Quantum Theory of Radiation (Sakurai, Advanced Quantum Mechanics, Ch. 2)

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Gauge Transformation

$$\mathrm{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \mathrm{A}}{\partial t} \; , \qquad \mathrm{B} = \nabla \times \mathrm{A}$$

$$\varphi \rightarrow \varphi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \; , \qquad \mathrm{A} \rightarrow \mathrm{A} + \nabla\Lambda$$

Hamiltonian

- Hamiltonian for a particle of electric change *e* (*e*<0 for the electron) subjected to the electromagnetic field.

$$H = \frac{1}{2m}\left(\mathrm{p} - \frac{e\mathrm{A}}{c}\right)^2 + e\varphi$$

$$\left(\mathrm{p} - \frac{e\mathrm{A}}{c}\right)^2 \rightarrow p^2 - \left(\frac{e}{c}\right)(\mathrm{p}\cdot\mathrm{A} + \mathrm{A}\cdot\mathrm{p}) + \left(\frac{e}{c}\right)^2 \mathrm{A}^2$$

$\mathrm{p}:$ canonical momentum

$$\Pi \equiv m\frac{d\mathrm{x}}{dt} = \mathrm{p} - \frac{e\mathrm{A}}{c}$$

kinematical (or mechanical) momentum

Transversality Condition

$$\nabla \cdot \mathrm{A} = 0$$

$$\mathrm{A} = \mathrm{A}_{\perp} + \mathrm{A}_{||}, \qquad \nabla \cdot \mathrm{A}_{\perp} = 0$$

$$\nabla \times \mathrm{A}_{||} = 0$$

$\mathrm{A}_{\perp}$: Transverse component of A

$\mathrm{A}_{||}$: Longitudinal component of A

$$H = \sum_{j} \frac{1}{2m_j}\left[\mathrm{p}^{(j)} - e_j \frac{\mathrm{A}_{\perp}\left(\mathrm{x}^{(j)}\right)}{c}\right]^2 + \sum_{i>j} \frac{e_i e_j}{4\pi\left|\mathrm{x}^{(i)} - \mathrm{x}^{(j)}\right|} + H_{\mathrm{rad}}$$

Fourier Decomposition

$$\mathrm{A}(\mathbf{x},t)=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\sum_{\alpha}\left(c_{\mathbf{k},\alpha}(t)\varepsilon^{(\alpha)}e^{i\mathbf{k}\cdot\mathbf{x}}+c^{*}_{\mathbf{k},\alpha}(t)\varepsilon^{(\alpha)}e^{-i\mathbf{k}\cdot\mathbf{x}}\right)$$

$$=\frac{1}{\sqrt{V}}\sum_{\mathrm{k}}\sum_{\alpha}\left(c_{\mathrm{k},\alpha}(0)\,\varepsilon^{(\alpha)}e^{ik\cdot x}+c^{*}_{\mathrm{k},\alpha}(0)\,\varepsilon^{(\alpha)}e^{-ik\cdot x}\right)$$

$$c_{\mathrm{k},\alpha}(t)=c_{\mathrm{k},\alpha}(0)e^{-i\omega t} \qquad c^{*}_{\mathrm{k},\alpha}(t)=c^{*}_{\mathrm{k},\alpha}(0)e^{i\omega t}$$

$$\omega=|\mathbf{k}|c$$

$$k\cdot x=\mathbf{k}\cdot\mathbf{x}-\omega t=\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|ct$$

Hamiltonian of Field

$$H = \tfrac{1}{2}\int\left(|\mathrm{B}|^2 + |\mathrm{E}|^2\right)d^3x = \tfrac{1}{2}\int\left[|\nabla\times\mathrm{A}|^2 + |(1/c)(\partial\mathrm{A}/\partial t)|^2\right]d^3x$$

$$H = \sum_{\mathbf{k}}\sum_{\alpha} 2(\omega/c)^2 c^*_{\mathbf{k},\alpha}\, c_{\mathbf{k},\alpha}$$

