[2008][05-2]

Computer aided ship design

Part 2. Ship Motion & Wave Load

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Prof. Kyu-Yeul Lee

Department of Naval Architecture and Ocean Engineering,
Seoul National University of College of Engineering

Advanced
Ship
Design
Automation
Laboratory

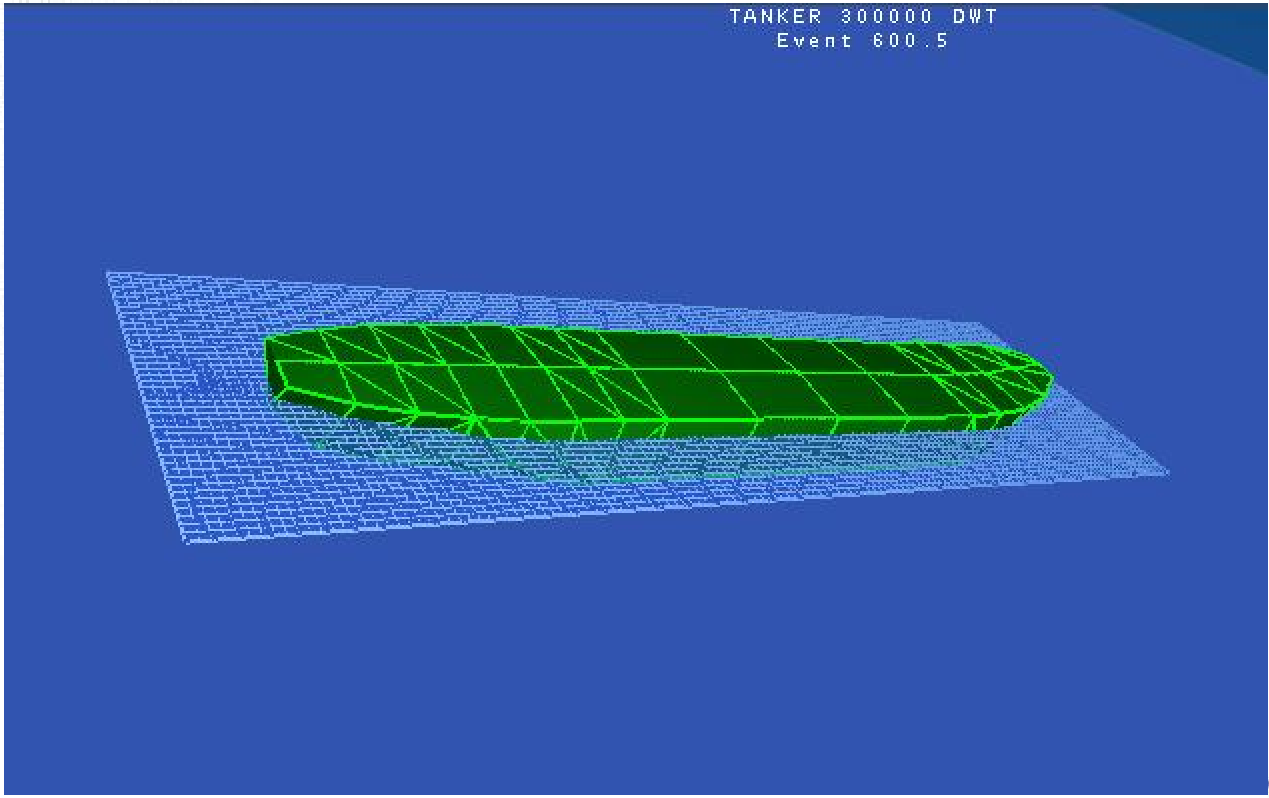
Part II. Ship Motion & Wave Load
: 강의 개요

- **유체 역학**(학부 2학년), **해양환경정보시스템**(학부 3학년), **선박운항제어론**(학부 3학년), **선체구조설계시스템**(학부 3학년)의 전공 과목과 **공학 수학**(학부 2학년), **기초구조정역학**(학부 2학년), **동역학**(학부 2학년)의 기초 교과목을 토대로 *선박의 6자유도 운동 방정식* (Ship Motion) 의 유도 과정을 이해한다.
- 선박의 6자유도 운동 방정식의 계산 결과를 *구조 설계* (Wave Load) 에 활용하는 방안에 대해 이해한다.

Part II. Ship Motion & Wave Load
: Term project (I) – 6DOF Ship motion simulation

- **선박의 6자유도 운동 방정식을 수치적으로 계산하여 시간에 따른 선박의 운동을 구하고, 이를 화면에 가시화 한다.**

ex) MOSES (Multi-Operational Structural Engineering Simulator)
: Ultramarine **에서 개발**(**미국**, **휴스턴**)



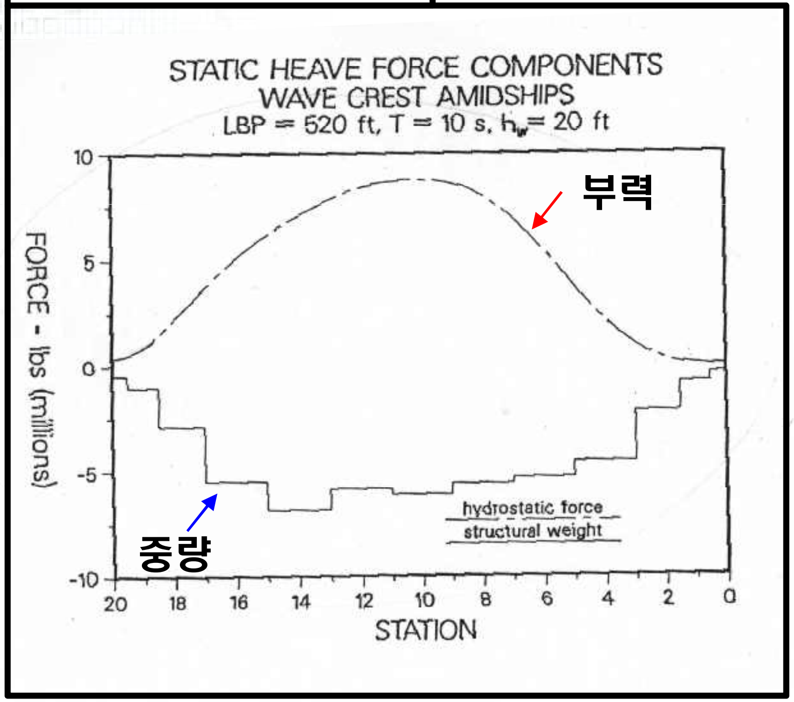
Part I. **곡선**,**곡면 모델링의 결과 사용**

1) VWBM : Vertical Wave Bending Moment
2) SWBM : Still Water Bending Moment

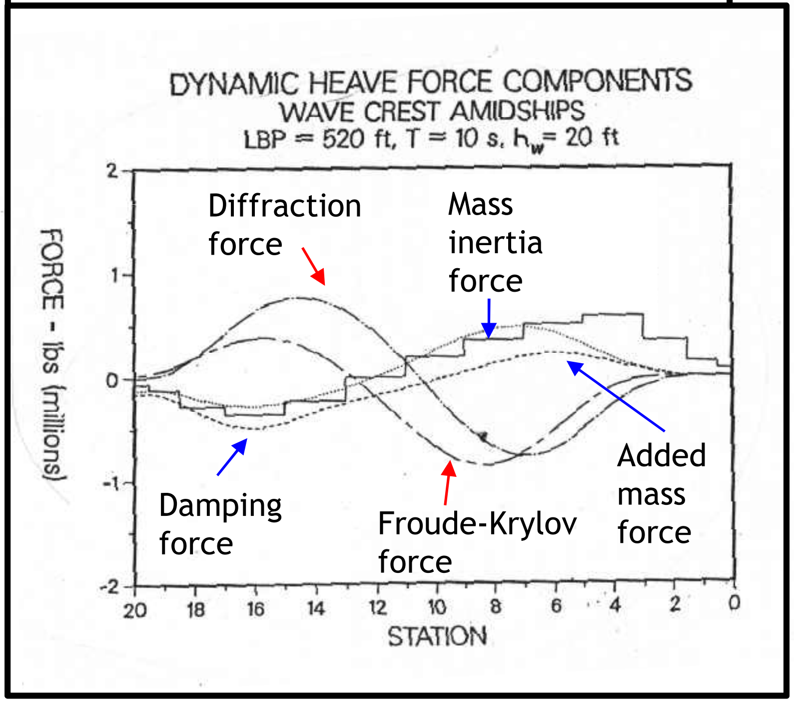
Part II. Ship Motion & Wave Load
: Term project (II) – Wave Load에 의한 VWBM[1] 계산

- 6자유도 운동 방정식의 풀이 결과를 사용하여 선박에 작용하는 Wave Load를 계산하고, Shear force 및 Bending moment를 계산한다.

중량-부력 하중 곡선



Wave와 선박 운동에 의한 하중 곡선



Part II. Ship Motion & Wave Load
: 참고 자료

- **참고 서적 및 자료**

1) 小山健夫 **저, 구종도 역, "선박과 해양구조물의 운동학", 연건문화사**, 1997
2) Bhattacharyya, R. , "Dynamics of Marine Vehicles", John Wiley & Sons, 1978
3) Faltinsen, O.M. , "Sea loads on ships and offshore structures", Cambridge Univ. Press, 1998
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5) Newman, J.N. , "Marine Hydrodynamics", The MIT Press, Cambridge, 1997
6) **이승건, "선박운동 조종론", 부산대학교 출판부**, 2004
7) Journee, J.M.J. , Massie, W.W. , "Offshore Hydrodynamics", Delft University of Technology, 2001 (http://www.shipmotions.nl/index.html)
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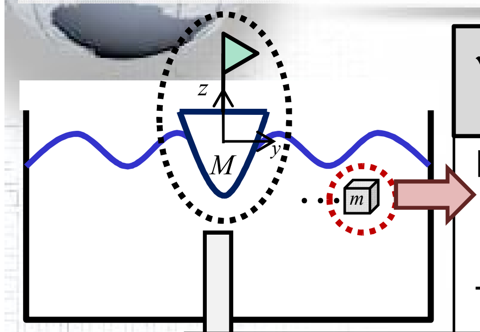
11) Erwin Kreyszig, "Advanced Engineering Mathematics", Wiley, 2005
12) Falnes, J. , "Ocean waves and oscillating systems", Cambridge Univ. Press, 2002

Introduction.

Introduction
: 유체 역학 및 선박 운동 방정식 개요

1) RTT : Reynold Transport Theorem
2) SWBM : Still Water Bending Moment
3) VWBM : Vertical Wave Bendidng Moment

✓ Assumption
① 뉴턴 유체 (Newtonian fluid)
② 비압축성 유동 (Incompressible flow)
③ 비점성 유동 (Invicid flow)
④ 비회전 유동 (Irrotational flow)



미시적 유도/
거시적 유도(RTT[1])

✓유체입자의 운동방정식

Lagrangian & Eulerian Description

Newton's 2nd Law

$$\mathbf{m}\ddot{\mathbf{x}} = \sum \mathbf{F} = \mathbf{F}_{body} + \mathbf{F}_{surface}$$

유체입자 6자유도

Cauchy equation → ①② → 전단 응력 → Navier-Stokes equation → ③ → Euler equation → ④ → Curl과 회전 → Bernoulli equation

유체입자 3자유도
(회전 3자유도 X)

Mass Conservation Law → ②④ → Laplace Equation

$$\nabla^2\Phi = 0$$

$$\rho\frac{\partial\Phi}{\partial t} + P + \frac{1}{2}\rho\left|\nabla\Phi\right|^2 + \rho g\,z = C$$

✓선박의 6자유도 운동 방정식

① Coordinate system 정의
(Global & Body-fixed coordinate)

② Newton's 2nd Law

$$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F} = \mathbf{F}_{body} + \boxed{\mathbf{F}_{surface}}$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{Fluid}$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{F.K} + \mathbf{F}_D + \mathbf{F}_R$$

$$= \mathbf{F}_{restoring} + \mathbf{F}_{exciting} - \mathbf{A}\ddot{\mathbf{x}} - \mathbf{B}\dot{\mathbf{x}}$$

$$(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} = \mathbf{F}_{restoring} + \mathbf{F}_{exciting}$$

cf) 선형화 된 복원력 ($\mathbf{F}_{restoring} = -\mathbf{C}\mathbf{x}$)

$$(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{F}_{exciting}$$

z방향 성분

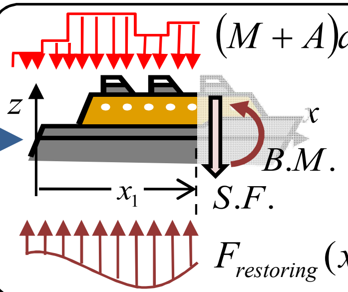
✓유체력 계산

$\Phi_T = \Phi_I$ (Incident wave potential)
$+\Phi_D$ (Diffraction velocity potential)
$+\Phi_R$ (Radiation velocity potential)

Linearization

$$P = -\rho g z - \rho\frac{\partial\Phi_T}{\partial t}$$

$$\mathbf{F}_{Fluid} = \iint_{S_B} P\mathbf{n}dS = \mathbf{F}_{static} + \mathbf{F}_{F.K} + \mathbf{F}_D + \mathbf{F}_R$$



$(M+A)a_z(x)$

(a_z : z방향 가속도)
(v_z : z방향 속도)

$F_{restoring}(x) + F_{exciting}(x) - Bv_z(x)$

✓Shear force(S.F.) 및 bending moment(B.M.)

Shear force($S.F.$) 계산

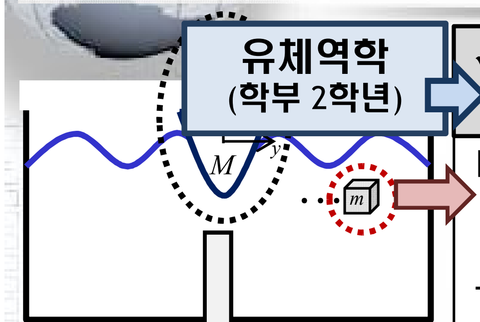
↓(적분)

Bending moment($B.M.$)

Introduction
: 학부 과목과의 연계

1) RTT : Reynold Transport Theorem
2) SWBM : Still Water Bending Moment
3) VWBM : Vertical Wave Bendidng Moment

✓ Assumption
① 뉴턴 유체 (Newtonian fluid)
② 비압축성 유동 (Incompressible flow)
③ 비점성 유동 (Invicid flow)
④ 비회전 유동 (Irrotational flow)



유체역학 (학부 2학년)

✓유체입자의 운동방정식

Lagrangian & Eulerian Description

Newton's 2nd Law

$$\mathbf{m\ddot{x}} = \sum \mathbf{F} = \mathbf{F}_{body} + \mathbf{F}_{surface}$$

유체입자 6자유도

Cauchy equation → ①② Navier-Stokes equation → ③ Euler equation → ④ Bernoulli equation

전단 응력

유체입자 3자유도 (회전 3자유도 X)

Curl과 회전

시적 유도/ 시적 유도(RTT[1])

Mass Conservation Law ②④

공학수학 (학부2학년)

$$\nabla^2\Phi = 0$$

$$\rho\frac{\partial\Phi}{\partial t} + P + \frac{1}{2}\rho\left|\nabla\Phi\right|^2 + \rho g z = C$$

✓선박의 6자유도 운동 방정식

운항제어론 (학부 3학년)

① Coordinate system 정의 (Global & Body-fixed coordinate)

② Newton's 2nd Law

$$\mathbf{M\ddot{x}} = \sum \mathbf{F} = \mathbf{F}_{body} + \boxed{\mathbf{F}_{surface}}$$

조선해양공학계획 (학부 2학년)

$$= \mathbf{F}_{gra} \ldots \mathbf{F}_D + \mathbf{F}_R$$

$$= \mathbf{F}_{restoring} + \mathbf{F}_{exciting} - \mathbf{A\ddot{x}} - \mathbf{B\dot{x}}$$

$$(\mathbf{M}+\mathbf{A})\mathbf{\ddot{x}} + \mathbf{B\dot{x}} = \mathbf{F}_{restoring} + \mathbf{F}_{exciting}$$

cf) 선형화 된 복원력 ($\mathbf{F}_{restoring} = -\mathbf{Cx}$)

$$(\mathbf{M}+\mathbf{A})\mathbf{\ddot{x}} + \mathbf{B\dot{x}} + \mathbf{Cx} = \mathbf{F}_{exciting}$$

공학수학 (학부2학년)

z방향 성분

✓유체력 계산

$\Phi_T = \Phi_I$ (Incident wave potential)
$+\Phi_D$ (Diffraction velocity potential)
$+\Phi_R$ (Radiation velocity potential)

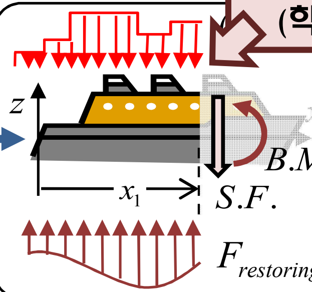
Linearization

$$P = -\rho g z - \rho\frac{\partial\Phi_T}{\partial t}$$

$$\mathbf{F}_{Fluid} = \iint_S P\mathbf{n}dS = \mathbf{F}_{static} + \mathbf{F}_{F.K} + \mathbf{F}_D + \mathbf{F}_R$$

선체구조설계시스템 (학부 3학년)

재료역학 (학부2학년)



(a_z : z방 …)
(v_z : z방향 속도)

$$F_{restoring}(x) + F_{exciting}(x) - Bv_z(x)$$

✓Shear force(S.F.) 및 bending moment(B.M.)

Shear force($S.F.$) 계산

↓(적분)

Bending moment($B.M.$)

Introduction
: 배울 내용

1) RTT : Reynold Transport Theorem
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✓ Assumption
① 뉴턴 유체 (Newtonian fluid)
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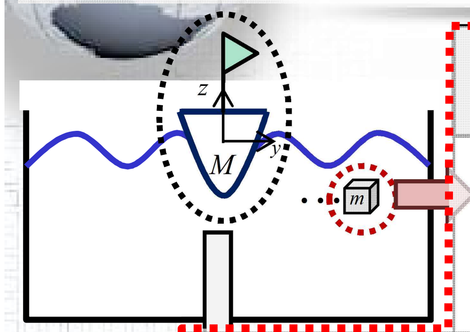


(1) 유체 입자의 운동 방정식

✓유체입자의 운동방정식

Lagrangian & Eulerian Description

전단 응력

Curl과 회전

Newton's 2nd Law

$m\ddot{x} = \sum F$

$= F_{body} + F_{surface}$

Cauchy equation → Navier-Stokes equation → Euler equation → Bernoulli equation

미시적 유도/ 거시적 유도(RTT[1])

Mass Conservation Law → ②④ Laplace Equation

$\nabla^2\Phi = 0$

$\rho\dfrac{\partial\Phi}{\partial t} + P + \dfrac{1}{2}\rho|\nabla\Phi|^2 + \rho g z = C$

(2) 선박의 6자유도 운동 방정식

✓선박의 6자유도 운동 방정식

① Coordinate system 정의
(Global & Body-fixed coordinate)

② Newton's 2nd Law

$M\ddot{x} = \sum F = F_{body} + F_{surface}$

$= F_{gravity} + F_{Fluid}$

$= F_{gravity} + F_{static} + F_{F.K} + F_D + F_R$

$= F_{restoring} + F_{exciting} - A\ddot{x} - B\dot{x}$

$(M + A)\ddot{x} + B\dot{x} = F_{restoring} + F_{exciting}$

cf) 선형화 된 복원력 ($F_{restoring} = -Cx$)

$(M + A)\ddot{x} + B\dot{x} + Cx = F_{exciting}$

z방향 성분

(3) Velocity potential & Strip Theory

✓유체력 계산

$\Phi_T = \Phi_I$ (Incident wave potential)

$+ \Phi_D$ (Diffraction velocity potential)

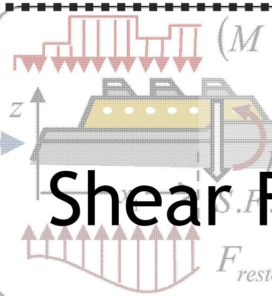
$+ \Phi_R$ (Radiation velocity potential)

Linearization

$P = -\rho g z - \rho\dfrac{\partial\Phi_T}{\partial t}$

$F_{Fluid} = \iint_{S_B} P\mathbf{n}dS = F_{static} + F_{F.K} + F_D + F_R$

(4) Wave Load & Shear Force/Bending Moment



$(M + A)a_z(x)$

$(a_z$: z방향 가속도)

$(v_z$: z방향 속도)

$F_{restoring}(x) + F_{exciting}(x) - Bv_z(x)$

✓Shear force(S.F.) 및 bending moment(B.M.)

Shear force(S.F.)

Bending moment(B.M.)

유체 입자의 운동 방정식

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유체 입자의 운동 방정식 정리
(Cauchy eq. ~ Bernoulli eq.)

1) 전단응력이 전단변형율(의 시간변화율)에 비례하는 유체
2) 선형 변형과 등방 팽창에 의한 점성 계수 μ,λ의 관계식을 정의함

Cauchy Equation : $\rho\dfrac{d\mathbf{V}}{dt}=\rho\mathbf{g}+\nabla\bullet\sigma \quad ,\left(\mathbf{V}=[u,v,w]^T\right)$

① 뉴턴 유체[1] (Newtonian fluid)
② Stokes assumption[2]

Navier-Stokes Equation : (in general form) $\rho\dfrac{d\mathbf{V}}{dt}=\rho\mathbf{g}-\nabla P+\mu\left(\dfrac{1}{3}\nabla(\nabla\bullet\mathbf{V})+\nabla^2\mathbf{V}\right)$

$(\mu=0)$ ③ 비점성 유동 (Invicid flow)

Euler Equation : $\rho\dfrac{d\mathbf{V}}{dt}=\rho\mathbf{g}-\nabla P$

$\rho=\rho(P)$ ④ barotropic flow

Euler Equation : (Another form) $\dfrac{\partial\mathbf{V}}{\partial t}+\nabla B=\mathbf{V}\times\boldsymbol{\omega} \quad ,\left(B=\dfrac{1}{2}q^2+gy+\int\dfrac{dP}{\rho}\ ,\ q^2=u^2+v^2+w^2\right)$

$\left(\dfrac{\partial\mathbf{V}}{\partial t}=0\right)$ ⑤ Steady flow

Bernoulli equation (case1) $B=\text{Constant}$

$\left(\dfrac{1}{2}q^2+gy+\int\dfrac{dP}{\rho}=C\right)$

along streamlines and vortex lines

$\left(\begin{array}{l}\mathbf{V}=\nabla\Phi\\ q^2=|\nabla\Phi|^2,\boldsymbol{\omega}=0\end{array}\right)$ ⑥ Unsteady, irrotational flow

Bernoulli equation (case2) $\dfrac{\partial\Phi}{\partial t}+\dfrac{1}{2}|\nabla\Phi|^2+gy+\int\dfrac{dP}{\rho}=F(t)$

$(\rho=constant)$ ⑦ Incompressible flow

Bernoulli equation (case3) $\dfrac{\partial\Phi}{\partial t}+\dfrac{1}{2}|\nabla\Phi|^2+gy+\dfrac{P}{\rho}=F(t)$

Continuity Equation(질량보존) $\dfrac{\partial\rho}{\partial t}+\nabla\bullet\rho\mathbf{V}=0$

⑦ incompressible flow

$\rho=constant\left(\dfrac{\partial\rho}{\partial t}=0\right)$

$\nabla\bullet\mathbf{V}=0$

⑥ irrotational flow $(\mathbf{V}=\nabla\Phi)$

Laplace Equation $\nabla^2\Phi=0$

Newtonian fluid
Stokes assumption
Invicid flow
Unsteady flow
Irrotational flow
Incompressible flow

Cauchy Equation의 유도

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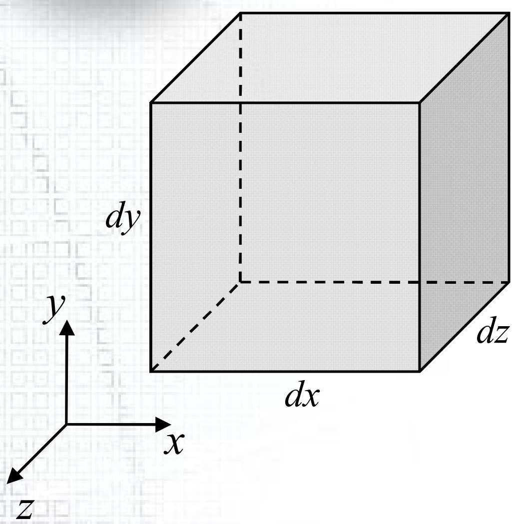
1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93

Cauchy Equation[1],[2] 유도

미소 유체 요소



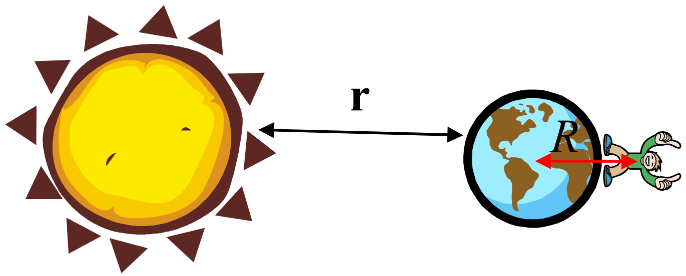
✓ **미소 유체 요소가 받는 힘** (Newton's 2nd Law)

질량 X 가속도 ➡ Lagrangian & Eulerian description

$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \mathbf{F}_{Surface}$$ **(체적력 + 표면력)**

Action at a distance
without physical Contact[2)]

ex) gravitational force , magnetic, electrostatic,...



