Lecture Note 4

B_1 Method

Nov. 2008

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Derivation of B_L Equation (1)

- Scattering Source Expansion

$$\Sigma(z,E'\to E,\mu_s)=\sum_{l=0}^{L}\frac{2l+1}{4\pi}\Sigma_l(z,E'\to E)P_l(\mu_s)$$

$$P_l(\mu_s)=P_l(\mu)P_l(\mu')+2\sum_{m=1}^{l}\tilde{P}_l^m(\mu)\tilde{P}_l^m(\mu')\cos m(\alpha-\alpha')\cdots\leftarrow \text{Addition Theorem}$$

$$\int_0^\infty\int_{\hat{\Omega}'}\sum_{l=0}^{L}\left(\frac{2l+1}{4\pi}\Sigma_l(z,E'\to E)(P_l(\mu)P_l(\mu')+2\sum_{m=1}^{l}\tilde{P}_l^m(\mu)\tilde{P}_l^m(\mu')\cos(\alpha-\alpha'))\right)\varphi(z,E',\mu')d\hat{\Omega}'dE'$$

$$=\int_0^\infty 2\pi\int_{-1}^{1}\sum_{l=0}^{L}\frac{2l+1}{4\pi}\Sigma_l(z,E'\to E)P_l(\mu)P_l(\mu')\varphi(z,E',\mu')d\mu'dE'$$

$$=\int_0^\infty\sum_{l=0}^{L}\frac{2l+1}{4\pi}\cdot 2\pi\int_{-1}^{1}P_l(\mu')\varphi(z,E',\mu')d\mu'\Sigma_l(z,E'\to E)P_l(\mu)dE'$$

$$=\sum_{l=0}^{L}\frac{2l+1}{4\pi}\int_0^\infty\Sigma_l(z,E'\to E)\phi_l(z,E')dE'P_l(\mu)$$

$$\approx\frac{1}{4\pi}\left[\int_0^\infty\Sigma_0(z,E'\to E)\phi(z,E')dE'+3\mu\int_0^\infty\Sigma_1(z,E'\to E)J(z,E')dE'\right]$$

Derivation of B_L Equation (2)

- Assume linear anisotropy in scattering

$$\Sigma(E' \to E, \mu_s) = \frac{1}{4\pi}\Sigma_0(E' \to E) + \frac{3}{4\pi}\mu\Sigma_1(E' \to E)$$

- 1-D Boltzmann Equation

$$\mu\frac{\partial\varphi(z,E,\mu)}{\partial z} + \Sigma_t\varphi = \frac{\chi}{4\pi}\psi + \frac{1}{4\pi}\int_0^\infty \Sigma_0(E' \to E)\phi(z,E')dE' + \frac{3}{4\pi}\mu\int_0^\infty \Sigma_1(E' \to E)J(z,E')dE' \quad \cdots ①$$

↑ Addition theorem

$$\phi(z,E) = 2\pi\int_{-1}^{1}\varphi(z,E,\mu)d\mu \quad : \text{ 0-th moment}$$

$$J(z,E) = 2\pi\int_{-1}^{1}\mu\varphi(z,E,\mu)d\mu \quad : \text{ 1-st moment}$$

- Seperation of Variables

$$\varphi(z,E,\mu) = \hat{\phi}(z)\tilde{\varphi}(E,\mu)$$

$$\phi(z,E) = \hat{\phi}(z)\cdot 2\pi\int_{-1}^{1}\tilde{\varphi}(E,\mu)d\mu = \hat{\phi}(z)\hat{\varphi}(E)$$

$$J(z,E) = \hat{\phi}(z)\cdot 2\pi\int_{-1}^{1}\mu\tilde{\varphi}(E,\mu)d\mu = \hat{\phi}(z)\hat{J}(E) \quad \cdots ②$$

Derivation of B_L Equation (3)