$$\ddot{c}_{\mathbf{k},\alpha} = -\omega^2 c_{\mathbf{k},\alpha}$$

$$Q_{\mathbf{K},\alpha} = \frac{1}{c}\left(c_{\mathbf{k},\alpha} + c^*_{\mathbf{k},\alpha}\right), \qquad P_{\mathbf{k},\alpha} = -\frac{i\omega}{c}\left(c_{\mathbf{k},\alpha} - c^*_{\mathbf{k},\alpha}\right)$$

$$H = \sum_{\mathbf{k}}\sum_{\alpha} 2\left(\frac{\omega}{c}\right)^2\left[\frac{c\left(\omega Q_{\mathbf{k},\alpha} - iP_{\mathbf{k},\alpha}\right)}{2\omega}\right]\left[\frac{c\left(\omega Q_{\mathbf{k},\alpha} + iP_{\mathbf{k},\alpha}\right)}{2\omega}\right] = \sum_{\mathbf{k}}\sum_{\alpha}\tfrac{1}{2}\left(P^2_{\mathbf{k},\alpha} + \omega^2 Q^2_{\mathbf{k},\alpha}\right)$$

Quantization of Radiation Oscillators

$$\left[Q_{\mathbf{k},\alpha}, P_{\mathbf{k}',\alpha'}\right] = ih\delta_{\mathbf{kk}'}\delta_{\alpha\alpha'}$$

$$\left[Q_{\mathbf{k},\alpha}, Q_{\mathbf{k}',\alpha'}\right] = 0$$

$$\left[P_{\mathbf{k},\alpha}, P_{\mathbf{k}',\alpha'}\right] = 0$$

$$a_{\mathbf{k},\alpha} = \left(1/\sqrt{2\hbar\omega}\right)\left(\omega Q_{\mathbf{k},\alpha} + iP_{\mathbf{k},\alpha}\right)$$

$$a^{\dagger}_{\mathrm{k},\alpha} = \left(1/\sqrt{2\hbar\omega}\right)\left(\omega Q_{\mathrm{k},\alpha} - iP_{\mathrm{k},\alpha}\right)$$

$$\left[a_{\mathrm{k},\alpha}, a^{\dagger}_{\mathrm{k}',\alpha'}\right] = -\frac{i}{2\hbar}\left[Q_{\mathrm{k},\alpha}, P_{\mathrm{k}',\alpha'}\right] + \frac{i}{2\hbar}\left[P_{\mathrm{k},\alpha}, Q_{\mathrm{k}',\alpha'}\right] = \delta_{\mathrm{kk}'}\delta_{\alpha\alpha'}$$

$$\left[a_{\mathrm{k},\alpha}, a_{\mathrm{k}',\alpha'}\right] = \left[a^{\dagger}_{\mathrm{k},\alpha}, a^{\dagger}_{\mathrm{k}',\alpha'}\right] = 0 \qquad N_{k,\alpha} = a^{\dagger}_{\mathrm{k},\alpha}a_{\mathrm{k},\alpha}$$

Photon Number State

$$\left| n_{\mathrm{k}1,\alpha 1}, n_{\mathrm{k}2,\alpha 2}, \ldots, n_{\mathrm{k}l,\alpha l}, \ldots \right\rangle = \left| n_{\mathrm{k}1,\alpha 1} \right\rangle \left| n_{\mathrm{k}2,\alpha 2} \right\rangle \ldots \left| n_{\mathrm{k}l,\alpha n} \right\rangle \ldots$$

$$\left| n_{\mathrm{k}1,\alpha 1}, n_{\mathrm{k}2,\alpha 2}, \ldots \right\rangle = \prod_{\mathrm{k}i,\alpha i} \frac{\left(a^{\dagger}_{\mathrm{k}i,\alpha i} \right)^{n_{\mathrm{k}i,\,\alpha i}}}{\sqrt{n_{\mathrm{k}i,\alpha i}!}} \left| 0 \right\rangle$$

