Lecture Note 1

Collision Probability Method

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Prof. Joo Han-gyu
Department of Nuclear Engineering

Neutron Transport Equation and Solution Difficulties

- Boltzmann Neutron Transport Equation

$$\hat{\Omega}\cdot\nabla\varphi(\vec{r},E,\hat{\Omega})+\Sigma_t(\vec{r},E)\varphi(\vec{r},E,\hat{\Omega})=\int_{\Omega'}\int_{E'}\Sigma_s(\hat{\Omega}'\rightarrow\hat{\Omega},E'\rightarrow E)\varphi(\vec{r},E',\hat{\Omega}')dE'd\hat{\Omega}'+\frac{1}{4\pi}\chi(E)\psi(\vec{r})$$

$\downarrow$

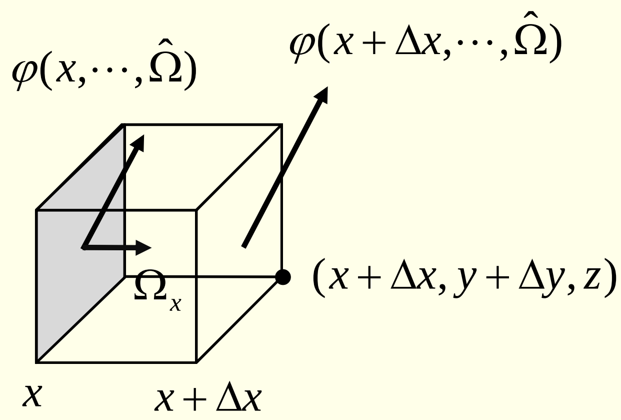
Streaming Term = Net outflow through surfaces $=\Omega_x\left(\varphi(x+\Delta x,\cdots,\hat{\Omega})-\varphi(x,\cdots,\hat{\Omega})\right)\Delta y\Delta z$

per unit volume $\rightarrow \lim_{\Delta x\rightarrow 0}\frac{\varphi(x+\Delta x,\cdots,\hat{\Omega})-\varphi(x,\cdots,\hat{\Omega})}{\Delta x}=\Omega_x\frac{\partial\varphi}{\partial x}$ $\rightarrow \hat{\Omega}\cdot\nabla\varphi$ in general

- Solution Difficulties

$$\Sigma(\vec{r},E)=\sum_i N_i(\vec{r},T(\vec{r}))\sigma_i(T(\vec{r}),E)$$

  - Strong energy dependence of Xsec
  - Angular dependence severe locally
  - Temperature dependence of Xsec which depends on flux through power



- Practical Solution Approaches
  - Multigroup
  - Angle discretization(S_n) or orthogonal expansion (P_L)
  - Iteration between flux and T/H field solutions

Three-Step Core Neutronics Calculation Procedure

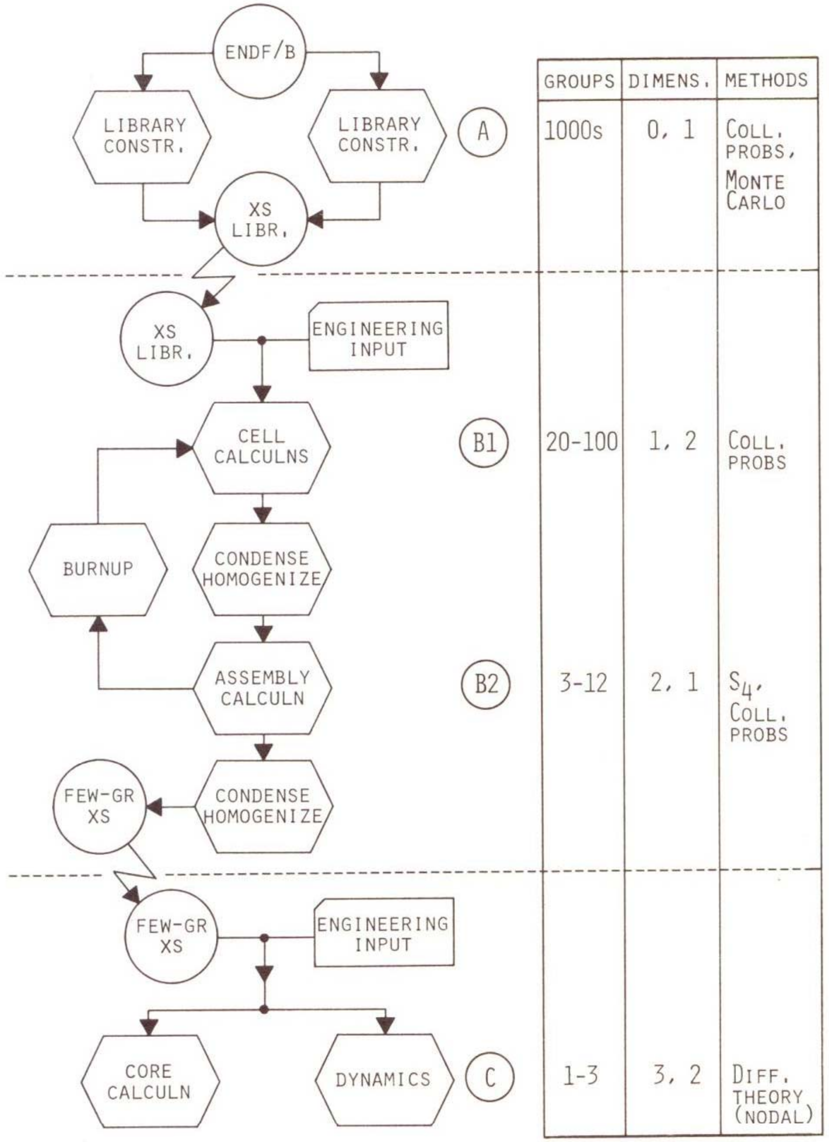


FIG. 3 Calculational flow scheme of reactor-physics design.

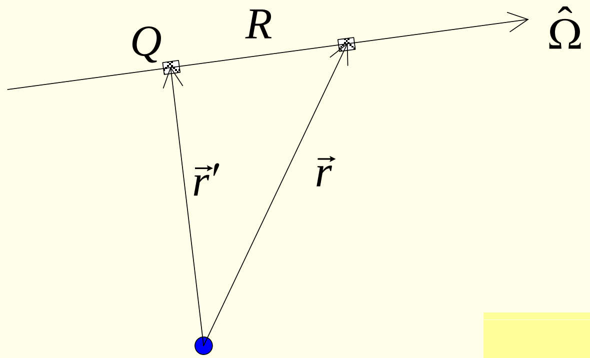
Two Forms of Multigroup Transport Equation

① Differential (Boltzmann) Transport Equation

$$\hat{\Omega}\cdot\nabla\varphi_g(\vec{r},\hat{\Omega})+\Sigma_{tg}(\vec{r})\varphi_g(\vec{r},\hat{\Omega})=Q_g$$

$$Q_g(\vec{r},\hat{\Omega})=\frac{1}{4\pi}\lambda\chi_g\psi+\sum_{g'=1}^{G}\int\Sigma_{g'g}(\vec{r},\hat{\Omega}'\rightarrow\hat{\Omega})\varphi_{g'}(\vec{r},\hat{\Omega}')d\hat{\Omega}'$$

② Integral Transport Equation (isotropic source)



$$R=\left|\vec{r}-\vec{r}'\right|\ [\text{cm}]$$

$$\varphi_g(\vec{r},\hat{\Omega})=\int_s\frac{Q_g(\vec{r}')}{4\pi R^2}e^{-\Sigma_t R}d\vec{r}')$$

$$\frac{R}{\lambda_t}\rightarrow\rho(R)$$ distance in mfp or optical length

$$\phi_g(\vec{r})=\int_V\frac{e^{-\rho(R)}}{4\pi R^2}Q_g(\vec{r}')dV'$$: Scalar Flux

$f(\phi)\rightarrow$ Integral Equation

Scattering Source in Multigroup Transport Equation

$$Q_g^s = \sum_{g'=1}^{G} \int_{\hat{\Omega}'} \Sigma_{g'g}(\vec{r},\hat{\Omega}' \to \hat{\Omega}) \varphi_g(\vec{r},\hat{\Omega}') d\hat{\Omega}'$$

$$= \int_{\hat{\Omega}'} \Sigma_{gg}(\vec{r},\hat{\Omega}' \to \hat{\Omega}) \varphi_g(\vec{r},\hat{\Omega}') d\hat{\Omega}'$$

Unknown :Self scattering, change only direction

$$+ \int_{\hat{\Omega}'} \sum_{g' \neq g} \Sigma_{g'g}(\vec{r},\hat{\Omega}' \to \hat{\Omega}) \varphi_g(\vec{r},\hat{\Omega}') d\hat{\Omega}'$$

Scattering from other groups, considered known under source iteration scheme to multigroup problems

- Differential Scattering Cross Section

$$\Sigma_{g'g}(\hat{\Omega}' \to \hat{\Omega}) = \Sigma_{g'g} f_{g'g}(\mu_s)$$

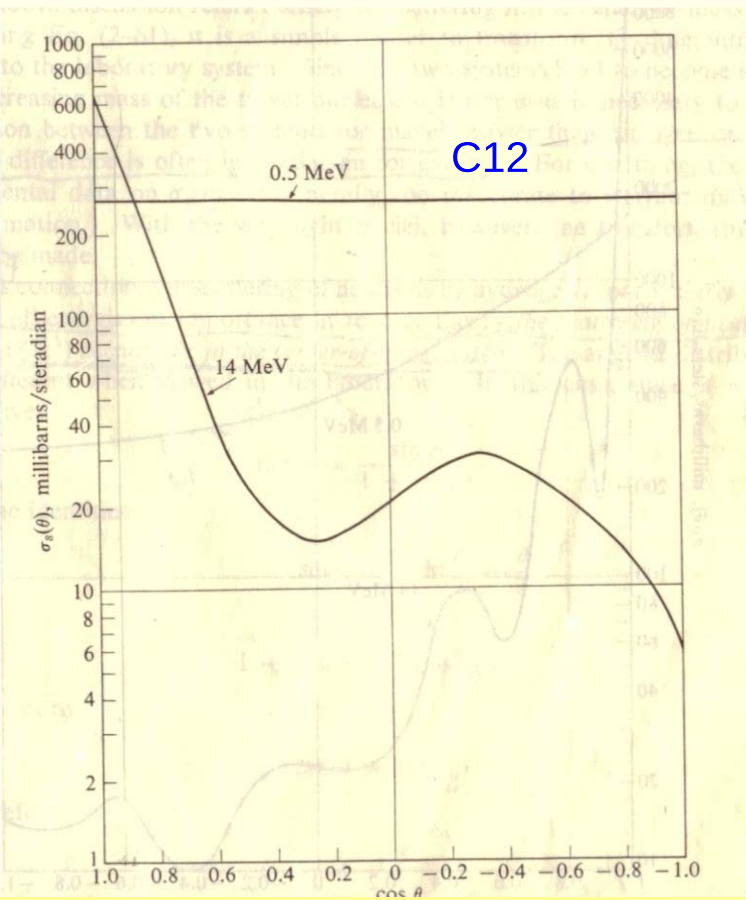
where $\mu_s = \cos(\theta_s)$ of the angle θ_s between $\hat{\Omega}'$ and $\hat{\Omega}$