Surface forces are exerted on an area element by the surroundings through direct contact[2)]

유체의 종류에 따라 다르지만, 저항하는 힘을 느낌

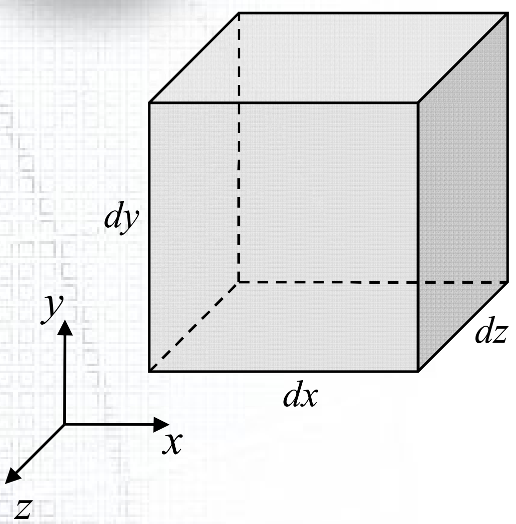
(Q) **공기중에서도 표면력이 있는가?**

(A) **있다.**
(밀도가 작아서 작용하는 힘을 못 느낄 뿐)

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93

Cauchy Equation[1),2)] 유도 – Body Force

미소 유체 요소



✓ Gravitational force is a body force on Earth.

$$\mathbf{F}_{Body} = \mathbf{F}_{Gravity} = m\mathbf{g} = \rho\mathbf{g}dxdydz \quad \left(m = \rho dV = \rho dxdydz\right)$$

$$\begin{cases} F_{x,Body} = F_{x,Gravity} = mg_x = \rho g_x dxdydz = 0 \\ F_{y,Body} = F_{y,Gravity} = mg_y = \rho g_y dxdydz = -mg \\ F_{z,Body} = F_{z,Gravity} = mg_z = \rho g_z dxdydz = 0 \end{cases}$$

지구상에서의 체적력

$g = 9.81$: 중력가속도

(cf) 중력에 의한 포텐셜 에너지 : $\Pi = gy$

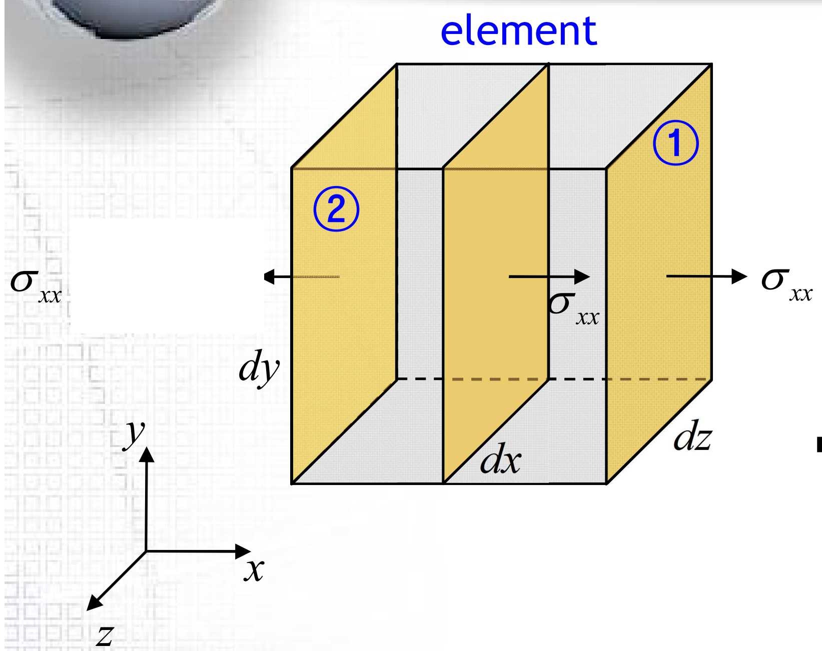
중력력은 보존력(conservative force)이므로,
중력에 의한 포텐셜 에너지의 Gradient로부터 구할 수 있음

$$\mathbf{g} = -\nabla\Pi \quad = -\nabla(gy)$$

$$= -\left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right](gy) = -g\mathbf{j}$$

Cauchy Equation[1],[2] 유도
– Stress at a point

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.30-32,88-93



✓ 한 점에서의 응력(Stress at a point)

➡ 어느 면에 어느 방향으로 힘이 작용하는지 정의해야 함

Orientation of the surface

Direction of the force

➔ 2개의 방향 필요

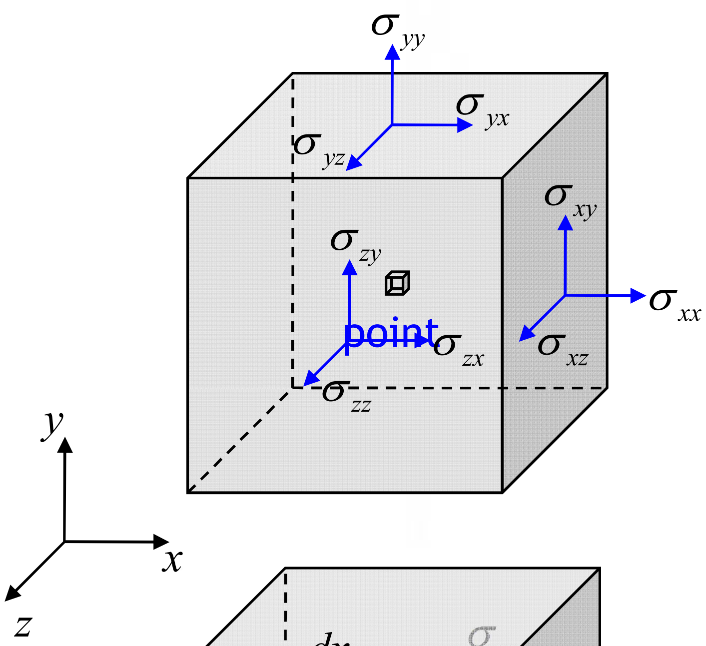
$$\sigma_{x\,x}$$

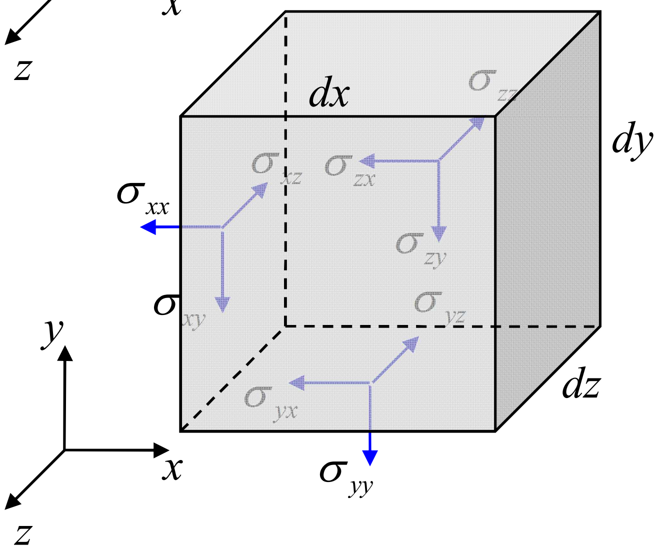
➡ point를 element로 간주하여 면에 작용하는 응력을 정의한 후, dx,dy,dz가 거의 0에 가깝다 ($dx,dy,dz \to 0$)는 것으로부터 응력을 정의

➡ $dx \to 0$일 때, 두 면이 일치하므로, 응력의 크기는 같고, 방향은 반대로 정의됨

Cauchy Equation[1),2)] 유도
– Stress at a point

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.30-32,88-93





✓ 한 점에서의 응력(Stress at a point)

➡ 어느 면에 어느 방향으로 힘이 작용하는지 정의해야 함

Orientation of the surface — Direction of the force ➔ 2개의 방향 필요

$$\sigma_{x\,y}$$

➡ 한 면에 수직인 성분과 2개의 접선 방향 성분으로 나타냄 (3개의 응력 성분이 존재함)

➡ 육면체의 세 면에 대해 총 9개의 응력 성분을 정의

$$\left(\sigma_{xx},\sigma_{xy},\sigma_{xz},\sigma_{yx},\sigma_{yy},\sigma_{yz},\sigma_{zx},\sigma_{zy},\sigma_{zz}\right)$$

3X3 Matrix 형태로 정리함

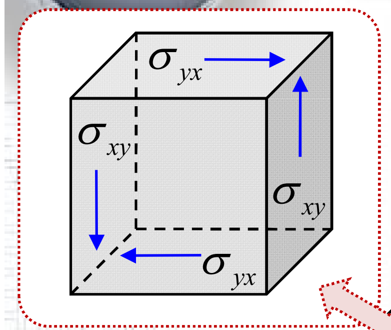
응력 텐서 : (stress tensor)

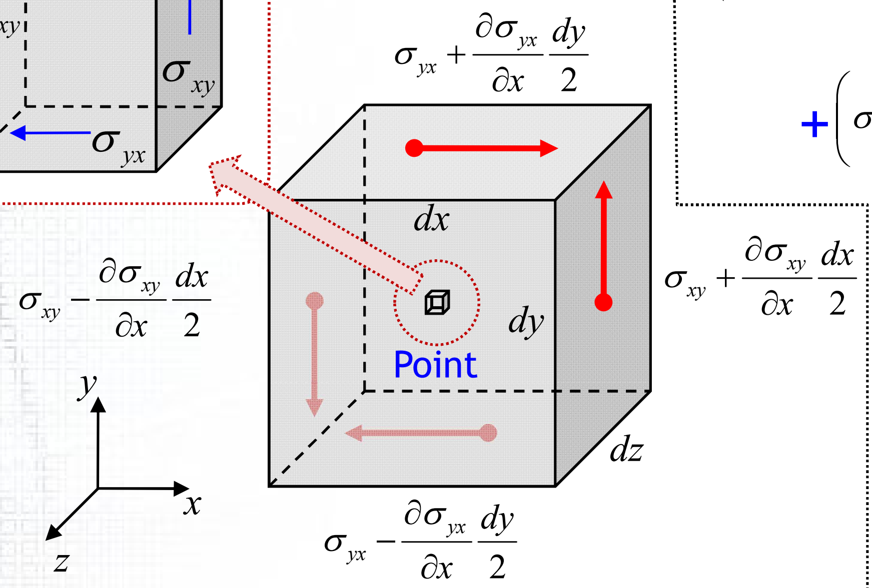
$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

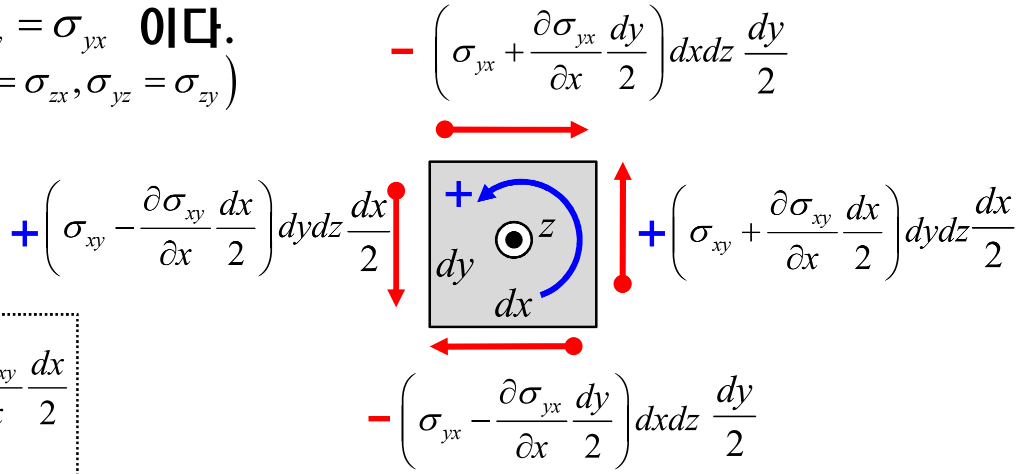
1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93

Cauchy Equation[1),2)] 유도
– Stress at a point

(pf) 응력 성분 중에서 $\sigma_{xy} = \sigma_{yx}$ 이다.
$\left(\sigma_{xz} = \sigma_{zx}, \sigma_{yz} = \sigma_{zy}\right)$







※ Total moment

$$M_z = \left(\sigma_{xy} + \cancel{\frac{\partial \sigma_{xy}}{\partial x}\frac{dx}{2}}\right)dydz\frac{dx}{2} - \left(\sigma_{yx} + \cancel{\frac{\partial \sigma_{yx}}{\partial x}\frac{dy}{2}}\right)dxdz\frac{dy}{2}$$
$$+\left(\sigma_{xy} - \cancel{\frac{\partial \sigma_{xy}}{\partial x}\frac{dx}{2}}\right)dydz\frac{dx}{2} - \left(\sigma_{yx} - \cancel{\frac{\partial \sigma_{yx}}{\partial x}\frac{dy}{2}}\right)dxdz\frac{dy}{2}$$
$$= \sigma_{xy}dxdydz - \sigma_{yx}dxdydz = \left(\sigma_{xy} - \sigma_{yx}\right)dxdydz$$

※ Mass moment of inertia

$$I_{zz} = \frac{m}{12}\left(dx^2 + dy^2\right) = \frac{\rho dxdydz}{12}\left(dx^2 + dy^2\right)$$

➡ rotational equilibrium of the element

$$I_{zz}\dot{\omega} = M_z$$
$$\frac{\rho dxdydz}{12}\left(dx^2 + dy^2\right)\dot{\omega} = \left(\sigma_{xy} - \sigma_{yx}\right)dxdydz$$
$$\frac{\rho}{12}\left(\cancel{dx^2 + dy^2}\right)\dot{\omega} = \sigma_{xy} - \sigma_{yx} \quad = 0$$

point이므로 $dx \rightarrow 0$, $dy \rightarrow 0$

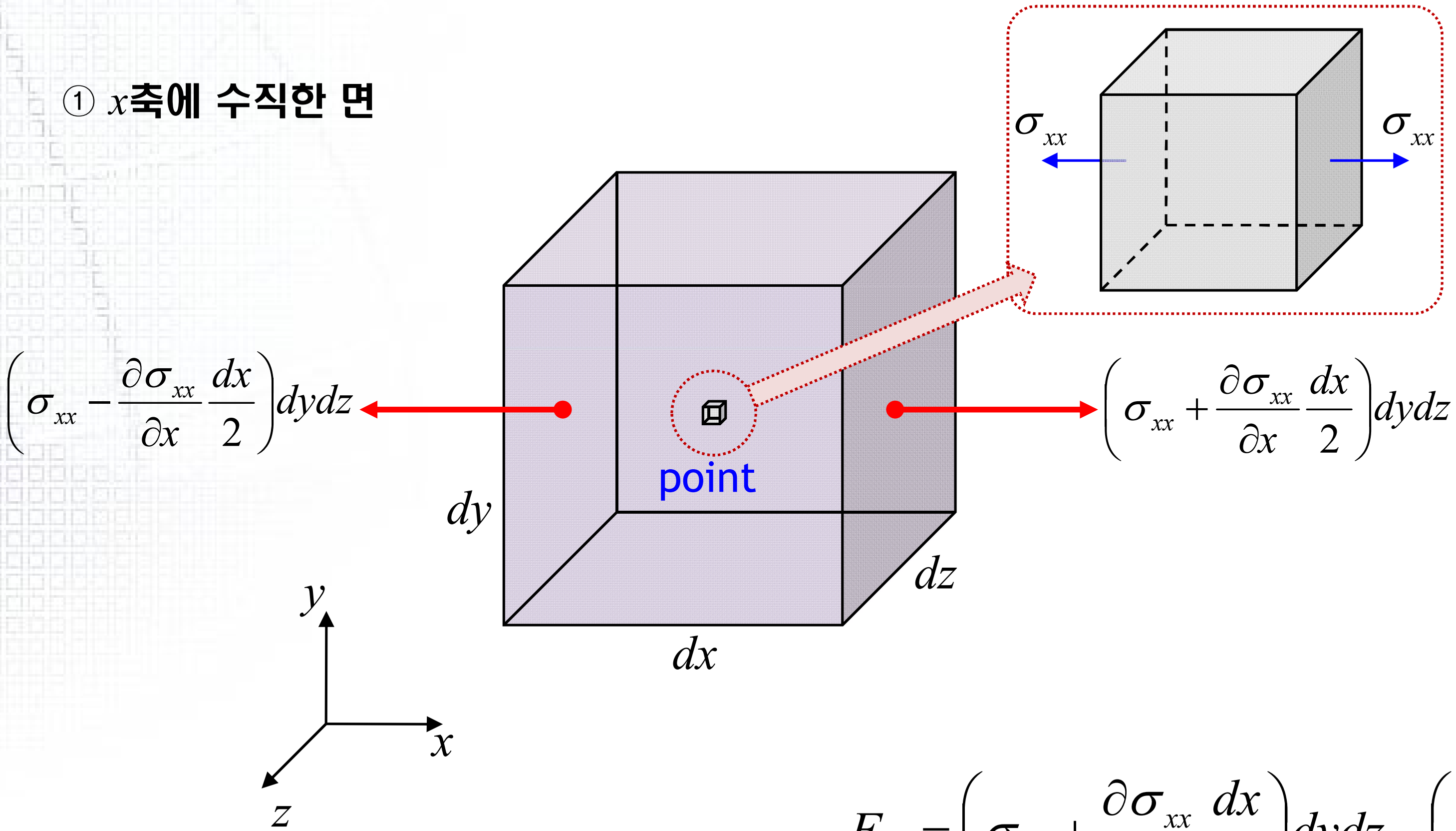
$$\boxed{\therefore \sigma_{xy} = \sigma_{yx} \quad \left(\sigma_{xz} = \sigma_{zx}, \sigma_{yz} = \sigma_{zy}\right)}$$

Cauchy Equation[1],[2] 유도
– Surface force on Element

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93

표면력 중 x축 방향 힘 : 중심면으로부터 $\pm\dfrac{dx}{2}$ 만큼 떨어진 곳에서 x축 방향으로 작용하는 힘

① x축에 수직한 면



$$F_{xx} = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{dx}{2}\right)dydz$$

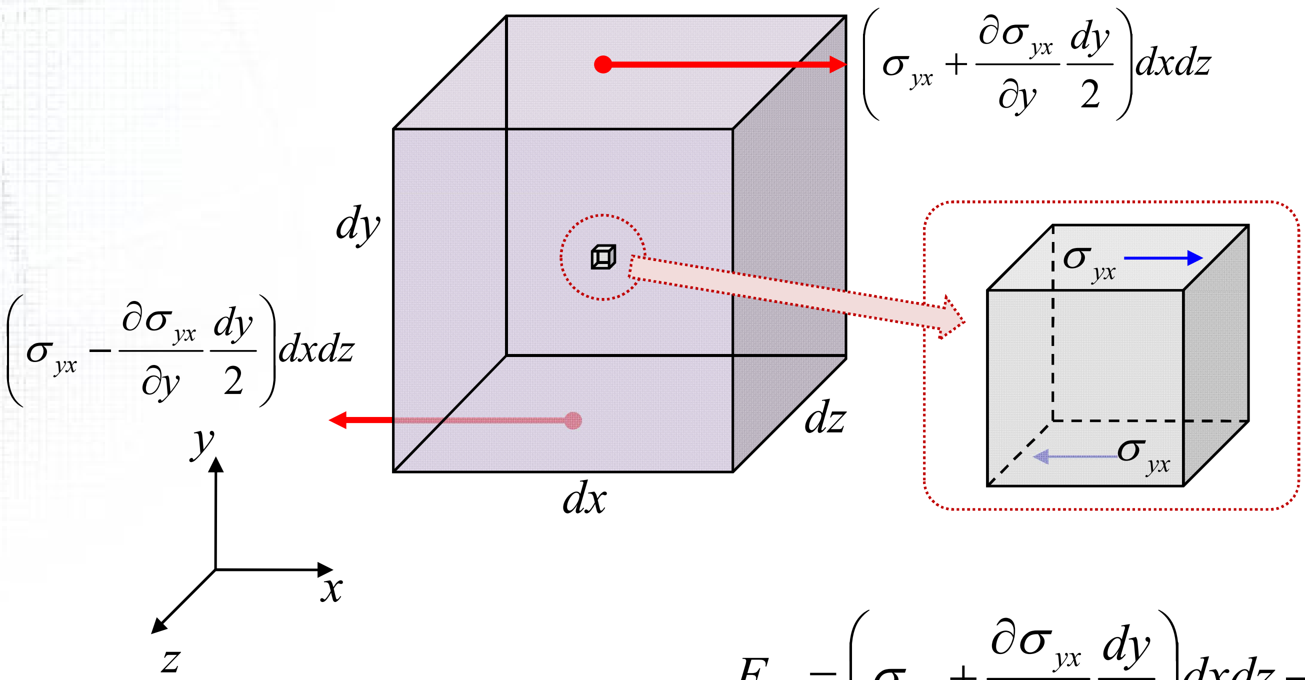
$$= \frac{\partial \sigma_{xx}}{\partial x}dxdydz$$

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93

Cauchy Equation[1],[2] 유도
– Surface force on Element

표면력 중 x축 방향 힘 : 중심면으로부터 $\pm\frac{dy}{2}$ 만큼 떨어진 곳에서 x축 방향으로 작용하는 힘

② y축에 수직한 면



$$F_{yx}=\left(\sigma_{yx}+\frac{\partial\sigma_{yx}}{\partial y}\frac{dy}{2}\right)dxdz-\left(\sigma_{yx}-\frac{\partial\sigma_{yx}}{\partial y}\frac{dy}{2}\right)dxdz$$

$$=\frac{\partial\sigma_{yx}}{\partial y}dxdydz$$

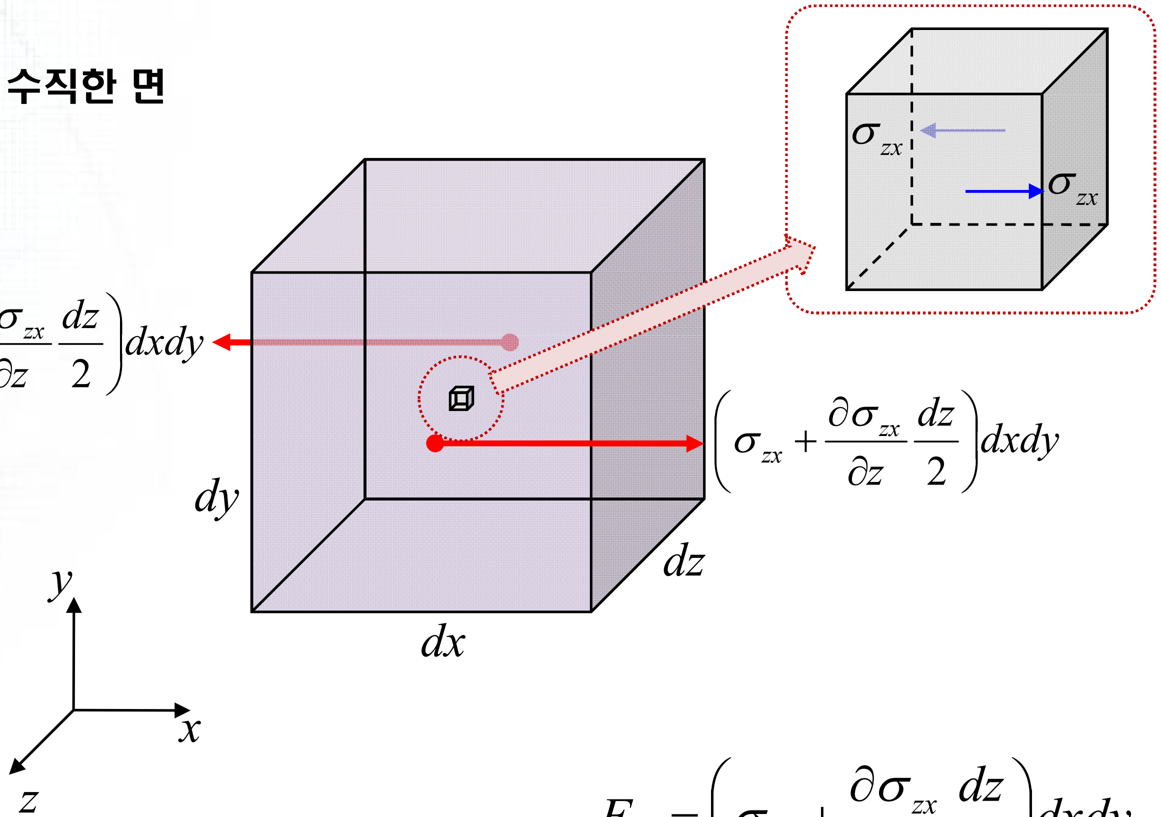
1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93

Cauchy Equation[1),2)] 유도
– Surface force on Element

표면력 중 x축 방향 힘 : 중심면으로부터 $\pm\dfrac{dz}{2}$ 만큼 떨어진 곳에서 x축 방향으로 작용하는 힘

③ z축에 수직한 면

$$\left(\sigma_{zx} - \frac{\partial \sigma_{zx}}{\partial z}\frac{dz}{2}\right)dxdy$$



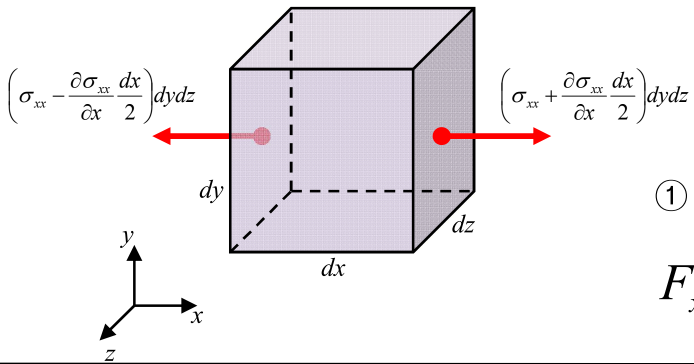
$$\left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\frac{dz}{2}\right)dxdy$$

$$F_{zx} = \left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z}\frac{dz}{2}\right)dxdy - \left(\sigma_{zx} - \frac{\partial \sigma_{zx}}{\partial z}\frac{dz}{2}\right)dxdy$$

$$= \frac{\partial \sigma_{zx}}{\partial z}dxdydz$$

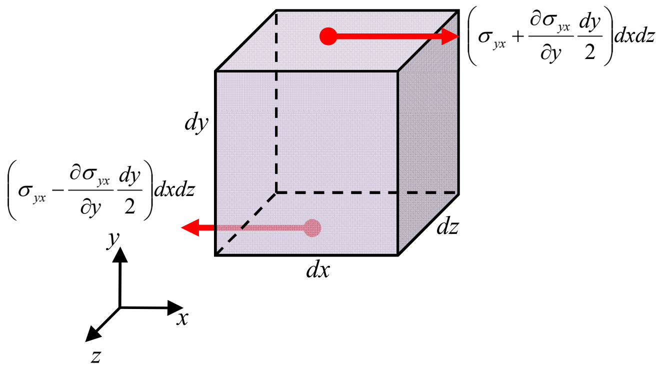
Cauchy Equation[1),2)] 유도
– Surface force on Element

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93



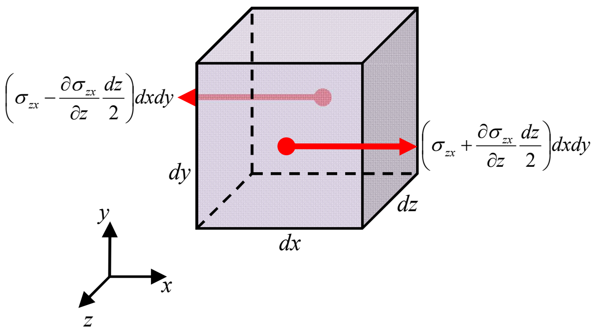
① x축에 수직한 면

$$F_{xx} = \frac{\partial \sigma_{xx}}{\partial x} dxdydz$$



② y축에 수직한 면

$$F_{yx} = \frac{\partial \sigma_{yx}}{\partial y} dxdydz$$



③ z축에 수직한 면

$$F_{zx} = \frac{\partial \sigma_{zx}}{\partial z} dxdydz$$

✓ 표면력 중 x축 방향 힘

$$F_{x,surface} = F_{xx} + F_{yx} + F_{zx}$$

$$= \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \right) dxdydz$$

✓ 표면력 중 y축 방향 힘

$$F_{y,surface} = F_{xy} + F_{yy} + F_{zy}$$

$$= \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \right) dxdydz$$

✓ 표면력 중 z축 방향 힘

$$F_{z,surface} = F_{xz} + F_{yz} + F_{zz}$$

$$= \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) dxdydz$$

Cauchy Equation[1],[2] 유도
– Surface force on Element

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93

(3) 미소 유체 요소에 작용하는 표면력

① 표면력 중 x축 방향 힘

$$F_{x,Surface} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \right) dxdydz$$

② 표면력 중 y축 방향 힘

$$F_{y,Surface} = \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \right) dxdydz$$

③ 표면력 중 z축 방향 힘

$$F_{z,Surface} = \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) dxdydz$$

벡터 형태로 나타내면

$$\mathbf{F}_{Surface} = \hat{i}\, F_{x,Surface} + \hat{j}\, F_{y,Surface} + \hat{k}\, F_{z,Surface}$$

$$= \left[\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \right) \hat{i} + \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \right) \hat{j} + \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) \hat{k} \right] dxdydz$$

$$= \left[\frac{\partial}{\partial x} \left(\sigma_{xx} \hat{i} + \sigma_{xy} \hat{j} + \sigma_{xz} \hat{k} \right) + \frac{\partial}{\partial y} \left(\sigma_{yx} \hat{i} + \sigma_{yy} \hat{j} + \sigma_{yz} \hat{k} \right) + \frac{\partial}{\partial z} \left(\sigma_{zx} \hat{i} + \sigma_{zy} \hat{j} + \sigma_{zz} \hat{k} \right) \right] dxdydz$$

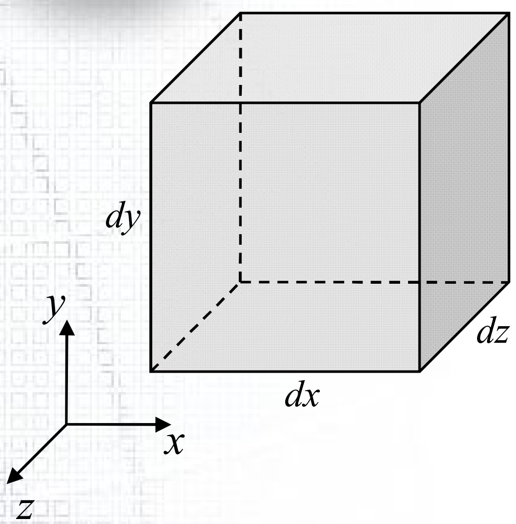
$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \boxed{\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} dxdydz = \underline{\left[\nabla \bullet \sigma \right] dxdydz}$$

응력 텐서 : Symmetric Matrix $\left(\because \sigma_{xy} = \sigma_{yx}, \sigma_{xz} = \sigma_{zx}, \sigma_{yz} = \sigma_{zy} \right)$

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.88-93

Cauchy Equation[1] 유도

미소 유체 요소



✓ 미소 유체 요소가 받는 힘 (Newton's 2nd Law)

$$m\frac{d\mathbf{V}}{dt} = \rho\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)dxdydz \quad (m = \rho dxdydz)$$

$$m\frac{d\mathbf{V}}{dt} = \sum \mathbf{F} = \mathbf{F}_{Body} + \mathbf{F}_{Surface}$$ (체적력 + 표면력)

$$\mathbf{F}_{Body} = \rho \mathbf{g} dxdydz$$

$$\mathbf{F}_{Surface} = \left[\nabla \bullet \sigma_{ij}\right]dxdydz$$

대입

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)\cancel{dxdydz} = \rho\,\mathbf{g}\,\cancel{dxdydz} + \left[\nabla \bullet \sigma\right]\cancel{dxdydz}$$

$dxdydz$ 로 나누면

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right) = \rho\,\mathbf{g} + \nabla \bullet \sigma$$ => Cauchy Equation

Navier-Stokes Equation

Advanced
Ship
Design
Automation
Laboratory

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

Navier-Stokes Equation 유도

✓ Cauchy Equation

$$\rho\frac{d\mathbf{V}}{dt}=\rho\left(\frac{\partial \mathbf{V}}{\partial t}+u\frac{\partial \mathbf{V}}{\partial x}+v\frac{\partial \mathbf{V}}{\partial y}+w\frac{\partial \mathbf{V}}{\partial z}\right)=\rho\mathbf{g}+\nabla\bullet\sigma$$

x축 성분 : $\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x+\boxed{\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{yx}}{\partial y}+\frac{\partial \sigma_{zx}}{\partial z}}$

y축 성분 : $\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y+\boxed{\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{zy}}{\partial z}}$

z축 성분 : $\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z+\boxed{\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}}$

(Q) 3개의 식인데, 미지수는 몇 개?