Quantized Radiation Field I

$$\mathbf{A}(\mathbf{x},t)=\left(1/\sqrt{V}\right)\sum_{\mathrm{k}}\sum_{\alpha}c\sqrt{h/2\omega}\left[a_{\mathrm{k},\alpha}(t)\varepsilon^{(\alpha)}e^{i\mathrm{k}\cdot\mathrm{x}}+a^{\dagger}_{\mathrm{k},\alpha}(t)\varepsilon^{(\alpha)}e^{-ik\cdot x}\right]$$

$$H=\frac{1}{2}\sum_{\mathrm{k}}\sum_{\alpha}\hbar\omega\left(a^{\dagger}_{\mathrm{k},\alpha}a_{\mathrm{k},\alpha}+a_{\mathrm{k},\alpha}a^{\dagger}_{\mathrm{k},\alpha}\right)=\sum_{\mathrm{k}}\sum_{\alpha}\left(N_{\mathrm{k},\alpha}+\frac{1}{2}\right)\hbar\omega$$

$$H=\sum_{\mathbf{k}}\sum_{\alpha}\hbar\omega N_{\mathbf{k},\alpha}$$

$$P=\sum_{\mathbf{k}}\sum_{\alpha}\hbar\mathbf{k}N_{\mathbf{k},\alpha}$$

Quantized Radiation Field II

$$a_{\mathrm{k},\alpha} = a_{\mathrm{k},\alpha}(0)e^{-i\omega t}, \qquad a^{\dagger}_{\mathrm{k},\alpha} = a^{\dagger}_{\mathrm{k},\alpha}(0)e^{i\omega t}$$

$$\mathrm{A}(\mathrm{x},t) = \frac{1}{\sqrt{V}}\sum_{\mathrm{k}}\sum_{\alpha} c\sqrt{\frac{\hbar}{2\omega}}[a_{\mathrm{k},\alpha}(0)\ \varepsilon^{(\alpha)}e^{\mathrm{ik}\cdot\mathrm{x}-i\omega t} + a^{+}_{\mathrm{k},\alpha}(0)\ \varepsilon^{(\alpha)}e^{-\mathrm{ik}\cdot\mathrm{x}+i\omega t}]$$

Vacuum Fluctuation

$$\langle 0|\mathrm{E}\cdot\mathrm{E}|0\rangle - \left|\langle 0|\mathrm{E}|0\rangle\right|^2 = \langle 0|\mathrm{E}\cdot\mathrm{E}|0\rangle = \infty$$

$$\overline{\mathrm{E}} = (1/\Delta V)\int_{\Delta V} \mathrm{E}d^3x$$

$$\langle 0|\overline{\mathrm{E}}\cdot\overline{\mathrm{E}}|0\rangle \sim \hbar c/(\Delta l)^4$$

Casimir effect

Uncertainty Relation I

$$\left[\mathrm{E}_x(\mathbf{x},t),\mathrm{B}_y(\mathbf{x}',t)\right]=ic\hbar\frac{\partial}{\partial z}\delta^3(\mathbf{x}-\mathbf{x}')$$

$$\left[\mathrm{E}_x(\mathbf{x},t),\mathrm{B}_y(\mathbf{x}',t')\right]=0 \quad \textbf{for} \quad (\mathbf{x}-\mathbf{x}')^2-\left(1/c^2\right)(t-t')^2 \neq 0$$

Uncertainty Relation II

$$a_{\mathrm{k},\alpha} = e^{i(\phi_{\mathrm{k},\alpha} - \omega t)} \sqrt{N_{\mathrm{k},\alpha}}$$

$$a^{\dagger}_{\mathrm{k},\alpha} = \sqrt{N_{\mathrm{k},\alpha}} \; e^{i(\phi_{\mathrm{k},\alpha} - \omega t)}$$

$$[N, \phi] = i$$

$$\Delta N \Delta \phi \gtrsim 1$$

Atom-Photon Interaction I

$$H_{\mathbf{int}} = \sum_{t} \left[-\frac{e}{2mc} \left(\mathbf{p}_i \cdot \mathrm{A}(\mathbf{x}_i, t) + \mathrm{A}(\mathbf{x}_i, t) \cdot \mathbf{p}_i \right) + \frac{e^2}{2mc^2} \mathrm{A}(\mathbf{x}_i, t) \cdot \mathrm{A}(\mathbf{x}_i, t) \right]$$