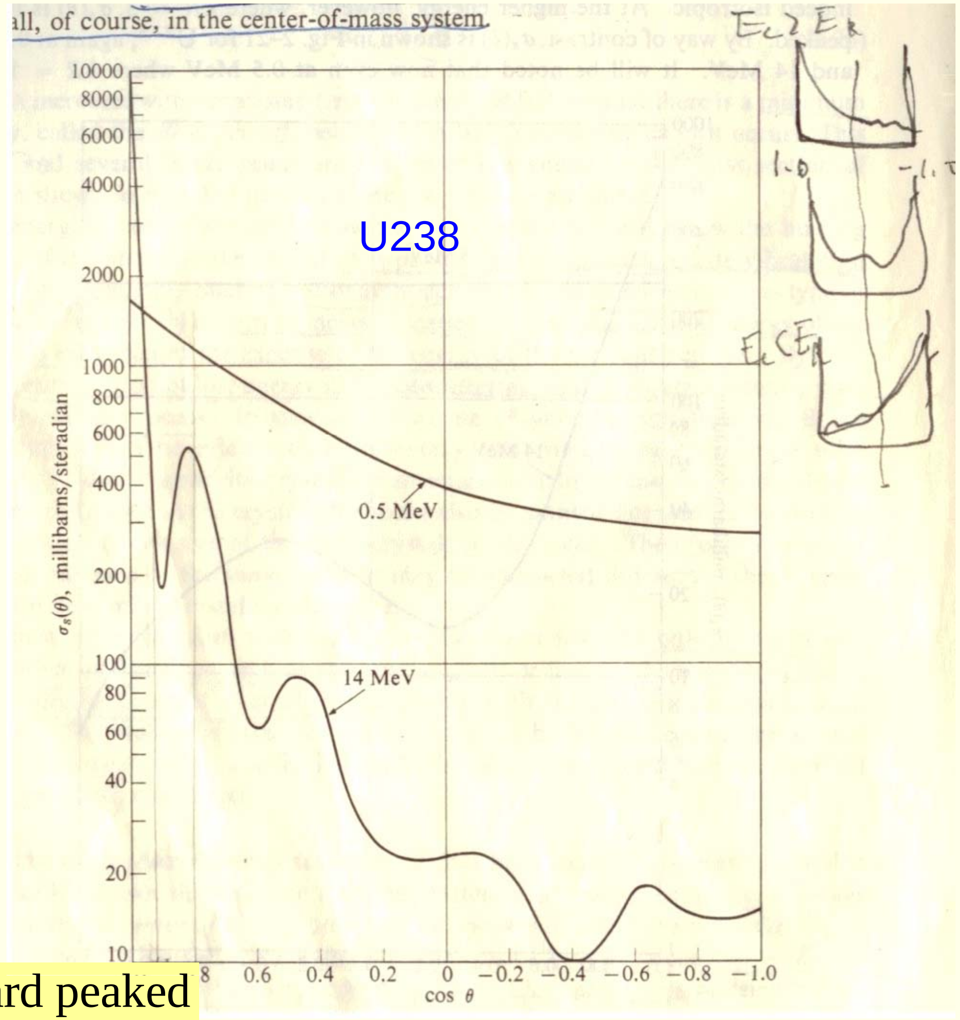
Scattering kernel ($f_{g'g}(\mu_s)$)

Differential Scattering Cross Section in CMS

- s wave scattering is isotropic in CMS ($\because l = 0 \rightarrow \tau = 0$)

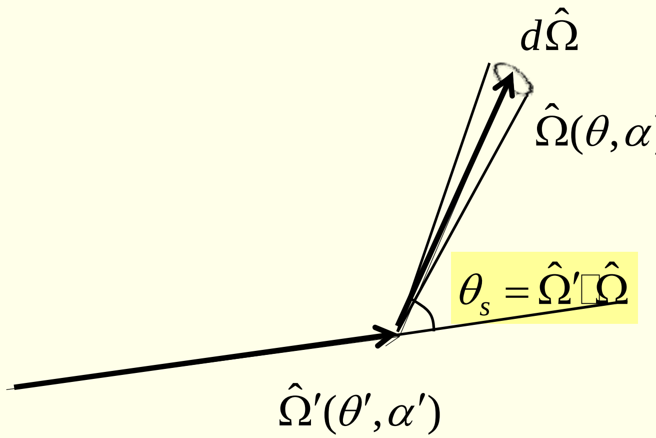
  low energy, light nucleus





- At higher energy or for heavier nuclei, forward peaked

Representation of Scattering Anisotropy

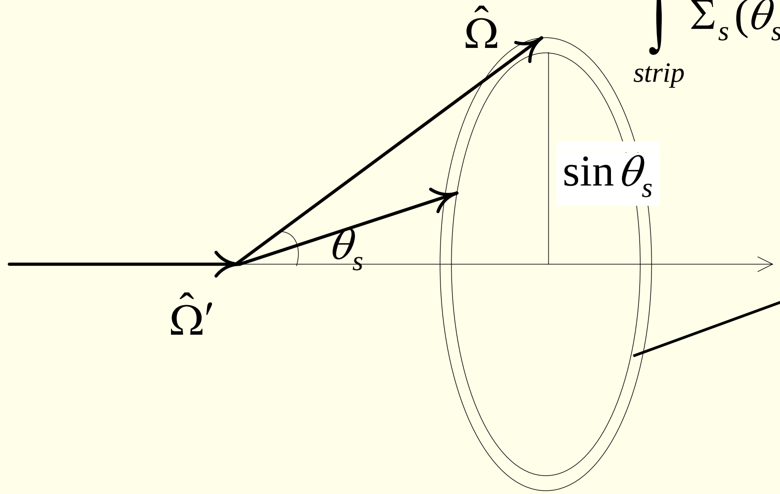


$$\Sigma_s(\theta_s, E' \rightarrow E)\, d\hat{\Omega}\, dE$$

$\Sigma_s(\theta_s, E' \rightarrow E)$ → Differential Scattering Xsec, 1/cm-steradian,eV

: Probability per unit distance of travel to scatter into angle $d\hat{\Omega}$ around $\hat{\Omega}$ and to dE from E

- Consideration of Azimuthal Symmetry



$$\int_{strip} \Sigma_s(\theta_s, E' \rightarrow E)\, d\hat{\Omega} = \int_{2\pi} \Sigma_s(\theta_s, E' \rightarrow E)\, d\theta_s \sin\theta_s\, d\alpha$$

$$= 2\pi\Sigma_s\left(\theta_s, E' \rightarrow E\right)\sin\theta_s\, d\theta_s\, dE$$

Legendre Polynomial for Scattering Anistropy

$-$Integrate over θ_0

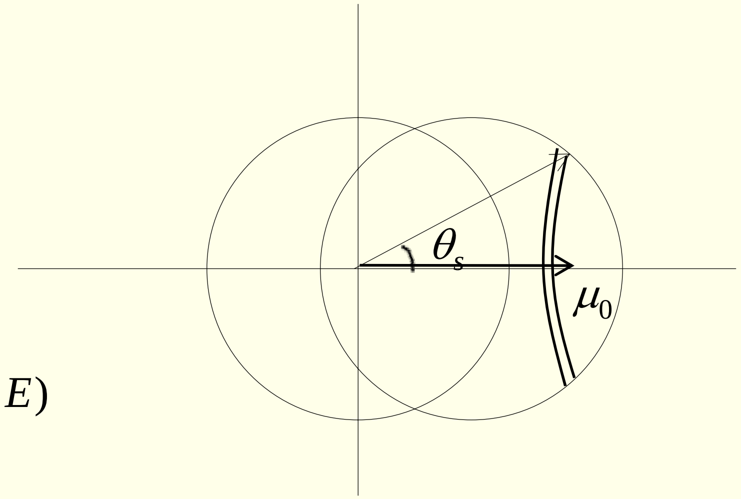
$$\int_0^{\pi} 2\pi\Sigma_s(\theta_s, E' \to E)\sin\theta_s d\theta_s$$

$$\mu = \cos\theta_s \quad \to \quad d\mu = -\sin\theta_s d\theta_s$$

$$\theta_s \; : \; 0 \to \pi$$

$$\mu : \; 1 \to -1$$

$$= \int_{-1}^{1} 2\pi\Sigma_s(\mu_s, E' \to E)d\mu_s \quad = \Sigma_s(E' \to E)$$



- Legendre Expansion of Angular Dependence

l-th Legendre moment of $f(x)$

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x) \quad \to \int_{-1}^{1} f(x)P_l(x)dx = a_l < P_l, P_l > \quad \to a_l = \frac{1}{< P_l, P_l >}\int_{-1}^{1} f(x)P_l(x)dx$$

Let $f(\mu_s) = \; 2\pi\Sigma_s(\mu_s, E' \to E)$

$$< P_l, P_l > = \int_{-1}^{1} P_l(x)P_l(x)dx = \left|P_l\right|^2$$

$$\text{Momont:} \int_{-1}^{1} 2\pi\Sigma_s(\mu_s, E' \to E)P_l(x)d\mu_s \equiv \tilde{\Sigma}_s^{(l)}(E' \to E) \to P_l \;\; \text{moment}$$

$$a_l = \frac{1}{< P_l, P_l >}\tilde{\Sigma}_s^{(l)}(E' \to E) \;\; \to 2\pi\Sigma_s(\mu_s, E' \to E) = \sum_{l=0}^{\infty}\frac{1}{< P_l, P_l >}\tilde{\Sigma}_s^{(l)}(E' \to E)P_l(\mu_s)$$

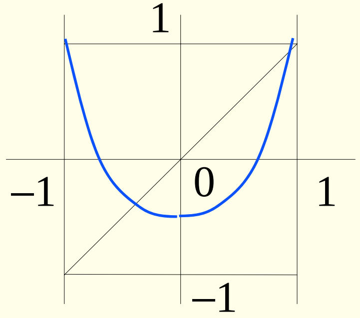
Legendre Functions

$$\int_{-1}^{1} P_l(x)dx = 0$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$



$$\int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$|P_0|^2 = \frac{2}{1}, \quad |P_1|^2 = \frac{2}{3}, \quad |P_2|^2 = \frac{2}{5}$$

$$|P_l|^2 = \frac{2}{2l+1}$$

- Final form of Legendre angular expansion

$$2\pi\Sigma_s(\mu_s, E' \to E) = \sum_{l=0}^{\infty} \frac{2l+1}{2} \tilde{\Sigma}_s^{(l)}(E' \to E) P_l(\mu_s)$$

$$\tilde{\Sigma}_s^{(0)}(E' \to E) = \int_{-1}^{1} 2\pi\Sigma_s(\mu_s, E' \to E) d\mu_s = \Sigma_s(E' \to E)$$

$$\tilde{\Sigma}_s^{(1)}(E' \to E) = \int_{-1}^{1} 2\pi\Sigma_s(\mu_s, E' \to E)\mu_s d\mu_s$$

– in 4π space

$$\Sigma_s(\mu_s, E' \to E) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \tilde{\Sigma}_s^{(l)}(E' \to E) P_l(\mu_s)$$