- Insert ② into ①

$$\mu\tilde{\varphi}(E,\mu)\frac{\partial\hat{\phi}(z)}{\partial z}+\Sigma_t\hat{\phi}(z)\tilde{\varphi}(E,\mu)=\frac{\chi}{4\pi}\int_0^\infty \nu\Sigma_f(E')\hat{\phi}(z)\hat{\varphi}(E')dE'$$
$$+\frac{1}{4\pi}\int_0^\infty \Sigma_0(E'\to E)\hat{\phi}(z)\hat{\varphi}(E')dE'$$
$$+\frac{3\mu}{4\pi}\int_0^\infty \Sigma_1(E'\to E)\hat{\phi}(z)\hat{J}(E')dE'$$

- Divide by $\mu\hat{\phi}(z)\tilde{\varphi}(E,\mu)$

$$\frac{1}{\hat{\phi}(z)}\frac{\partial\hat{\phi}(z)}{\partial z}=\frac{1}{\mu}[-\Sigma_t+\frac{1}{4\pi}\frac{1}{\tilde{\varphi}(E,\mu)}\int_0^\infty(\chi(E)\nu\Sigma_f(E')\hat{\varphi}(E')$$
$$+\Sigma_0(E'\to E)\hat{\varphi}(E')$$
$$+3\mu\Sigma_1(E'\to E)\hat{J}(E'))dE'] = \pm iB$$

$$B^2=-\frac{1}{\phi}\frac{d^2\phi}{dz^2},\quad \frac{d^2\phi}{dz^2}=-B^2\phi\to$$

$$\frac{d\phi}{dz}=\pm\sqrt{-B^2}\phi=\pm iB\phi,\quad \frac{1}{\hat{\phi}}\frac{d\hat{\phi}}{dz}=\pm iB$$

$$\hat{\phi}(z)=a_+e^{iBz}+a_-e^{-iBz}$$

$$B^2=\begin{cases} >0\to B\text{ real} & \text{sine, cosine}\\ =0\to\text{zero }B & \text{linear}\\ <0\to B\text{ imaginary} & \text{exponential}\end{cases}$$

Derivation of B_L Equation (4)

$$(\Sigma_t \pm iB\mu)\tilde{\varphi}(E,\mu) = \frac{1}{4\pi}[\chi(E)\int_0^\infty \nu\Sigma_f(E')\hat{\varphi}(E')dE' + \int_0^\infty \Sigma_0(E' \to E)\hat{\varphi}(E')dE' + 3\mu\int_0^\infty \Sigma_1(E' \to E)\hat{J}(E')dE'] \quad \cdots ③$$

- Divide by $\Sigma_t \pm iB\mu$

$$\tilde{\varphi}(E,\mu) = \frac{1}{\Sigma_t \pm iB\mu}\frac{1}{4\pi}[\mathbf{A}]$$

where $\mathbf{A} = \underbrace{\chi(E)\int_0^\infty \nu\Sigma_f(E')\hat{\varphi}(E')dE'}_{FS} + \underbrace{\int_0^\infty \Sigma_0(E' \to E)\hat{\varphi}(E')dE'}_{SS_0} + 3\mu\underbrace{\int_0^\infty \Sigma_1(E' \to E)\hat{J}(E')dE'}_{SS_1}$

- Legendre Expansion of Angular Flux (No spatial dependence)

$$\tilde{\varphi}(E,\mu) = \sum_{l=0}^{L}\hat{\varphi}_l(E)\frac{2l+1}{4\pi}P_l(\mu) \qquad \begin{cases} \hat{\varphi}_0(E) = \hat{\varphi}(E) & \text{Spectrum of flux} \\ \hat{\varphi}_1(E) = \hat{J}(E) & \text{Spectrum of current} \end{cases}$$

Derivation of B_L Equation (5)

- B_1 method

Let L=1

$$\frac{1}{4\pi}\hat{\varphi}(E)+\frac{3}{4\pi}\mu\hat{J}(E)\,\square\,\frac{1}{\Sigma_t \pm iB\mu}\frac{1}{4\pi}[\mathbf{A}] \quad \cdots ④$$