- Photon Absorption

$$\left\langle B; n_{\mathbf{k},\alpha} - 1 \middle| H_{\mathbf{int}} \middle| A; n_{\mathbf{k},\alpha} \right\rangle = -\frac{e}{mc} \left\langle B; n_{\mathbf{k},\alpha} - 1 \middle| \sum_i c \sqrt{\frac{\hbar}{2\omega V}} a_{\mathbf{k},\alpha}(0) e^{i\mathbf{k}\cdot\mathbf{x}_l - i\omega t} \mathbf{p}_i \varepsilon^{(\alpha)} \middle| A; n_{\mathbf{k},\alpha} \right\rangle$$

$$= -\frac{e}{mc} \sqrt{\frac{n_{\mathbf{k},\alpha} \hbar}{2\omega V}} \sum_i \left\langle B \middle| e^{i\mathbf{k}\cdot\mathbf{x}_l} \mathbf{p}_i \cdot \varepsilon^{(\alpha)} \middle| A \right\rangle e^{-i\omega t}$$

- Photon Emission

$$\left\langle B; n_{\mathbf{k},\alpha} + 1 \middle| H_{\mathbf{int}} \middle| A; n_{\mathbf{k},\alpha} \right\rangle == -\frac{e}{m} \sqrt{\frac{(n_{\mathbf{k},\alpha} + 1)\hbar}{2\omega V}} \sum_i \left\langle B \middle| e^{i\mathbf{k}\cdot\mathbf{x}_l} \mathbf{p}_i \cdot \varepsilon^{(\alpha)} \middle| A \right\rangle e^{i\omega t}$$

Atom-Photon Interaction II

absorption : $c\sqrt{\frac{n_{\mathbf{k},\alpha}\hbar}{2\omega V}}\varepsilon^{(\alpha)}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$

emission : $c\sqrt{\frac{(n_{\mathbf{k},\alpha}+1)\hbar}{2\omega V}}\varepsilon^{(\alpha)}e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}$

Time-Dependent Perturbation Theory I

$$\psi = \sum_k c_k(t) u_k(\mathbf{x}) e^{-iE_k t/\hbar}$$

$$H_0 u_k(\mathbf{x}) = E_k u_k(\mathbf{x})$$

$$(H_0 + H_I)\psi = i\hbar(\partial\psi/\partial t) = i\hbar\sum_k \left(\dot{c}_k u_k e^{-iE_k t/\hbar} - i(E_k/\hbar) c_k u_k e^{-iE_k t/\hbar}\right)$$

$$\sum_k H_I c_k u_k e^{-iE_k t/\hbar} = i\hbar\sum_l \dot{c}_l u_l e^{-iE_l t/\hbar}$$

Time-Dependent Perturbation Theory II

$$\dot{c}_m = \sum_k \left(1/i\hbar\right)\langle m|H_I(t)|k\rangle e^{i(E_m-E_k)t/\hbar}c_k(t)$$

$$c_k(0) = \delta_{kl}$$

$$c_m^{(1)}(t) = \frac{1}{i\hbar}\int_0^t dt'\langle m|H_I(t')|l\rangle e^{i(E_m-E_l)t'/\hbar}$$

$$c_m \simeq c_m^{(1)} + c_m^{(2)}$$

$$c_m^{(2)}(t) = \frac{1}{(i\hbar)^2}\sum_n \int_0^t dt''\int_0^{t''} dt'\langle m|H_I(t'')|n\rangle e^{i(E_m-E_n)t''/\hbar}\langle n|H_I(t')|l\rangle e^{i(E_n-E_l)t'/\hbar}$$