Linear Scattering Anisotropy

- Neglect second and higher order terms in Legendre expansion

$$2\pi\Sigma(\mu_s, E' \to E) = \frac{1}{2}\Sigma_s(E' \to E) + \frac{3}{2}\tilde{\Sigma}_s^{(1)}(E' \to E)\mu_s + \frac{5}{2}\tilde{\Sigma}_s^{(2)}(E' \to E)\mu_s^2 + \cdots$$

- Average Cosine

$$\bar{\mu}_s(E' \to E) = \int_{-1}^{1} p(\mu_s, E' \to E)\mu_s d\mu_s$$

$$= \int_{-1}^{1} \frac{\Sigma_s(\mu_s, E' \to E)}{\Sigma_s(E' \to E)}\mu_s \cdot d\mu_s = \frac{\tilde{\Sigma}_s^{(1)}(E' \to E)}{\Sigma_s(E' \to E)}$$

$$\bar{\mu}_s(E') = \frac{1}{\Sigma_s(E')}\int_0^{\infty}\int_{-1}^{1} 2\pi\Sigma_s(\mu_s, E' \to E) d\mu_s dE$$

$$= \frac{\tilde{\Sigma}_s^{(1)}(E')}{\Sigma_s(E')}$$

$$\to \tilde{\Sigma}_s^{(1)}(E) = \bar{\mu}_s(E)\Sigma_s(E)$$

$$\Sigma_{s1} = \bar{\mu}_s \Sigma_{s0}$$

- In Multigroup Approach

$$S = \begin{bmatrix} \Sigma_{11} & \cdots & \Sigma_{g'1} & \Sigma_{G1} \\ \vdots & \ddots & \Sigma_{g'2} & \\ \Sigma_{1g} & \Sigma_{2g} & \ddots & \Sigma_{Gg} \\ & & \Sigma_{g'g} & \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \Sigma_{11} & \Sigma_{21} & \\ \Sigma_{12} & \Sigma_{22} & \\ \Sigma_{13} & & \end{bmatrix} + \frac{3}{2}\mu_s \begin{bmatrix} \tilde{\Sigma}_{11}^{(1)} & \tilde{\Sigma}_{21}^{(1)} & \\ \tilde{\Sigma}_{12}^{(1)} & \tilde{\Sigma}_{22}^{(1)} & \\ \tilde{\Sigma}_{13}^{(1)} & & \end{bmatrix}$$

$$(\Sigma_{g'g} \equiv \Sigma_{g' \to g})$$

$$\Sigma_{sg} = \sum_{g'=1}^{G}\Sigma_{gg'}$$

$$\bar{\mu}_{sg} = \frac{\tilde{\Sigma}_{sg}^{(1)}}{\Sigma_{sg}} = \frac{\sum_{g'=1}^{G}\tilde{\Sigma}_{gg'}^{(1)}}{\sum_{g'=1}^{G}\Sigma_{gg'}}$$

← defined for out-scattering

How to Construct Scattering Source with Linear Scattering Anisotropy

$$\Sigma_{g'g}(\mu_s, \alpha) = \left(\frac{1}{2}\Sigma_{g'g} + \frac{3}{2}\tilde{\Sigma}^{(1)}_{g'g} \right) \times \frac{1}{2\pi} = \frac{1}{4\pi}\Sigma_{g'g} + \frac{3}{4\pi}\mu_s \tilde{\Sigma}^{(1)}_{g'g}$$

- Scattering Source

$$Q_{g'g}(\hat{\Omega}) = \int_{\hat{\Omega}} \Sigma_{g'g}(\mu_s)\varphi_{g'}(\hat{\Omega}')d\hat{\Omega}'$$

$$= \int_{4\pi} \left(\frac{1}{4\pi}\Sigma_{g'g} + \frac{3}{4\pi}\tilde{\Sigma}^{(1)}_{g'g}\mu_s \right)\varphi_{g'}(\hat{\Omega}')d\hat{\Omega}'$$

μ_s: $\hat{\Omega}' \cdot \hat{\Omega}$

$$= \frac{\Sigma_{g'g}}{4\pi}\int_{4\pi}\varphi_{g'}(\hat{\Omega}')d\hat{\Omega}' + \frac{3\tilde{\Sigma}^{(1)}_{g'g}}{4\pi}\int_{4\pi}\hat{\Omega}' \cdot \hat{\Omega}\varphi_{g'}(\hat{\Omega}')d\hat{\Omega}'$$

$$= \frac{1}{4\pi}\Sigma_{g'g}\phi_{g'} + \frac{3}{4\pi}\tilde{\Sigma}^{(1)}_{g'g}\vec{J}_{g'} \cdot \hat{\Omega}$$

$$\int_{4\pi}\varphi_{g'}\left(\hat{\Omega}'\right)\hat{\Omega}'d\hat{\Omega}' \cdot \hat{\Omega} = \vec{J}_{g'} \cdot \hat{\Omega}$$

$\rightarrow$ need the current info to construct scattering source under P_1 representation of scattering

Transport Cross Section

- Suppose first absorpion free medium

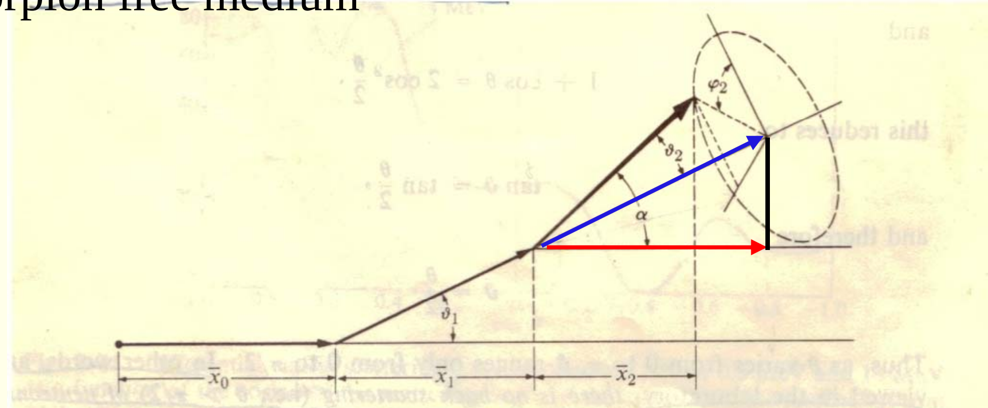


Fig. 2-22. Diagram for calculating the transport cross section.

$$\bar{x}_1 = \lambda_s \bar{\mu} \qquad \bar{x}_2 = \lambda_s \bar{\mu}^2 \qquad \bar{x}_n = \lambda_s \bar{\mu}^n \qquad \lambda = \lambda_s \left(1 + \bar{\mu} + \bar{\mu}^2 + \cdots\right) = \frac{\lambda_s}{1-\bar{\mu}} = \frac{1}{\Sigma_s \left(1-\bar{\mu}\right)}$$

$\Sigma_s^{tr} = \left(1-\bar{\mu}\right)\Sigma_s$ $\rightarrow$ reduced scattering cross section to consider anisotropic scattering
isotroic scattering treatment, transport corrected scattering Xsec

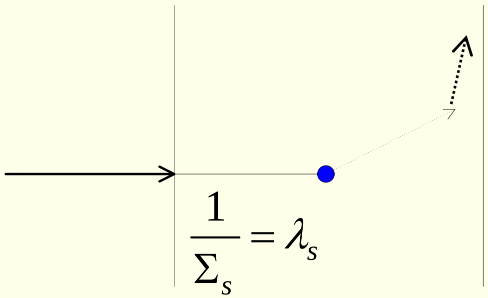
- With now absorption

$\Sigma_{tr} = \Sigma_a + \Sigma_s^{tr}$ $\rightarrow$ transport cross section, later to be used to define diffusion coefficient

Projection by fraction μ to the incoming direction
Projection to the inital direction after n collision= μ^n

Transport Correction

- Transport Correction



$$\frac{1}{\Sigma_s} = \lambda_s$$

$$\lambda_{tr} = \frac{\lambda_s}{1-\bar{\mu}_s} = \frac{1}{\Sigma_s(1-\bar{\mu}_s)} = \frac{1}{\Sigma_s^{tr}}$$

$\Sigma_s^{tr} = \Sigma_s(1-\bar{\mu}_s)$: transport corrected scattering xsec

$$\Sigma_t' = \Sigma_a + \Sigma_s^{tr} = \Sigma_a + \Sigma_s(1-\bar{\mu}_s) = \Sigma_{tr}$$

–Anisotropic Scattering

Forward peak $\bar{\mu}_s > 0$

$$\bar{\mu}_s \Sigma_{sg} = \frac{\sum_{g'=1}^{G} \tilde{\Sigma}_{g'g}^{(1)} J_{g'}}{\sum_{g'=1}^{G} J_{g'}}$$

- Transport Correction in Multigroup Approach for Isotropic Scattering Treatment

$$S = \left[\Sigma_{g'g}\right] = \frac{1}{2}\begin{bmatrix} \Sigma_{11} & \Sigma_{21} & \Sigma_{31} \\ \Sigma_{12} & \Sigma_{22} & \Sigma_{32} \\ \Sigma_{13} & \Sigma_{23} & \Sigma_{33} \end{bmatrix} + \frac{3}{2}\mu_s \begin{bmatrix} \tilde{\Sigma}_{11}^{(1)} & \tilde{\Sigma}_{21}^{(1)} & \tilde{\Sigma}_{31}^{(1)} \\ \tilde{\Sigma}_{12}^{(1)} & \tilde{\Sigma}_{22}^{(1)} & \tilde{\Sigma}_{32}^{(1)} \\ \tilde{\Sigma}_{13}^{(1)} & \tilde{\Sigma}_{23}^{(1)} & \tilde{\Sigma}_{22}^{(1)} \end{bmatrix} \approx \begin{bmatrix} \Sigma_{11}^{tr} & \Sigma_{21} & \Sigma_{31} \\ \Sigma_{12} & \Sigma_{22}^{tr} & \Sigma_{32} \\ \Sigma_{13} & \Sigma_{23} & \Sigma_{33}^{tr} \end{bmatrix}$$

$\Sigma_{gg}^{tr} = \Sigma_{gg} - \bar{\mu}_s \Sigma_{sg}$

correction only diagonal

$\bar{\mu}_s \Sigma_{sg} = \tilde{\Sigma}_{sg}^{(1)} = \sum_{g'=1}^{G} \tilde{\Sigma}_{gg'}^{(1)}$: Column sum(out-scat) in principle!, but usually replaced by row sum with inscattering current weighting for high energy groups (consistent P_1 for g<G$_{epithermal}$)

1G Integral Transport Equation with Transport Correction

$$\phi_g(\vec{r}) = \int_V \frac{e^{-\rho_g(R)}}{4\pi R^2}\left(\Sigma_{gg}^{tr}\phi(\vec{r}) + Q_g'(\vec{r})\right)dV'$$

$\rho_g = \Sigma_{tr,g}R = \frac{R}{\lambda_{tr,g}}$ $\quad \Sigma_{tr,g} = \Sigma_{tg} - \bar{\mu}\Sigma_{sg}$

$Q_g'(\vec{r})$: Isotropic Source After Transport Correction

$\Sigma_{gg}^{tr} = \Sigma_{gg} - \bar{\mu}\Sigma_{gg}$: for self scattering source

$Q_g'(\vec{r}) = \lambda\chi_g\psi + \sum_{g'\neq g}\Sigma_{g'g}(\vec{r})\phi_{g'}(\vec{r})$: Fission + Scattering from other groups