Apply $2\pi\int_{-1}^{1} d\mu$

$$\hat{\varphi}(E)=2\pi\int_{-1}^{1}\frac{1}{\Sigma_t \pm iB\mu}\frac{1}{4\pi}[\mathbf{A}]\,d\mu$$

$$=\underbrace{\boxed{\frac{1}{2}\int_{-1}^{1}\frac{1}{\Sigma_t \pm iB\mu}d\mu}}_{A_{00}}\cdot(\mathrm{FS}+\mathrm{SS}_0)+\underbrace{\boxed{\frac{1}{2}\int_{-1}^{1}\frac{\mu}{\Sigma_t \pm iB\mu}d\mu}}_{A_{01}}\cdot(3\mathrm{SS}_1)$$

Apply $2\pi\int_{-1}^{1} \mu d\mu$

$$\hat{J}(E)=2\pi\int_{-1}^{1}\frac{1}{\Sigma_t \pm iB\mu}\frac{\mu}{4\pi}[\mathbf{A}]\,d\mu$$

$$=\underbrace{\boxed{\frac{1}{2}\int_{-1}^{1}\frac{\mu}{\Sigma_t \pm iB\mu}d\mu}}_{A_{10}=A_{01}}\cdot(\mathrm{FS}+\mathrm{SS}_0)+\underbrace{\boxed{\frac{1}{2}\int_{-1}^{1}\frac{\mu^2}{\Sigma_t \pm iB\mu}d\mu}}_{A_{11}}\cdot(3\mathrm{SS}_1)$$

Calculation of Coefficients (1)

- A_{00}

$$A_{00} = \frac{1}{2}\int_{-1}^{1}\frac{1}{\Sigma_t \pm iB\mu}d\mu \qquad \begin{array}{l} B^2>0 \;\; B\to real \\ B^2<0 \;\; B\to imaginary \end{array}$$

$i)\ B^2 > 0$

$$A_{00} = \frac{1}{2}\int_{-1}^{1}\frac{\Sigma_t \mp iB\mu}{\Sigma_t^{\ 2}+B^2\mu^2}d\mu = \frac{1}{2}\int_{-1}^{1}\frac{1}{1+\left(\frac{B}{\Sigma_t}\mu\right)^2}\frac{d\mu}{\Sigma_t} = \frac{1}{2B}\int_{-\frac{B}{\Sigma_t}}^{\frac{B}{\Sigma_t}}\frac{1}{1+\tau^2}d\tau = \frac{1}{B}\int_{0}^{\frac{B}{\Sigma_t}}\frac{1}{1+\tau^2}d\tau$$

$$= \frac{1}{B}\tan^{-1}\frac{B}{\Sigma_t}$$

$ii)\ B^2 < 0\ or\ -B^2 = +k^2 > 0,\ B = ik$

$$A_{00} = \frac{1}{2}\int_{-1}^{1}\frac{1}{\Sigma_t \mp k\mu}d\mu = \frac{1}{2}\frac{1}{(\mp k)}\ln(\Sigma_t \mp k\mu)\Big|_{-1}^{1} = \frac{1}{2}\frac{1}{(\mp k)}\left[\ln(\Sigma_t \mp k) - \ln(\Sigma_t \pm k)\right]$$

$$= \frac{1}{2}\frac{1}{(\mp k)}\ln\frac{\Sigma_t \mp k}{\Sigma_t \pm k} = \frac{1}{2}\frac{1}{k}\ln\frac{\Sigma_t + k}{\Sigma_t - k}$$

Calculation of Coefficients (2)

