta \Delta p}{\delta x_i} , \tag{7.28}$$

which does satisfy continuity.

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods
  - If the pressure gradient term is not included in the source term,

$$A_{\mathrm{P}}^{u_i} u_{i,\mathrm{P}}^{n+1} + \sum_l A_l^{u_i} u_{i,l}^{n+1} = Q_{u_i}^{n+1} - \left(\frac{\delta p^{n+1}}{\delta x_i}\right)_{\mathrm{P}} . \tag{7.29}$$

  - On each outer iteration.

$$A_{\mathrm{P}}^{u_i} u_{i,\mathrm{P}}^{m*} + \sum_l A_l^{u_i} u_{i,l}^{m*} = Q_{u_i}^{m-1} - \left(\frac{\delta p^{m-1}}{\delta x_i}\right)_{\mathrm{P}} . \tag{7.30}$$

  - u_i^m: current estimate of the solution u_i^{n+1}
  - Implicit methods are preferred for steady and slow-transient flows

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - Since the pressure used in these iterations was obtained from the previous outer iteration or time step, the velocities computed from Eq. (7.30) do not normally satisfy the discretized continuity eqn → the velocities need to be corrected; this requires modification of the pressure field
  - The velocity at node P obtained from Eq. (7.30)

$$u_{i,\mathrm{P}}^{m*} = \boxed{\frac{Q_{u_i}^{m-1} - \sum_l A_l^{u_i} u_{i,l}^{m*}}{A_\mathrm{P}^{u_i}}} - \frac{1}{A_\mathrm{P}^{u_i}} \left(\frac{\delta p^{m-1}}{\delta x_i} \right)_\mathrm{P} . \quad (7.31)$$

  - u_{iP}^{m*} is not the final value for iteration m; it is a predicted value

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - Rewrite 앞 페이지 붉은 박스

$$u_{i,\mathrm{P}}^{m*} = \boxed{\tilde{u}_{i,\mathrm{P}}^{m*}} - \frac{1}{A_\mathrm{P}^{u_i}} \left(\frac{\delta p^{m-1}}{\delta x_i} \right)_\mathrm{P} . \quad (7.32)$$

  - $\tilde{u}_i^{m*}$: velocity field from which the contribution of pressure gradient has been removed
  - Next, correct the velocities so that they satisfy the continuity eqn by correcting the pressure field

$$\frac{\delta(\rho u_i^m)}{\delta x_i} = 0 , \quad (7.33)$$

  - The corrected velocities and pressure are linked by

$$u_{i,\mathrm{P}}^{m} = \tilde{u}_{i,\mathrm{P}}^{m*} - \boxed{\frac{1}{A_\mathrm{P}^{u_i}} \left(\frac{\delta p^{m}}{\delta x_i} \right)_\mathrm{P}} . \quad (7.34)$$

p^m that satisfies $\frac{\delta(\rho u_i^m)}{\delta x_i} = 0$

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - Continuity enforced by inserting Eq. (7.34) into (7.33) → discrete Poisson eqn for pressure

$$\frac{\delta}{\delta x_i}\left[\frac{\rho}{A_{\mathrm{P}}^{u_i}}\left(\frac{\delta p^m}{\delta x_i}\right)\right]_{\mathrm{P}} = \left[\frac{\delta(\rho \tilde{u}_i^{m*})}{\delta x_i}\right]_{\mathrm{P}} . \qquad (7.35)$$

  - After solving for pressure, Eq. (7.35), the final velocity field at the new iteration u_i^m, is calculated from Eq. (7.34)
  - Now the velocity field satisfies the continuity eqn, but the velocity and pressure fields do not satisfy the momentum eqn (7.30) → begin another outer iteration and the process is continued until a velocity field that satisfies both the momentum and continuity is obtained.
  - Projection methods – divergence-producing part of the field is projected out.
  - Iteration: First construct velocity field that does not satisfy continuity, then correct it by subtracting pressure gradient.