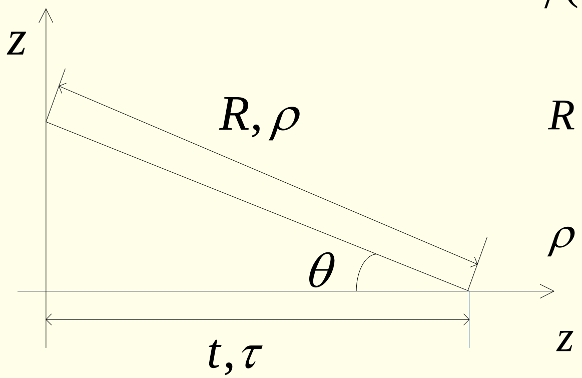
where $\psi(\vec{r}) = \sum_{g'=1}^{G}\nu\Sigma_{fg'}(\vec{r})\phi_{g'}(\vec{r})$

- Kernels of Integral Transport Equation
  - For point source

$n(\vec{r}' \to \vec{r}) = \frac{e^{-\rho(R)}}{4\pi R^2}$: flux due to unit source

Kernels of Integral Transport Equation in 2D Problems

- Flux at point away R from a unit line source
  - line source



$$\phi(t)=\int_{-\infty}^{\infty}\frac{e^{-\rho(R)}}{4\pi R^2}dz$$

$$R=f(t,z)=\frac{t}{\cos\theta},\quad \frac{z}{t}=\tan\theta,\quad dz=t\tan'\theta d\theta=\frac{t}{\cos^2\theta}d\theta$$

$$\rho=\frac{\tau}{\cos\theta}$$

$$z:-\infty\to\infty$$

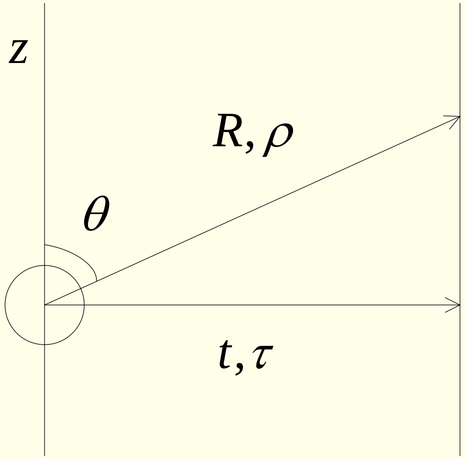
$$\theta:-\frac{\pi}{2}\to\frac{\pi}{2}$$

$$\phi(t)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{-\frac{\tau}{\cos\theta}}}{4\pi\frac{t^2}{\cos^2\theta}}t\frac{1}{\cos^2\theta}d\theta=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{-\frac{\tau}{\cos\theta}}}{4\pi t}d\theta=\frac{1}{2\pi t}\int_0^{\frac{\pi}{2}}e^{-\frac{\tau}{\cos\theta}}d\theta=\frac{ki_1(\tau)}{2\pi t}$$

Bickely Function of order n: $Ki_n(\tau)\equiv\int_0^{\frac{\pi}{2}}\cos^{n-1}\theta e^{-\frac{\tau}{\cos\theta}}d\theta$

Probability of Horizontal Uncollided Movement

- Probability of a source neutron to move t horizontally uncollided $\rightarrow$ Emitted in $d\hat{\Omega}$, then travel R



$$d\hat{\Omega} = \sin\theta d\theta d\alpha$$

$$p(\tau) = \int_0^{\pi}\int_0^{2\pi} \frac{d\Omega}{4\pi} e^{-\rho(R)} = \int_0^{\pi} \frac{1}{2} e^{-\rho(R)} \sin\theta d\theta$$

$$= \int_0^{\pi} \frac{1}{2} e^{-\frac{\tau}{\sin\theta}} \sin\theta d\theta$$

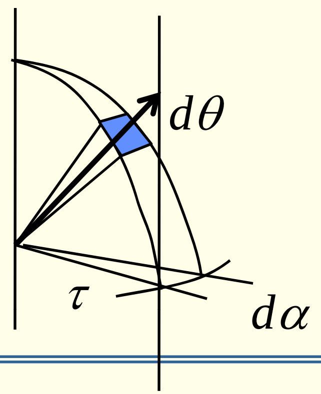
$$\theta' = \frac{\pi}{2} - \theta \qquad d\theta' = -d\theta$$

$$\theta : 0 \rightarrow \pi \qquad \theta' : \frac{\pi}{2} \rightarrow -\frac{\pi}{2}$$

$$sin\theta = sin(\frac{\pi}{2} - \theta') = cos\theta'$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} e^{-\frac{\tau}{\cos\theta'}} \cos\theta' d\theta'$$

$$= \int_0^{\frac{\pi}{2}} e^{-\frac{\tau}{\cos\theta}} \cos\theta d\theta \qquad = Ki_2(\tau)$$

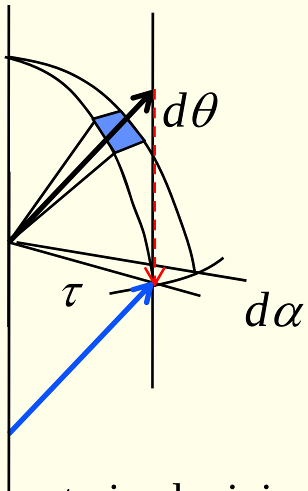


$$\text{or } p(\tau) = \frac{1}{\frac{d\alpha}{2\pi}} \int_0^{\pi} e^{-\rho(R)} \frac{sin\theta d\theta d\alpha}{4\pi} = \int_0^{\pi} \frac{1}{2} e^{-\rho(R)} \sin\theta d\theta$$

fraction to reach τ out of the neutrons emitted to dα

Optimum Polar Angle Set in 2D Transport Calculation

- If we want to describe the neutron motion with discrete angles in 2D transport calculations, what would be the the appropriate polar angles ?
  - Converse the uncollided movement probabilty as closely as possible



$$p(\tau) = Ki_2(\tau) = \int_0^{\frac{\pi}{2}} e^{-\frac{\tau}{\cos\theta}} \cos\theta d\theta \quad = \sum_{m=1}^{M} w_m e^{-\frac{\tau}{\cos\theta_m}} \cos\theta_m$$

- Quadrature representation of integral

$$Or \quad p(\tau) = Ki_2(\tau) = \int_0^{\frac{\pi}{2}} e^{-\frac{\tau}{\sin\theta}} \sin\theta d\theta = \int_0^1 e^{-\frac{\tau}{\sqrt{1-\mu^2}}} d\mu \quad = \sum_{m=1}^{M} w_m e^{-\frac{\tau}{\sqrt{1-\mu_m}}}$$

- Constrained minimization problem to find w_m and μ_m ?

$$E_{max} = \underset{0<\tau<\tau_{max}}{Max} \left| Ki_2(\tau) - \sum_{m=1}^{M} w_m e^{-\frac{\tau}{\sqrt{1-\mu_m}}} \right|$$

can be solved by IMSL routine NCNLS

Minimize E_{max} $\quad subject\ to \sum_{m=1}^{M} w_m = 1$ and $0 < \mu_m < 1$

Properties of Bickley Function

- Derivative and Integral of Bickley Function
  - Definition of Bickley Function

$$Ki_n(x) = \int_0^{\frac{\pi}{2}} \cos^{n-1}\theta e^{-\frac{x}{\cos\theta}} d\theta$$

  - Derivative of Bickley Function

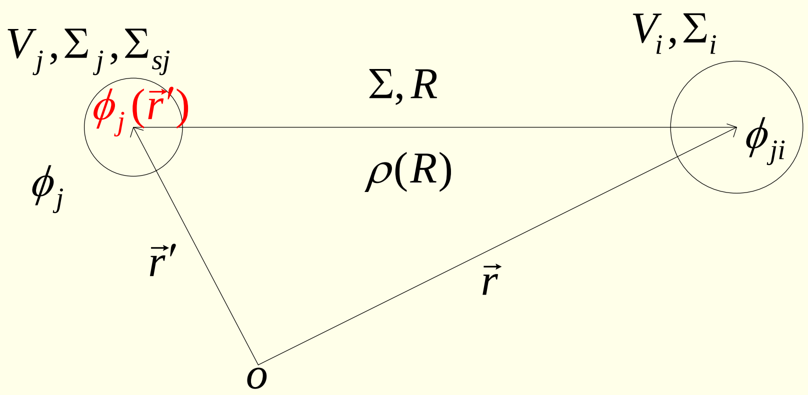
$$\frac{dKi_n(x)}{dx} = \int_0^{\frac{\pi}{2}} \cos^{n-1}\theta \left(-\frac{1}{\cos\theta}\right) e^{-\frac{x}{\cos\theta}} d\theta$$

$$= -\int_0^{\frac{\pi}{2}} \cos^{n-2}\theta e^{-\frac{x}{\cos\theta}} d\theta \quad = -Ki_{n-1}(x)$$

  - Integral of Bickley Function

Integral from 0 to x	$Ki_n(x) - Ki_n(0) = -\int_0^x Ki_{n-1}(y)dy$ $\int_0^x Ki_n(y)dy = Ki_{n+1}(0) - Ki_{n+1}(x)$
Integral from x to ∞	$Ki_n(\infty) - Ki_n(x) = -\int_x^\infty Ki_{n-1}(y)dy$ $\int_x^\infty Ki_n(y)dy = Ki_{n+1}(x)$

Transport Kernel



$$\phi_j \equiv \frac{1}{V_j}\int_{V_j} \phi_j(\vec{r}_j) dV_j$$ Flat Flux Approximation

- What would be the flux(ϕ_{ji}) in V_i induced by ϕ_j and Q'_j in V_j?