(A) 밀도(1개), 속도 (3개), 응력 텐서(6개) **총 10개의 미지수**가 존재

$$\left(\rho,u,v,w,\underline{\sigma_{xx},\sigma_{xy},\sigma_{xz},\sigma_{yy},\sigma_{yz},\sigma_{zz}}\right)$$

응력 텐서 9개 성분 중 대칭성으로 인해 6개의 미지수 존재 $\left(\sigma_{ij}=\sigma_{ji}\right)$

Great Idea !!! **응력 텐서를 속도로 표현하자!!!**

➡ 유체 입자의 운동과 변형, 뉴턴 유체의 가정으로부터 유도할 수 있음

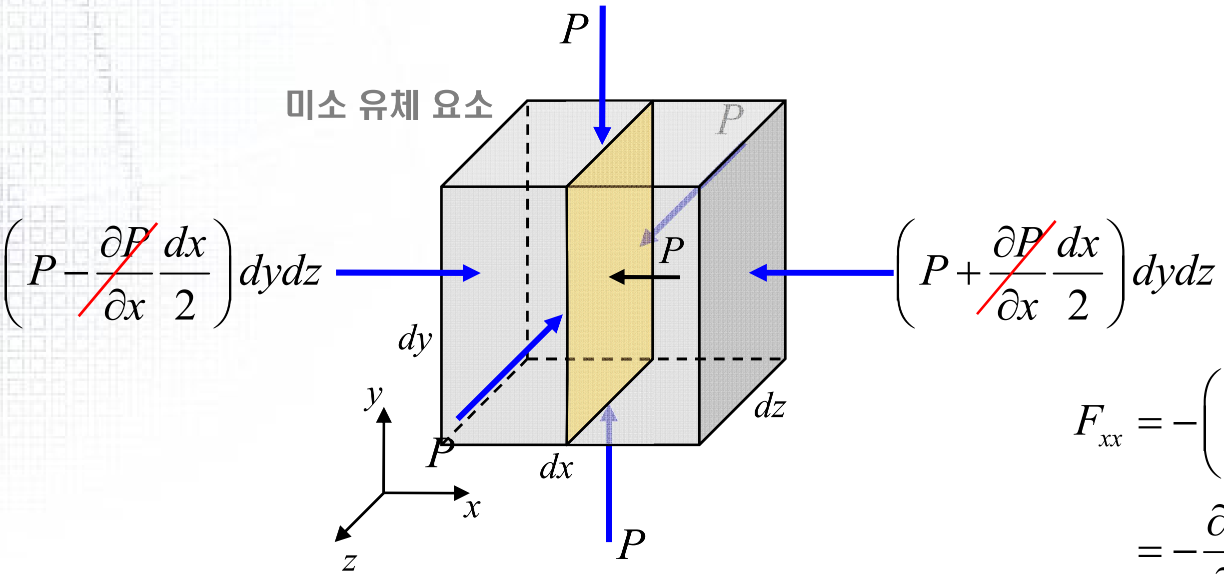
Navier-Stokes Equation 유도

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

✓ 응력 텐서 성분의 분해 : 정지 상태의 유체인 경우

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}$$

P : 이상기체의 상태방정식과 같은 열역학 방정식으로부터 구해지는 **열역학적 압력**(thermodynamic pressure)



$$F_{xx} = -\left(P + \frac{\partial P}{\partial x}\frac{dx}{2}\right)dydz + \left(P - \frac{\partial P}{\partial x}\frac{dx}{2}\right)dydz$$

$$= -\frac{\partial P}{\partial x}dxdydz = m\ddot{x} = 0$$

➡ 정지 상태에서의 미소 유체 입자의 양쪽 면에 작용하는 열역학적 압력은 같다.

➡ 동일한 방법으로 다른 두 축에 대한 열역학적 압력도 고려하면, 모든 면에 작용하는 열역학적 압력은 동일함

Navier-Stokes Equation 유도

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

✓ 응력 텐서 성분의 분해 : 유체가 운동하는 경우

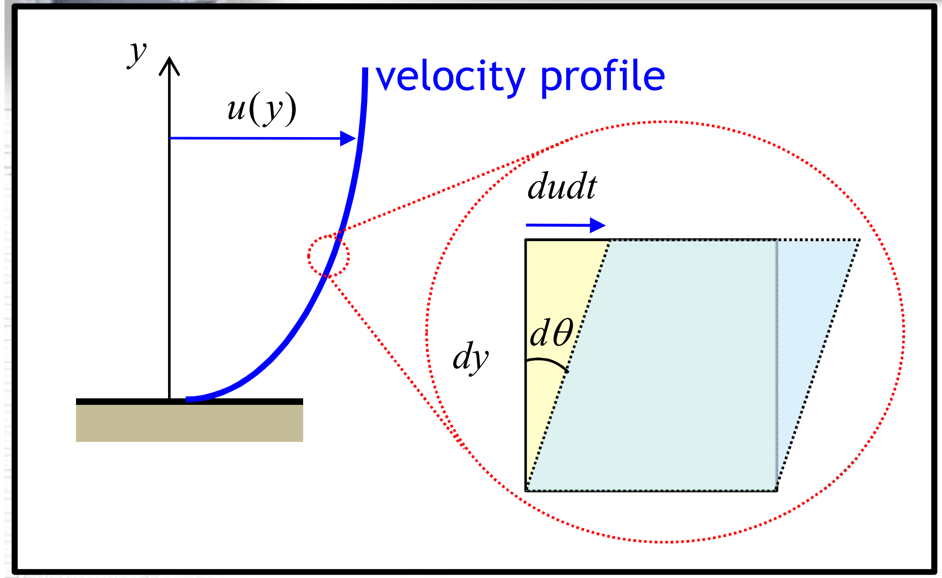
$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \underline{\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}}$$

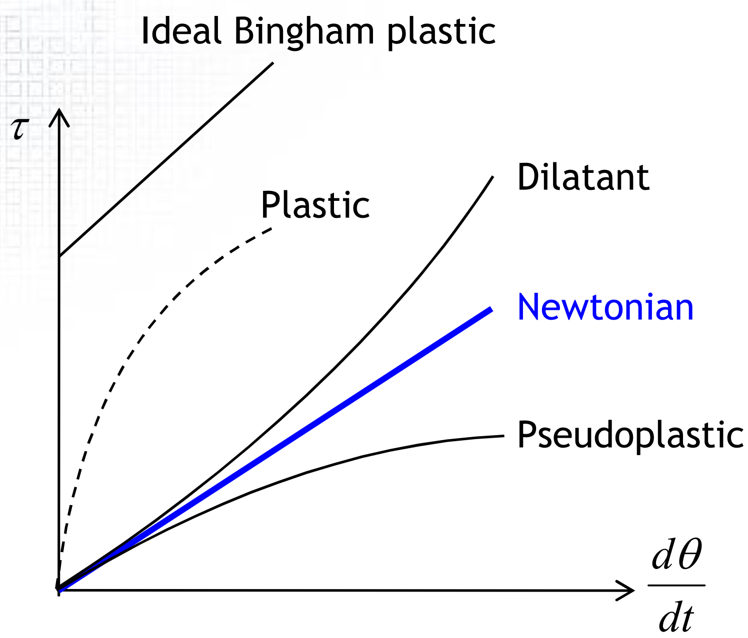
유체 입자가 운동하면, 점성에 의해 각 면에 응력이 발생함

➡ 유체 입자의 운동과 변형, 뉴턴 유체의 가정으로부터 유도할 수 있음

뉴턴 유체[1] (Newtonian Fluid)

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264





✓ 미소 구간에서의 전단변형율

$$d\theta \approx \tan d\theta = \frac{dudt}{dy} = \frac{du}{dy}dt$$

✓ 전단변형율의 시간변화율은 속도 구배와 같음

$$\frac{d\theta}{dt} = \frac{du}{dy} \quad \text{---①}$$

✓ 뉴턴 유체(Newtonian fluid)
: 전단응력이 전단변형율의 시간변화율에 비례

$$\tau \propto \frac{d\theta}{dt} \quad \text{---②}$$

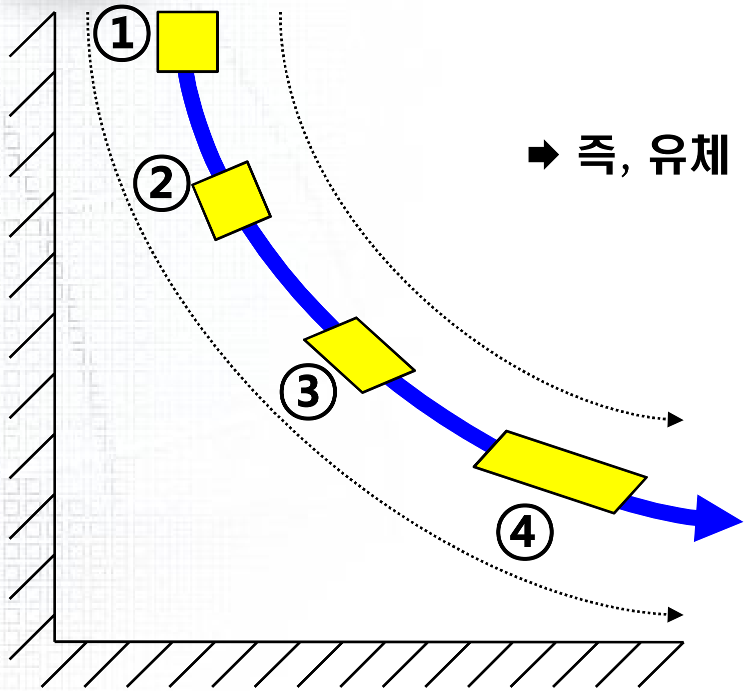
✓ 뉴턴 유체(Newtonian fluid)의 특징
: ①,②에 의해, 전단응력은 속도구배에 비례함
(비례 상수 μ : 점성 계수)

$$\tau \propto \frac{du}{dy} \Rightarrow \tau = \mu\frac{du}{dy}$$

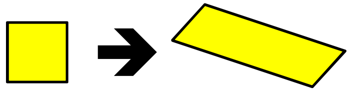
Navier-Stokes Equation 유도
- 유체 입자의 운동과 변형

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

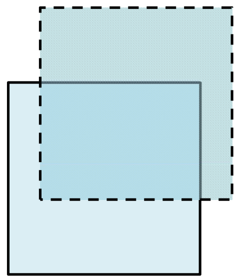
미소 유체 요소

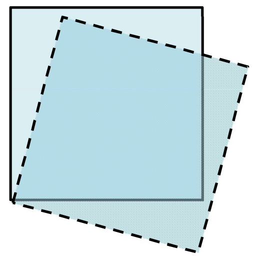


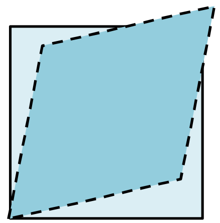
① ➔ ④ : 유체 입자의 이동 경로

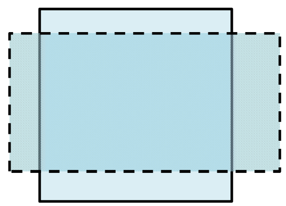

: 유체 입자의 변형

➡ 즉, 유체 입자는 운동(병진,회전)하면서, 변형(전단변형,선형변형)도 함


병진 운동 (translation)

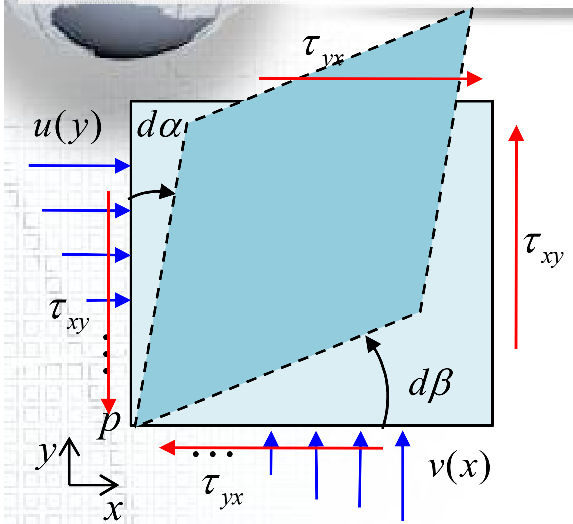

회전 (rotation)


전단변형(angular distortion)

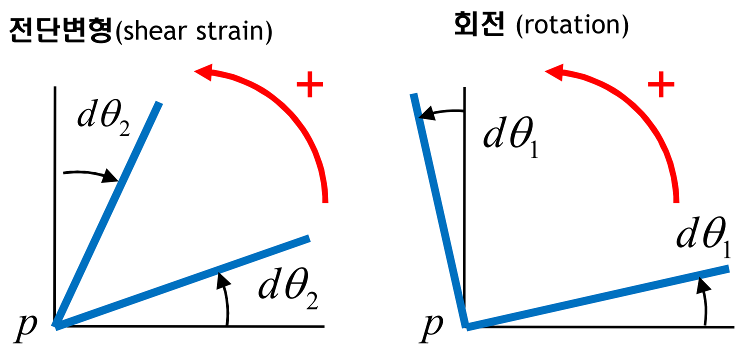

선형변형(volume distortion)

Navier-Stokes Equation 유도 – 전단변형에 의한 전단력

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, “공학도를 위한 길잡이 유체공학입문” , 문운당, 2002, pp258-264



회전과 전단변형으로 나타낼 수 있음



$$-d\alpha = -d\theta_2 + d\theta_1 \quad \cdots ①$$

$$d\beta = d\theta_2 + d\theta_1 \quad \cdots ②$$

Given : $d\alpha, d\beta$

Find : 전단응력 $\tau_{xy} \left(= \tau_{yx}\right)$

(1) ②에서 ①을 빼면, $d\beta + d\alpha = 2d\theta_2$

$$d\theta_2 = \frac{1}{2}(d\alpha + d\beta)$$

➡ 전단변형율

(2) dt로 나누면, $\dfrac{d\theta_2}{dt} = \dfrac{1}{2}\left(\dfrac{d\alpha}{dt} + \dfrac{d\beta}{dt}\right)$

➡ 전단변형율(의 시간변화율)

(3) 뉴턴 유체라 가정하면, 전단응력은 전단변형율의 시간변화율에 비례함 (비례상수 2μ)

$$\tau = 2\mu\frac{d\theta_2}{dt} = 2\mu\frac{1}{2}\left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right) = \mu\left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right)$$

(4) 전단변형율의 시간 변화율은 속도 구배와 같음

$$\tau = \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) = \tau_{xy} = \tau_{yx}$$

(5) 동일한 방법에 의해 다음도 성립함

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{du}{dz} + \frac{dw}{dx}\right), \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right)$$

Navier-Stokes Equation 유도 – 선형 변형에 의한 수직응력

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
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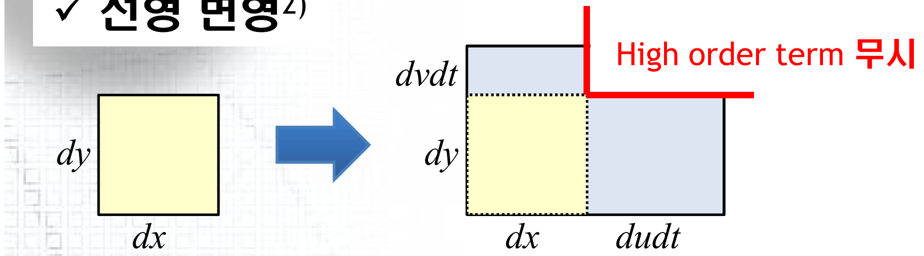
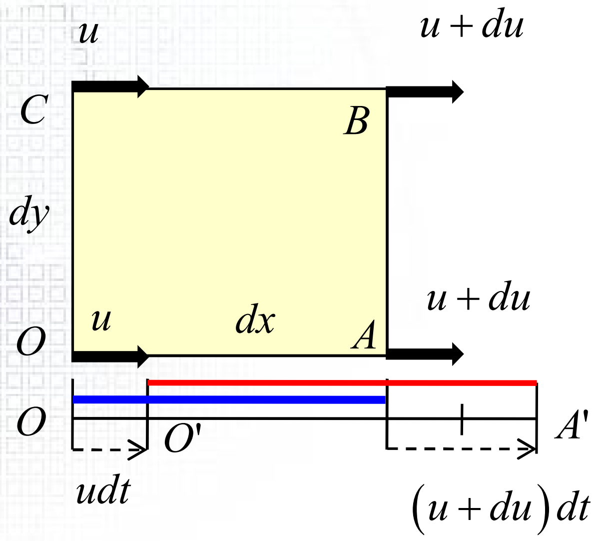
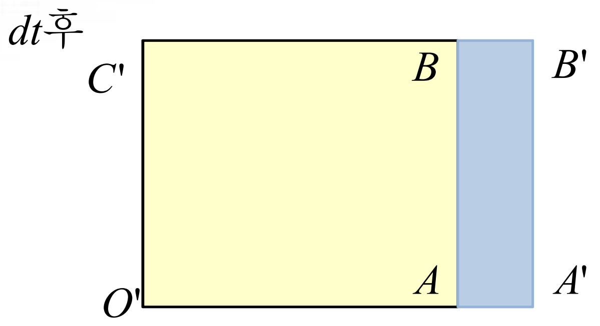
등방 팽창* : 각 축방향으로 동일한 길이만큼 팽창

✓ 선형 변형2)







➡ 시간 t에서의 부피 : $V = dxdydz$

➡ 시간 t+dt에서의 부피 :

$$V + \boxed{dV} = (dx + dudt)(dy + dvdt)(dz + dwdt)$$

High order term 무시

$$= dxdydz + (dudydz + dvdxdz + dwdxdy)dt + \cancel{(\cdots)dt^2 + (dudvdw)dt^3}$$

$$\cong dxdydz + (dudydz + dvdxdz + dwdxdy)dt$$

$$= dxdydz + dxdydz\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)dt \quad = V + \boxed{V\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)dt}$$

$$\boxed{\therefore dV = V\Theta dt} \quad \left(\Theta = \nabla \cdot \mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$$

$$AA' = (u + du)dt - udt = \underline{dudt}$$

속도차이에 의해 선형적으로 늘어난 길이

Navier-Stokes Equation 유도 – 선형 변형에 의한 수직응력

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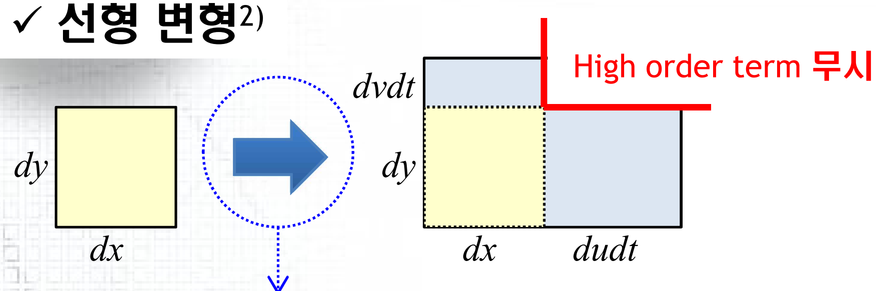
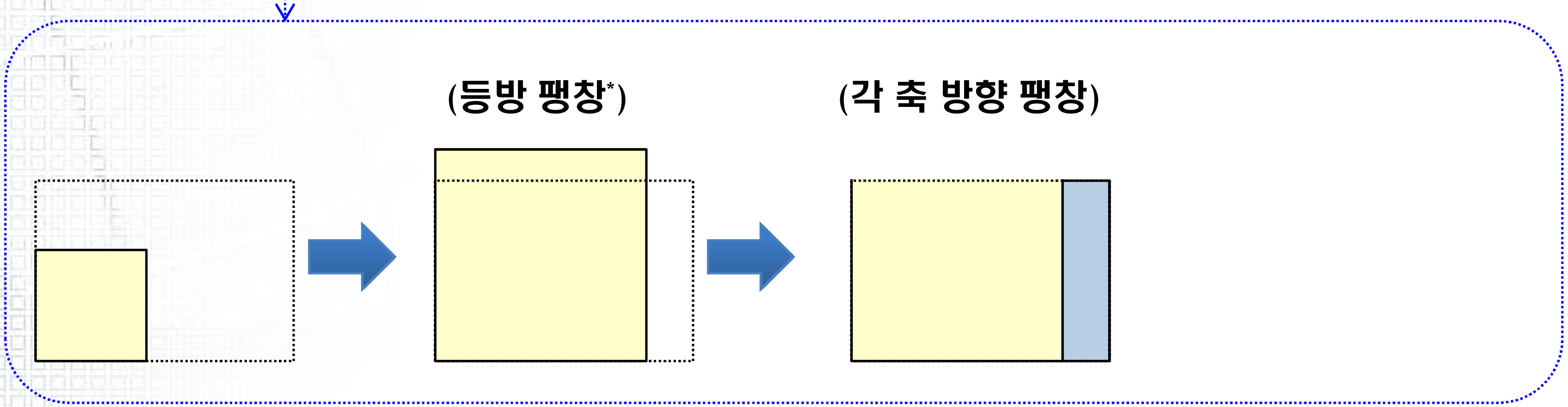
등방 팽창* : 각 축방향으로 동일한 길이만큼 팽창

✓ 선형 변형[2]





Navier-Stokes Equation 유도 – 선형 변형에 의한 수직응력

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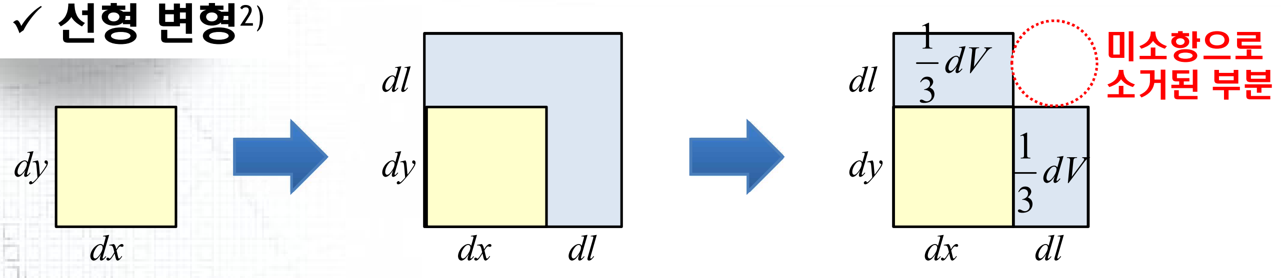
등방 팽창* : 각 축방향으로 동일한 길이만큼 팽창

✓ 선형 변형[2)]



(등방 팽창*)

➡ 등방 팽창이므로, $t+dt$에서 증가한 부피 dV 중 1/3이 각 축방향으로 증가한 부피

$$dV = V\Theta dt \quad \left(\Theta = \nabla \bullet \mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right)$$

$$\frac{1}{3}dV = \frac{1}{3}V\Theta dt \quad \underline{= dldydz}$$

(x축으로 등방 팽창한 부피)

dl로 정리

$$dl = \frac{1}{3}\frac{V}{dydz}\Theta dt = \frac{1}{3}\frac{dxdydz}{dydz}\Theta dt = \frac{1}{3}\Theta dxdt$$

$dxdt$로 나눔

$$\frac{1}{dt}\left(\frac{dl}{dx}\right) = \frac{1}{3}\Theta$$

➡ 등방팽창에 의한 변형율

➡ 등방팽창에 의한 변형율의 시간변화율

➡등방팽창에 의한 수직응력은 등방팽창에 의한 변형율의 시간변화율에 비례함 (비례상수 : 3γ)

$$\tau_x^{dilation,isotrpic} = 3\gamma\frac{1}{3}\Theta = \gamma\Theta$$

(분수를 소거하기 위해 3γ 를 택한 것)

Navier-Stokes Equation 유도 – 선형 변형에 의한 수직응력

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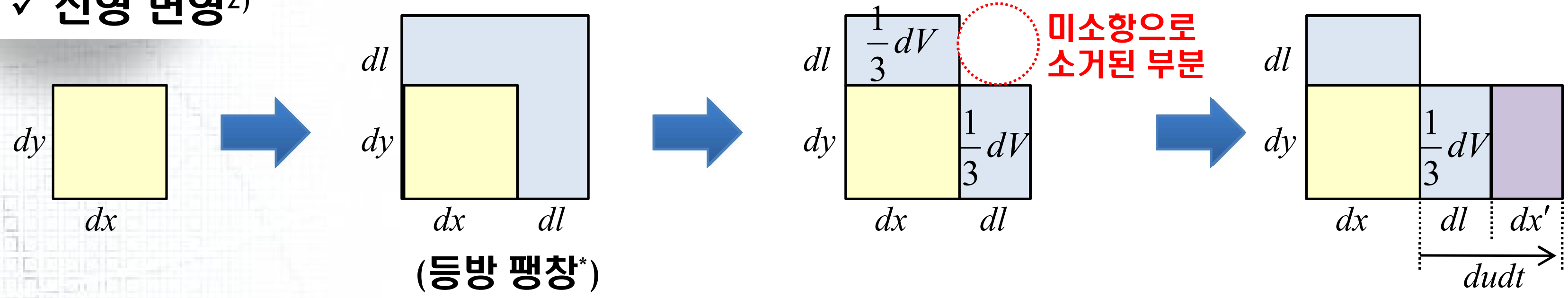
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

$$\Theta = \nabla \bullet \mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$$

✓ **선형 변형**[2)]



➡ **등방 팽창을 제외한 x축방향의 팽창에 의한 변형율의 시간 변화율**

- x축방향의 변형율 : $\frac{dx'}{dx} = \frac{dudt - dl}{dx} = \frac{dudt}{dx} - \frac{dl}{dx}$

- x축방향의 변형율의 시간변화율 : $\frac{1}{dt}\frac{dx'}{dx} = \frac{du}{dx} - \frac{1}{dt}\frac{dl}{dx} = \frac{du}{dx} - \frac{1}{3}\Theta$

➡ **등방 팽창을 제외한 x축방향의 팽창에 의한 전단응력** (비례상수 : 2μ)

$$\tau_x^{dilation,x} = 2\mu\left(\frac{\partial u}{\partial x} - \frac{1}{3}\Theta\right)$$

➡ **등방팽창에 의한 변형율의 시간변화율**

$$\frac{1}{dt}\left(\frac{dl}{dx}\right) = \frac{1}{3}\Theta$$

λ,μ는 물질의 특성에 따라 달라지는 값

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Title

Navier-Stokes Equation 유도
- 선형 변형에 의한 수직응력

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
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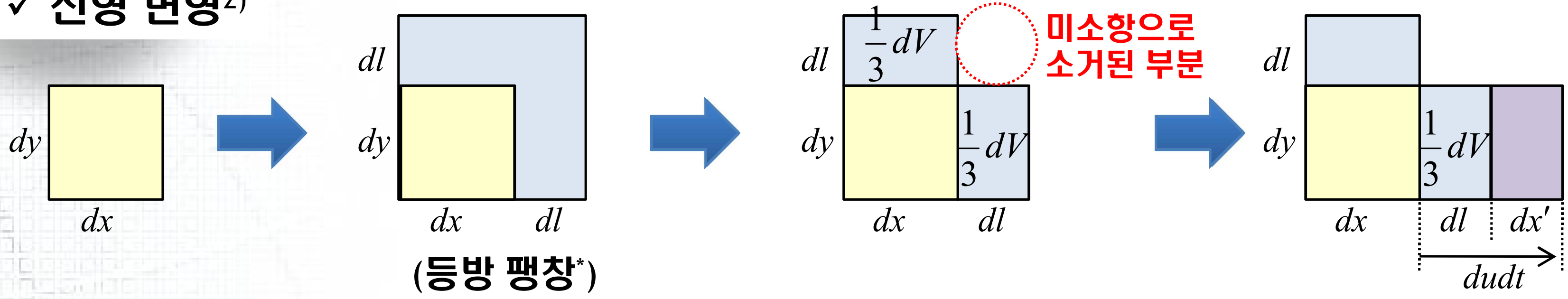
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

$$\left(\Theta = \nabla \bullet \mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$$

✓ 선형 변형[2)]



➡ 등방팽창에 의한 변형율의 시간변화율

$$\frac{1}{dt}\left(\frac{dl}{dx}\right) = \frac{1}{3}\Theta$$

λ, μ는 물질의 특성에 따라 달라지는 값

➡ x축 방향의 팽창에 의한 전단응력 ($\tau_{xx} = \tau_x^{dilation,isotropic} + \tau_x^{dilation,x}$)

$$\tau_{xx} = \tau_x^{dilation,isotropic} + \tau_x^{dilation,x}$$

$$= \gamma\Theta + 2\mu\left(\frac{\partial u}{\partial x} - \frac{1}{3}\Theta\right) = \left(\gamma - \frac{2\mu}{3}\right)\Theta + 2\mu\frac{\partial u}{\partial x} = \lambda\Theta + 2\mu\frac{\partial u}{\partial x}$$

$$= \lambda(\nabla \bullet \mathbf{V}) + 2\mu\frac{\partial u}{\partial x} \qquad \left(\lambda = \gamma - \frac{2\mu}{3}\right)$$