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - A pressure-correction is used instead of the actual pressure

$$u_i^m = u_i^{m*} + u' \quad \text{and} \quad p^m = p^{m-1} + p' \qquad (7.36)$$

  - Substitute the correction into the momentum eq (7.30)

$$u'_{i,\mathrm{P}} = \tilde{u}'_{i,\mathrm{P}} - \frac{1}{A_{\mathrm{P}}^{u_i}}\left(\frac{\delta p'}{\delta x_i}\right)_{\mathrm{P}} , \qquad (7.37)$$

$$\boxed{\begin{aligned} u_i^{m*} &= u_i^m - u' \\ p^{m-1} &= p^m - p' \end{aligned}}$$

where

$$\tilde{u}'_{i,\mathrm{P}} = -\frac{\sum_l A_l^{u_i} u'_{i,l}}{A_{\mathrm{P}}^{u_i}} . \qquad (7.38)$$

No contribution of Q

  - Application of Eq. (7.33) to corrected velocities and use of Eq. (7.37) produces pressure-correction eqn HW #7

$$\frac{\delta}{\delta x_i}\left[\frac{\rho}{A_{\mathrm{P}}^{u_i}}\left(\frac{\delta p'}{\delta x_i}\right)\right]_{\mathrm{P}} = \left[\frac{\delta(\rho u_i^{m*})}{\delta x_i}\right]_{\mathrm{P}} + \cancel{\left[\frac{\delta(\rho \tilde{u}'_i)}{\delta x_i}\right]_{\mathrm{P}}} . \qquad (7.39)$$

Unknown → neglected

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - Once the pressure correction is obtained, the velocities are updated using Eqs (7.36) and (7.37) – SIMPLE algorithm

SNUTT
Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - A more gentle way of treating the last term in Eq. (7.39) – approximate it rather than neglecting it

$$u'_{i,\mathrm{P}} \approx \frac{\sum_l A_l^{u_i} u'_{i,l}}{\sum_l A_l^{u_i}} \, . \qquad (7.40)$$

Weighted mean of the neighbor values

  - Approximate it from Eq. (7.38)

$$\tilde{u}'_{i,\mathrm{P}} \approx -u'_{i,\mathrm{P}} \frac{\sum_l A_l^{u_i}}{A_{\mathrm{P}}^{u_i}} \, , \quad \boxed{\leftarrow \frac{\sum_l A_l^{u_i} u'_{i,l}}{\sum_l A_l^{u_i}} \cdot \frac{\sum_l A_l^{u_i}}{A_P^{u_i}}} \qquad (7.41)$$

  - Insert it in Eq. (7.37) and rearrange

$$u'_{i,\mathrm{P}} = -\frac{1}{A_{\mathrm{P}}^{u_i} + \sum_l A_l^{u_i}} \left(\frac{\delta p'}{\delta x_i} \right)_{\mathrm{P}} \, . \qquad (7.42)$$

SNUTT
Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - With this approximation, the coeff $A_P^{u_i}$ in Eq. (7.39) is replaced by $A_P^{u_i} + \sum_l A_l^{u_i}$ and the last term disappears – SIMPLEC algorithm

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - Another method is derived by neglecting $\tilde{u}'_i$ in the first correction step as in SIMPLE, but following the correction with another corrector step – second correction to the velocity u''

Eq. (7.38)

$$u''_{i,\mathrm{P}} = \tilde{u}'_{i,\mathrm{P}} - \frac{1}{A_\mathrm{P}^{u_i}} \left(\frac{\delta p''}{\delta x_i} \right)_\mathrm{P} ; \quad \boxed{\text{Corresponding to (7.37)}} \tag{7.43}$$

  - 2nd pressure-correction eqn

$$\frac{\delta}{\delta x_i} \left[\frac{\rho}{A_\mathrm{P}^{u_i}} \left(\frac{\delta p''}{\delta x_i} \right) \right]_\mathrm{P} = \left[\frac{\delta(\rho \tilde{u}'_i)}{\delta x_i} \right]_\mathrm{P} . \quad \boxed{\text{Corresponding to (7.39)}} \tag{7.44}$$

  - Essentially an iterative method for solving Eq. (7.39) with the last term treated explicitly – PISO algorithm