① Source in V_j (per unit volume)

$$\Sigma_{sj}\phi_j + Q'_j$$

② Flux at $\vec{r}_i$ due to source in dV_j at $\vec{r}_j$

$$\phi(\vec{r}_j \to \vec{r}_i) dV_j = \frac{e^{-\rho(R)}}{4\pi R^2}(\Sigma_{sj}\phi_j + Q'_j) dV_j$$ ($Q'_j \to Q_j$ from now on for simplicity)

Transport Kernel

③ Total flux at $\vec{r}_i$ due to whole source in V_j

$$\phi_{ji}(\vec{r}_i) = \int_{V_j} \phi(\vec{r}_j \to \vec{r}_i) dV_j$$

$$= \int_{V_j} \frac{e^{-\rho(R)}}{4\pi R^2} (\Sigma_{sj}\phi_j + Q_j) dV_j$$

④ Total flux in V_i

$$\int_{V_i} \phi_{ji}(\vec{r}_i) dV_i \equiv \phi_{ji} V_i = \int_{V_i}\int_{V_j} \frac{e^{-\rho(R)}}{4\pi R^2} (\Sigma_{sj}\phi_j + Q_j) dV_j dV_i$$

$$\therefore \phi_{ji} = \frac{1}{V_i} \int_{V_i}\int_{V_j} \frac{e^{-\rho(R)}}{4\pi R^2} (\Sigma_{sj}\phi_j + Q_j) dV_j dV_i$$

$$= \underbrace{\frac{1}{V_i} \int_{V_i}\int_{V_j} \frac{e^{-\rho(R)}}{4\pi R^2} dV_j dV_i}_{T_{ji}} (\Sigma_{sj}\phi_j + Q_j) = T_{ji}(\Sigma_{sj}\phi_j + Q_j)$$

T_{ji} $\to$ Source to flux conversion factor

= Transport Kernel

Reciprocity in Transport Kernel

$$T_{ji} = \frac{1}{V_i}\int_{V_i}\int_{V_j} n(\vec{r}_j \to \vec{r}_i)\,dV_j dV_i$$

$$T_{ij} = \frac{1}{V_j}\int_{V_j}\int_{V_i} n(\vec{r}_i \to \vec{r}_j)\,dV_i dV_j$$

$$\int_{V_j}\int_{V_i} n(\vec{r}_i \to \vec{r}_j)\,dV_i dV_j = \int_{V_i}\int_{V_j} n(\vec{r}_j \to \vec{r}_i)\,dV_j dV_i$$

$\Rightarrow T_{ji}V_i = T_{ij}V_j$: Reciprocity Relation

Total flux due to all V_j

$$\phi_i = \sum_j \phi_{ji} = \sum_j T_{ji}(\Sigma_{sj}\phi_j + Q_j)$$

First Collision Probability

- What would be the collision rate of $\vec{r}_i$ caused by the unit source at $\vec{r}_j$?

$$n(\vec{r}_j \to \vec{r}_i)\Sigma_i = P(\vec{r}_j \to \vec{r}_i) \quad \leftarrow R_{ji} = \text{reaction rate at } \vec{r}_i \text{ per unit source at } \vec{r}_j$$

= Probability that a neutron born isotropically at $\vec{r}_j$ has the first collision at $\vec{r}_i$

- Total collision rate in V_i due to source in V_j

$$T_{ji} = \frac{1}{V_i}\int_{V_i}\int_{V_j} n(\vec{r}_j \to \vec{r}_i)\,dV_j\,dV_i$$

$$\Sigma_i\phi_{ji}V_i\left(=\int \Sigma_i\phi(\vec{r}_i)\,dV_i\right) = \Sigma_i V_i T_{ji}(\Sigma_{sj}\phi_j + Q_j)$$

$$= \Sigma_i\int_{V_i}\int_{V_j}\frac{e^{-\rho(R)}}{4\pi R^2}(\Sigma_{sj}\phi_j + Q_j)\,dV_j\,dV_i$$

$$= \int_{V_i}\int_{V_j}\Sigma_i n(\vec{r}_j \to \vec{r}_i)\,dV_j\,dV_i(\Sigma_{sj}\phi_j + Q_j)$$

$$= \underbrace{\int_{V_i}\int_{V_j}P(\vec{r}_j \to \vec{r}_i)\,dV_j\,dV_i\frac{1}{V_j}}_{=\tilde{P}_{ji}}\underbrace{(\Sigma_{sj}\phi_j + Q_j)V_j}_{\text{Total source in } V_j} = \tilde{P}_{ji}(\Sigma_{sj}\phi_j + Q_j)V_j$$

First Collision Probability

$$\tilde{P}_{ji} = \frac{1}{V_j}\int_{V_i}\int_{V_j} P(\vec{r}_j \to \vec{r}_i)\,dV_j\,dV_i$$

$\frac{1}{V_j}$ → souce density if there is one source in V_j

$\tilde{P}_{ji}$ → Probability that a neutron born isotropically in V_j suffers the first collision in V_i

$\Rightarrow$ Collision probability for volume j, i

$$\Sigma_i \phi_{ji} V_i = \tilde{P}_{ji} V_j (\Sigma_{sj}\phi_j + Q_j)$$

$$= \tilde{P}_{ji} V_j \Sigma_j \left(\frac{\Sigma_{sj}}{\Sigma_j}\phi_j + \frac{Q_j}{\Sigma_j}\right)$$

$$= P_{ji}\left(c_j\phi_j + \frac{Q_j}{\Sigma_j}\right)$$

→ For the advantage in calculation

$\rightarrow$ collision rate=flux-to-collision factor $\times$ flux

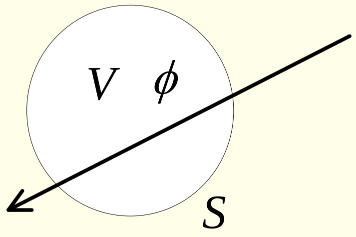
$$P_{ji} = \tilde{P}_{ji} V_j \Sigma_j$$

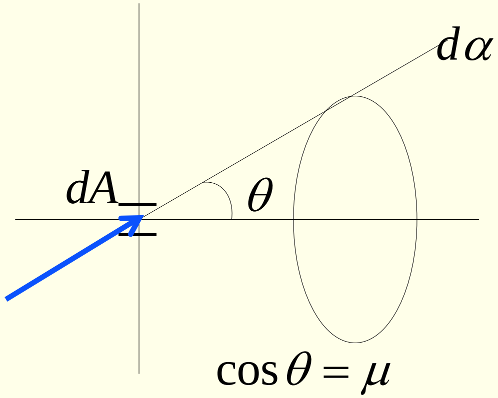
$c_j = \frac{\Sigma_{sj}}{\Sigma_j}$: ratio of self-scattering to total collision

$\frac{Q_j}{\Sigma_j}$: source driven flux $\left(\Sigma_j\phi_j = q_j \rightarrow \phi_j = \frac{q_j}{\Sigma_j}\right)$

$$\Sigma_i \phi_i V_i = \sum_j P_{ji}\left(c_j\phi_j + \frac{Q_j}{\Sigma_j}\right)$$

Cosine Current





For infinite uniform composition

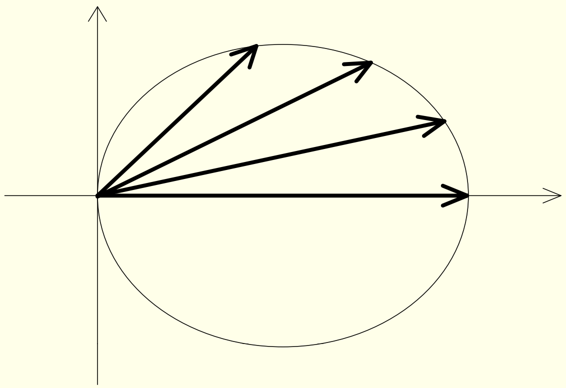
$\Rightarrow$ Uniform and isotropic angular flux

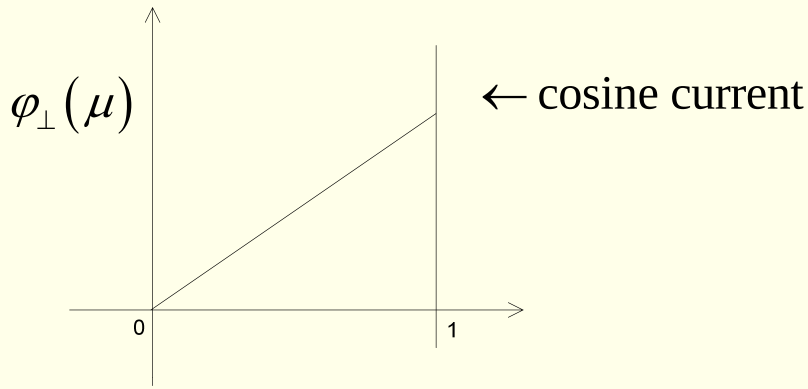
$$\varphi(\mu) = \text{const}$$

$$\varphi(\hat{\Omega}) = \frac{\phi}{4\pi} \qquad \varphi(\mu) = \frac{1}{2}\phi$$

Neutrons passing through unit surface area at the boundary wall $= \varphi_{\perp}(\mu) = \frac{1}{2}\phi\mu$

$$J_{out} = \int_0^1 \varphi(\mu)\cdot\mu \, d\mu = \int_0^1 \frac{1}{2}\phi\mu \, d\mu = \frac{1}{4}\phi$$

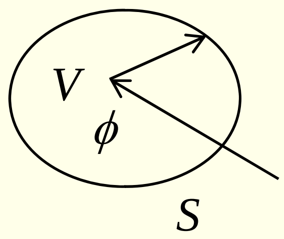




SNURPL

Escape Probability

- For uniform source Q in V, what would be ϕ?



$$\Sigma_{tg}\phi_g = \Sigma_{gg}\phi_g + Q'_g \quad \rightarrow \Sigma_{rg}\phi_g = Q'_g \qquad \Sigma_{rg} = \Sigma_{tg} - \Sigma_{gg} \text{ (Removal Xsec)}$$

$$\rightarrow \phi_g = \frac{Q'_g}{\Sigma_{rg}}$$

For 1 source (excluding self scattering source) in $V \rightarrow Q' = \frac{1}{V}$ $\quad \rightarrow \phi = \frac{1}{\Sigma_r V}$

- P : Probability that a neutron born in V escape throught S, escape probability

Γ: Absorption Blackness - Probability that a neutron entering uniformly through surface S with cosine current distribution is absorbed in V

– Suppose 1 source neutron in V, what is the number of neutrons escaping from V through S? P

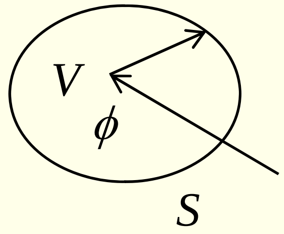
It should be balanced by the neutron to be absorbed in V after entering through S

$(= in - current \times surface\ area \times \Gamma)$

$$\frac{1}{4}\phi S\Gamma = 1 \cdot P \qquad \rightarrow \Gamma = \frac{4}{\phi S}P = \frac{4V}{S}\Sigma_r P = \bar{\ell}\Sigma_r P \qquad \bar{\ell} = \frac{4V}{S} : \text{mean chord length}$$

First Collision Escape Probability

- p : First collision escape probability - escape V without having any collision in V
  γ: First collision probability in V for neutron coming through S with cosine current



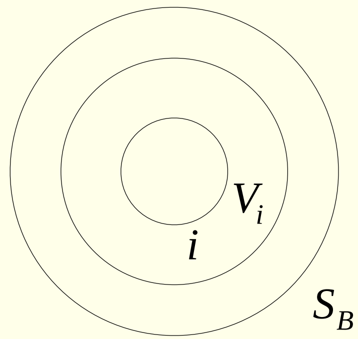
- Collision Rate: $\Sigma_{tg}\phi = \left(\Sigma_{rg} + \Sigma_{gg}\right)\phi_g = \Sigma_{gg}\phi_g + Q'_g$

  For 1 source neutron in V, $Q'_g = \dfrac{1}{V}$, $\phi = \dfrac{1}{\Sigma_r V}$ How many collision in V?

  No collision on the path!