- A_{01}

$$A_{01} = \frac{1}{2}\int_{-1}^{1}\frac{\mu}{\Sigma_t \pm iB\mu}d\mu = \frac{1}{2}\int_{-1}^{1}\left\{\left(\pm\frac{1}{iB}\right)+\frac{\mp\frac{\Sigma_t}{iB}}{\Sigma_t \pm iB\mu}\right\}d\mu = \pm\frac{1}{iB}\mp\frac{\Sigma_t}{iB}A_{00} = \mp\frac{i}{B}\left(1-\Sigma_t A_{00}\right)$$

$$\boxed{A_{00} = \frac{1}{2}\int_{-1}^{1}\frac{1}{\Sigma_t \pm iB\mu}d\mu}$$

- A_{11}

$$A_{11} = \frac{1}{2}\int_{-1}^{1}\frac{\mu^2}{\Sigma_t \pm iB\mu}d\mu = \frac{1}{2}\int_{-1}^{1}\frac{\mu^2\left(\Sigma_t \pm iB\mu\right)}{\Sigma_t^{\ 2}+B^2\mu^2}d\mu = \frac{1}{2}\int_{-1}^{1}\frac{\Sigma_t\mu^2}{\Sigma_t^{\ 2}+B^2\mu^2}d\mu$$

$$= \frac{1}{2}\int_{-1}^{1}\left(\frac{-\frac{\Sigma_t^{\ 3}}{B^2}}{\Sigma_t^{\ 2}+B^2\mu^2}+\frac{\Sigma_t}{B^2}\right)d\mu$$

$$= \frac{\Sigma_t}{B^2}\left(1-\Sigma_t A_{00}\right)$$

$$\boxed{A_{00} = \frac{1}{2}\int_{-1}^{1}\frac{\Sigma_t}{\Sigma_t^{\ 2}+B^2\mu^2}d\mu,\ \ B^2>0}$$

- Spectra of flux and current

$$\hat{\varphi}(E) = A_{00}(\mathrm{FS}+\mathrm{SS}_0)+3A_{01}(\mathrm{SS}_1)$$

$$\hat{J}(E) = A_{10}(\mathrm{FS}+\mathrm{SS}_0)+3A_{11}(\mathrm{SS}_1)$$

B_1 Equation

$$\hat{\varphi}(E) = A_{00}(\text{FS+SS}_0) \mp \frac{3i}{B}\left(1-\Sigma_t A_{00}\right)(\text{SS}_1) = A_{00}(\text{FS+SS}_0) \pm \frac{3}{iB}\left(1-\Sigma_t A_{00}\right)(\text{SS}_1)$$

$$\hat{J}(E) = \pm\frac{1}{iB}\left(1-\Sigma_t A_{00}\right)(\text{FS+SS}_0) + 3\frac{\Sigma_t}{B^2}\left(1-\Sigma_t A_{00}\right)(\text{SS}_1)$$

- Eliminate SS_1 after removing iB from the 0-th order term $\quad \pm iB\hat{J}(E) = \left(1-\Sigma_t A_{00}\right)(\text{FS+SS}_0) \pm 3i\frac{\Sigma_t}{B}\left(1-\Sigma_t A_{00}\right)(\text{SS}_1)$

$$\Sigma_t\hat{\varphi}(E) \pm iB\hat{J}(E) = \Sigma_t A_{00}(\text{FS+SS}_0) + (1-\Sigma_t A_{00})\cdot(\text{FS+SS}_0) = \text{FS+SS}_0$$

- Eliminate FS+SS_0 after removing iB from 1-st order terms $\quad \pm iB\hat{\varphi}(E) = \pm iBA_{00}(\text{FS+SS}_0) + 3\left(1-\Sigma_t A_{00}\right)(\text{SS}_1)$

$$\pm iB\hat{\varphi}(E) + \frac{B^2}{1-\Sigma_t A_{00}} A_{00}\hat{J}(E) = (3(1-\Sigma_t A_{00}) + 3\Sigma_t A_{00})\text{SS}_1 = 3\text{SS}_1$$

- B_1 Equation

$$\text{Let } \frac{B^2}{1-\Sigma_t A_{00}} A_{00} \equiv 3\Sigma_t \alpha \quad \Longrightarrow \quad \begin{cases} \Sigma_t\hat{\varphi}(E) \pm iB\hat{J}(E) = \text{FS+SS}_0 \\ \pm iB\hat{\varphi}(E) + 3\alpha\Sigma_t\hat{J}(E) = 3\text{SS}_1 \end{cases}$$

$$\alpha = f(B,E)$$

$$\Longrightarrow \begin{cases} \Sigma_t\hat{\varphi}(E) \pm iB\hat{J}(E) = \chi(E)\hat{\psi} + \int_0^\infty \Sigma_0(E' \to E)\hat{\varphi}(E')dE' \\ \pm iB\hat{\varphi}(E) + 3\alpha\Sigma_t\hat{J}(E) = 3\int_0^\infty \Sigma_1(E' \to E)\hat{J}(E')dE' \end{cases}$$