Time-Dependent Perturbation Theory III

$$H_I(t) = H_I' e^{\mp i\omega t} \qquad \textbf{for} \quad \begin{cases} \textbf{absorption} \\ \textbf{emission} \end{cases}$$

$$c_m^{(1)} = \frac{1}{i\hbar}\langle m|H_I'|l\rangle \int_0^t dt' e^{i(E_m - E_l \mp \hbar\omega)t'/\hbar}$$

$$\left|c_m^{(1)}(t)\right|^2 = (2\pi/\hbar)\left|\langle m|H_I'|l\rangle\right|^2 t\delta(E_m - E_l \mp \hbar\omega)$$

Fermi's Golden Rule

$$\rho_{\hbar\omega,d\Omega} = \frac{V|k|^2}{(2\pi)^3}\frac{d|k|d\Omega}{d(\hbar\omega)} = \frac{V\omega^2}{(2\pi)^3}\frac{d\Omega}{\hbar c^3}$$

$$w_{d\Omega} = \int \left(\left| c_m^{(1)} \right|^2 \big/ t \right) \rho_{\hbar\omega,d\Omega} d(\hbar\omega) = (2\pi/\hbar) \left| \langle m | H_I' | l \rangle \right|^2 \rho_{\hbar\omega,d\Omega}$$

Fermi's Golden Rule

$$E_m - E_l + \hbar\omega = 0$$

Spontaneous Emission in the Dipole Approximation I

$$w_{d\Omega} = \frac{2\pi}{\hbar}\frac{e^2\hbar}{2m^2\omega V}\left|\sum_i \left\langle B\right| e^{-i\mathbf{k}\cdot\mathbf{x}i}\varepsilon^{(\alpha)}\cdot\mathbf{p}_i\left| A\right\rangle\right|^2 \frac{V\omega^3 d\Omega}{(2\pi)^3\hbar c^3}$$

$$\aleph_{\text{photon}} = 1/|k| \rangle\rangle \gamma_{\text{atom}}$$

$$e^{-i\mathbf{k}\cdot\mathbf{x}i} = 1 - i\mathbf{k}\cdot\mathbf{x}_i - (\mathbf{k}\cdot\mathbf{x}_i)^2/2 + \cdots \approx 1$$

$$w_{d\Omega} = \frac{e^2\omega}{8\pi^2 m^2\hbar c^3}\left|\left\langle B\right|\mathbf{p}\left| A\right\rangle\cdot\varepsilon^{(\alpha)}\right|^2 d\Omega$$

Spontaneous Emission in the Dipole Approximation II

$$\left[\mathrm{p}^2, \mathrm{x}\right] = -2i\hbar \mathrm{p}$$

$$\langle B|\mathbf{p}|A\rangle = \left\langle B\left|\frac{im}{\hbar}[H_0, \mathbf{x}]\right|A\right\rangle = \frac{im(E_B - E_A)}{\hbar}\langle B|\mathbf{x}|A\rangle = -im\omega \mathbf{x}_{BA}$$

$$-\frac{e\mathrm{A}\cdot\mathbf{p}}{mc} \rightarrow \frac{e}{c}\mathbf{x}\cdot\frac{\partial \mathrm{A}}{\partial t} = -e\mathbf{x}\cdot\mathrm{E}$$

Spontaneous Emission in the Dipole Approximation III

$$w_{d\Omega} = \frac{e^2\omega^3}{8\pi^2\hbar c^3}\left|\mathbf{x}_{BA}\right|^2 \cos^2\Theta^{(\alpha)} d\Omega$$

$$\cos\Theta^{(\alpha)} = \left|\mathbf{x}_{BA}\cdot\varepsilon^{(\alpha)}\right| / \mathbf{x}_{BA}$$

$$\cos\Theta^{(1)} = \sin\theta\cos\phi\ , \quad \cos\Theta^{(2)} = \sin\theta\sin\phi$$

$$2\pi\int_{-1}^{1}\sin^2\theta d(\cos\theta) = 8\pi/3$$

$$w = \frac{e^2\omega^3}{3\pi\hbar c^3}\left|\mathbf{x}_{BA}\right|^2 = \left(\frac{e^2}{4\pi\hbar c}\right)\frac{4}{3}\frac{\omega^3}{c^2}\left|\mathbf{x}_{BA}\right|^2$$