  Collision rate per unit volume: $\Sigma\phi = \dfrac{\Sigma}{\Sigma_r V}$

  Total collisions in V = $\Sigma\phi V = \dfrac{\Sigma}{\Sigma_r}$



- Balance between first collisions for incoming and exiting neutrons

$$\frac{1}{4}\phi S\gamma = \frac{\Sigma}{\Sigma_r} p \qquad \rightarrow \gamma = \frac{4V}{S}\Sigma p = \bar{\ell}\Sigma p$$

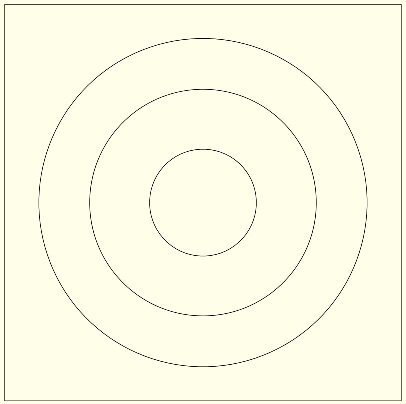
or $\dfrac{1}{4}\phi S\gamma = \Sigma\phi V p$

HW3: Prove the following rigorously:

$$\gamma_i = \frac{4V_i}{S_B}\Sigma_i p_i \ (Eq.24d)$$

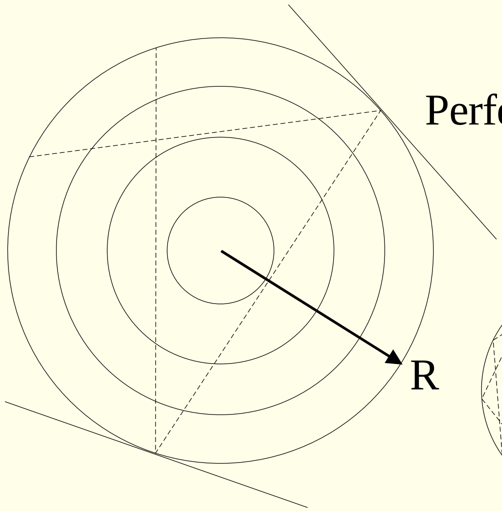
Pin-cell Problem

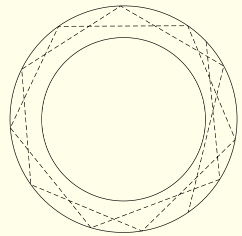
- Wigner-Seitz Approximation



$$\pi R^2 = P^2$$

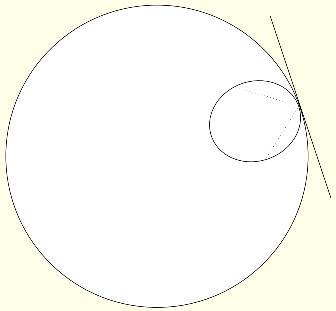
$$R = \frac{P}{\sqrt{\pi}}$$





Newmarch effect:
some neutrons born at an outer ring can't reach the innermost ring (fuel) resulting abnormal high flux in coolant

– White Boundary Condition



Cosine current:
incoming current
with cosine dist.

collect all outgoing neutrons then
shoot back with cosine current!

Albedo

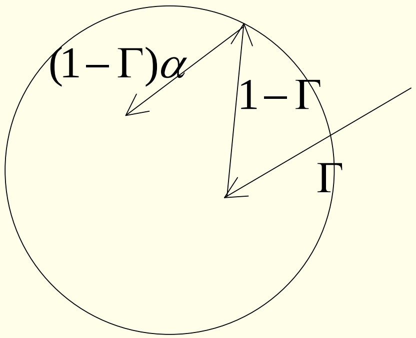
- Partial Reflection with Albedo

$$\alpha = \frac{J_{net}}{\phi_s}$$

$$\alpha = \frac{J_{in}}{J_{out}} \quad \rightarrow J_{in} = \alpha J_{out}$$

$$\begin{cases} \alpha = 0 : \text{Black} \\ \alpha \quad : \text{Gray} \\ \alpha = 1 : \text{Reflective} \end{cases}$$

- Boundary Multiplication
  - for single incoming neutron



$$1 + (1-\Gamma)\alpha + \left((1-\Gamma)\alpha\right)^2 + \cdots = \frac{1}{1-(1-\Gamma)\alpha} \equiv n_\alpha^\Gamma$$

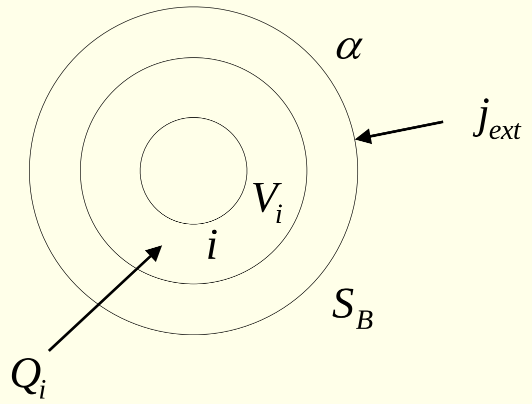
$\rightarrow$ Boundary multiplication factor

– Equivalent to n_α^Γ incoming neutrons

Circular Pin-cell Problem

- Problem Statement

Find $\phi_{i,}$ j^{+} and j^{-} for given α, j_{ext} and Q_i



– Total Absorption Blanckness for Multiple Interior Regions

$$\Gamma = \sum_{i=1}^{n} \Gamma_i$$

$X_i^k(\alpha)$: Flux at i due to unit source density at k

$Y_i\ (\alpha)$: Flux at i due to unit incoming neutron current through surface S_B

$$\phi_i = Y_i(\alpha) j_{ext} + \sum_{k=1}^{n} X_i^k(\alpha) Q_k \qquad \begin{cases} Y_i(\alpha) = f(Y_i), & Y_i \equiv Y_i(0) \\ X_i^k(\alpha) = f(X_i), & X_i^k \equiv X_i^k(0) \end{cases}$$

Circular Pin-cell Problem

- Relation between patial flux due to incoming neutrons (Y_i) and absorption blackness

  Total removal rate in V_i per unit incoming neutron $(\Sigma_{ri}Y_i\square V_i)$

  =Absorption per unit incoming neutron (Γ_i) $\rightarrow \Gamma_i = \Sigma_{ri}Y_iV_i$

- Flux due to incoming current in case of multiple reflection

$$Y_i(\alpha) = Y_i n_\alpha^\Gamma = \frac{Y_i}{1-(1-\Gamma)\alpha}$$

- Flux due to internal source in case of multiple reflection

  x_k: # of neutrons reaching the surface at first escape for unit source density in V_k

$$① \; x_k = V_k - \sum_{j=1}^{n} \Sigma_{rj} X_j^k V_j$$

$$② \; x_k = V_k P_k = \frac{S_B}{4\Sigma_{rk}}\Gamma_k = \frac{S_B}{4}V_kY_k \quad (\because \Gamma_k = \Sigma_{rk}V_kY_k)$$

  –what is the flux due to the returning neutrons per unit source density in k? $\alpha x_k Y_i(\alpha)$

$$X_i^k(\alpha) = X_i^k + \alpha x_k Y_i(\alpha) = X_i^k + \alpha x_k \frac{Y_i}{1-(1-\Gamma)\alpha}$$

Circular Pin-cell Problem

- Coupled equation for $\alpha = 0$

$$\Sigma_i V_i \phi_i = \sum_j P_{ji}(\underset{\uparrow}{c_i} \phi_j + \frac{Q_j}{\Sigma_j})$$

Scattering @ j

- Flux due to unit source density at $k \rightarrow Q_j = \delta_{jk}$

$$\Sigma_i V_i X_i^k = \sum_j P_{ji}(\underset{\uparrow}{c_j} X_j^k + \frac{\delta_{jk}}{\Sigma_j}) = \sum_j P_{ji} c_j X_j^k + \frac{P_{ki}}{\Sigma_k} \quad \forall i$$

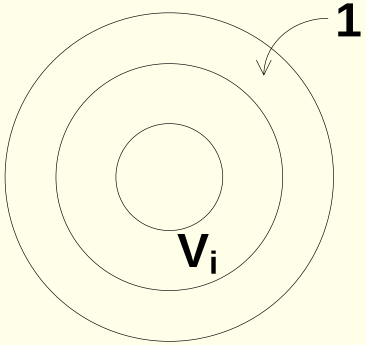
induced flux from source @ k

- Coupled Linear System for X_i^k

$$\begin{bmatrix} \Sigma_1 V_1 - P_{11}c_1 & -P_{21}c_2 & \cdots & -P_{n1}c_n \\ -P_{12}c_1 & \Sigma_2 V_2 - P_{22}c_2 & \cdots & -P_{n2}c_n \\ \vdots & \vdots & \ddots & \vdots \\ -P_{1n}c_1 & -P_{2n}c_n & \cdots & \Sigma_n V_n - P_{nn}c_n \end{bmatrix} \begin{bmatrix} X_1^k \\ X_2^k \\ \vdots \\ X_n^k \end{bmatrix} = \begin{bmatrix} \frac{P_{k1}}{\Sigma_k} \\ \frac{P_{k2}}{\Sigma_k} \\ \vdots \\ \frac{P_{kn}}{\Sigma_k} \end{bmatrix}$$

Circular Pin-cell Problem

- Flux due to 1 incoming neutron



γ_i:Collision rate due to the first collision of the neutron coming from the surface

$$\Sigma_i V_i Y_i = \sum_{i=1}^{n} P_{ij} c_j Y_j + \gamma_i$$

← first collision from incoming neutrons uncollided elsewhere

$$\underbrace{\begin{bmatrix} \Sigma_1 V_1 - P_{11}c_1 & -P_{21}c_2 & \cdots & -P_{n1}c_n \\ -P_{12}c_1 & \Sigma_2 V_2 - P_{22}c_2 & \cdots & -P_{n2}c_n \\ \vdots & \vdots & \ddots & \vdots \\ -P_{1n}c_1 & -P_{2n}c_n & \cdots & \Sigma_n V_n - P_{nn}c_n \end{bmatrix}}_{A} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$$

- Relation btwn collision prob. and first collision escape probability

$$p_i = 1 - \sum_{j=1}^{n} \tilde{P}_{ij} \quad \rightarrow \gamma_i = \frac{4V_i}{S_B}\Sigma_i p_i = \frac{4}{S_B}(\Sigma_i V_i - \sum_{j=1}^{n} \Sigma_i V_i \tilde{P}_{ij}) = \frac{4}{S_B}(\Sigma_i V_i - \sum_{j=1}^{n} P_{ij})$$

- Linear System

$$Ax = b \qquad b_i = \frac{P_{ki}}{\Sigma_k} \quad \text{for } X_i^k \text{ and } b_i = \frac{4}{S_B}(\Sigma_i V_i - \sum_{j=1}^{n} P_{ij}) \quad \text{for } Y_i$$