Multi group form of B_1 equation

- Multi group form

$$\begin{cases}\Sigma_{tg}\hat{\varphi}_g \pm iB\hat{J}_g = \chi_g\hat{\psi} + \sum_{g'=1}^{G}\Sigma_{g'g}\hat{\varphi}_{g'} \\ \pm iB\hat{\varphi}_g + 3\alpha_g\Sigma_{tg}\hat{J}_g = 3\sum_{g'=1}^{G}\Sigma^{(1)}_{g'g}\hat{J}_{g'}\end{cases} \Rightarrow \begin{cases}\Sigma_{tg}\varphi_g \mp BJ_g = \chi_g\psi + \sum_{g'=1}^{G}\Sigma_{g'g}\varphi_{g'} \\ \pm B\varphi_g + 3\alpha_g\Sigma_{tg}J_g = 3\sum_{g'=1}^{G}\Sigma^{(1)}_{g'g}J_{g'}\end{cases} \quad \because \begin{cases}\hat{J}_g = iJ_g \\ \hat{\varphi}_g = \varphi_g\end{cases}$$

for real values

$$3\left(\alpha_g(B)\Sigma_{tg}J_g - \sum_{g'=1}^{G}\Sigma^{(1)}_{g'g}J_{g'}\right) = \mp B\varphi_g \qquad \mathbf{J} = \begin{bmatrix} J_1 \\ \vdots \\ J_G \end{bmatrix} \quad \mathbf{\Phi} = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_G \end{bmatrix} \Rightarrow \mathbf{D}^{-1}\mathbf{J} = \mp B\mathbf{\Phi}$$

$\mathbf{J} = \mp\mathbf{D}B\mathbf{\Phi} \rightarrow$ Similar to Fick's Law

$$\mathbf{D}^{-1} = 3\begin{bmatrix} \alpha_1\Sigma_{t1} - \Sigma^{(1)}_{11} & -\Sigma^{(1)}_{21} & -\Sigma^{(1)}_{31} & \cdot \\ -\Sigma^{(1)}_{12} & \alpha_2\Sigma_{t2} - \Sigma^{(1)}_{22} & -\Sigma^{(1)}_{32} & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -\Sigma^{(1)}_{1G} & \cdot & \cdot & \alpha_G\Sigma_{tG} - \Sigma^{(1)}_{GG} \end{bmatrix} \qquad \mathbf{\Sigma_t} = \begin{bmatrix} \Sigma_{t1} & & \\ & \Sigma_{t2} & \\ & & \\ & & \Sigma_{tG} \end{bmatrix}$$

$$\mathbf{\Sigma_t\Phi} \mp B(\mp\mathbf{D}B\mathbf{\Phi}) = \chi\psi + \mathbf{S\Phi} \Rightarrow \left(\mathbf{\Sigma_t} + \mathbf{D}B^2 - \mathbf{S}\right)\mathbf{\Phi} = \chi\psi \qquad \cdots(*)$$

B_1 Search Algorithm and Determination of Diffusion Coeff.

- Buckling Search Algorithm

$$\left(\boldsymbol{\Sigma}_t + \mathbf{D}B^2 - \mathbf{S}\right)\boldsymbol{\Phi} = \chi\psi \cdots (*)$$

① Assume $B^2 = 0$

② Solve (*) to obtain φ_g with normalized fission source $\psi=1$

③ Obtain new FS, $\psi' = \sum_{g=1}^{G} \nu\Sigma_{fg}\varphi_g$. Set $k=\psi'$.