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - Another similar method – the pressure-correction eqn (7.39) is solved first with the last term neglected as in SIMPLE. The pressure correction so obtained is used only to correct the velocity field so that it satisfies continuity, i.e., to obtain u_i^m. The new pressure field is calculated from the pressure eqn (7.35) using $\tilde{u}_i^m$ instead of $\tilde{u}_i^{m*}$. - SIMPLER

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - Due to the neglect of the last term in Eq. (7.39), SIMPLE does not converge rapidly.
  - Convergence can be improved if one adds only a portion of *p'* to p^{m-1}

$$p^m = p^{m-1} + \alpha_p p' \qquad (7.45)$$

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Implicit Pressure-Correction Methods – Cont.
  - Solution algorithm for this class of methods

1. Start calculation of the fields at the new time t_{n+1} using the latest solution u_i^n and p^n as starting estimates for u_i^{n+1} and p^{n+1}.
2. Assemble and solve the linearized algebraic equation systems for the velocity components (momentum equations) to obtain u_i^{m*}.
3. Assemble and solve the pressure-correction equation to obtain p'.
4. Correct the velocities and pressure to obtain the velocity field u_i^m, which satisfies the continuity equation, and the new pressure p^m.
   For the PISO algorithm, solve the second pressure-correction equation and correct both velocities and pressure again.
   For SIMPLER, solve the pressure equation for p^m after u_i^m is obtained above.
5. Return to step 2 and repeat, using u_i^m and p^m as improved estimates for u_i^{n+1} and p^{n+1}, until all corrections are negligibly small.
6. Advance to the next time step.

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Other Methods
  - Fractional step methods
    - In projection methods, the pressure is used to enforce continuity. The pressure is also used in computing the velocity field in the fist step of the method explicitly.
    - The fractional step method provides an approach that does not use pressure in the predictor step. → Essentially an approximate factorization.
    - Euler explicit advancement of the NS eqns

$$u_i^{n+1} = u_i^n + (C_i + D_i + P_i)\Delta t \qquad (7.51)$$

C_i: convective terms

D_i: diffusive terms

P_i: pressure terms

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Other Methods – Cont.
  - Fractional step methods – Cont.

$$u_i^* = u_i^n + (C_i)\Delta t \qquad (7.52)$$

$$u_i^{**} = u_i^* + (D_i)\Delta t \qquad (7.53)$$

$$u_i^{n+1} = u_i^{**} + (P_i)\Delta t \qquad (7.54)$$

  - In the 3rd step, P_i is the gradient of a quantity (pseudo-pressure or pressure-like variable) that obeys a Poisson eqn.
  - A particular fractional step method

Crank-Nicolson

$$\frac{(\rho u_i)^* - (\rho u_i)^n}{\Delta t} = \boxed{\frac{1}{2}\left[H(u_i^n) + H(u_i^*)\right]} - \frac{\delta p^n}{\delta x_i}\,, \qquad (7.55)$$

- $H(u_i)$: operator representing the discretized convective, diffusive, and source terms

Calculation of the Pressure – Cont.

- Other Methods – Cont.
  - Fractional step methods – Cont.

Zero divergence

$$\frac{(\rho u_i)^{**} - (\rho u_i)^*}{\Delta t} = \frac{1}{2}\frac{\delta p^n}{\delta x_i}\,. \qquad (7.56)$$

$$\frac{(\rho u_i)^{n+1} - (\rho u_i)^{**}}{\Delta t} = -\frac{1}{2}\frac{\delta p^{n+1}}{\delta x_i}\,. \qquad (7.57)$$

- Pressure eqn for the new pressure

$$\frac{\delta}{\delta x_i}\left(\frac{\delta p^{n+1}}{\delta x_i}\right) = \frac{2}{\Delta t}\frac{\delta(\rho u_i)^{**}}{\delta x_i}\,. \qquad (7.58)$$

- Upon solution of the pressure eqn, the new velocity field is obtained from Eq. (7.57)

$$\frac{(\rho u_i)^{n+1} - (\rho u_i)^n}{\Delta t} = \frac{1}{2}\left[H(u_i^n) + H(u_i^*)\right] - \frac{1}{2}\left(\frac{\delta p^n}{\delta x_i} + \frac{\delta p^{n+1}}{\delta x_i}\right)\,. \qquad (7.59)$$

Calculation of the Pressure – Cont.