Circular Pin-cell Problem

- Total number of Neutrons reaching boundary first time

$$x = \sum_{i=1}^{n} Q_i x_i$$

- Total number of neutrons escaping

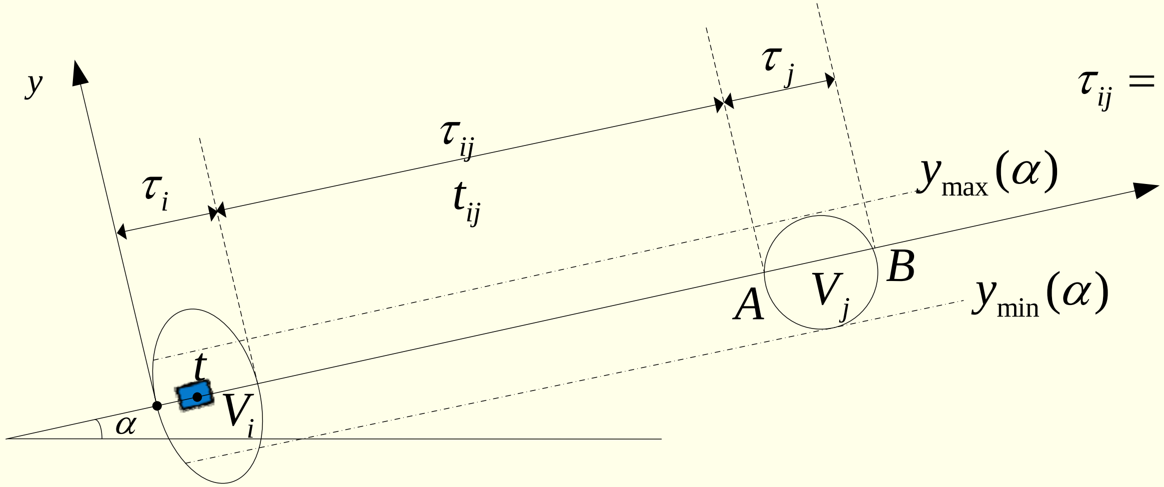
$$J^+(\alpha) = \frac{x + (1-\Gamma) J^{ext}}{1-\alpha(1-\Gamma)}$$

- Total number of neutrons incoming

$$J^-(\alpha) = \frac{\alpha x + J^{ext}}{1-\alpha(1-\Gamma)}$$

Calculation of Collision Probability

$\tau_{ij} = \Sigma_{ij} t_{ij}$: Optical length

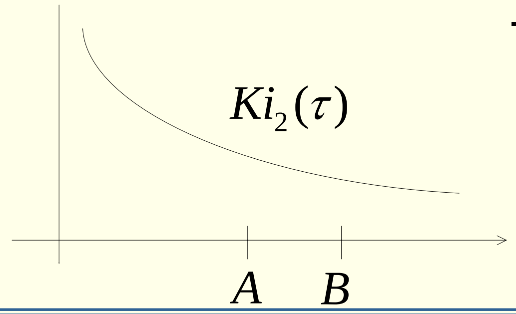


-Probability to move from point t to the left side of V_j without collision

$$P_A(t) = Ki_2(\tau_{ij} + \Sigma_i (t_i - t))$$

-Probability to move from point t to the right side of V_j without collision

$$P_B(t) = Ki_2(\tau_{ij} + \tau_j + \Sigma_i (t_i - t))$$



-Probability for collision between A and B

$$\tilde{P}_{ij}(t; y, \alpha) = P_A(t) - P_B(t)$$

$$= Ki_2(\underbrace{\tau_{ij} + \Sigma_i (t_i - t)}_{\tau_{ij} + \tau_i - \Sigma_i \tau}) - Ki_2(\underbrace{\tau_{ij} + \Sigma_i (t_i - t) + \tau_j}_{\tau_{ij} + \tau_i + \tau_j - \Sigma_i \tau})$$

Transport Kernel

- for unit source density in t_i $\rightarrow$ t_i neutrons in t_i

$$\tau_i = \tau(y), \tau_{ij} = \tau_{ij}(y)$$

$$\tilde{P}_{ij}(y,\alpha) = \frac{1}{t_i}\int_0^{t_i} \tilde{P}_{ij}(t;y,\alpha)dt \;\; = \frac{1}{t_i}\int_0^{t_i} (Ki_2(\tau_{ij}+\tau_i-\Sigma_i t) - Ki_2(\tau_{ij}+\tau_i+\tau_j-\Sigma_i t))dt$$

$$= \frac{1}{\Sigma_i t_i}\int_{\tau_{ij}}^{\tau_{ij}+\tau_i} (Ki_2(\tau) - Ki_2(\tau+\tau_j))d\tau$$

$$\tau_{ij}+\tau_i-\Sigma_i t = \tau, \quad d\tau = -\Sigma_i dt, \;\; dt = \frac{1}{-\Sigma_i}d\tau$$

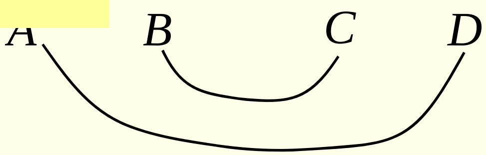
$$t = 0 \rightarrow t_i$$

$$\tau = \tau_{ij}+\tau_i \rightarrow \tau_{ij}$$

$$\int_a^b K_i(x)dx = K_{i+1}(a) - K_{i+1}(b)$$

$$= \frac{1}{\Sigma_i t_i}\left[(Ki_3(\tau_{ij}) - Ki_3(\tau_{ij}+\tau_i)) - (Ki_3(\tau_{ij}+\tau_j) - Ki_3(\tau_{ij}+\tau_i+\tau_j))\right]$$

$$= \frac{1}{\Sigma_i t_i}\left[\left(Ki_3(\tau_{ij}) + Ki_3(\tau_{ij}+\tau_i+\tau_j)\right) - \left(Ki_3(\tau_{ij}+\tau_i) + (Ki_3(\tau_{ij}+\tau_j)\right)\right]$$



- for uniform isotropic source in volume element $t_i dy$ in V_i

$$l_{AD} + l_{BC} - l_{AC} - l_{BD}$$

$$\tilde{P}_{ij} = \int_0^{y_{\max}(\alpha)} \tilde{P}_{ij}(y,\alpha)\frac{t_i dy}{V_i}$$

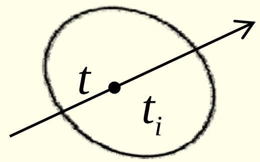
$$= \frac{1}{\Sigma_i V_i}\int_0^{y_{\max}(\alpha)} \left[\left(Ki_3(\tau_{ij}) + Ki_3(\tau_{ij}+\tau_i+\tau_j)\right) - \left(Ki_3(\tau_{ij}+\tau_i) + (Ki_3(\tau_{ij}+\tau_j)\right)\right]dy$$

$$P_{ij} = \Sigma_i V_i \tilde{P}_{ij} \;\; = \int_0^{y_{\max}(\alpha)} \left[\left(Ki_3(\tau_{ij}) + Ki_3(\tau_{ij}+\tau_i+\tau_j)\right) - \left(Ki_3(\tau_{ij}+\tau_i) + (Ki_3(\tau_{ij}+\tau_j)\right)\right]dy \;\; = P_{ji}$$

Transport Kernel

- What if $j = i$?

Given $Ki_2(\tau)$, what is the probability to have collision within τ $=1-Ki_2(\tau)$



$$\rightarrow \tilde{P}_{ii}(t;y,\alpha) = 1 - Ki_2(\Sigma_i(t_i - t))$$

$$\tilde{P}_{ii}(y,\alpha) = \frac{1}{t_i}\int_0^{t_i} \tilde{P}_{ii}(t;y,\alpha)dt = \frac{1}{t_i}\int_0^{t_i}\left[1 - Ki_2(\Sigma_i(t_i - t))\right]dt$$

$$= 1 - \frac{1}{\Sigma_i t_i}\left[Ki_3(0) - Ki_3(t_i)\right]$$

$$\tilde{P}_{ii} = \int_0^{y_{\max}} \tilde{P}_{ii}(y,\alpha)\frac{t_i dy}{V_i}$$

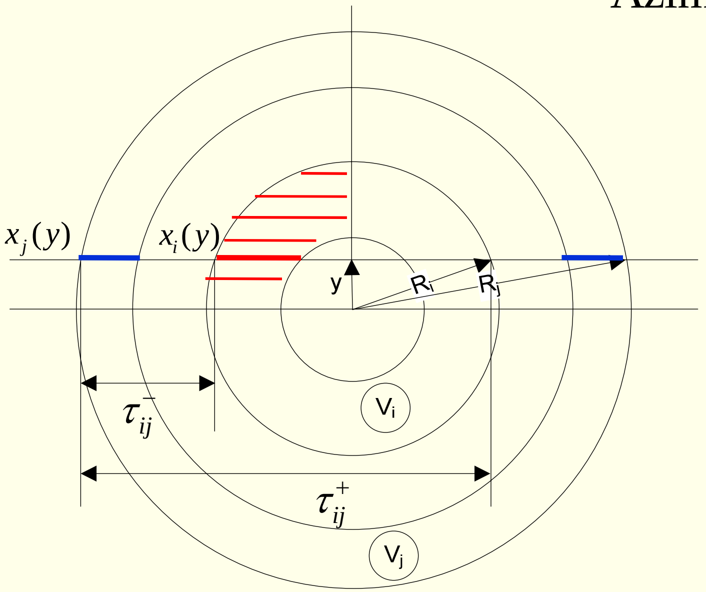
$$= 1 - \frac{1}{\Sigma_i V_i}\int_0^{y_{\max}}\left[Ki_3(0) - Ki_3(\tau_i)\right]dy$$

$$P_{ii} = \Sigma_i V_i \tilde{P}_{ii}$$

$$= \Sigma_i V_i - \int_0^{y_{\max}}\left[Ki_3(0) - Ki_3(\tau_i)\right]dy$$

Annular Geometry

- Collision Probability in Annular Geometry



Azimuthal symmetry $\rightarrow$ no need for consideration of α

$$x_i = (\sqrt{R_i^2 - y^2})_{mfp} \qquad P_{ij} = P_{ji}$$

$$\tau_{ij}^- = x_j - x_i$$

$$\tau_{ij}^+ = x_j + x_i$$

– for a line source located left in V_i

left moving

$$P_{ij}^-(y) = Ki_3(\tau_{i\,j-1}^-) + Ki_3(\tau_{i-1\,j}^-) - \left(Ki_3(\tau_{i\,j}^-) + Ki_3(\tau_{i-1\,j-1}^-)\right)$$

right moving

$$P_{ij}^+(y) = Ki_3(\tau_{i-1\,j-1}^+) + Ki_3(\tau_{i\,j}^+) - \left(Ki_3(\tau_{i-1\,j}^+) + Ki_3(\tau_{i\,j-1}^+)\right)$$

$$P_{ij}(y) = 2\left(P_{ij}^-(y) + P_{ij}^+(y)\right)$$:2 for two sources (1st and 2nd Quadrants)

$$= 2[Ki_3(\tau_{i\,j}^+) - Ki_3(\tau_{i\,j}^-) + Ki_3(\tau_{i-1\,j-1}^+) - Ki_3(\tau_{i-1\,j-1}^-)$$

$$-\left(Ki_3(\tau_{i-1\,j}^+) - Ki_3(\tau_{i-1\,j}^-) + Ki_3(\tau_{i\,j-1}^+) - Ki_3(\tau_{i\,j-1}^-)\right)]$$

Let $P_{ij} = \int_0^{R_i} P_{ij}(y)dy$ $\quad S_{ij} = \int_0^{R_i}\left(Ki_3(\tau_{ij}^+) - Ki_3(\tau_{ij}^-)\right)dy$ $\quad \rightarrow P_{ij} = 2\left(S_{ij} + S_{i-1\,j-1} - (S_{i\,j-1} + S_{i-1\,j})\right)$