→ 등방팽창을 제외한 각 축방향의 수직 응력

→ 등방팽창에 의한 수직 응력

➡ 동일한 방법에 의해 다음도 성립함 : $\tau_{yy} = \lambda\Theta + 2\mu\frac{\partial v}{\partial y}$, $\tau_{zz} = \lambda\Theta + 2\mu\frac{\partial w}{\partial z}$

Navier-Stokes Equation 유도

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3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

✓ 응력 텐서 성분의 분해 : 유체가 운동하는 경우

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

유체 입자가 운동하면, 점성에 의해 각 면에 응력이 발생함

P : 열역학적 압력
(thermodynamic pressure)

➡ 유체 입자의 운동과 변형, 뉴턴 유체의 가정으로부터 유도할 수 있음

✓수직응력

$$\tau_{xx} = \lambda\Theta + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yy} = \lambda\Theta + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zz} = \lambda\Theta + 2\mu\frac{\partial w}{\partial z}$$

$$\left(\Theta = \nabla\bullet\mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$$

✓전단응력

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{du}{dz} + \frac{dw}{dx}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right)$$

Navier-Stokes Equation 유도

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3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

✓ 응력 텐서 성분의 분해 : 유체가 운동하는 경우 (P : 열역학적 압력)
(thermodynamic pressure)

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

➡ 응력 텐서의 대각 성분의 합을 구해보면,

$$\sigma_{xx} = -P + \tau_{xx} = -P + \lambda\Theta + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -P + \tau_{yy} = -P + \lambda\Theta + 2\mu\frac{\partial v}{\partial y}$$

$$+)\quad \sigma_{zz} = -P + \tau_{zz} = -P + \lambda\Theta + 2\mu\frac{\partial w}{\partial z}$$

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3P + 3\lambda\Theta + 2\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= -3P + (3\lambda + 2\mu)\Theta$$

✓수직응력

$$\tau_{xx} = \lambda\Theta + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yy} = \lambda\Theta + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zz} = \lambda\Theta + 2\mu\frac{\partial w}{\partial z}$$

$$\left(\Theta = \nabla \bullet \mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$$

Navier-Stokes Equation 유도

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3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264
4) 矢川元基, 이형직 역,유한요소법에 의한 유체역학,열전도 해석 입문, 피어슨 에듀케이션 코리아, pp8-10

✓ 응력 텐서 성분의 분해 : 유체가 운동하는 경우 (P : 열역학적 압력) (thermodynamic pressure)

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

✓ 수직응력

$$\tau_{xx} = \lambda\Theta + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yy} = \lambda\Theta + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zz} = \lambda\Theta + 2\mu\frac{\partial w}{\partial z}$$

$$\left(\Theta = \nabla\bullet\mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$$

➡ 응력 텐서의 대각 성분의 합을 구해보면,

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3P + (3\lambda + 2\mu)\Theta$$

➡ *평균압력 or *기계적 압력을 정의

$$-\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \overline{P}$$

$$\boxed{\overline{P} = P - \left(\lambda + \frac{2}{3}\mu\right)\Theta = P - \kappa\Theta}$$

➔ 평균압력과 열역학적 압력은 일반적으로 같지 않음

(참고) 비압축성 유동을 가정할 경우,
열역학적 압력을 결정할 수 있는 상태 방정식이 존재하지 않음[2)의 p102]

($\rho = const.$ 가 상태방정식을 대체함[1)의 p427])

(참고) 비압축성 유동(incompressible flow)일 경우, $\Theta = 0$이므로
비로소 평균압력과 열역학적 압력은 같아짐

*평균 압력(mean pressure) or *기계적 압력(mechanical pressure) : 유체요소의 안쪽으로 작용하는 평균 수직 응력

Navier-Stokes Equation 유도

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문", 문운당, 2002, pp258-264

✓ 응력 텐서 성분의 분해 : 유체가 운동하는 경우 (P : 열역학적 압력) (thermodynamic pressure)

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

➡ Stokes assumption : $\kappa = \lambda + \frac{2}{3}\mu = 0 \Rightarrow \lambda = -\frac{2}{3}\mu$

$$\left(\bar{P} = P - \left(\lambda + \frac{2}{3}\mu \right)\Theta = P - \kappa\Theta \right)$$

$$\sigma_{xx} = -P + \tau_{xx} = -P + \lambda\Theta + 2\mu\frac{\partial u}{\partial x} = -P - \frac{2}{3}\mu\Theta + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yx} = \tau_{yx} = \mu\left(\frac{du}{dy} + \frac{dv}{dx} \right)$$

$$\sigma_{zx} = \tau_{zx} = \mu\left(\frac{du}{dz} + \frac{dw}{dx} \right)$$

✓수직응력

$$\tau_{xx} = \lambda\Theta + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yy} = \lambda\Theta + 2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zz} = \lambda\Theta + 2\mu\frac{\partial w}{\partial z}$$

$$\left(\Theta = \nabla \bullet \mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right)$$

✓전단응력

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{du}{dy} + \frac{dv}{dx} \right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{du}{dz} + \frac{dw}{dx} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{dv}{dz} + \frac{dw}{dy} \right)$$

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

Navier-Stokes Equation 유도

$$\left(\Theta = \nabla \cdot \mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$$

$$\boxed{\begin{aligned} \sigma_{xx} &= -P - \frac{2}{3}\mu\Theta + 2\mu\frac{\partial u}{\partial x} \\ \sigma_{yx} &= \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) \\ \sigma_{zx} &= \mu\left(\frac{du}{dz} + \frac{dw}{dx}\right) \end{aligned}}$$

➡ Navier-Stokes Equation의 유도 ($\underline{x}$축 성분)

$$\rho\frac{du}{dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}$$

$$= \rho g_x + \frac{\partial}{\partial x}\left(-P - \frac{2}{3}\mu\Theta + 2\mu\frac{\partial u}{\partial x}\right) + \mu\frac{\partial}{\partial y}\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \mu\frac{\partial \sigma_{zx}}{\partial z}\left(\frac{du}{dz} + \frac{dw}{dx}\right)$$

$$= \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\mu\frac{\partial}{\partial x}(\Theta) + \mu\left(2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial z \partial x}\right)$$

$$= \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\mu\frac{\partial}{\partial x}(\Theta) + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$= \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\mu\frac{\partial}{\partial x}(\Theta) + \mu\left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right\}$$

$$= \rho g_x - \frac{\partial P}{\partial x} - \frac{2}{3}\mu\frac{\partial}{\partial x}(\Theta) + \mu\left\{\frac{\partial}{\partial x}\Theta + \nabla^2 u\right\} = \rho g_x - \frac{\partial P}{\partial x} + \frac{\mu}{3}\frac{\partial}{\partial x}(\Theta) + \mu\nabla^2 u$$

Navier-Stokes Equation 유도

(Continue)

$$\left(\Theta = \nabla \bullet \mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$$

- x축 성분 :

$$\rho \frac{du}{dt} = \rho g_x - \frac{\partial P}{\partial x} + \frac{\mu}{3}\frac{\partial}{\partial x}(\Theta) + \mu \nabla^2 u$$

- y축 성분

$$\rho \frac{dv}{dt} = \rho g_y - \frac{\partial P}{\partial y} + \frac{\mu}{3}\frac{\partial}{\partial y}(\Theta) + \mu \nabla^2 v$$

- z축 성분

$$\rho \frac{dw}{dt} = \rho g_z - \frac{\partial P}{\partial z} + \frac{\mu}{3}\frac{\partial}{\partial z}(\Theta) + \mu \nabla^2 w$$

벡터 형태로 정리하면 $\left(\mathbf{V} = [u, v, w]^T\right)$

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla P + \frac{\mu}{3}\nabla\Theta + \mu \nabla^2 \mathbf{V}$$

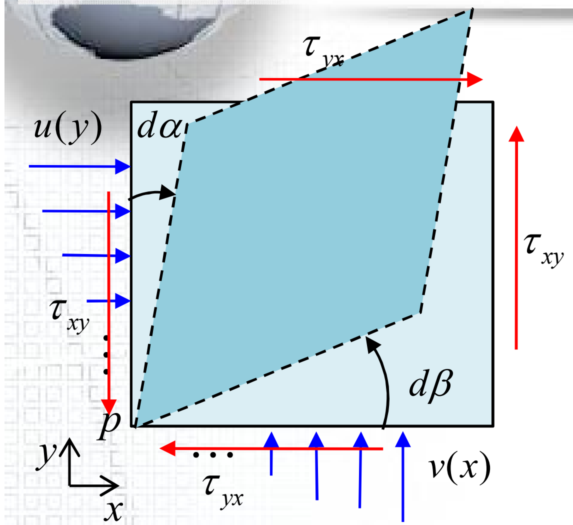
$$= \rho \mathbf{g} - \nabla P + \mu\left(\frac{1}{3}\nabla\Theta + \nabla^2 \mathbf{V}\right)$$

=> Navier-Stokes' Equation
(General form)

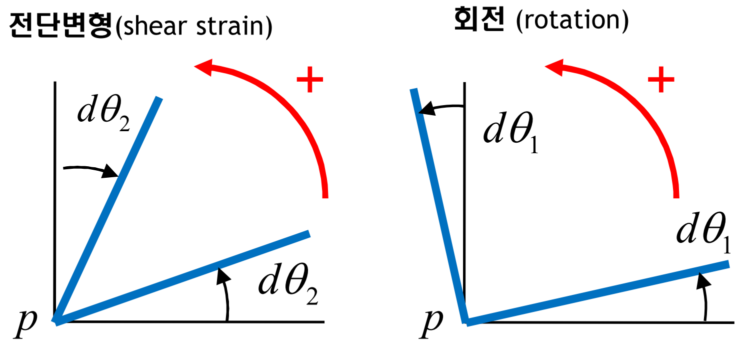
유체 입자의 회전

유체 입자의 회전과 Curl

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, “공학도를 위한 길잡이 유체공학입문” , 문운당, 2002, pp258-264



회전과 전단변형으로 나타낼 수 있음



$$-d\alpha = -d\theta_2 + d\theta_1 \quad \cdots ①$$

$$d\beta = d\theta_2 + d\theta_1 \quad \cdots ②$$

Given : $d\alpha, d\beta$

Find : 회전각속도 ω_z

(1) ②와 ①을 더하면, $d\beta - d\alpha = 2d\theta_1$

$$d\theta_1 = \frac{1}{2}(d\beta - d\alpha)$$

➡ z축에 대한 회전 각도

(2) dt로 나누면, $$\frac{d\theta_1}{dt} = \frac{1}{2}\left(\frac{d\beta}{dt} - \frac{d\alpha}{dt}\right)$$

➡ z축에 대한 회전 각속도

(3) 전단변형율의 시간 변화율은 속도 구배와 같음

$$\omega_z = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$$

(5) 동일한 방법에 의해 다음도 성립함

➡ y축에 대한 회전각속도 : $\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

➡ x축에 대한 회전각속도 : $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$

유체 입자의 회전과 Curl

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

✓ curl의 정의

$$\mathrm{curl}\,\mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

$$= 2\omega_x \mathbf{i} + 2\omega_y \mathbf{j} + 2\omega_z \mathbf{k}$$

=> curl V 는 유체 입자의 (회전 각속도/2)에 해당하는 값

✓ x축에 대한 각속도

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

✓ y축에 대한 각속도

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

✓ z축에 대한 각속도

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

✓ Irrotational Flow : $\nabla \times \mathbf{v} = 0 \left(\dfrac{\partial w}{\partial y} = \dfrac{\partial v}{\partial z},\ \dfrac{\partial u}{\partial z} = \dfrac{\partial w}{\partial x},\ \dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial y}\right)$

✓ $\nabla \times \mathbf{v} = 0$ 일 때, $\mathbf{v} = \nabla \Phi$ 인 Scalar Function Φ가 존재함

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, p179~182
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.118~122

Euler Equation & Bernoulli Equation[1,2] 유도

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, p179~182
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.118~122

Euler Equation[1],[2] 유도

$$\mathbf{V} = [u, v, w]^T$$

$$\Theta = \nabla \bullet \mathbf{V} = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$$

✓ Navier-Stokes Equation (general form)

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla P + \mu \left(\frac{1}{3} \nabla \Theta + \nabla^2 \mathbf{V} \right)$$

비점성 유동(invicid flow)**라 가정하면**

$$(\mu = 0)$$

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla P$$

양변을 ρ로 나눔

$$\boxed{\frac{d\mathbf{V}}{dt} = \mathbf{g} - \frac{\nabla P}{\rho}}$$

=> Euler Equation

x**축 성분 :** $$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

y**축 성분 :** $$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = g_y - \frac{1}{\rho}\frac{\partial P}{\partial y}$$

z**축 성분 :** $$\frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = g_z - \frac{1}{\rho}\frac{\partial P}{\partial z}$$

Bernoulli Equation[1),2)]의 유도 – Euler equation의 변경

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, p179~182
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.118~122

$$\frac{d\mathbf{V}}{dt}=\mathbf{g}-\frac{\nabla P}{\rho} \xrightarrow{x\text{축 성분}} \frac{\partial u}{\partial t}+\boxed{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}}=g_x-\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\boxed{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}}=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\left(v\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial x}\right)+\left(w\frac{\partial w}{\partial x}-w\frac{\partial w}{\partial x}\right)$$

$$=\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}\right)+\left(v\frac{\partial u}{\partial y}-v\frac{\partial v}{\partial x}\right)+\left(w\frac{\partial u}{\partial z}-w\frac{\partial w}{\partial x}\right)$$

$$=\left[\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}w^2\right)\right]+v\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{1}{2}\underbrace{\left(u^2+v^2+w^2\right)}_{=q^2}\right)+v\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{1}{2}q^2\right)-v\omega_z+w\omega_y$$

$$\boxed{\operatorname{curl}\mathbf{v}=\nabla\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\mathbf{i}+\left(\frac{\partial u}{\partial z}-\frac{\partial v}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\mathbf{k}=\omega_x\mathbf{i}+\omega_y\mathbf{j}+\omega_z\mathbf{k}}$$

Bernoulli Equation[1),2)]의 유도 – Euler equation의 변경

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, p179~182
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, p17 ,pp.118~122

*등엔트로피 : 유체의 엔트로피가 변하지 않음, 열의 공급이 일어나지 않고(adiabaitc), 마찰이 없음(frictionless)

$$\frac{d\mathbf{V}}{dt} = \mathbf{g} - \frac{\nabla P}{\rho}$$

x축 성분 →

$$\frac{\partial u}{\partial t} + \underline{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}} = g_x - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\frac{\partial u}{\partial t} + \overline{\frac{\partial}{\partial x}\left(\frac{1}{2}q^2\right) - v\omega_z + w\omega_y} = g_x - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

Body force를 포텐셜 에너지의 gradient로 표현 $(\mathbf{g} = -\nabla(gy))$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}q^2\right) + \frac{\partial}{\partial x}(gy) + \frac{1}{\rho}\frac{\partial P}{\partial x} = v\omega_z - w\omega_y$$

① barotropic flow

(유체의 밀도가 압력만의 함수로 표현)

$$\rho = \rho(P)$$

① 에 의해 다음이 성립함! ➡ $\frac{1}{\rho(P)}\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\int \frac{dP}{\rho(P)}$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}q^2\right) + \frac{\partial}{\partial x}(gy) + \frac{\partial}{\partial x}\int \frac{dP}{\rho} = v\omega_z - w\omega_y$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}q^2 + gy + \int \frac{dP}{\rho}\right) = v\omega_z - w\omega_y$$

Bernoulli Equation[1],[2]의 유도 – Euler equation의 변경

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, p179~182
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.118~122

✓ Euler Equation

$$\frac{d\mathbf{V}}{dt} = \mathbf{g} - \frac{\nabla P}{\rho}$$

x축 성분

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}q^2 + gy + \int \frac{dP}{\rho}\right) = v\omega_z - w\omega_y$$

y축 성분

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial y}\left(\frac{1}{2}q^2 + gy + \int \frac{dP}{\rho}\right) = w\omega_x - u\omega_z$$

z축 성분

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial z}\left(\frac{1}{2}q^2 + gy + \int \frac{dP}{\rho}\right) = u\omega_y - v\omega_x$$

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla B = \mathbf{V} \times \boldsymbol{\omega}$$: Another form of Euler equation

$$\mathbf{V} = [u, v, w]^T$$

$$\boldsymbol{\omega} = \left[\omega_x, \omega_y, \omega_z\right]^T$$

$$\mathbf{V} \times \boldsymbol{\omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u & v & w \\ \omega_x & \omega_y & \omega_z \end{vmatrix} = \mathbf{i}\left(v\omega_z - w\omega_y\right) + \mathbf{j}\left(w\omega_x - u\omega_z\right) + \mathbf{k}\left(u\omega_y - v\omega_x\right)$$

$$B = \frac{1}{2}q^2 + gy + \int \frac{dP}{\rho}$$: Bernoulli function

(단위 질량당 운동에너지,포텐셜에너지,유동에너지의 합)

Bernoulli Equation[1],[2]의 유도 – 현재까지의 가정 정리

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, p179~182
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.118~122

Cauchy Equation : $$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} + \nabla \bullet \sigma$$

① **뉴턴 유체** (Newtonian fluid)

② Stokes assumption

Navier-Stokes Equation : (in general form) $$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla P + \mu \left(\frac{1}{3} \nabla \Theta + \nabla^2 \mathbf{V} \right)$$

③ **비점성 유동** (Invicid flow)

Euler Equation : $$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla P$$

③ barotropic flow

Euler Equation : (Another form) $$\frac{\partial \mathbf{V}}{\partial t} + \nabla B = \mathbf{V} \times \boldsymbol{\omega}$$

※ Euler equation**을 유도하기 까지**
비압축성 유동(Incompressible flow)**은 가정하지 않았음**

다음에 대해 Bernoulli equation**을 유도**

1) Steady flow / rotational or irrotational flow
2) Unsteady flow / irrotational flow

Bernoulli Equation[1),2)]의 유도

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, p179~182
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.118~122

1) Steady flow에서의 Bernoulli equation

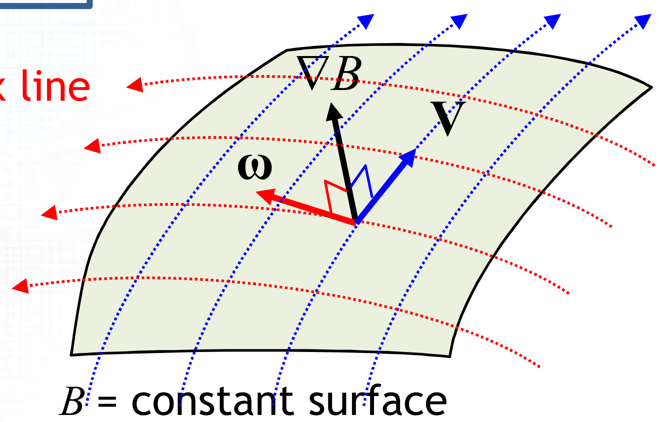
Euler Equation : $$\frac{\partial \mathbf{V}}{\partial t} + \nabla B = \mathbf{V} \times \boldsymbol{\omega}$$

$$B = \frac{1}{2}q^2 + gy + \int \frac{dP}{\rho}$$

steady flow
(유체 입자 운동이 시간과 무관) $\left(\frac{\partial \mathbf{V}}{\partial t} = 0 \right)$

$$\nabla B = \mathbf{V} \times \boldsymbol{\omega}$$

식의 의미



$B = \text{constant}$: **곡면**(surface)

∇B : **곡면**(surface)**에 수직인 벡터**

$\mathbf{V}$: **유체입자의 속력**

$\boldsymbol{\omega}$: **유체입자의 회전 (회전축 벡터)**

∇B **는** $\mathbf{V}$ **와** $\boldsymbol{\omega}$ **에 수직**

➡ B = constant **인 곡면 위에** stream line**과** vortex line**을 포함하고 있음**

➡ **즉,** stream line**과** vortex line **이 이루는 곡면에서** B=constant**를 만족함**

$$\therefore B = \frac{1}{2}q^2 + gy + \int \frac{dP}{\rho} = \text{ constant along streamlines and vortex lines}$$

➡ Invicid,steady,barotropic flow

Bernoulli Equation[1],[2]의 유도

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, p179~182
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.118~122

2) Unsteady irrotational flow 에서의 Bernoulli equation

✓ irrotational flow의 경우 다음을 만족함

➡ $\mathbf{V} = \nabla\phi$

➡ $\text{curl}\,\mathbf{V} = \boldsymbol{\omega} = 0$

✓ Euler equation에 대입함

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla B = \mathbf{V} \times \boldsymbol{\omega} \Rightarrow \frac{\partial}{\partial t}(\nabla\phi) + \nabla B = 0 \Rightarrow \nabla\frac{\partial \phi}{\partial t} + \nabla B = 0$$

$$\Rightarrow \nabla\left(\frac{\partial \phi}{\partial t} + B\right) = \nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + gy + \int \frac{dP}{\rho}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + gy + \int \frac{dP}{\rho} = F(t)}$$

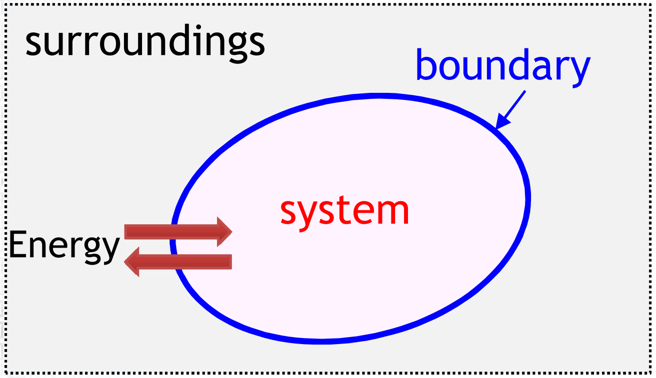
➡ Invicid,unsteady,barotropic,irrotational flow

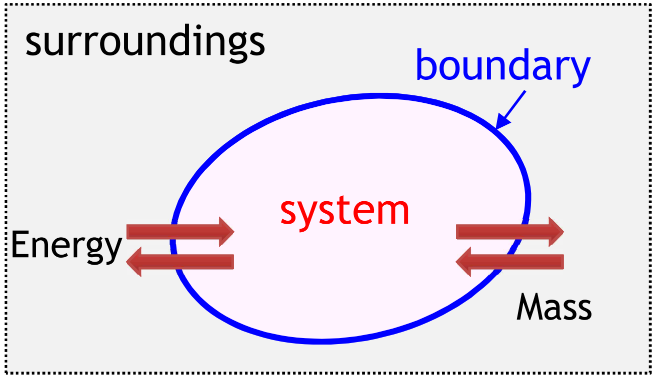
1) Young,D.F., Munson,B.R., Okiishi,T.H., 고형종 역,A brief introduction to Fluid Mechanics, 홍릉과학출판사,2004,pp.219-233

에너지 방정식과 Bernoulli equation의 비교[1)]

에너지 방정식

✓ 시스템(system)의 정의 : 연구의 대상이 되는 공간 내의 영역이나 영역 내에 포함된 물질의 양





➡ **밀폐 시스템**(closed system)
: **에너지(열,일)의 출입**은 가능하나
질량의 통과가 없음

➡ **개방 시스템**(open system)
: **에너지(열,일)의 출입** 및
질량의 유출입도 가능

➡ **시스템의 단위 질량당 에너지**

$$e = u + \frac{\mathbf{V}^2}{2} + gz$$

내부E 운동E 포텐셜E

(E:Energy)

➡ **시스템의 단위 질량당 에너지**

$$e = u + \boxed{\frac{P}{\rho}} + \frac{\mathbf{V}^2}{2} + gz$$

유동E
:질량의 유출입이 가능한 경우,
유동을 유지시키기 위한 에너지

에너지 방정식

✓ 에너지 방정식 :

검사체적을 통과해 빠져나가는 에너지

검사체적을 통과해 들어오는 에너지

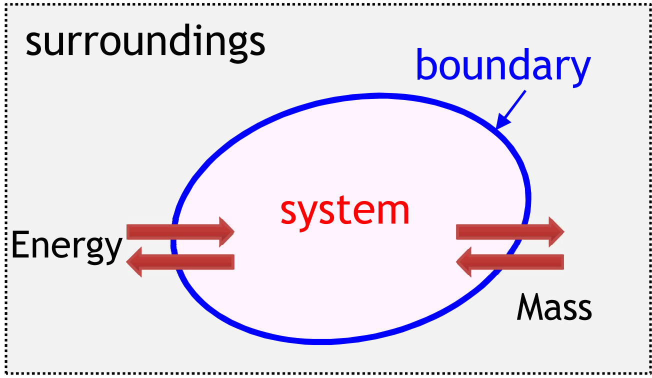
$$\frac{d}{dt}\int_{CV} e\rho dV + \dot{m}_{out}e_{out} - \dot{m}_{in}e_{in} - \dot{Q} - \dot{W} = 0$$

control volume(CV) 내의 전체 에너지 변화

외부에서 가해준 열

외부에서 공급한 일

$$\frac{d}{dt}\int_{CV} e\rho dV + \dot{m}_{out}\left(u_{out} + \left(\frac{P}{\rho}\right)_{out} + \frac{\mathbf{V}_{out}^{\ 2}}{2} + gz_{out}\right) - \dot{m}_{in}\left(u_{in} + \left(\frac{P}{\rho}\right)_{in} + \frac{\mathbf{V}_{in}^{\ 2}}{2} + gz_{in}\right) = \dot{Q} + \dot{W}$$



➡ 시스템의 단위 질량당 에너지

$$e = u + \frac{P}{\rho} + \frac{\mathbf{V}^2}{2} + gz$$

에너지 방정식

✓ 정상 상태 (steady state)라고 가정하면, CV 내부의 에너지 변화 없음

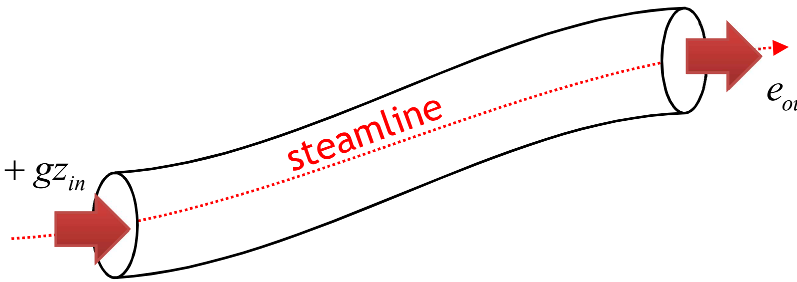
$$\cancel{\frac{d}{dt}\int_{CV} e\rho dV} + \dot{m}_{out}\left(u_{out} + \left(\frac{P}{\rho}\right)_{out} + \frac{\mathbf{V}_{out}^{\ 2}}{2} + gz_{out}\right) - \dot{m}_{in}\left(u_{in} + \left(\frac{P}{\rho}\right)_{in} + \frac{\mathbf{V}_{in}^{\ 2}}{2} + gz_{in}\right) = \dot{Q} + \dot{W}$$

✓ 유선을 따라 이동하는 경우, 유선을 포함하는 관을 CV으로 선택하면, <u>질량 유량이 동일함</u> $\left(\dot{m}_{out} = \dot{m}_{in} = \dot{m}\right)$

$$\dot{m}\left(u_{out} + \left(\frac{P}{\rho}\right)_{out} + \frac{\mathbf{V}_{out}^{\ 2}}{2} + gz_{out}\right) - \dot{m}\left(u_{in} + \left(\frac{P}{\rho}\right)_{in} + \frac{\mathbf{V}_{in}^{\ 2}}{2} + gz_{in}\right) = \dot{Q} + \dot{W}$$

$$\dot{m}\left(u_{out} - u_{in} + \frac{P_{out}}{\rho} - \frac{P_{in}}{\rho} + \frac{\mathbf{V}_{out}^{\ 2} - \mathbf{V}_{in}^{\ 2}}{2} + g\left(z_{out} - z_{in}\right)\right) = \dot{Q} + \dot{W}$$

$$e_{in} = u_{in} + \frac{P_{in}}{\rho} + \frac{\mathbf{V}_{in}^{\ 2}}{2} + gz_{in}$$



$$e_{out} = u_{out} + \frac{P_{out}}{\rho} + \frac{\mathbf{V}_{out}^{\ 2}}{2} + gz_{out}$$

에너지 방정식과 Bernoulli equation 비교

✓ 에너지 방정식에서 외부로부터의 열의 공급이나 일이 없다고 하면,

$$\dot{m}\left(u_{out}-u_{in}+\frac{P_{out}}{\rho}-\frac{P_{in}}{\rho}+\frac{\mathbf{V}_{out}{}^2-\mathbf{V}_{in}{}^2}{2}+g(z_{out}-z_{in})\right)=\cancel{\dot{Q}+\dot{W}}=0$$

$$u_{out}-u_{in}+\frac{P_{out}}{\rho}-\frac{P_{in}}{\rho}+\frac{\mathbf{V}_{out}{}^2-\mathbf{V}_{in}{}^2}{2}+g(z_{out}-z_{in})=0$$

$$\underline{\frac{P_{out}}{\rho}+\frac{\mathbf{V}_{out}{}^2}{2}+gz_{out}=\frac{P_{in}}{\rho}+\frac{\mathbf{V}_{in}{}^2}{2}+gz_{in}}-(u_{out}-u_{in})$$

➡ 내부에너지 변화(마찰에 의한 에너지 손실)

➡ steady,incompressible Bernoulli equation

마찰에 의한 에너지 손실이 없다고 가정하면,

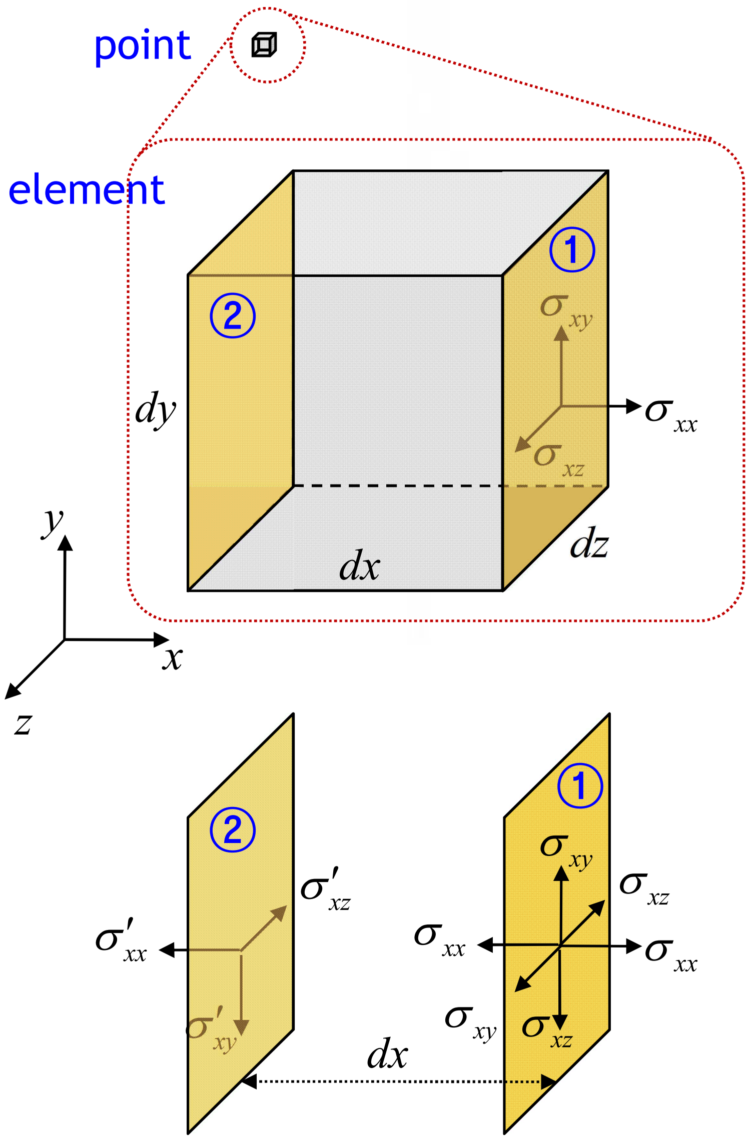
$$\frac{P_{out}}{\rho}+\frac{\mathbf{V}_{out}{}^2}{2}+gz_{out}=\frac{P_{in}}{\rho}+\frac{\mathbf{V}_{in}{}^2}{2}+gz_{in}$$

보충 슬라이드

Advanced
Ship
Design
Automation
Laboratory

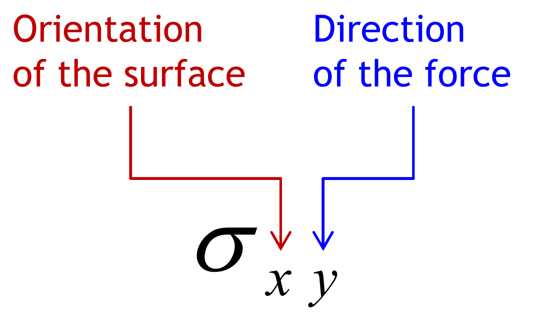
Cauchy Equation[1],[2] 유도
– Stress at a point

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005,pp.396~401
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.30-32,88-93



✓ 한 점에서의 응력(Stress at a point)

➡ 어느 면에 어느 방향으로 힘이 작용하는지 정의해야 함



➔ 2개의 방향 필요

➡ 한 면에 수직인 성분과 2개의 접선 방향 성분으로 나타남 (3개의 응력 성분이 존재함)

➡ 면 ②에 작용하는 응력 성분은 ?