④ Change Buckling (B^2) and solve for φ_g and k

⑤ Predict B^2 to make k equal to 1

⑥ Iterate ④ and ⑤ until $|k-1| < \varepsilon$

- B^2 and φ_g are obtained as the result.

$\rightarrow$ group condensation

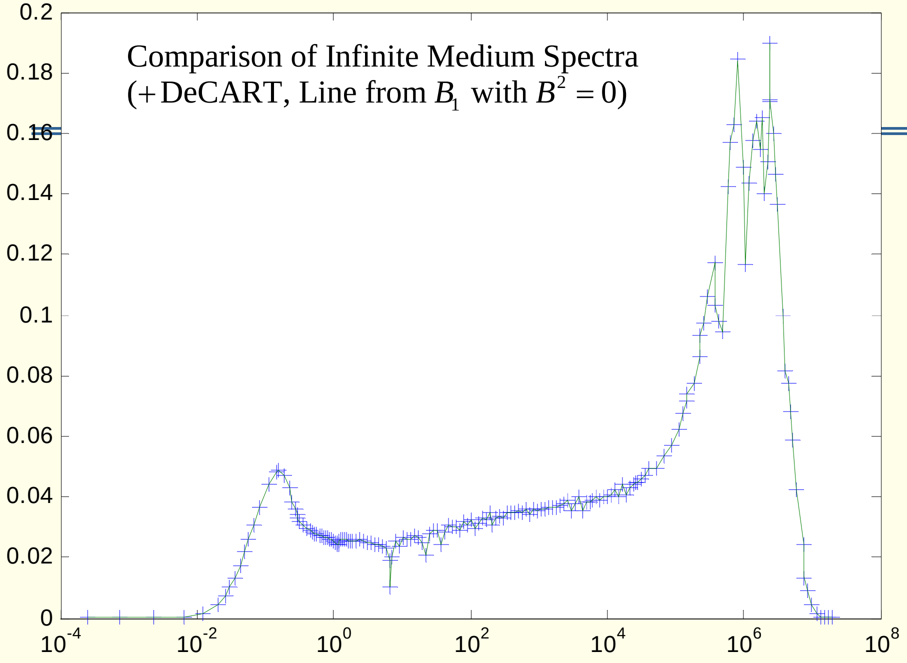
- Determination of Group Diffusion Coefficient

$$\mathbf{J} = -\mathbf{D}B\boldsymbol{\Phi}$$

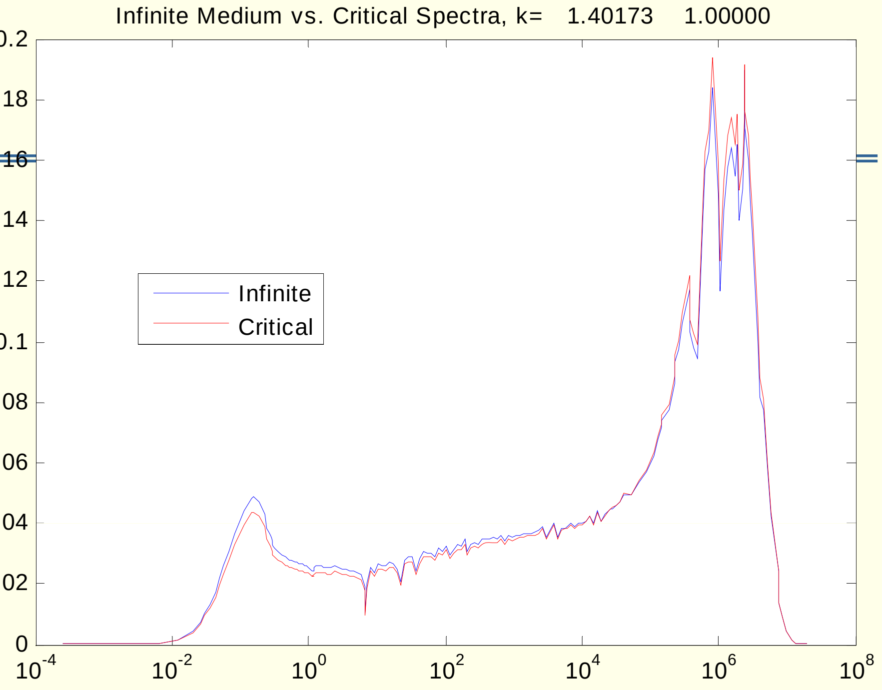
Approximate $J_g = -B\sum_{g'=1}^{G} D_{gg'}\phi_{g'} \cong -D_g B\phi_g$