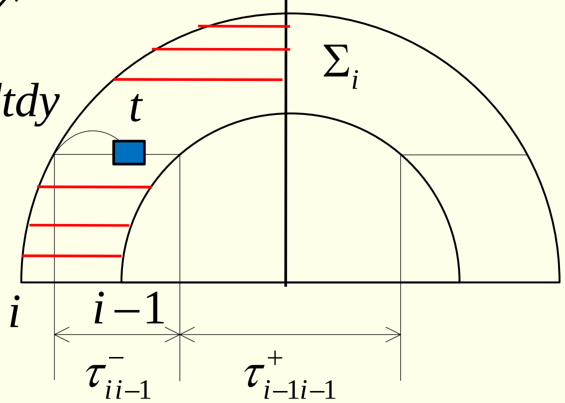
Self Collision Probability

for uniform source density $\rightarrow$ total $\frac{1}{4}V_i$ source neutrons

Source moving to both directions

- What if $j = i$? $\quad 1 - Ki_2(x)$: Prob. to have collision in x

Source in $dtdy$: $\frac{1}{2}dtdy$



right moving $\quad$ right moving

$$\left(1 - Ki_2(\tau^-_{i\,i-1} - \Sigma_i t)\right) + \left(1 - Ki_2(\Sigma_i t)\right) + \left(Ki_2(\tau^-_{i\,i-1} - \Sigma_i t + \tau^+_{i-1\,i-1}) - Ki_2(\tau^-_{i\,i-1} - \Sigma_i t + \tau^+_{i-1\,i})\right) \Rightarrow C$$

– # of neutrons to have collision in upper half of V_i for unit source neutron in quadrant

$$n^Q_{ii} = \frac{1}{2}\int_0^{R_i}\int_0^{t_i} C\,dt\,dy = \frac{1}{2}\left[2\cdot\frac{V_i}{4} + \frac{1}{\Sigma_i}\int_0^{R_i}\left\{\begin{array}{l}\left(Ki_3(\tau^-_{i\,i-1}) - Ki_3(0)\right) + \left(Ki_3(\tau^-_{i\,i-1}) - Ki_3(0)\right)\\ +\left(Ki_3(\tau^+_{i-1\,i-1}) - Ki_3(\tau^+_{i\,i-1}) - \left(Ki_3(\tau^+_{i-1\,i}) - Ki_3(\tau^+_{i\,i})\right)\right)\end{array}\right\}dy\right]$$

from two 1's above

$$= \frac{1}{2}\left[\frac{V_i}{2} + \frac{1}{\Sigma_i}\int_0^{R_i}\left\{\begin{array}{l}\left(Ki_3(\tau^+_{ii}) - Ki_3(\tau^-_{ii})\right) + \left(Ki_3(\tau^+_{i-1\,i-1}) - Ki_3(\tau^-_{i-1\,i-1})\right)\\ -\left(Ki_3(\tau^+_{i\,i-1}) - Ki_3(\tau^-_{i\,i-1})\right) - \left(Ki_3(\tau^+_{i-1\,i}) - Ki_3(\tau^-_{i-1\,i})\right)\end{array}\right\}dy\right]$$

$$\tilde{P}_{ii} = \frac{n^Q_{ii}}{\frac{V_i}{4}} \qquad = \frac{1}{2}\left[\frac{V_i}{2} + \frac{1}{\Sigma_i}\left(S_{ii} + S_{i-1\,i-1} - (S_{i\,i-1} + S_{i-1\,i})\right)\right]$$

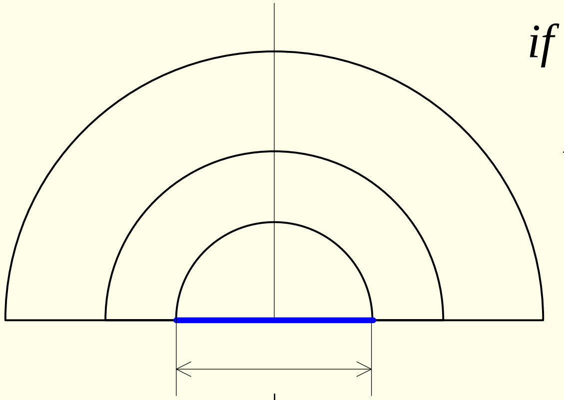
$$\tau_i = \tau^-_{ii-1}, \quad \tau^-_{ii} = 0$$

$$P_{ii} = \Sigma_i V_i + 2\left(S_{ii} + S_{i-1\,i-1} - (S_{i\,i-1} + S_{i-1\,i})\right)$$

Generalized Collision Probability Kernel

–For the innermost region (i=1)

$$S_{ij} = \int_0^{R_i} \left(Ki_3(\tau_{ij}^+) - Ki_3(\tau_{ij}^-) \right) dy$$



$$if \;\; i = 0 \rightarrow \tau_{oj}^+ = R_j, \;\; \tau_{0j}^- = R_j. \;\; S_{0j} = 0?$$

$$P_{0j} = Ki_3(\tau_{1j}^+) + Ki_3(\tau_{1j-1}^-) - \left(Ki_3(\tau_{1j-1}^+) + Ki_3(\tau_{1j}^-) \right)$$

$$= S_{1j} - S_{1j-1}$$

$$= S_{1j} + S_{0j-1} - \left(S_{0j} + S_{1j-1} \right)$$

Set $S_{0j} = 0$, then apply the general formula!

• Generalized Collision Probability Kernel

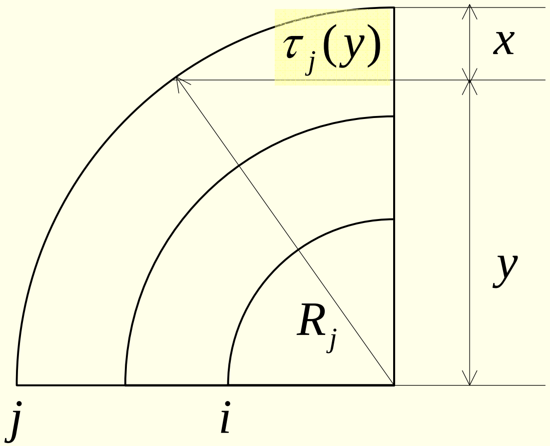
$$P_{ij} = \delta_{ij} \Sigma_i V_i + 2\left(S_{ii} + S_{i-1\,i-1} - (S_{i\,i-1} + S_{i-1\,i}) \right)$$

Calculation of Sij

$$S_{ij}^k = \int_{R_{k-1}}^{R_k} \left(Ki_3(\tau_{ij}^+(y)) - Ki_3(\tau_{ij}^-(y)) \right) dy \quad \rightarrow S_{ij} = \sum_{k=1}^{i} S_{ij}^k$$

$$p = 2\frac{y - R_{k-1}}{\underbrace{R_k - R_{k-1}}_{\Delta_k}} - 1 \qquad dp = \frac{2}{\Delta_k} dy \longrightarrow S_{ij}^k = \frac{\Delta_k}{2}\int_{-1}^{1} \tilde{f}(p)dp \; = \frac{\Delta_k}{2}\omega_i f(x_i)$$

$$\int_{-1}^{1} f(x)dx = \Sigma \omega_i f_i$$: Gauss Quadrature



$$\tau_j = \sqrt{R_j^2 - y^2} = \sqrt{R_j^2 - (R_j - x)^2} = \sqrt{2R_i x - x^2} \approx \sqrt{2R_i}\sqrt{x}$$

$$= \sqrt{2R_i}\,\tilde{p} = \sqrt{2R_i \Delta R_i}\, p$$

$$\boxed{\sqrt{x} = \tilde{p} \rightarrow \tilde{p}^2 = x}$$

to normalize

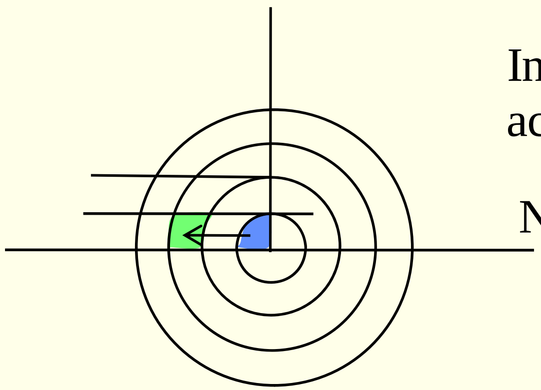
$$\frac{x}{R_j - R_{j-1}} = \frac{\tilde{p}^2}{\Delta R_j} = p^2 \rightarrow \frac{R_j - y}{\Delta R_j} = p^2$$

$$\frac{-dy}{\Delta R_j} = 2pdp \qquad \rightarrow S_{ij}^k = \int_0^1 \left(Ki_3(\tau_{ij}^+(p)) - Ki_3(\tau_{ij}^-(p)) \right) \frac{2}{\Delta R_j} pdp$$

$$\int_0^1 xf(x)dx = \Sigma \omega_i f_i \quad : \text{ Gauss-Jacobi Quadrature}$$

$$\rightarrow S_{ij}^k = \frac{2}{\Delta R_j} \int_0^1 \left(Ki_3(\tau_{ij}^+(p)) - Ki_3(\tau_{ij}^-(p)) \right) pdp$$

$$= \frac{2}{\Delta R_j} \sum_k w_k \left(Ki_3(\tau_{ij}^+(p_k)) - Ki_3(\tau_{ij}^-(p_k)) \right)$$



Instead of calculating S_{ij} for whole annular region, accummulate contributions from each sector

Nested loops required:

- Loop over R_k (y direction)
  - Loop over i(*source*)
    - Loop over j($j \geq i$)

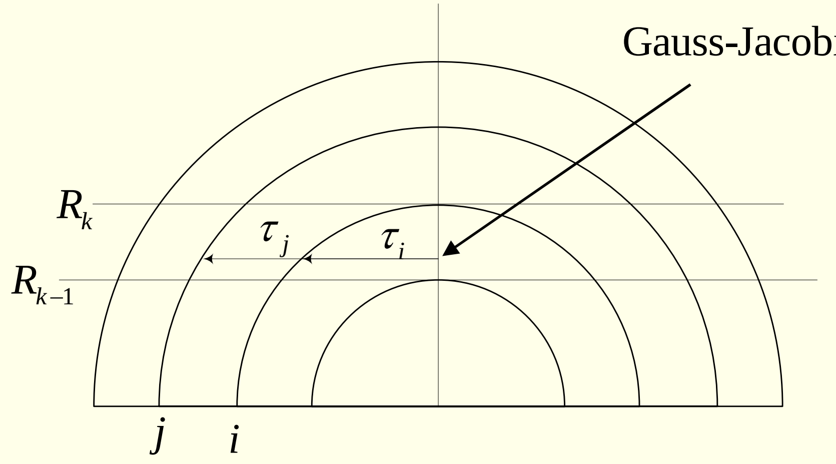
Actual Implementation

- Actual implementation

① Calculate S_{ij}

② Calculate P_{ij}

$$P_{ij} = \delta_{ij}\Sigma_i V_i + 2\left(S_{ii} + S_{i-1\,i-1} - (S_{i\,i-1} + S_{i-1\,i})\right)$$



③ Construct the linear system

$$\Sigma_i V_i \phi_i = \sum_j p_{ji}(c_i\phi_j + \frac{Q_j}{\Sigma_j})$$

$$\Sigma_i V_i X_i^k = \sum_j P_{ji}(c_j X_j^k + \frac{Q_j}{\Sigma_j}) = \sum_j P_{ji} c_j X_j^k + \frac{P_{ki}}{\Sigma_k}$$

④ Solve the linear system and find flux and current