면 ②에서는 Normal vector의 방향이 면 ①과 반대이므로, 응력의 방향을 반대로 정의함 (sign convention)

면 ①과 ②는 dx만큼 떨어져 있기 때문에 일반적으로 응력이 다름

point이므로 $dx \rightarrow 0$ ➔ 즉, 두 면은 서로 일치함.

따라서, Point에서 두 면의 응력의 크기는 같고 방향은 반대

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, p179~182

Bernoulli Equation 유도

$$\mathrm{curl}\,\mathbf{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k} = 0$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z},\ \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x},\ \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$(\mathbf{V} = \nabla \Phi)$$

✓ Euler Equation

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla P \xrightarrow{\text{x축 성분}} \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial P}{\partial x}$$

$$\downarrow\ u = \frac{\partial \Phi}{\partial x},\ \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x},\ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}\ \text{대입}$$

$$\rho\left(\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial x}\right) + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right) = \rho g_x - \frac{\partial P}{\partial x}$$

$$\downarrow$$

$$\rho\left(\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(v^2\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(w^2\right)\right) = \rho g_x - \frac{\partial P}{\partial x}$$

$$\downarrow$$

$$\rho\left(\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2 + v^2 + w^2\right)\right) = \rho g_x - \frac{\partial P}{\partial x}$$

Bernoulli Equation[1] 유도

(continue) $$\rho\left(\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)\right)=\rho g_x-\frac{\partial P}{\partial x}$$

양변을 x로 적분해 주면

$$\int \rho\left[\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)\right]dx=\int\left[\rho g_x-\frac{\partial P}{\partial x}\right]dx$$

$$\rho\frac{\partial \Phi}{\partial t}+\frac{1}{2}\rho\left(u^2+v^2+w^2\right)=\rho g_x x-P+f_1(y,z,t)$$

$$\rho\frac{\partial \Phi}{\partial t}+\frac{1}{2}\rho\left(u^2+v^2+w^2\right)+P=\rho g_x x+f_1(y,z,t)$$

Bernoulli Equation[1] 유도

x축 성분 : $\boxed{\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2}\rho\left(u^2 + v^2 + w^2\right) + P} = \rho g_x x + f_1(y, z, t)$

y축 성분 : $\boxed{\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2}\rho\left(u^2 + v^2 + w^2\right) + P} = \rho\, g_y y + f_2(x, z, t)$

z축 성분 : $\boxed{\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2}\rho\left(u^2 + v^2 + w^2\right) + P} = \rho\, g_z z + f_3(x, y, t)$

① **좌변이 모두 같으므로, 우변도 같다.**

$$\rho g_x x + f_1(y, z, t) = \rho g_y y + f_2(x, z, t) = \rho g_z z + f_3(x, y, t)$$

② **중력장만을 고려한다면,** $g_x = g_y = 0, g_z = -g$ **라 할 수 있다.**

$$f_1(y, z, t) = f_2(x, z, t) = -\rho g z + f_3(x, y, t)$$

③ f_1**과** f_2**를 비교하면,** x,y**의 함수가 아님을 알 수 있다.** f_3**은** t**의 함수가 된다.**

$$f_1(z, t) = f_2(z, t) = -\rho g z + f(t)$$

④ **원래 식에 대입하면,**

$$\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2}\rho\left(u^2 + v^2 + w^2\right) + P + \rho\, g z = f(t) \longrightarrow$$

⑤ **속도항을 속도 포텐셜로 표현하면,**

$$\boxed{\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2}\rho\left|\nabla \Phi\right|^2 + P + \rho\, g z = f(t)}$$

=> Bernoulli Equation

Bernoulli Equation[1)]

$$\rho \frac{\partial \Phi}{\partial t} + \frac{1}{2}\rho \left|\nabla \Phi\right|^2 + P + \rho g z = f(t)$$

=> Bernoulli Equation

정상 유동 (Steady flow)**라 하면,**

$$\frac{\partial \Phi}{\partial t} = 0 \ , \quad f(t) = C \quad (C : const)$$

$$\frac{1}{2}\rho \left|\nabla \Phi\right|^2 + P + \rho g z = C$$

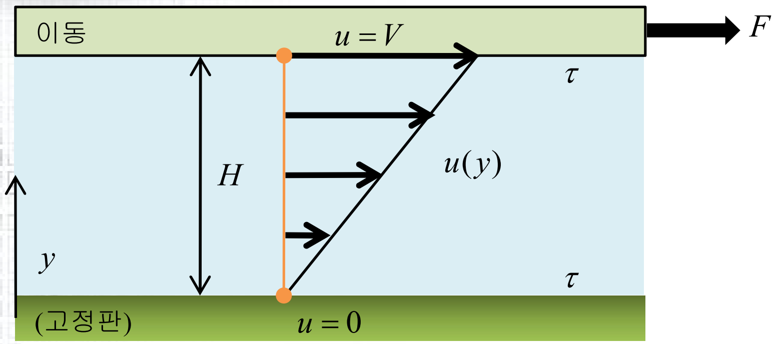
Kinetic Energy + Pressure Energy + Potential Energy = Constant

=> Mechanical Energy conservation

뉴턴 유체[1] (Newtonian Fluid)

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

✓ 뉴턴 유체[1] (Newtonian Fluid)



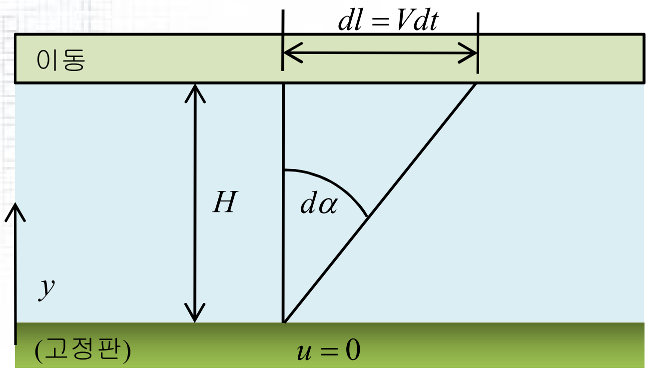
No-Slip condition
정상상태

전단응력 : $\tau = \dfrac{F}{A}$

유체의 속도분포 : $u(y) = \dfrac{V}{H} y$

시간이 무한이 지나면, 높이에 따라 속도 분포가 선형적으로 변함

유체의 속도변화 (속도구배) : $\dfrac{du}{dy} = \dfrac{V}{H}$



(assump. $d\alpha \langle\langle 1$)

전단변형율(shear strain rate) : $d\alpha \approx \tan d\alpha = \dfrac{dl}{H} = \dfrac{V}{H} dt = \dfrac{du}{dy} dt$

전단변형율(의 시간변화율)* : $\dfrac{d\alpha}{dt} = \dfrac{du}{dy}$

전단응력은 전단변형율에 비례함 : $\tau \propto \dfrac{d\alpha}{dt}$
(실험적으로 증명 가능함 - 구체적으로)

즉, 전단응력은 속도 구배에 비례함 : $\tau \propto \dfrac{du}{dy}$ ➡ $\tau = \mu \dfrac{du}{dy}$
(Newtonian Fluid)

*[김찬중, 유체공학입문,문운당,2004]에서는 이것을 전단변형율의 시간변화율이라고 명기
[Cengel, Fluid Mechanics, Mc Graw Hill,2006],[Whilt,Fluid Mechanics 6th edition]은 전단변형율이라는 용어사용

Bernoulli equation의 상수 'C' 의 의미와 계기 압력[1)]

Bernoulli Equation

$$\rho \cancel{\frac{\partial \Phi}{\partial t}} + P + \cancel{\frac{1}{2}\rho\left|\nabla\Phi\right|^2} + \rho g\, z = \mathrm{C}$$

☞ **항등식**

✓ **정적인 상태 :** $\Phi = 0 \quad \left(\frac{\partial \Phi}{\partial t} = 0, \nabla\Phi = 0\right)$

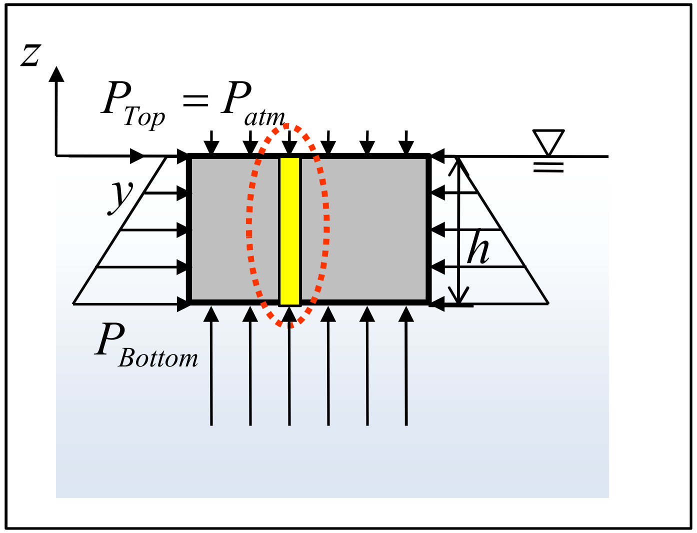
$$P + \rho g\, z = \mathrm{C}$$

항등식 이므로, $z = 0$ **일 때도 성립**

$$P_{atm} = \mathrm{C}$$

$z=0$일 때 압력 = 대기압(P_{atm})

$$\therefore \rho \frac{\partial \Phi}{\partial t} + P + \frac{1}{2}\rho\left|\nabla\Phi\right|^2 + \rho g\, z = \mathrm{P}_{atm}$$



✓ **물체 바닥에서의 압력은?**

$$\rho \frac{\partial \Phi}{\partial t} + P_{Bottom} + \frac{1}{2}\rho\left|\nabla\Phi\right|^2 + \rho g z = P_{atm}$$

$$\rho \frac{\partial \Phi}{\partial t} + \left(\cancel{P_{atm}} + P_{Fluid}\right) + \frac{1}{2}\rho\left|\nabla\Phi\right|^2 + \rho g z = \cancel{P_{atm}}$$

$$\therefore \rho \frac{\partial \Phi}{\partial t} + \boxed{P_{Fluid}} + \frac{1}{2}\rho\left|\nabla\Phi\right|^2 + \rho g z = 0$$

'계기 압력'

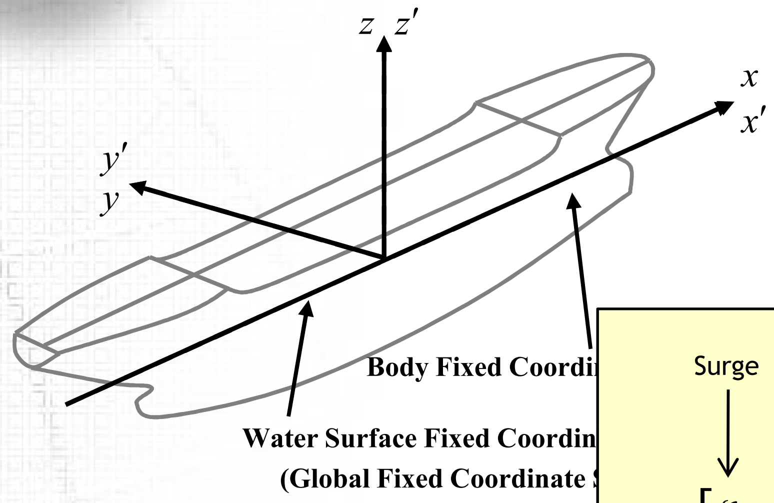
※ Bernoulli equation에서 우변의 상수 C=0으로 표기한 경우, 압력 P는 대기압을 뺀 유체만의 압력임을 의미함

선박의 6자유도 운동 방정식
(6DOF Equations of Ship Motion)

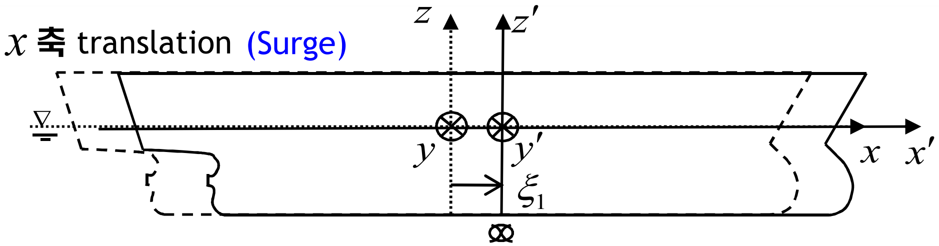
Coordinate System

x' 축 - 원점: Midship, (+): 선수
y' 축 - 원점: Centerline, (+): 좌현
z' 축 - 원점: 수선면, (+): 선박의 위

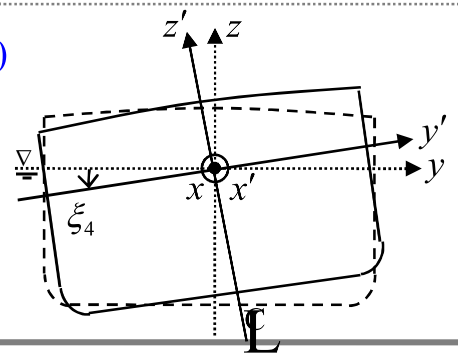
x축 - 원점: Midship, (+): x'축을 포함하고 수선면과 직교인 평면과 수선면 사이의 교선
y축 - 원점: Centerline, (+): z축과 x축의 외적 방향
z축 - 원점: 수면, (+): 수선면에 수직한 위 방향



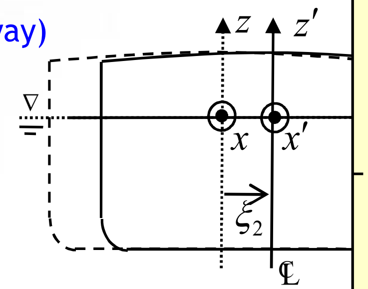
x 축 translation (Surge)



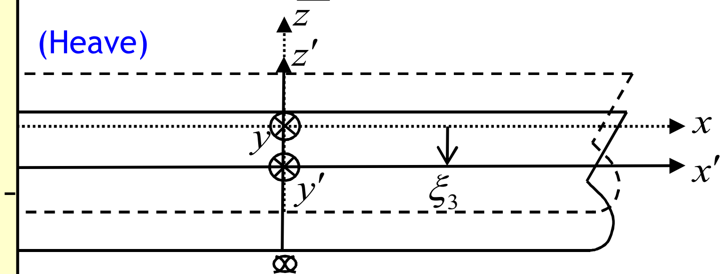
x 축 rotation (Roll)



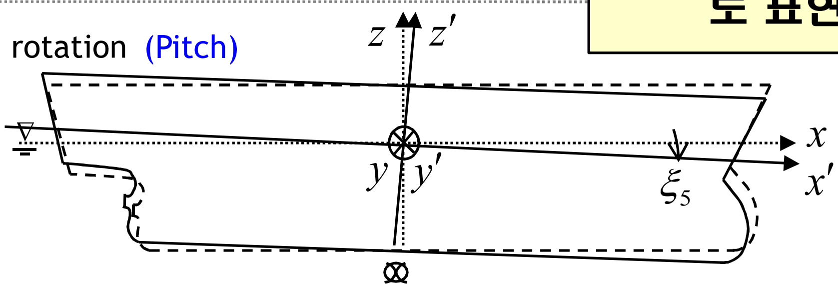
y 축 translation (Sway)

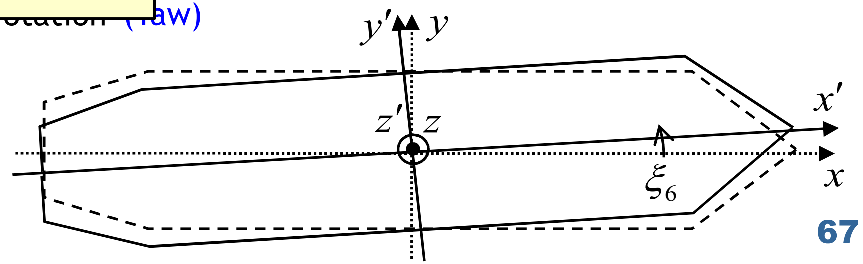


(Heave)



y 축 rotation (Pitch)





$$\mathbf{x} = \left[\overset{\text{Surge}}{\xi_1}, \underset{\text{Sway}}{\xi_2}, \overset{\text{Heave}}{\xi_3}, \underset{\text{Roll}}{\xi_4}, \overset{\text{Pitch}}{\xi_5}, \underset{\text{Yaw}}{\xi_6}\right]^T$$

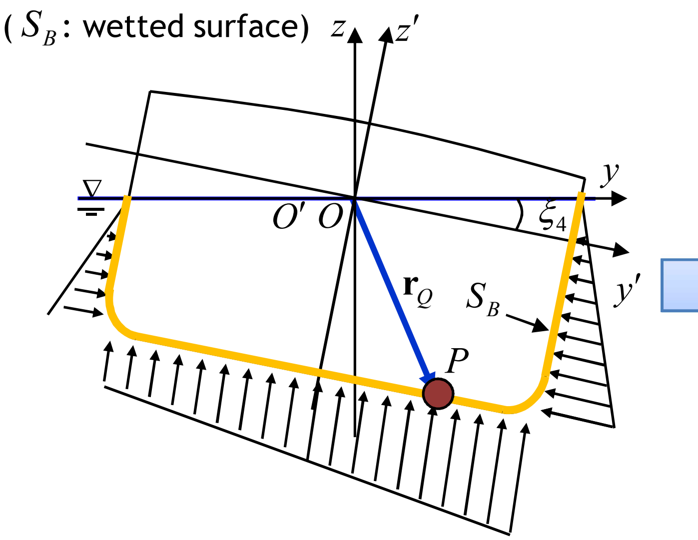
$$\mathbf{x} = \left[x, y, z, \phi, \theta, \psi\right]^T$$

로 표현하기도 함

선박의 6자유도 운동 방정식
: Force & moment acting on the surface

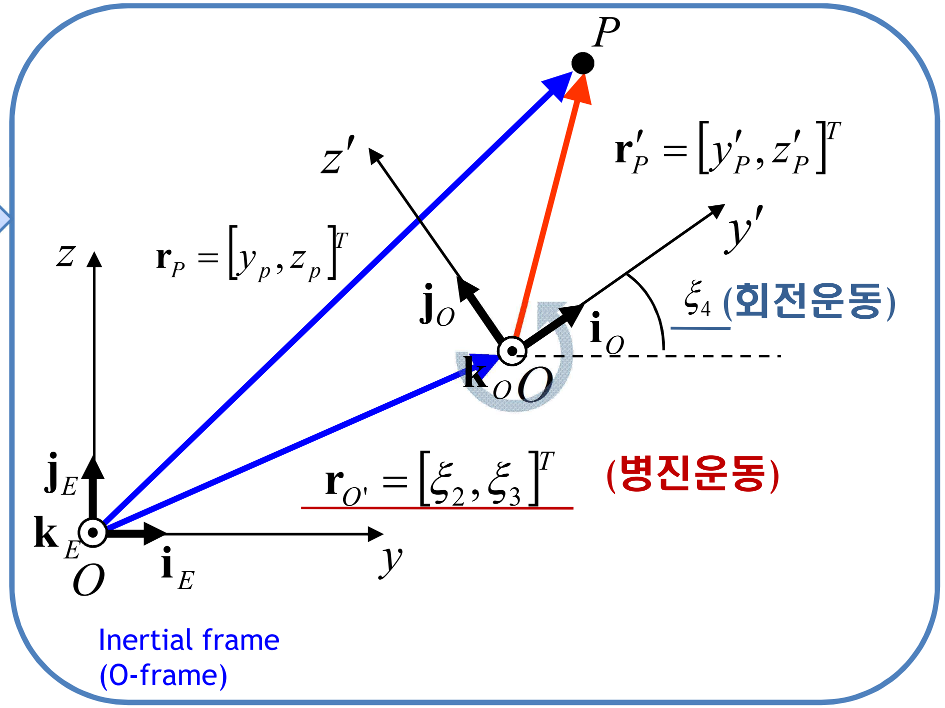
(S_B : wetted surface)

좌현으로 기울어진 상태
(선박을 정면에서 바라봄)



$\left(\mathbf{r} = \left[x_p, y_p, z_p\right]^T\right)$

✓ **지구 고정 좌표계** $O\text{-}xyz$ **에서 표현된 선박 표면의 점 P와 선박 운동 변위** $\mathbf{x} = \left[\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\right]^T$ **의 관계**



전후동요 변위　　상하동요 변위　　종동요 변위

$$\mathbf{x} = \left[\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6\right]^T$$

좌우동요 변위　　횡동요 변위　　선수동요 변위

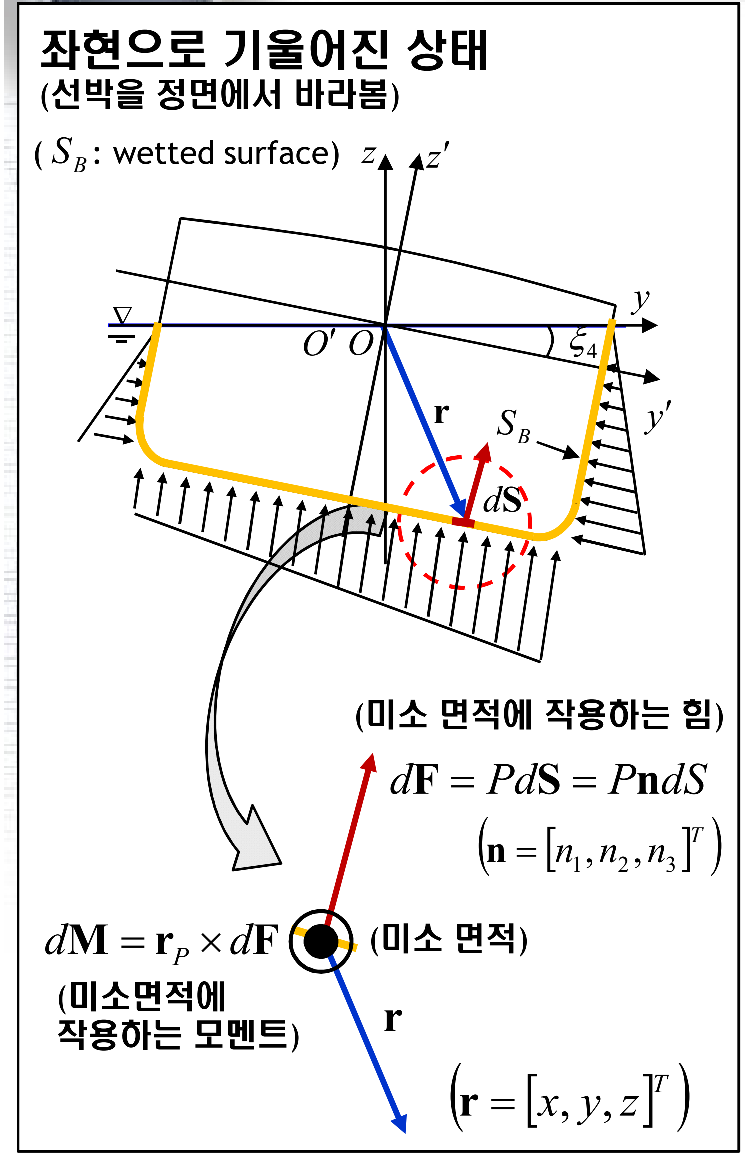
$$\mathbf{r}_P = \mathbf{r}_{O'} + Rot(\xi_4)\mathbf{r}'_P$$

$$\begin{bmatrix} y_P \\ z_P \end{bmatrix} = \underbrace{\begin{bmatrix} \xi_2 \\ \xi_3 \end{bmatrix}}_{\text{(병진운동)}} + \underbrace{\begin{bmatrix} \cos\xi_4 & -\sin\xi_4 \\ \sin\xi_4 & \cos\xi_4 \end{bmatrix}}_{\text{(회전운동)}} \begin{bmatrix} y'_P \\ z'_P \end{bmatrix}$$

지구고정 좌표계 $O\text{-}xyz$ 에서 선박 표면의 점 P는 운동 변위로 표현됨

선박의 6자유도 운동 방정식
: Force & moment acting on the surface

(S_B : wetted surface)



- Force : **표면에 작용하는 모든 힘을 적분하여 구함**
  - ✓ **미소 면적에 작용하는 단위 길이당 힘 :**

$$d\mathbf{F} = P \cdot d\mathbf{S} = P \cdot \mathbf{n}dS = -\rho g z \cdot \mathbf{n}dS$$

  - ✓ Total force

$$\boxed{\mathbf{F} = \iint_{S_B} P\mathbf{n}dS}$$

- Moment : **(모멘트)=(거리) X (힘)**
  - ✓ **미소 면적에 작용하는 단위 길이당 모멘트 :**

$$d\mathbf{M} = \mathbf{r} \times d\mathbf{F} = \mathbf{r} \times P\mathbf{n}dS = (\mathbf{r} \times \mathbf{n})PdS$$

  - ✓ Total moment

$$\boxed{\mathbf{M} = \iint_{S_B} P(\mathbf{r} \times \mathbf{n})dS}$$

왜 r이 먼저 오는가? (좌표축에서 양의 방향을 고려함)