$$D_g = \frac{J_g}{B\phi_g} = \frac{\sum_{g'=1}^{G} D_{gg'}\phi_{g'}}{\phi_g}$$

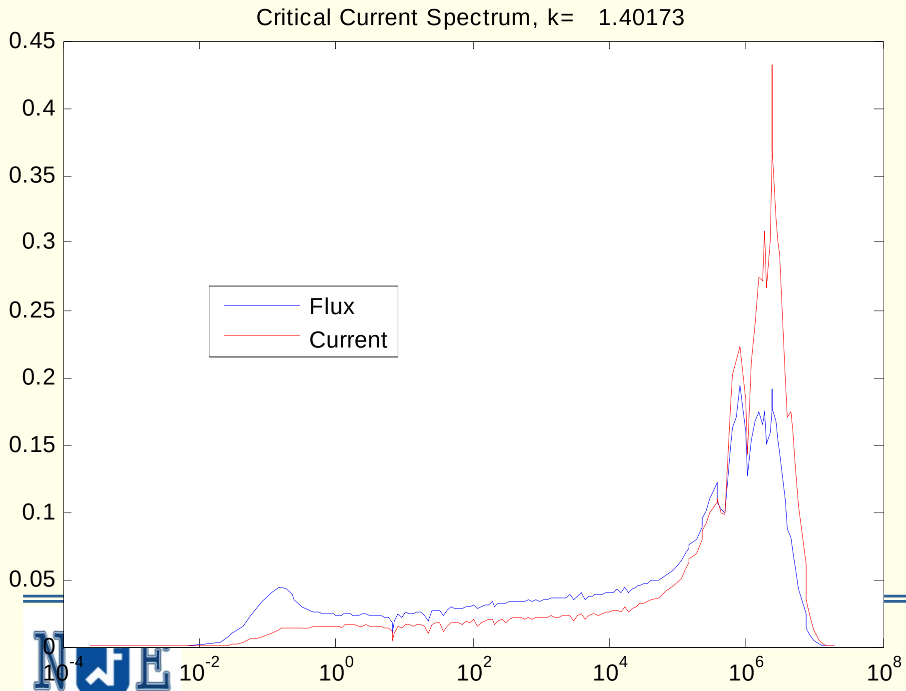
: weighted average of $D_{gg'}$



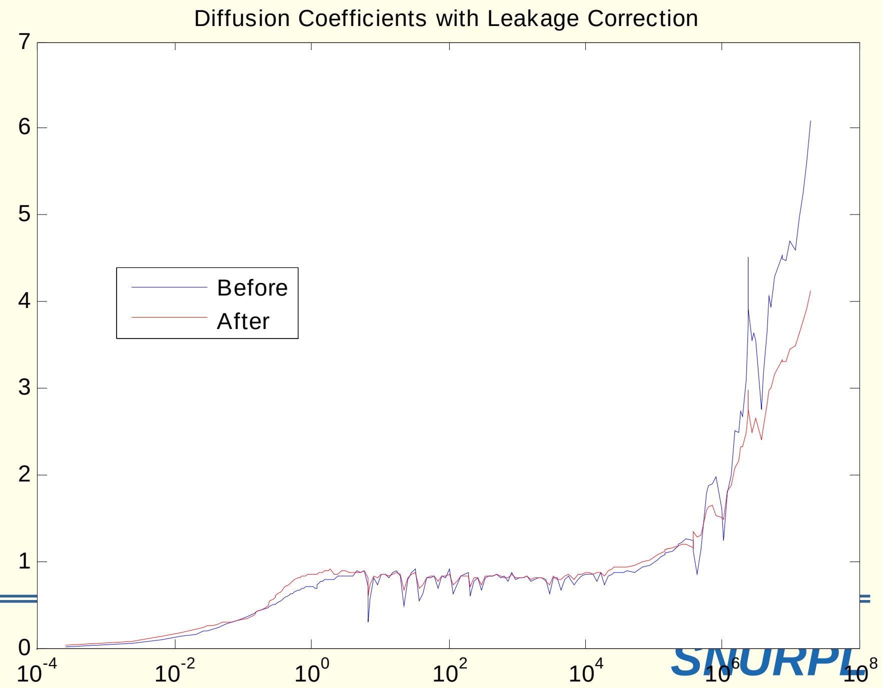
Comparison of Infinite Medium Spectra
(+DeCART, Line from B_1 with $B^2 = 0$)
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
10^{-4} 10^{-2} 10^{0} 10^{2} 10^{4} 10^{6} 10^{8}