- Other Methods – Cont.
  - Fractional step methods – Cont.
    - SIMPLE-type preferred for steady flow, while fractional step for unsteady flow

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Other Methods – Cont.
  - Streamfunction-vorticity methods
    - For incompressible 2D flows with constant fluid properties, N-S eqns can be simplified by introducing the streamfunction ψ and vorticity ω

$$\frac{\partial \psi}{\partial y} = u_x , \qquad \frac{\partial \psi}{\partial x} = -u_y , \tag{7.62}$$

$$\omega = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} . \tag{7.63}$$

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} . \tag{7.64}$$

    - Reason for introducing the streamfunction is that for flows with constant ρ, μ, and $\boldsymbol{g}$, the continuity eqn is identically satisfied and need not be dealt with explicitly
    - Subsitute Eqs. (7.62) into (7.63) leads to a kinematic eqn

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega . \tag{7.65}$$

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Other Methods – Cont.
  - Streamfunction-vorticity methods – Cont.
    - Differentiate the x and y momentum eqns w.r.t. y and x, respectively, and subtract from each other → dynamic eqn for vorticity HW #7

$$\rho\frac{\partial\omega}{\partial t} + \rho u_x\frac{\partial\omega}{\partial x} + \rho u_y\frac{\partial\omega}{\partial y} = \mu\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) . \qquad (7.66)$$

    - Pressure disappears in either of these eqns → N-S eqns have been eliminated by a set of just 2 PDEs
    - The 2 eqns are coupled through the appearance of u_x and u_y in the vorticity eqn and by the vorticity ω acting as the source term in the Poisson eqn for ψ

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Other Methods – Cont.
  - Streamfunction-vorticity methods – Cont.
    - Given an initial velocity field, the vorticity is computed by differentiation
    - The dynamic vorticity eqn is then used to compute the vorticity at the new time step
    - Having the vorticity, it is possible to compute the streamfunction at the new time step by solving the Poisson eqn
    - Having the streamfunction, the velocity components are easily obtained by differentiation
    - Repeat for the next time step

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Other Methods – Cont.
  - Streamfunction-vorticity methods – Cont.
    - Problem with BCs
    - Less popular in recent years because its extension to 3D flows is difficult
    - Difficulties in dealing with variable fluid properties, compressibility, and BCs

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Other Methods – Cont.
  - Artificial compressibilty methods
    - Whether methods developed for compressible flows can be adapted to the solution of the incompressible flows
    - Compressible flow eqns – hyperbolic, while incompressible flow eqns – mixed parabolic-elliptic
    - Difference from the lack of time derivative term in incompressible continuity eqn – compressible version contains the time derivative of density
    - Most straight-forward means of giving the incomressible eqns hyperbolic character is to append a time derivative to the continuity eqn – Time derivative of pressure is a clear choice, because density is constant and velocity appears in momentum eqns

SNUTT Seoul National University Towing Tank

Calculation of the Pressure – Cont.

- Other Methods – Cont.
  - Artificial compressibilty methods – Cont.

$$\frac{1}{\beta}\frac{\partial p}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \, , \qquad (7.67)$$

- The larger β, the more incompressible the eqns
- Fully implicit Euler scheme

$$\frac{p_P^{n+1} - p_P^n}{\beta \, \Delta t} + \left[\frac{\delta(\rho u_i)}{\delta x_i}\right]_P^{n+1} = 0 \, . \qquad (7.69)$$

- Linearize the velocity field at the new time level

$$(\rho u_i)^{n+1} \approx (\rho u_i^*)^{n+1} + \left[\frac{\partial(\rho u_i^*)}{\partial p}\right]^{n+1} (p^{n+1} - p^n) \, . \qquad (7.70)$$

- Insert Eq. (7.70) into (7.69) → obtain an eqn for the new pressure p^{n+1}

SNUTT Seoul National University Towing Tank

Remaining Sections

- Practical implementation of the methods
- Read through!

SNUTT Seoul National University Towing Tank