선박의 6자유도 운동 방정식
:Notation

$$\left(\mathbf{r}\times\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ n_1 & n_2 & n_3\end{vmatrix}=\mathbf{i}(yn_3-zn_2)+\mathbf{j}(zn_1-xn_3)+\mathbf{k}(xn_2-yn_1)\right)$$

✓ Fluid force acting on the surface

$$\mathbf{F}=\iint_{S_B}P\mathbf{n}dS$$

$$\left(\mathbf{n}=[n_1,n_2,n_3]^T\right)$$

성분별로 나눠쓰면,

$$\left(\begin{aligned}F_1&=\iint_{S_B}Pn_1dS\\ F_2&=\iint_{S_B}Pn_2dS\\ F_3&=\iint_{S_B}Pn_3dS\end{aligned}\right)$$

$$(j=1,\cdots,6)$$

$$\boxed{F_j=\iint_{S_B}Pn_jdS}$$

✓ Fluid moment acting on the surface

$$\mathbf{M}=\iint_{S_B}P(\mathbf{r}\times\mathbf{n})dS$$

$$\left(\begin{aligned}\mathbf{n}&=[n_1,n_2,n_3]^T\\ \mathbf{r}&=[x,y,z]^T\end{aligned}\right)$$

성분별로 나눠쓰면,

$$M_1=\iint_{S_B}P(yn_3-zn_2)dS\ ,\quad \left(F_4=\iint_{S_B}Pn_4dS\right)$$

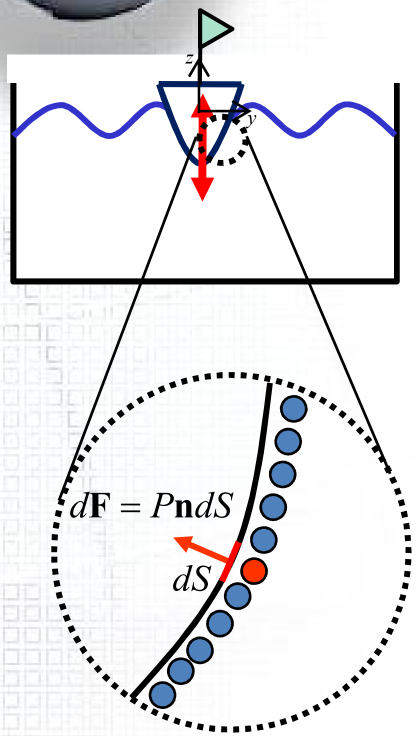
$$M_2=\iint_{S_B}P(zn_1-xn_3)dS\ ,\quad \left(F_5=\iint_{S_B}Pn_5dS\right)$$

$$M_3=\iint_{S_B}P(xn_2-yn_1)dS\ ,\quad \left(F_6=\iint_{S_B}Pn_6dS\right)$$

선박의 6자유도 운동 방정식 유도

$\mathbf{F}_{F.K}$: Froude- krylov force
$\mathbf{F}_D$: Diffraction force
$\mathbf{F}_R$: Radiation force

Φ_I : Incident wave velocity potential
Φ_D : Diffraction velocity potential
Φ_R : Radiation velocity potential



$d\mathbf{F}$: 하나의 유체 입자가 선박 표면에 가하는 힘

dS : 미소 면적

$\mathbf{n}$: 미소 면적의 Normal 벡터

✓ Bernoulli Equation

$$\rho\frac{\partial\Phi}{\partial t}+P+\cancel{\frac{1}{2}\rho|\nabla\Phi|^2}+\rho gz=0$$

Linearization

✓ Laplace Equation

$$\nabla^2\Phi=0$$

$$\underbrace{\Phi}_{\text{Linear combination of basic solutions}}=\underbrace{\Phi_I+\Phi_D+\Phi_R}_{\text{Basic solutions}}$$

$$P_{Fluid}=-\rho gz-\rho\frac{\partial\Phi}{\partial t}=\boxed{-\rho gz}-\rho\left(\frac{\partial\Phi_I}{\partial t}+\frac{\partial\Phi_D}{\partial t}+\frac{\partial\Phi_R}{\partial t}\right)$$

유체입자가 표면에 작용하는 압력

$$=\boxed{P_{static}}+\underbrace{P_{F.K}+P_D+P_R}_{P_{dynamic}}$$

$$\mathbf{F}_{Fluid}=\iint_{S_B}P_{Fluid}\,\mathbf{n}dS=\mathbf{F}_{static}+\mathbf{F}_{F.K}+\mathbf{F}_D+\mathbf{F}_R$$

(S_B: wetted surface)

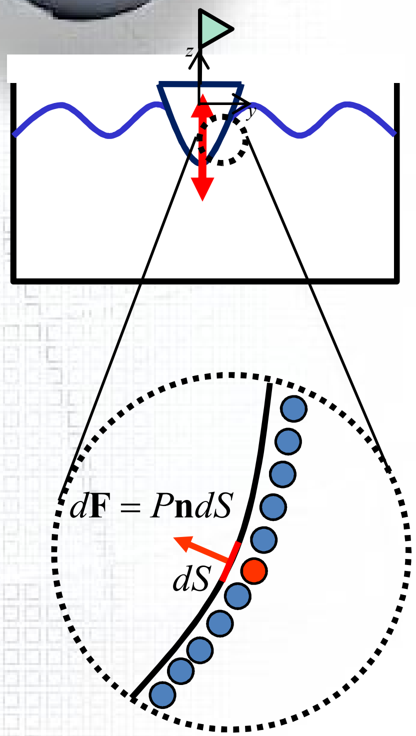
선박의 침수 표면 전체에 대하여 적분 (표면력)
(유체입자가 선박에 작용하는 힘과 모멘트)

✓ 유체입자 하나에 작용하는 Body force 와 Surface force로부터 구한 압력을 선박의 침수 표면 전체에 대해 적분하여 선박에 작용하는 유체력을 계산함

선박의 6자유도 운동 방정식 유도

$\mathbf{F}_{F.K}$: Froude- krylov force
$\mathbf{F}_D$: Diffraction force
$\mathbf{F}_R$: Radiation force

Φ_I : Incident wave velocity potential
Φ_D : Diffraction velocity potential
Φ_R : Radiation velocity potential



$d\mathbf{F}$: 하나의 유체 입자가 선박 표면에 가하는 힘

dS : 미소 면적

$\mathbf{n}$: 미소 면적의 Normal 벡터

$$\mathbf{x} = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]^T$$

ξ_1 : *surge* ξ_3 : *roll*
ξ_2 : *sway* ξ_4 : *pitch*
ξ_3 : *heave* ξ_5 : *yaw*

$\mathbf{M}_A$: 6×6 added mass matrix
$\mathbf{B}$: 6×6 damping coeff. matrix
$\mathbf{C}$: 6×6 restoring coeff. matrix

✓ 선박에 작용하는 유체력

$$\mathbf{F}_{Fluid} = \iint_{S_B} P\mathbf{n}dS = \mathbf{F}_{static} + \mathbf{F}_{F.K} + \mathbf{F}_D + \mathbf{F}_R$$

: 유체입자 하나에 작용하는 (Body force) 와 (Surface force)로부터 구한 압력을 선박의 침수 표면 전체에 대해 적분하여 선박에 작용하는 유체력을 계산함

✓ 선박의 6자유도 운동방정식

선박의 Surface force로 작용

Newton's 2nd Law

$$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F} = \text{(Body Force)} + \boxed{\text{(Surface Force)}}$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{Fluid} + \mathbf{F}_{external}$$

Body force　　Surface force　　기타 외부에서 작용하는 외력

$$\mathbf{M}\ddot{\mathbf{x}} = \boxed{\mathbf{F}_{Gravity}} + \boxed{\mathbf{F}_{static} + \mathbf{F}_{F.K} + \mathbf{F}_D + \mathbf{F}_R} \boxed{+\mathbf{F}_{external,\,dynamic} + \mathbf{F}_{external,\,static}}$$

$\mathbf{F}_{\text{Restoring}}$　　$\mathbf{F}_{\text{wave exciting}}$　　$\mathbf{F}_R = -\mathbf{A}\ddot{\mathbf{x}} - \mathbf{B}\dot{\mathbf{x}}$

added mass　　Damping Coefficient

$$\mathbf{M}\ddot{\mathbf{x}} = (\mathbf{F}_{gravity} + \mathbf{F}_{static}) + \mathbf{F}_{wave\ exciting} - \mathbf{A}\ddot{\mathbf{x}} - \mathbf{B}\dot{\mathbf{x}} + \mathbf{F}_{external,\,dynamic} + \mathbf{F}_{external,\,static}$$

Linearization $,\ (F_{restoring} = (\mathbf{F}_{gravity} + \mathbf{F}_{static}) \approx -\mathbf{C}\mathbf{x})$

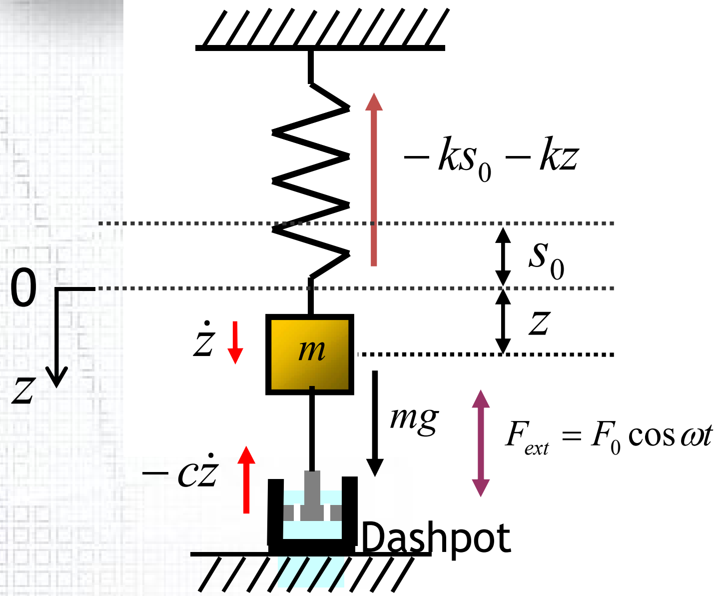
복원력 계수 C는 어떻게 구할까?

$$(\mathbf{M} + \mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{F}_{wave\ exciting} + \mathbf{F}_{external,\,dynamic} + \mathbf{F}_{external,\,static}$$

선박의 6자유도 운동 방정식 : Heave 운동에서의 복원력 계수 C구하기

$$\mathbf{F}_{restoring} = (\mathbf{F}_{gravity} + \mathbf{F}_{static}) \approx -\mathbf{C}\mathbf{x}$$

✓ Mass-Spring-Damper system



$$m\ddot{z} = F$$

$$= \cancel{mg - ks_0} - kz - c\dot{z} + F_0 \cos \omega t$$

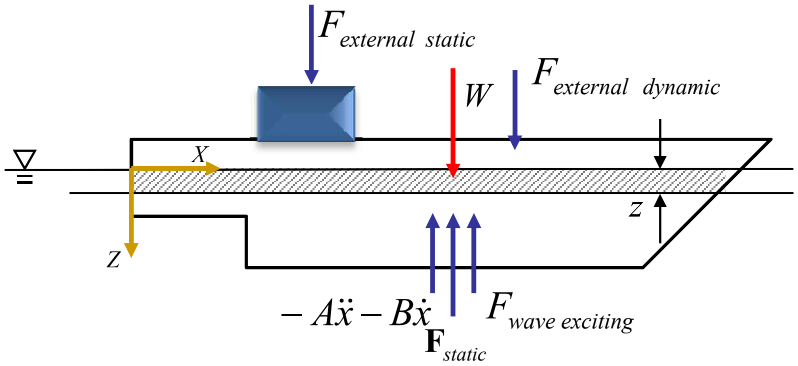
$$= -kz - c\dot{z} + F_0 \cos \omega t$$

$$m\ddot{z} + c\dot{z} + kz = F_0 \cos \omega t$$

ρ : *density of sea water*

A_{WP} : *Waterplane Area*

✓ **선박의 heave 운동방정식 예**



$$\mathbf{M}\ddot{z} = \sum \mathbf{F} = \underbrace{(\mathbf{F}_{Gravity} + \mathbf{F}_{static})}_{\mathbf{F}_{restoring}} + \mathbf{F}_{wave\ exciting} - \mathbf{A}\ddot{z} - \mathbf{b}\dot{z} + \mathbf{F}_{external,dynamic} + \mathbf{F}_{external,static}$$

✓ **선박의 heave운동에서의 복원력의 예**

$$\mathbf{F}_{restoring} = (\mathbf{F}_{Gravity} + \mathbf{F}_{static})$$

$$\mathbf{F}_{gravity} = Mg\mathbf{k}$$

M: 선박 중량

선박이 heave운동시 변위 z가 작다고 가정

$$\mathbf{F}_{static} = \left(-\rho g V_0 \underbrace{- \rho g A_{wp} \cdot z}_{\text{: heave 변위 z에 의한 추가 부력}}\right)\mathbf{k}$$

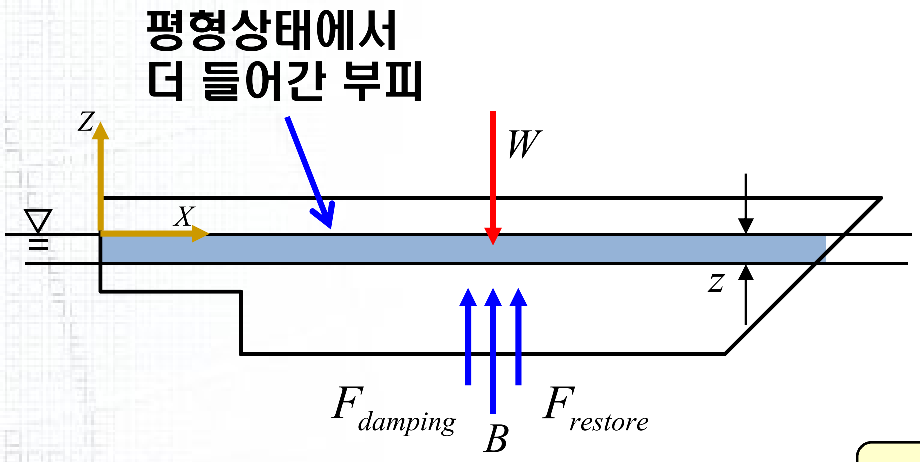
변위에 대해 반대방향으로 부력이 작용함을 의미

$$\mathbf{F}_{restoring} = (\cancel{Mg - \rho g V_0} - \rho g A_{WP} \cdot z)\mathbf{k}$$

$$= -(\rho g A_{WP}) \cdot z\mathbf{k} = -C\, z\mathbf{k} \quad ,(C = \rho g A_{WP})$$

선박의 6자유도 운동 방정식 : Uncoupled equation of motion

✓ Heave 운동 방정식 유도



$$M\ddot{\xi}_3 = \sum F = F_{Body} + F_{Surface}$$

$$= F_{gravity} + F_{Fluid}$$

$$= \underbrace{F_{gravity}}_{-Mg} + \underbrace{F_{static}}_{\rho g V_0 - \rho g A_{wp} \cdot \xi_3} + \underbrace{F_{F.K} + F_D}_{F_{\text{exciting},3}} + \underbrace{F_R}_{-A_{33}\ddot{\xi}_3 - B_{33}\dot{\xi}_3}$$

$$= \cancel{-Mg} + (\cancel{\rho g V_0} - \rho g A_{wp}\xi_3) + F_{exciting,3} - A_{33}\ddot{\xi}_3 - B_{33}\dot{\xi}_3$$

$$\boxed{\therefore (M + A_{33})\ddot{\xi}_3 + B_{33}\dot{\xi}_3 + \rho g A_{wp}\xi_3 = F_{exciting,3}}$$

✓ Surge 운동 방정식 유도
(Heave에서 복원력 성분만 제외)

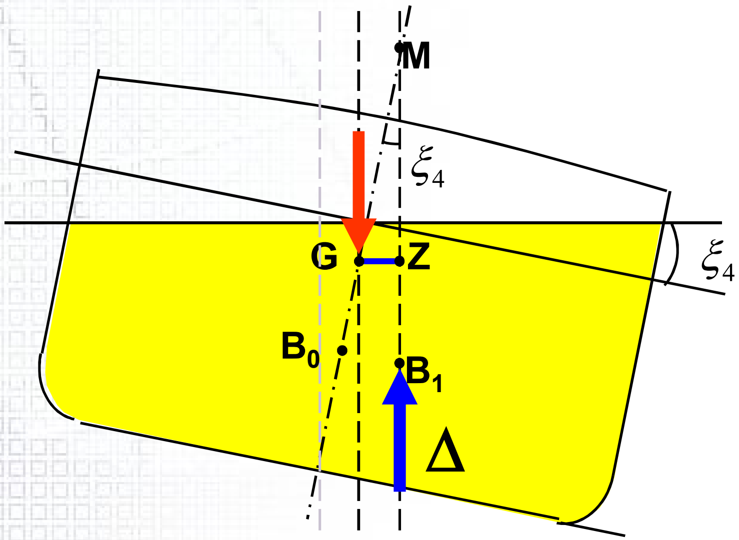
$$\boxed{\therefore (M + A_{11})\ddot{\xi}_1 + B_{11}\dot{\xi}_1 = F_{exciting,1}}$$

✓ Sway 운동 방정식 유도
(Heave에서 복원력 성분만 제외)

$$\boxed{\therefore (M + A_{22})\ddot{\xi}_2 + B_{22}\dot{\xi}_2 = F_{exciting,2}}$$

선박의 6자유도 운동 방정식
: Uncoupled equation of motion

✓ Roll 운동 방정식 유도



$$I_{xx}\ddot{\xi}_4 = \sum M = M_{Body} + M_{Surface}$$

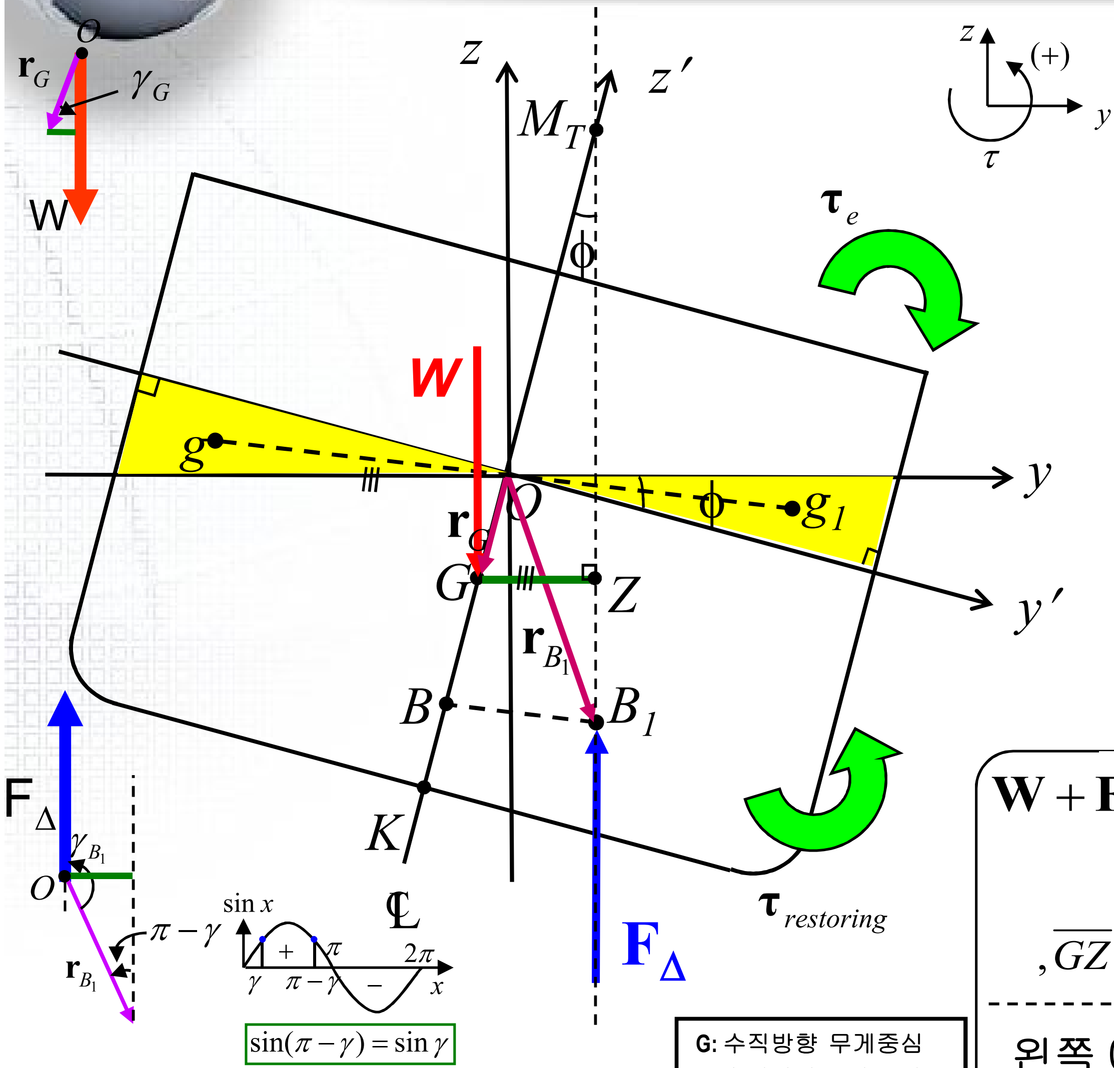
$$= M_{gravity} + M_{Fluid}$$

$$= \underbrace{M_{gravity}}_{\mathbf{r}_G \times W\mathbf{k}} + \underbrace{M_{static}}_{\mathbf{r}_{B_1} \times \Delta\mathbf{k}} + \underbrace{M_{F.K} + M_D}_{M_{\text{exciting},4}} + \underbrace{M_R}_{-A_{44}\ddot{\xi}_4 - B_{44}\dot{\xi}_4}$$

$$= \Delta \cdot \overline{GZ} + M_{exciting,4} - A_{44}\ddot{\xi}_4 - B_{44}\dot{\xi}_4$$

5. 횡 복원력

– 선박의 횡 복원 안정성(Transv. Stability) – 안정 상태 (3)



$oy'z'$: *Body fixed coordinate*

oyz : *Global coordinate*

M_T: B_1을 지나는 부력 작용선과 선체 중심선과의 교점

G: 수직방향 무게중심
B: 수직방향 부력 중심
W: 선박 무게
$\mathbf{F}_\Delta$: 부력

⑥-2)회전축 O를 지나고, y-z 평면에 수직한 축에 대한 중력과 부력에 의한 모멘트를 구해보면,

$$\boldsymbol{\tau} = \boldsymbol{\tau}_G + \boldsymbol{\tau}_\Delta$$

$$= \mathbf{r}_G \times \mathbf{W} + \mathbf{r}_{B_1} \times \mathbf{F}_\Delta$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_G & z_G \\ 0 & 0 & W_z \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & y_{B_1} & z_{B_1} \\ 0 & 0 & \mathbf{F}_{\Delta,z} \end{vmatrix}$$

$$= y_G \cdot W_z\,\mathbf{i} + y_{B_1} \cdot \mathbf{F}_{\Delta,z}\,\mathbf{i}$$

$$\mathbf{W} + \mathbf{F}_\Delta = 0 \Rightarrow W_z + F_{\Delta,z} = 0 \Rightarrow W_z = -F_{\Delta,z}$$

$$\boldsymbol{\tau} = (-y_G + y_{B_1}) \cdot F_{\Delta,z}\,\mathbf{i}$$

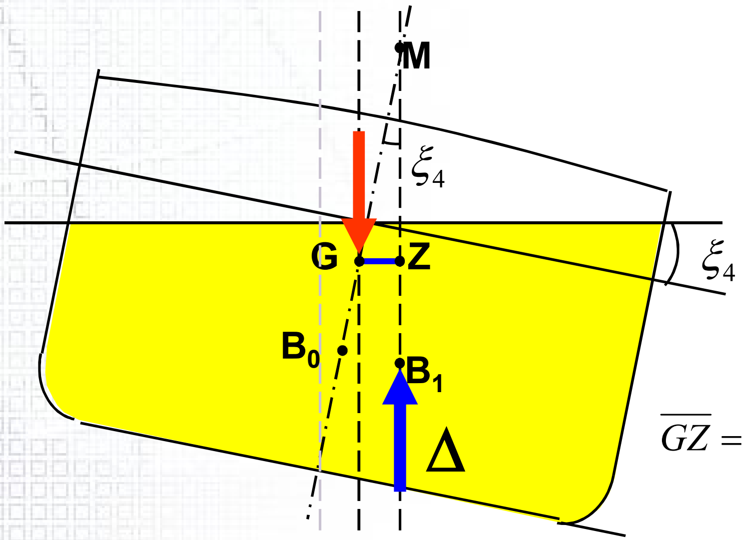
$$, \overline{GZ} = \left| -y_G + y_{B_1} \right|$$

Z: B_1을 통한 부력작용선과 G를 지나고 y축과 평행한 선이 만나는 점

왼쪽 예제의 경우, $\boldsymbol{\tau}$ 가 (+)이면(무게중심이 부력 작용선에 대해 왼쪽에 위치) $\boldsymbol{\tau}$ 는 복원 모멘트 $\boldsymbol{\tau}_{restoring}$ 로 작용함 $\boldsymbol{\tau} = \boldsymbol{\tau}_{restoring}$

선박의 6자유도 운동 방정식
: Uncoupled equation of motion

✓ Roll 운동 방정식 유도



$$I_{xx}\ddot{\xi}_4 = \sum M = M_{Body} + M_{Surface}$$
$$= M_{gravity} + M_{Fluid}$$

$$= \underbrace{M_{gravity}}_{\mathbf{r}_G \times W\mathbf{k}} + \underbrace{M_{static}}_{\mathbf{r}_{B_1} \times \Delta\mathbf{k}} + \underbrace{M_{F.K} + M_D}_{M_{\text{exciting},4}} + \underbrace{M_R}_{-A_{44}\ddot{\xi}_4 - B_{44}\dot{\xi}_4}$$

$$= \Delta \cdot \overline{GZ} + M_{exciting,4} - A_{44}\ddot{\xi}_4 - B_{44}\dot{\xi}_4$$

$\overline{GZ} = -\overline{GM}\sin\xi_4$

$$= -\Delta \cdot \overline{GM}\sin\xi_4 + M_{exciting,4} - A_{44}\ddot{\xi}_4 - B_{44}\dot{\xi}_4$$

$\sin\xi_4 \approx \xi_4$

$$\approx -\Delta \cdot \overline{GM}\xi_4 + M_{exciting,4} - A_{44}\ddot{\xi}_4 - B_{44}\dot{\xi}_4$$

$$\boxed{\therefore (I_{xx} + A_{44})\ddot{\xi}_4 + B_{44}\dot{\xi}_4 + \Delta \cdot \overline{GM}_T\xi_4 = M_{exciting,4}}$$

✓ Pitch 운동 방정식 유도
(Roll 운동 방정식과 동일)

$$\boxed{\therefore (I_{yy} + A_{55})\ddot{\xi}_5 + B_{55}\dot{\xi}_5 + \Delta \cdot \overline{GM}_L\xi_5 = M_{exciting,5}}$$

✓ Yaw 운동 방정식 유도
(RollHeave에서 복원력 성분만 제외)

$$\boxed{\therefore (I_{zz} + A_{66})\ddot{\xi}_6 + B_{66}\dot{\xi}_6 = M_{exciting,6}}$$

1) Journee, J.M.J. , Adegeest, L.J.M. ,Theoretical Manual of Strip Theory program" Seaway for Windows", Delft University of Technology, 2003, pp38~42
2) Newman, J.N. , Marine Hydrodynamics, The MIT Press, Cambridge, 1997, pp 307~311
3) Journee, J.M.J. , Massie, W.W. , Offshore Hydrodynamics, Delft University of Technology, 2001,pp8-1~4

선박의 6자유도 운동 방정식

(변위 : $\mathbf{x}=[\xi_1,\cdots,\xi_6]^T$)
$(\mathbf{M},\mathbf{A},\mathbf{B},\mathbf{C}: 6\times 6\, Matrix)$

✓ 6DOF Equations of Ship Motion : 6 coupled equation

(단, 계수 $\mathbf{A},\mathbf{B},\mathbf{C}$ 및 외력 $\mathbf{F}_{exciting}$은 주어져 있다고 가정)

Given

$$(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}}+\mathbf{B}\dot{\mathbf{x}}+\mathbf{C}\mathbf{x}=\mathbf{F}_{exciting}$$