Infinite Medium vs. Critical Spectra, k= 1.40173 1.00000
Infinite
Critical
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
10^{-4} 10^{-2} 10^{0} 10^{2} 10^{4} 10^{6} 10^{8}



Critical Current Spectrum, k= 1.40173
Flux
Current
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
10^{-4} 10^{-2} 10^{0} 10^{2} 10^{4} 10^{6} 10^{8}



Diffusion Coefficients with Leakage Correction
Before
After
7
6
5
4
3
2
1
0
10^{-4} 10^{-2} 10^{0} 10^{2} 10^{4} 10^{6} 10^{8}

Alternate Method by P_1

$$(\Sigma_t \pm iB\mu)\tilde{\varphi}(E,\mu) = \frac{1}{4\pi}[\chi(E)\int_0^\infty \nu\Sigma_f(E')\hat{\varphi}(E')dE' + \int_0^\infty \Sigma_0(E' \to E)\hat{\varphi}(E')dE' + 3\mu\int_0^\infty \Sigma_1(E' \to E)\hat{J}(E')dE']$$

B_1 :General higher order Lengendre expansion after division
then take only upto first order inner product,
1 not for fundamental mode

- Insert expansion and take inner product before division by $(\Sigma_t \pm iB\mu)$

$$\tilde{\varphi}(E,\mu) = \frac{1}{4\pi}\hat{\varphi}(E) + \frac{3}{4\pi}\mu\hat{J}(E) + \frac{5}{4\pi}\left(\frac{3}{2}\mu^2 - \frac{1}{2}\right)\hat{\varphi}_2(E) + \cdots$$

LHS: $\frac{1}{4\pi}\Sigma_t\hat{\varphi}(E) + \frac{1}{4\pi}\left(3\Sigma_t\hat{J}(E) \pm iB\hat{\varphi}(E)\right)\mu \pm \frac{3}{4\pi}iB\hat{J}(E)\mu^2 + \left(\Sigma_t\hat{\varphi}(E) \pm iB\mu\hat{J}(E)\right)\left(P_2(\mu)\hat{\varphi}_2(E) + \cdots\right)$

- After inner product with 1 and μ in MG form

$$\begin{cases} \Sigma_{tg}\hat{\varphi}_g \pm iB\hat{J}_g = \chi_g\hat{\psi} + \sum_{g'=1}^{G}\Sigma_{g'g}\hat{\varphi}_{g'} \\ \pm iB\hat{\varphi}_g + 3\Sigma_{tg}\hat{J}_g + \beta_g\hat{\varphi}_{2,g} = 3\sum_{g'=1}^{G}\Sigma_{g'g}^{(1)}\hat{J}_{g'} \end{cases}$$

(the term $+\beta_g\hat{\varphi}_{2,g}$ is struck out)

$\because \int_{-1}^{1} iB\mu^2 P_2(\mu)d\mu \neq 0$

$\rightarrow$ same as B_1 other than missing α in front of J in the second Eq.

$\rightarrow$ No need for repeated calc. of $\mathbf{D}^{-1}$