Find

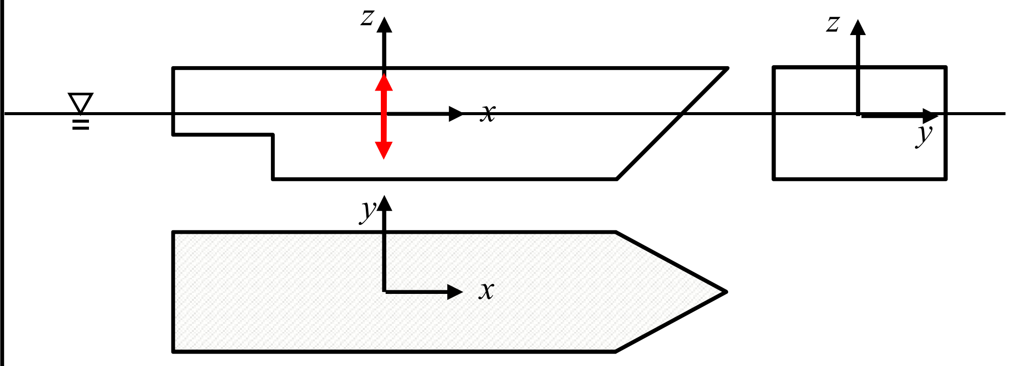
$$\mathbf{x},\dot{\mathbf{x}},\ddot{\mathbf{x}}$$

✓ Assumption

1. Small amplitude water wave (파장에 비해 파고가 작음)
2. small amplitude motion (선박의 운동이 작음)
3. Slender body (선박의 길이에 비해 폭이 작음, Strip theory에서 자세히 설명)
   ➔ 물체의 전후 동요(Surge)는 독립적으로 취급 (Coupling 고려 안함)
4. Lateral symmetry (symmetric about xz-plane)
   ➔ 물체 운동이 종운동(Longitudinal motion) 과 횡운동(Transverse motion)으로 나뉨

surge,heave,pitch ⟺ sway,roll,yaw

서로 영향을 주지 않음



(ex) 좌우 대칭 선박

Heave운동에 의해 영향을 받는 운동은?

surge , pitch

~~sway~~ , ~~roll~~ , ~~yaw~~

1) Journee, J.M.J. , Adegeest, L.J.M. ,Theoretical Manual of Strip Theory program" Seaway for Windows", Delft University of Technology, 2003, pp38~42
2) Newman, J.N. , Marine Hydrodynamics, The MIT Press, Cambridge, 1997, pp 307~311
3) Journee, J.M.J. , Massie, W.W. , Offshore Hydrodynamics, Delft University of Technology, 2001,pp8-1~4

선박의 6자유도 운동 방정식

✓ 6DOF Equations of Ship Motion
: 3 kinds of coupled motions (surge / heave-pitch / sway-roll-yaw)

$$\mathbf{x} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} M & 0 & 0 & 0 & Mz_C & -My_C \\ 0 & M & 0 & -Mz_C & 0 & Mx_C \\ 0 & 0 & M & My_C & -Mx_C & 0 \\ 0 & -Mz_C & My_C & I_{xx} & 0 & -I_{xz} \\ Mz_C & 0 & -Mx_C & 0 & I_{yy} & 0 \\ -My_C & Mx_C & 0 & -I_{zx} & 0 & I_{zz} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & B_{22} & 0 & B_{24} & 0 & B_{26} \\ B_{31} & 0 & B_{33} & 0 & B_{35} & 0 \\ 0 & B_{42} & 0 & B_{44} & 0 & B_{46} \\ B_{51} & 0 & B_{53} & 0 & B_{55} & 0 \\ 0 & B_{62} & 0 & B_{64} & 0 & B_{66} \end{bmatrix}$$

$$(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{F}_{exciting}$$

$$\mathbf{F}_{exciting} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 & A_{15} & 0 \\ 0 & A_{22} & 0 & A_{24} & 0 & A_{26} \\ A_{31} & 0 & A_{33} & 0 & A_{35} & 0 \\ 0 & A_{42} & 0 & A_{44} & 0 & A_{46} \\ A_{51} & 0 & A_{53} & 0 & A_{55} & 0 \\ 0 & A_{62} & 0 & A_{64} & 0 & A_{66} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & C_{53} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Journee, J.M.J. , Adegeest, L.J.M. ,Theoretical Manual of Strip Theory program" Seaway for Windows", Delft University of Technology, 2003, pp38~42
2) Newman, J.N. , Marine Hydrodynamics, The MIT Press, Cambridge, 1997, pp 307~311
3) Journee, J.M.J. , Massie, W.W. , Offshore Hydrodynamics, Delft University of Technology, 2001,pp8-1~4

선박의 6자유도 운동 방정식

✓ 6DOF Equations of Ship Motion
: 3 kinds of coupled motions (surge / heave-pitch / sway-roll-yaw)

Surge, Heave, Pitch

$$\mathbf{x} = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6]^T$$

Sway, Roll, Yaw

$$\mathbf{M} = \begin{bmatrix} M & 0 & 0 & 0 & Mz_C & -My_C \\ 0 & M & 0 & -Mz_C & 0 & Mx_C \\ 0 & 0 & M & My_C & -Mx_C & 0 \\ 0 & -Mz_C & My_C & I_{xx} & 0 & -I_{xz} \\ Mz_C & 0 & -Mx_C & 0 & I_{yy} & 0 \\ -My_C & Mx_C & 0 & -I_{zx} & 0 & I_{zz} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & B_{22} & 0 & B_{24} & 0 & B_{26} \\ B_{31} & 0 & B_{33} & 0 & B_{35} & 0 \\ 0 & B_{42} & 0 & B_{44} & 0 & B_{46} \\ B_{51} & 0 & B_{53} & 0 & B_{55} & 0 \\ 0 & B_{62} & 0 & B_{64} & 0 & B_{66} \end{bmatrix}$$

운동 방정식 : $(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{F}_{exciting}$

$$\mathbf{F}_{exciting} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 & A_{15} & 0 \\ 0 & A_{22} & 0 & A_{24} & 0 & A_{26} \\ A_{31} & 0 & A_{33} & 0 & A_{35} & 0 \\ 0 & A_{42} & 0 & A_{44} & 0 & A_{46} \\ A_{51} & 0 & A_{53} & 0 & A_{55} & 0 \\ 0 & A_{62} & 0 & A_{64} & 0 & A_{66} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & C_{53} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

➔ heave-pitch motion of equation :

$$\begin{bmatrix} M + A_{33} & -Mx_C + A_{35} \\ -Mx_C + A_{53} & A_{55} + I_{yy} \end{bmatrix} \begin{bmatrix} \ddot{\xi}_3 \\ \ddot{\xi}_5 \end{bmatrix} + \begin{bmatrix} B_{33} & B_{35} \\ B_{53} & B_{55} \end{bmatrix} \begin{bmatrix} \dot{\xi}_3 \\ \dot{\xi}_5 \end{bmatrix} + \begin{bmatrix} C_{33} & C_{35} \\ C_{53} & C_{55} \end{bmatrix} \begin{bmatrix} \xi_3 \\ \xi_5 \end{bmatrix} = \begin{bmatrix} F_3 \\ F_5 \end{bmatrix}$$

($y_c = 0$ 으로 가정)

1) Journee, J.M.J. , Adegeest, L.J.M. ,Theoretical Manual of Strip Theory program“ Seaway for Windows”, Delft University of Technology, 2003, pp38~42
2) Newman, J.N. , Marine Hydrodynamics, The MIT Press, Cambridge, 1997, pp 307~311
3) Journee, J.M.J. , Massie, W.W. , Offshore Hydrodynamics, Delft University of Technology, 2001,pp8-1~4

선박의 6자유도 운동 방정식

✓ 6DOF Equations of Ship Motion
: 3 kinds of coupled motions (surge / heave-pitch / sway-roll-yaw)

$$\mathbf{x} = [\overset{\text{전후동요 변위}}{\xi_1}, \underset{\text{좌우동요 변위}}{\xi_2}, \overset{\text{상하동요 변위}}{\xi_3}, \underset{\text{횡동요 변위}}{\xi_4}, \overset{\text{종동요 변위}}{\xi_5}, \underset{\text{선수동요 변위}}{\xi_6}]^T$$

$$\mathbf{M} = \begin{bmatrix} M & 0 & 0 & 0 & Mz_C & -My_C \\ 0 & M & 0 & -Mz_C & 0 & Mx_C \\ 0 & 0 & M & My_C & -Mx_C & 0 \\ 0 & -Mz_C & My_C & I_{xx} & 0 & -I_{xz} \\ Mz_C & 0 & -Mx_C & 0 & I_{yy} & 0 \\ -My_C & Mx_C & 0 & -I_{zx} & 0 & I_{zz} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} B_{11} & 0 & B_{13} & 0 & B_{15} & 0 \\ 0 & B_{22} & 0 & B_{24} & 0 & B_{26} \\ B_{31} & 0 & B_{33} & 0 & B_{35} & 0 \\ 0 & B_{42} & 0 & B_{44} & 0 & B_{46} \\ B_{51} & 0 & B_{53} & 0 & B_{55} & 0 \\ 0 & B_{62} & 0 & B_{64} & 0 & B_{66} \end{bmatrix}$$

$$(\mathbf{M} + \mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{F}_{exciting}$$

$$\mathbf{F}_{exciting} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 & A_{15} & 0 \\ 0 & A_{22} & 0 & A_{24} & 0 & A_{26} \\ A_{31} & 0 & A_{33} & 0 & A_{35} & 0 \\ 0 & A_{42} & 0 & A_{44} & 0 & A_{46} \\ A_{51} & 0 & A_{53} & 0 & A_{55} & 0 \\ 0 & A_{62} & 0 & A_{64} & 0 & A_{66} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & C_{53} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

➔ sway-roll-yaw :

$$\begin{bmatrix} m + A_{22} & -mz_c + A_{24} & mx_c + A_{26} \\ -mz_c + A_{42} & I_{xx} + A_{44} & -I_{xz} + A_{46} \\ mx_c + A_{62} & -I_{zx} + A_{64} & I_{zz} + A_{66} \end{bmatrix} \begin{bmatrix} \ddot{\xi}_2 \\ \ddot{\xi}_4 \\ \ddot{\xi}_6 \end{bmatrix} + \begin{bmatrix} B_{22} & B_{24} & B_{26} \\ B_{42} & B_{44} & B_{46} \\ B_{62} & B_{64} & B_{66} \end{bmatrix} \begin{bmatrix} \dot{\xi}_2 \\ \dot{\xi}_4 \\ \dot{\xi}_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_{44} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_4 \\ \xi_6 \end{bmatrix} = \begin{bmatrix} F_2 \\ F_4 \\ F_6 \end{bmatrix}$$

1) Newman, J.N. , Marine Hydrodynamics, The MIT Press, Cambridge, 1977, pp 307~310

선박의 6자유도 운동 방정식
: Heave 운동 방정식 풀이 (Assuption : Harmonic motion)

✓ 선박의 6자유도 운동 방정식

$$(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}}+\mathbf{B}\dot{\mathbf{x}}+\mathbf{C}\mathbf{x}=\mathbf{F}_{exciting}$$

ex) Heave

$$(M+A_{33})\ddot{\xi}_3+B_{33}\dot{\xi}_3+C_{33}\xi_3=F_{exciting,3}$$

시간이 충분이 흘러 외력의 주기와 같은 운동을 함 (Harmonic motion)
=> **초기** Transient motion**은 고려하지 않음**

$$\xi_3(t)=\xi_3^A e^{i\omega t}$$
$$\dot{\xi}_3(t)=i\omega\xi_3^A e^{i\omega t}$$
$$\ddot{\xi}_3(t)=-\omega^2\xi_3^A e^{i\omega t}$$

$$F_{exciting,3}=F_3^A e^{i\omega t}=\eta_0 f_3^A e^{i\omega t}$$

(η_0 : Wave Amplitude, Real)
(f_3^A: Wave exciting force Amplitude/1m wave amplitude, Complex)

ex) If ξ_3^A is not a complex (real)

Let $\xi_3^A=a$

$$\xi_3=\xi_3^A e^{i\omega t}$$ (Euler 공식)
$$=a(\cos\omega t+i\sin\omega t)$$
$$=a\cos\omega t+ia\sin\omega t$$
$$\mathrm{Re}\{\xi_3\}=a\cos\omega t$$

If ξ_3^A is a complex

Let $\xi_3^A=a+ib$

$$\xi_3=\xi_3^A e^{i\omega t}$$ (Euler 공식)
$$=(a+ib)(\cos\omega t+i\sin\omega t)$$
$$=(a\cos\omega t-b\sin\omega t)+i(b\cos\omega t+a\sin\omega t)$$
$$\mathrm{Re}\{\xi_3\}=a\cos\omega t-b\sin\omega t=c\cos(\omega t-\varepsilon)$$

Phase가 나타남

1) Newman, J.N. , Marine Hydrodynamics, The MIT Press, Cambridge, 1977, pp 307~310

선박의 6자유도 운동 방정식
: Heave 운동 방정식 풀이 (Assuption : Harmonic motion)

✓ 선박의 6자유도 운동 방정식

$$(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}}+\mathbf{B}\dot{\mathbf{x}}+\mathbf{C}\mathbf{x}=\mathbf{F}_{exciting}$$

ex) Heave

$$(M+A_{33})\ddot{\xi}_3+B_{33}\dot{\xi}_3+C_{33}\xi_3=F_{exciting,3}$$

$$\xi_3(t)=\xi_3^A e^{i\omega t}$$
$$\dot{\xi}_3(t)=i\omega\xi_3^A e^{i\omega t}$$
$$\ddot{\xi}_3(t)=-\omega^2\xi_3^A e^{i\omega t}$$

$$F_{exciting,3}=F_3^A e^{i\omega t}=\eta_0 f_3^A e^{i\omega t}$$

(η_0 : Wave Amplitude, Real)

(f_3^A : Wave exciting force Amplitude, Complex)

$$(M+A_{33})(-\omega^2\xi_3^A e^{i\omega t})+B_{33}(i\omega\xi_3^A e^{i\omega t})+C_{33}(\xi_3^A e^{i\omega t})=\eta_0 f_3^A e^{i\omega t}$$

$$\{-\omega^2(M+A_{33})+i\omega B_{33}+C_{33}\}\xi_3^A e^{i\omega t}=\eta_0 f_3^A e^{i\omega t}$$

$$\{\underbrace{-\omega^2(M+A_{33})+i\omega B_{33}+C_{33}}_{=\mathbf{D}}\}\xi_3^A=\eta_0 f_3^A \Rightarrow \xi_3^A=\eta_0 f_3^A D^{-1} \Rightarrow \frac{\xi_3^A}{\eta_0}=f_3^A\mathbf{D}^{-1}$$

$=\mathbf{D}$ (➔ Complex)

✓ RAO(Response Amplitude Operator)
: 1m wave Amplitude 를 가지는 주파수 ω인 wave에 대한 선박의 6자유도 운동 변위

1) Bhattacharyya, R. , "Dynamics of Marine Vehicles", John Wiley & Sons, 1978, pp.183-191

선박의 6자유도 운동 방정식
: Heave-pitch 연성 운동 방정식 풀이

✓ Assumption : **운동 주기가 외력(해양파)의 주기 ω와 동일함**

✓ Given : heave-pitch motion of equation **의 계수**

파고 (real)

Wave exciting force amplitude (complex)

ω : Wave frequency

$$\begin{bmatrix} M+A_{33} & -Mx_C+A_{35} \\ -Mx_C+A_{53} & A_{55}+I_{yy} \end{bmatrix}\begin{bmatrix} \ddot{\xi}_3 \\ \ddot{\xi}_5 \end{bmatrix}+\begin{bmatrix} B_{33} & B_{35} \\ B_{53} & B_{55} \end{bmatrix}\begin{bmatrix} \dot{\xi}_3 \\ \dot{\xi}_5 \end{bmatrix}+\begin{bmatrix} C_{33} & C_{35} \\ C_{53} & C_{55} \end{bmatrix}\begin{bmatrix} \xi_3 \\ \xi_5 \end{bmatrix}=\begin{bmatrix} F_{ext,3} \\ F_{ext,5} \end{bmatrix}=\begin{bmatrix} \eta_0\, F_3^A e^{i\omega t} \\ \eta_0\, F_5^A e^{i\omega t} \end{bmatrix}$$

✓ Find : heave, pitch **운동 변위, 속도, 가속도**

Motion amplitude(complex)

$$\begin{aligned} \xi_3(t) &= \xi_3^A e^{i\omega t} \\ \xi_5(t) &= \xi_5^A e^{i\omega t} \end{aligned} \quad \left(\begin{aligned} \dot{\xi}_3(t) &= i\omega\xi_3^A e^{i\omega t}, \ddot{\xi}_3(t) = -\omega^2\xi_3^A e^{i\omega t} \\ \dot{\xi}_5(t) &= i\omega\xi_5^A e^{i\omega t}, \ddot{\xi}_5(t) = -\omega^2\xi_5^A e^{i\omega t} \end{aligned}\right)$$

⇒ ξ_3^A, ξ_5^A **만 구하면 됨**

① 운동 방정식에 변위,속도,가속도 대입

$$\begin{bmatrix} M+A_{33} & -Mx_C+A_{35} \\ -Mx_C+A_{53} & A_{55}+I_{yy} \end{bmatrix}\underbrace{\begin{bmatrix} -\omega^2\xi_3^A e^{i\omega t} \\ -\omega^2\xi_5^A e^{i\omega t} \end{bmatrix}}_{\text{(가속도)}}+\begin{bmatrix} B_{33} & B_{35} \\ B_{53} & B_{55} \end{bmatrix}\underbrace{\begin{bmatrix} i\omega\xi_3^A e^{i\omega t} \\ i\omega\xi_5^A e^{i\omega t} \end{bmatrix}}_{\text{(속도)}}+\begin{bmatrix} C_{33} & C_{35} \\ C_{53} & C_{55} \end{bmatrix}\underbrace{\begin{bmatrix} \xi_3^A e^{i\omega t} \\ \xi_3^A e^{i\omega t} \end{bmatrix}}_{\text{(변위)}}=\begin{bmatrix} \eta_0 F_3^A e^{i\omega t} \\ \eta_0 F_5^A e^{i\omega t} \end{bmatrix}$$

선박의 6자유도 운동 방정식
: Heave-pitch 연성 운동 방정식 풀이

(continue)

① 운동 방정식에 변위,속도,가속도 대입

$$\begin{bmatrix} M+A_{33} & -Mx_C+A_{35} \\ -Mx_C+A_{53} & A_{55}+I_{yy} \end{bmatrix}\begin{bmatrix} -\omega^2\xi_3^A\cancel{e^{i\omega t}} \\ -\omega^2\xi_5^A\cancel{e^{i\omega t}} \end{bmatrix}+\begin{bmatrix} B_{33} & B_{35} \\ B_{53} & B_{55} \end{bmatrix}\begin{bmatrix} i\omega\xi_3^A\cancel{e^{i\omega t}} \\ i\omega\xi_5^A\cancel{e^{i\omega t}} \end{bmatrix}+\begin{bmatrix} C_{33} & C_{35} \\ C_{53} & C_{55} \end{bmatrix}\begin{bmatrix} \xi_3^A\cancel{e^{i\omega t}} \\ \xi_3^A\cancel{e^{i\omega t}} \end{bmatrix}=\begin{bmatrix} \eta_0F_3^A\cancel{e^{i\omega t}} \\ \eta_0F_5^A\cancel{e^{i\omega t}} \end{bmatrix}$$

② 양변을 $e^{i\omega t}$로 나눔

$$-\omega^2\begin{bmatrix} M+A_{33} & -Mx_C+A_{35} \\ -Mx_C+A_{53} & A_{55}+I_{yy} \end{bmatrix}\begin{bmatrix} \xi_3^A \\ \xi_5^A \end{bmatrix}+i\omega\begin{bmatrix} B_{33} & B_{35} \\ B_{53} & B_{55} \end{bmatrix}\begin{bmatrix} \xi_3^A \\ \xi_5^A \end{bmatrix}+\begin{bmatrix} C_{33} & C_{35} \\ C_{53} & C_{55} \end{bmatrix}\begin{bmatrix} \xi_3^A \\ \xi_3^A \end{bmatrix}=\begin{bmatrix} \eta_0F_3^A \\ \eta_0F_5^A \end{bmatrix}$$

③ $\begin{bmatrix} \xi_3^A \\ \xi_5^A \end{bmatrix}$로 묶어서 정리

$$\left(-\omega^2\begin{bmatrix} M+A_{33} & -Mx_C+A_{35} \\ -Mx_C+A_{53} & A_{55}+I_{yy} \end{bmatrix}+i\omega\begin{bmatrix} B_{33} & B_{35} \\ B_{53} & B_{55} \end{bmatrix}+\begin{bmatrix} C_{33} & C_{35} \\ C_{53} & C_{55} \end{bmatrix}\right)\begin{bmatrix} \xi_3^A \\ \xi_3^A \end{bmatrix}=\begin{bmatrix} \eta_0F_3^A \\ \eta_0F_5^A \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} -\omega^2(M+A_{33})+i\omega B_{33}+C_{33} & -\omega^2(-Mx_C+A_{35})+i\omega B_{35}+C_{35} \\ -\omega^2(-Mx_C+A_{53})+i\omega B_{53}+C_{53} & -\omega^2(A_{55}+I_{yy})+i\omega B_{55}+C_{55} \end{bmatrix}\begin{bmatrix} \xi_3^A \\ \xi_3^A \end{bmatrix}=\begin{bmatrix} \eta_0F_3^A \\ \eta_0F_5^A \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix}\begin{bmatrix} \xi_3^A \\ \xi_3^A \end{bmatrix}=\begin{bmatrix} \eta_0F_3^A \\ \eta_0F_5^A \end{bmatrix}\qquad\begin{pmatrix} P=-\omega^2(M+A_{33})+i\omega B_{33}+C_{33} \\ Q=-\omega^2(-Mx_C+A_{35})+i\omega B_{35}+C_{35} \\ R=-\omega^2(-Mx_C+A_{53})+i\omega B_{53}+C_{53} \\ S=-\omega^2(I_{yy}+A_{55})+i\omega B_{55}+C_{55} \end{pmatrix}$$

1) Bhattacharyya, R. , "Dynamics of Marine Vehicles", John Wiley & Sons, 1978, pp.183-191

선박의 6자유도 운동 방정식
: Heave-pitch 연성 운동 방정식 풀이

(continue)

③ $\begin{bmatrix} \xi_3^A \\ \xi_5^A \end{bmatrix}$ 로 묶어서 정리

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} \xi_3^A \\ \xi_3^A \end{bmatrix} = \begin{bmatrix} \eta_0 F_3^A \\ \eta_0 F_5^A \end{bmatrix}$$

$$\left(\begin{aligned} P &= -\omega^2 (M + A_{33}) + i\omega B_{33} + C_{33} \\ Q &= -\omega^2 (-Mx_C + A_{35}) + i\omega B_{35} + C_{35} \\ R &= -\omega^2 (-Mx_C + A_{53}) + i\omega B_{53} + C_{53} \\ S &= -\omega^2 (I_{yy} + A_{55}) + i\omega B_{55} + C_{55} \end{aligned} \right)$$

④ 역행렬을 곱하여 변위 를 ξ_3^A, ξ_5^A 구함

$$\begin{bmatrix} \xi_3^A \\ \xi_5^A \end{bmatrix} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}^{-1} \begin{bmatrix} \eta_0 F_3^A \\ \eta_0 F_5^A \end{bmatrix} = \frac{1}{PS - QR} \begin{bmatrix} S & -Q \\ -R & P \end{bmatrix} \begin{bmatrix} \eta_0 F_3^A \\ \eta_0 F_5^A \end{bmatrix}$$

$$= \frac{1}{PS - QR} \begin{bmatrix} \eta_0 \left(F_3^A S - F_5^A Q \right) \\ \eta_0 \left(-F_3^A R + F_5^A P \right) \end{bmatrix} = \begin{bmatrix} \eta_0 \dfrac{F_3^A S - F_5^A Q}{PS - QR} \\ \eta_0 \dfrac{F_5^A P - F_3^A R}{PS - QR} \end{bmatrix}$$

⑤ 1m 파고에 대한 운동 변위 (*RAO)

$$\therefore \frac{\xi_3^A}{\eta_0} = \frac{F_3^A S - F_5^A Q}{PS - QR}$$

$$\frac{\xi_5^A}{\eta_0} = \frac{F_5^A P - F_3^A R}{PS - QR}$$

*RAO(Response Amplitude Operator) : 1m v에 대한 선박의 운동 응답

1) Newman, J.N. , Marine Hydrodynamics, The MIT Press, Cambridge, 1977, pp 307~310

선박의 6자유도 운동 방정식
: 주파수 영역(Frequency domain)에서의 6자유도 운동 방정식 풀이

선박의 6자유도 운동 방정식

$$(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}}+\mathbf{B}\dot{\mathbf{x}}+\mathbf{C}\mathbf{x}=\mathbf{F}_{exciting}$$

General Case

$$\mathbf{x}=\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\\ \xi_5\\ \xi_6\end{bmatrix}=\begin{bmatrix}\xi_1^A\\ \xi_2^A\\ \xi_3^A\\ \xi_4^A\\ \xi_5^A\\ \xi_6^A\end{bmatrix}e^{i\omega t}=\mathbf{x}^A e^{i\omega t},\quad \dot{\mathbf{x}}=i\omega\mathbf{x}^A e^{i\omega t},\quad \ddot{\mathbf{x}}=-\omega^2\mathbf{x}^A e^{i\omega t},\quad \mathbf{F}_{exciting}=\eta_0\begin{bmatrix}f_1^A\\ f_2^A\\ f_3^A\\ f_4^A\\ f_5^A\\ f_6^A\end{bmatrix}e^{i\omega t}=\eta_0\mathbf{f}^A e^{i\omega t}$$

Complex amplitude

$$(\mathbf{M}+\mathbf{A})\left(-\omega^2\mathbf{x}^A e^{i\omega t}\right)+\mathbf{B}\left(i\omega\mathbf{x}^A e^{i\omega t}\right)+\mathbf{C}\left(\mathbf{x}^A e^{i\omega t}\right)=\eta_0\mathbf{f}^A e^{i\omega t}$$

$$\left\{-\omega^2(\mathbf{M}+\mathbf{A})+i\omega\mathbf{B}+\mathbf{C}\right\}\mathbf{x}^A e^{i\omega t}=\eta_0\mathbf{f}^A e^{i\omega t}$$

$$\underbrace{\left\{-\omega^2(\mathbf{M}+\mathbf{A})+i\omega\mathbf{B}+\mathbf{C}\right\}}_{=\mathbf{D}}\mathbf{x}^A=\eta_0\mathbf{f}^A \;\Rightarrow\; \mathbf{x}^A=\eta_0\mathbf{D}^{-1}\mathbf{f}^A \;\Rightarrow\; \frac{\mathbf{x}^A}{\eta_0}=\mathbf{D}^{-1}\mathbf{f}^A$$

✓ RAO(Response Amplitude Operator)
: 1m wave Amplitude 를 가지는 주파수 ω인 wave에 대한 선박의 6자유도 운동 변위

선박 6자유도 운동 방정식
: 시간 영역(Time domain)에서의 운동 방정식 풀이

$$(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}}+\mathbf{B}\dot{\mathbf{x}}+\mathbf{C}\mathbf{x}=\mathbf{F}_{exciting}$$

6자유도 선박의 운동 방정식

⇩ 변경

$$\ddot{\mathbf{x}}=(\mathbf{M}+\mathbf{A})^{-1}\left(-\mathbf{B}\dot{\mathbf{x}}-\mathbf{C}\mathbf{x}+\mathbf{F}_{exciting}\right)$$

⇩ 초기 위치 및 자세와 초기 속도 대입 $(\dot{\mathbf{x}}_0, \mathbf{x}_0)$

$$\ddot{\mathbf{x}}_1=(\mathbf{M}+\mathbf{A})^{-1}\left(-\mathbf{B}\dot{\mathbf{x}}_0-\mathbf{C}\mathbf{x}_0+\mathbf{F}_{exciting}\right)$$

⇩ 수치 적분[1])을 통해 가속도로부터 속도, 변위를 구함

$$\ddot{\mathbf{x}}_1 \to \dot{\mathbf{x}}_1 \to \mathbf{x}_1$$

⇩ 속도,변위 대입

$$\ddot{\mathbf{x}}_2=(\mathbf{M}+\mathbf{A})^{-1}\left(-\mathbf{B}\dot{\mathbf{x}}_1-\mathbf{C}\mathbf{x}_1+\mathbf{F}_{exciting}\right)$$

⇩ 수치 적분

$$\ddot{\mathbf{x}}_2 \to \dot{\mathbf{x}}_2 \to \mathbf{x}_2$$

Simulation Loop

① n번째 속도,변위 입력

② n+1번째 가속도 계산

$$\ddot{\mathbf{x}}_{n+2}=(\mathbf{M}+\mathbf{A})^{-1}\left(-\mathbf{B}\dot{\mathbf{x}}_{n+1}-\mathbf{C}\mathbf{x}_{n+1}+\mathbf{F}_{exciting}\right)$$

$$\ddot{\mathbf{x}}_{n+1} \to \boxed{\dot{\mathbf{x}}_{n+1} \to \mathbf{x}_{n+1}}$$

③ 수치적분하여 n+1번째 속도,변위 구함

Term Project#4 Ship motion simulation 프로그램에서는 시간 영역에서의 풀이 방법을 사용하여 매시간 선박의 운동을 계산함

Cf) Strip theory를 사용하여 Added mass, Damping coefficient, Wave excitng force 계산

안의 값은 Library File에서 구할 수 있음

$$\left(A_{jk}^{0} = \int_L a_{jk}dx, B_{jk}^{0} = \int_L b_{jk}dx \right)$$

$$A_{33} = \int_L a_{33}dx - \frac{U}{\omega^2} b_{33}^{A}$$

$$B_{33} = \int_L b_{33}dx + U a_{33}^{A}$$

$$A_{35} = -\int_L x a_{33}dx - \frac{U}{\omega^2} B_{33}^{0} + \frac{U}{\omega^2} x_A b_{33}^{A} - \frac{U^2}{\omega^2} a_{33}^{A}$$

$$B_{35} = -\int_L x b_{33}dx + U A_{33}^{0} - U x_A a_{33}^{A} - \frac{U^2}{\omega^2} b_{33}^{A}$$

$$A_{53} = -\int_L x a_{33}dx + \frac{U}{\omega^2} B_{33}^{0} + \frac{U}{\omega^2} x_A b_{33}^{A}$$

$$B_{53} = -\int_L x b_{33}dx - U A_{33}^{0} - U x_A a_{33}^{A}$$

$$A_{55} = \int_L x^2 a_{33}dx + \frac{U^2}{\omega^2} A_{33}^{0} - \frac{U}{\omega^2} x_A b_{33}^{A} + \frac{U^2}{\omega^2} x_A a_{33}^{A}$$

$$B_{55} = \int_L x^2 b_{33}dx + \frac{U^2}{\omega^2} B_{33}^{0} + U x_A{}^2 a_{33}^{A} + \frac{U^2}{\omega^2} x_A b_{33}^{A}$$

$$F_3 = \rho\alpha \int_L (f_3 + h_3)dx + \rho\alpha \frac{U}{i\omega} h_3^{A}$$

$$F_5 = -\rho\alpha \int_L \left[x(f_3 + h_3) + \rho\alpha \frac{U}{i\omega} h_3 \right] dx - \rho\alpha \frac{U}{i\omega} x_A h_3^{A}$$

U : 선박의 전진 속도	h_j : Sectional Diffraction force (jth mode)
ρ : 유체의 밀도	ω : Encounter wave frequency
α : Wave amplitude	$x_A, a_{jk}^{A}, b_{jk}^{A}$: Values at the aftermost section
f_j : Sectional Froude Krylov force (jth mode)	

Cf) Strip theory를 사용하여 Added mass, Damping coefficient, Wave excitng force 계산

$$\left(A_{jk}^{0}=\int_{L}a_{jk}dx,\ B_{jk}^{0}=\int_{L}b_{jk}dx\right)$$

$$A_{22}=\int_{L}a_{22}dx-\frac{U}{\omega^{2}}b_{22}^{A}$$

$$B_{22}=\int_{L}b_{22}dx+Ua_{22}^{A}$$

$$A_{24}=A_{42}=\int_{L}a_{24}dx-\frac{U}{\omega^{2}}b_{24}^{A}$$

$$B_{24}=B_{42}=\int_{L}b_{24}dx+Ua_{24}^{A}$$

$$A_{26}=\int_{L}xa_{22}dx+\frac{U}{\omega^{2}}B_{22}^{0}-\frac{U}{\omega^{2}}x_{A}b_{22}^{A}+\frac{U^{2}}{\omega^{2}}a_{22}^{A}$$

$$B_{26}=\int_{L}xb_{22}dx-UA_{22}^{0}+Ux_{A}a_{22}^{A}+\frac{U^{2}}{\omega^{2}}b_{22}^{A}$$

$$A_{44}=\int_{L}a_{44}dx-\frac{U}{\omega^{2}}b_{44}^{A}$$

$$B_{44}=\int_{L}b_{44}dx+Ua_{44}^{A}+B_{44}^{*}$$

$$A_{46}=\int_{L}xa_{24}dx+\frac{U}{\omega^{2}}B_{24}^{0}-\frac{U}{\omega^{2}}x_{A}b_{24}^{A}+\frac{U^{2}}{\omega^{2}}a_{24}^{A}$$

$$B_{46}=\int_{L}xb_{24}dx-UA_{24}^{0}+Ux_{A}a_{24}^{A}+\frac{U^{2}}{\omega^{2}}b_{24}^{A}$$

$$A_{62}=\int_{L}xa_{22}dx-\frac{U}{\omega^{2}}B_{22}^{0}-\frac{U}{\omega^{2}}x_{A}b_{22}^{A}$$

$$B_{62}=\int_{L}xb_{22}dx+UA_{22}^{0}+Ux_{A}a_{22}^{A}$$

$$A_{64}=\int_{L}xa_{24}dx-\frac{U}{\omega^{2}}B_{24}^{0}-\frac{U}{\omega^{2}}x_{A}b_{24}^{A}$$

$$B_{64}=\int_{L}xb_{24}dx+UA_{24}^{0}+Ux_{A}a_{24}^{A}$$

$$A_{66}=\int_{L}x^{2}a_{22}dx+\frac{U^{2}}{\omega^{2}}A_{22}^{0}-\frac{U}{\omega^{2}}x_{A}{}^{2}b_{22}^{A}+\frac{U^{2}}{\omega^{2}}x_{A}a_{22}^{A}$$

$$B_{66}=\int_{L}x^{2}b_{22}dx+\frac{U^{2}}{\omega^{2}}B_{22}^{0}+Ux_{A}{}^{2}a_{22}^{A}+\frac{U^{2}}{\omega^{2}}x_{A}b_{22}^{A}$$

U : **선박의 전진 속도**

ρ : **유체의 밀도**

α : Wave amplitude

f_j : Sectional Froude Krylov force (j^{th} mode)

h_j : Sectional Diffraction force (j^{th} mode)

ω : Encounter wave frequency

$x_A, a_{jk}^{A}, b_{jk}^{A}$: Values at the aftermost section

B_4^{*} : Roll Damping

Cf) Strip theory를 사용하여 Added mass, Damping coefficient, Wave excitng force 계산

$$\left(A^0_{jk} = \int_L a_{jk} dx, B^0_{jk} = \int_L b_{jk} dx \right)$$

$$F_2 = \rho\alpha \int_L (f_2 + h_2) dx + \rho\alpha \frac{U}{i\omega} h_2^A$$

$$F_4 = \rho\alpha \int_L (f_4 + h_4) dx + \rho\alpha \frac{U}{i\omega} h_4^A$$

$$F_6 = \rho\alpha \int_L \left[x(f_2 + h_2) + \rho\alpha \frac{U}{i\omega} h_2 \right] dx + \rho\alpha \frac{U}{i\omega} x_A h_2^A$$

U : 선박의 전진 속도	h_j : Sectional Diffraction force (j[th] mode)
ρ : 유체의 밀도	ω : Encounter wave frequency
α : Wave amplitude	x_A, a^A_{jk}, b^A_{jk} : Values at the aftermost section
f_j : Sectional Froude Krylov force (j[th] mode)	B_4^* : Roll Damping

1) Dennis G.Zill, Michael R. Cullen, "Advanced Engineering Mathematics" 3rd Edition, Jones and Bartlett publishers,2006, pp 281~284
2) Erwin Kreyszig, "Advanced Engineering Mathematics" 9th edition, Wiley, 2006,892~895

[cf] Fourth-Order Runge-Kutta Method

The most commonly used set of values for the parameters

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_{n,} y_n)$$

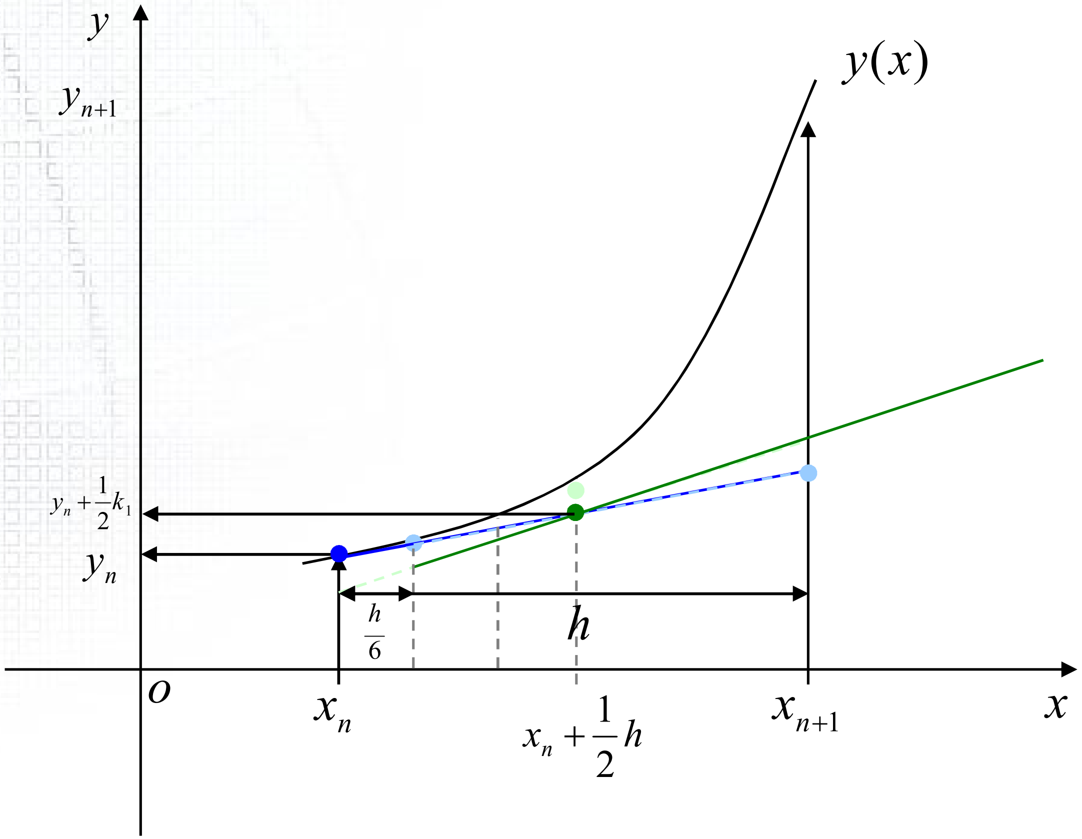
$$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$$

$$k_4 = f(x_n + h, y_n + \beta_6 hk_3)$$

[cf] Fourth-Order Runge-Kutta Method

x_n과 $x_{n+1} = x_n + h$ 사이를 6등분 하여 Improved Euler method와 비슷한 방법으로 $y(x)$를 구하는 방법.



① $x_n, y_n, f(x_n, y_n)$

② $h \cdot f(x_n, y_n) = k_1$

③ $\frac{1}{6}k_1$, $y_{n+1} = y_n + \frac{1}{6}k_1$

④ $h \cdot f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) = k_2$

⑤ $\frac{2}{6}k_2 = \frac{1}{3}k_2$, $y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2$

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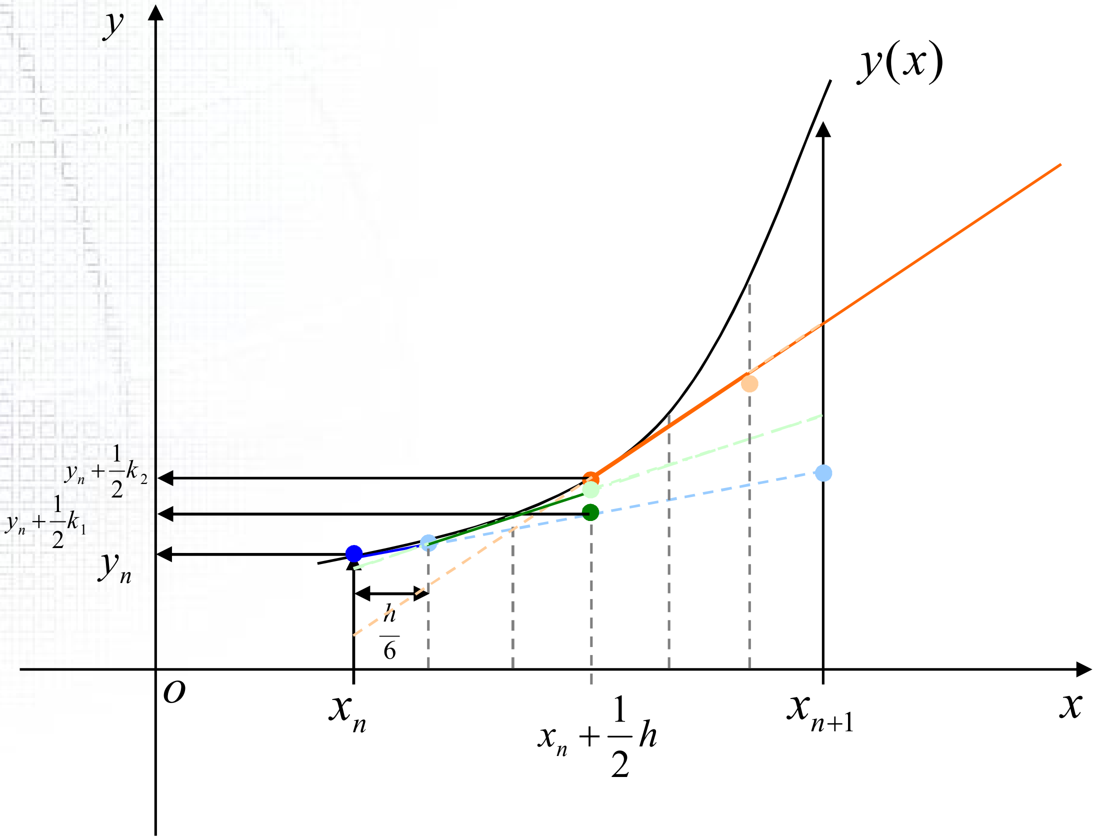

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⑤ $\frac{2}{6}k_2 = \frac{1}{3}k_2, \; y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2$

⑥ $y_n + \frac{1}{2}k_2$

⑦ $h \cdot f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2) = k_3$

⑧ $\frac{2}{6}k_3 = \frac{1}{3}k_3, y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3$

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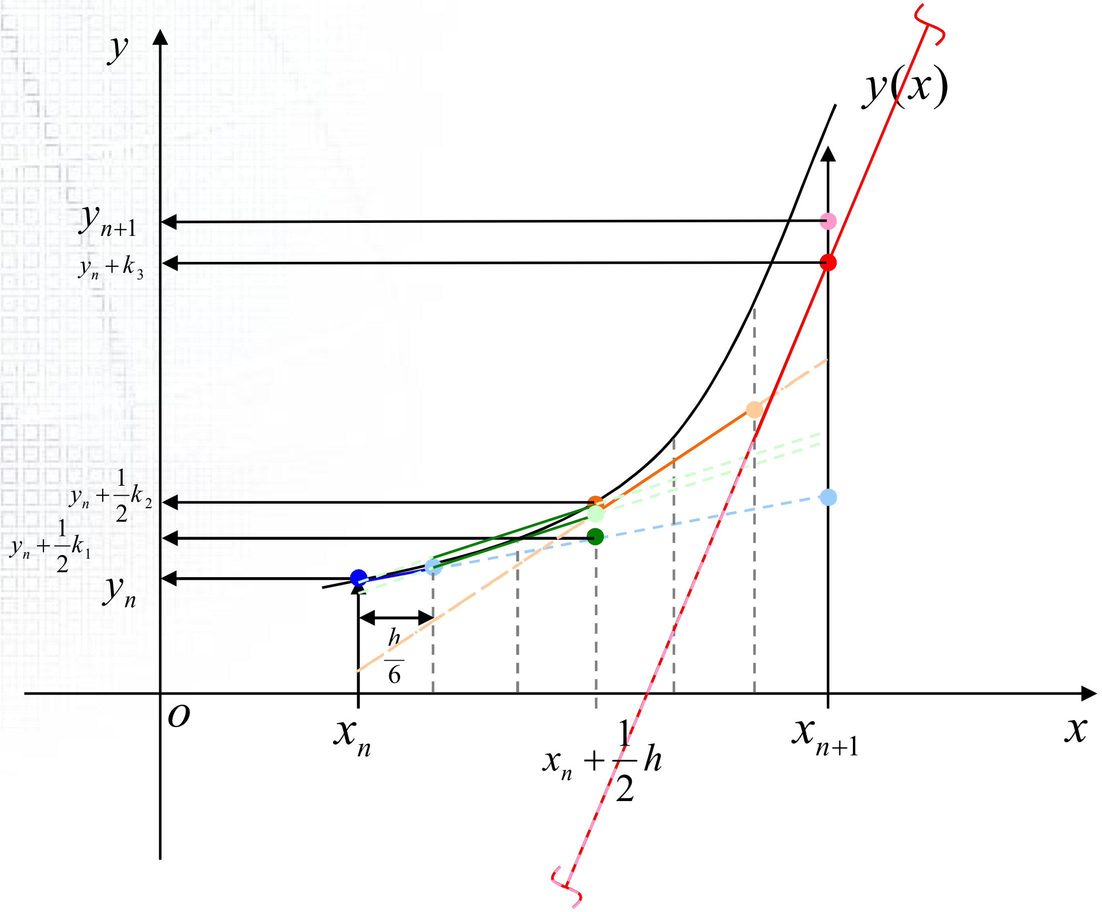
[cf] Fourth-Order Runge-Kutta Method

x_n과 $x_{n+1}=x_n+h$ 사이를 6등분 하여 Improved Euler method와 비슷한 방법으로 $y(x)$를 구하는 방법.



① $x_n, y_n, f(x_n, y_n)$

② $h\cdot f(x_n, y_n)=k_1$

③ $\frac{1}{6}k_1$, $y_{n+1}=y_n+\frac{1}{6}k_1$

④ $h\cdot f(x_n+\frac{1}{2}h, y_n+\frac{1}{2}k_1)=k_2$

⑤ $\frac{2}{6}k_2=\frac{1}{3}k_2$, $y_{n+1}=y_n+\frac{1}{6}k_1+\frac{1}{3}k_2$

⑥ $y_n+\frac{1}{2}k_2$

⑦ $h\cdot f(x_n+\frac{1}{2}h, y_n+\frac{1}{2}k_2)=k_3$

⑧ $\frac{2}{6}k_3=\frac{1}{3}k_3$, $y_{n+1}=y_n+\frac{1}{6}k_1+\frac{1}{3}k_2+\frac{1}{3}k_3$

⑨ y_n+k_3

⑩ $h\cdot f(x_n+h, y_n+k_3)=k_4$

⑪ $\frac{1}{6}k_4$

⑫ $y_{n+1}=y_n+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

[참고] 선박의 Heave 운동과 Spring-mass-damping system의 비교

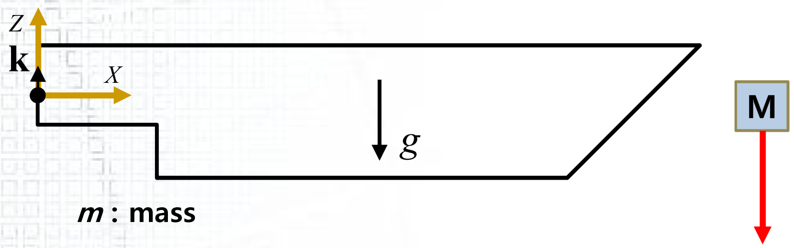
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

↓ **Linearization**

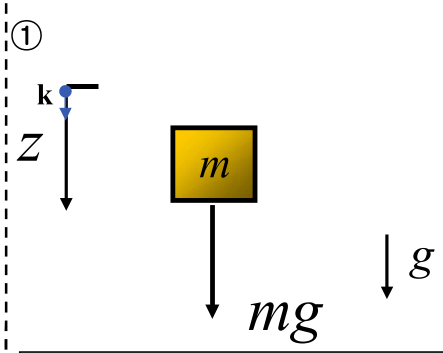
Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 1



$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity}$$
$$= -mg\mathbf{k}$$

✓ Mass-Spring-Damper system



By Newton's 2nd law,

$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k}$$

Nonlinearity

Nonlinearity of the nature ⇒ Nonlinear Mathematical Model ⇒ Numerical Method

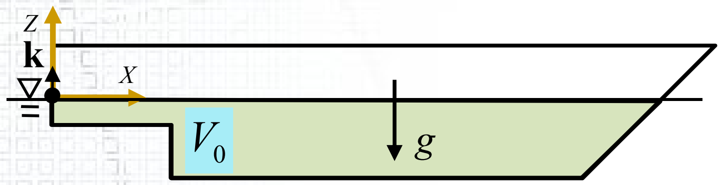
⇓ **Linearization**

Linear Mathematical Model ⇒ Analytic Solution

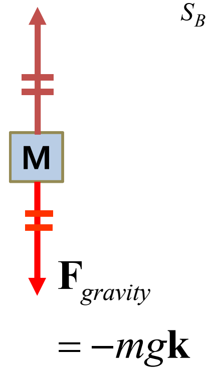
Ex) Heave Motion of a Ship – step 2

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k}$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

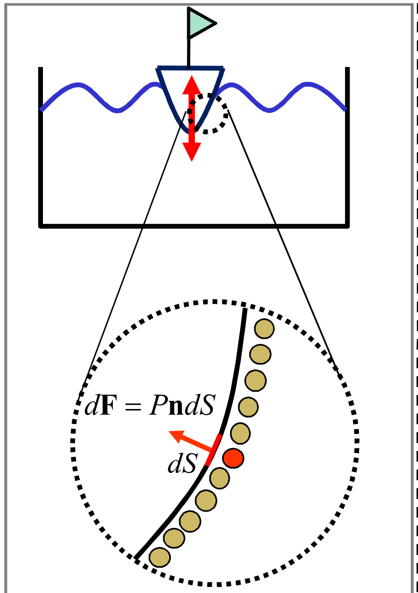


m : mass
V_0 : submerged volume
S_B : submerged surface area
ρ : density of sea water



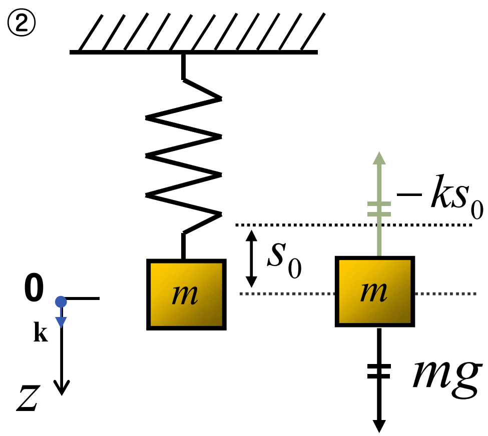
$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k}$$
$$= 0 \quad (\because \ddot{z} = 0)$$

: static equilibrium



dS : infinitesimal submerged surface area
$d\mathbf{F}$: force exerted by the infinitesimal fluid element on dS
$\mathbf{n}$: normal vector of dS

✓ Mass-Spring-Damper system



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k}$$
$$= 0 \quad (\because z'' = 0)$$

: static equilibrium

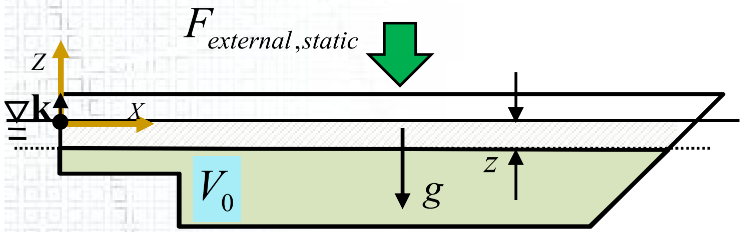
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

↓ **Linearization**

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 3



m : mass *ρ* : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0 \mathbf{k} + \mathbf{F}_{addtional\,bouyancy}$$

additional buoyancy caused by additional displacement z

M

$\mathbf{F}_{external,static}$

$\mathbf{F}_{gravity}$

$= -mg\mathbf{k}$

if, z is small

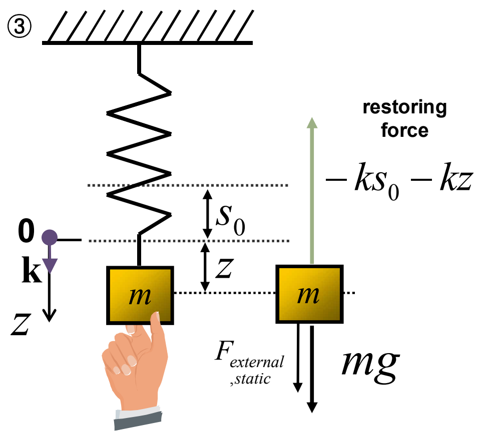
$$\mathbf{F}_{addtional\,bouyancy}$$
$$= -\rho g A_{WP}\mathbf{z}$$
$$= -k\mathbf{z}$$
$$, k = \rho g A_{WP}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{external,static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -\rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -k\mathbf{z} + \mathbf{F}_{external,static}$$
$$= 0 \ (\because \ddot{z} = 0)$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system

③



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= 0 \quad (\because z'' = 0)$$

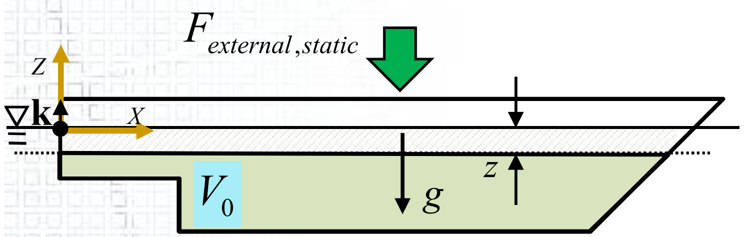
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

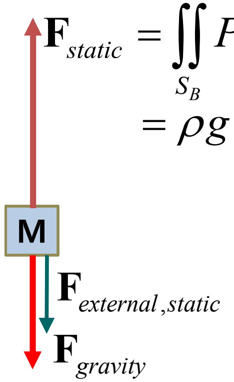
↓ **Linearization**

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 4



m : mass ρ : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k} + \mathbf{F}_{addtional\,bouyancy}$$

additional buoyancy caused by additional displacement z

$= -mg\mathbf{k}$

if, z is small

$$\mathbf{F}_{addtional\,bouyancy} = -\rho g A_{WP}\mathbf{z} = -k\mathbf{z} \quad , k = \rho g A_{WP}$$

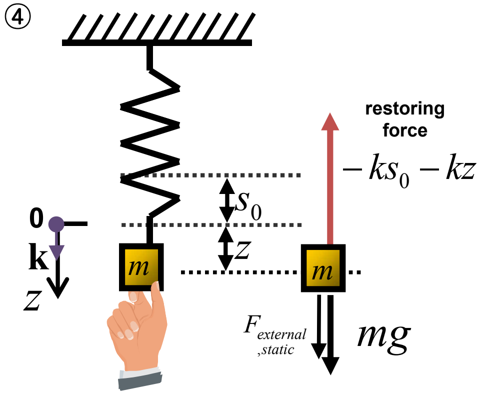
$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{external,static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -\rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -k\mathbf{z} + \mathbf{F}_{external,static} \quad , k = \rho g A_{WP}$$
$$= 0 \quad (\because \ddot{z} = 0)$$

Linearized Restoring Force

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system

④



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$

$m\mathbf{z}'' + k\mathbf{z} = 0$ Oscillation by the restoring force

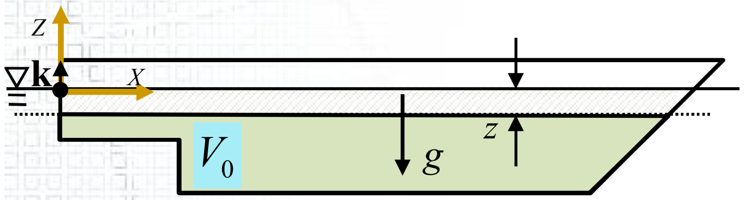
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

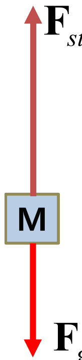
Nonlinear Mathematical Model ↓ **Linearization**

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 4



m : mass
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area
ρ : density of sea water



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0\mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0\mathbf{k} - k\mathbf{z}$$

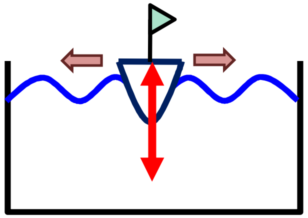
$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F} = \mathbf{F}_{gravity} + \mathbf{F}_{static} = -mg\mathbf{k} + \rho g V_0\mathbf{k} - \rho g A_{wp}\mathbf{z} = -\rho g A_{wp}\mathbf{z} = -k\mathbf{z}$$

Ship will oscillate forever?

Energy is dissipated by radiation wave →

정수 중 선박의 강제
운동에 의해 발생한 힘



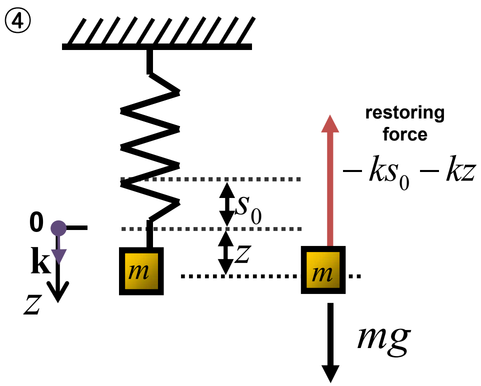
Radiation Force

$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0\mathbf{k}$

✓ Mass-Spring-Damper system

④



$$mz'' = \mathbf{F} = mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} = -kz\mathbf{k}$$

$$\boxed{mz'' + k\mathbf{z} = 0}$$ Oscillation by the restoring force

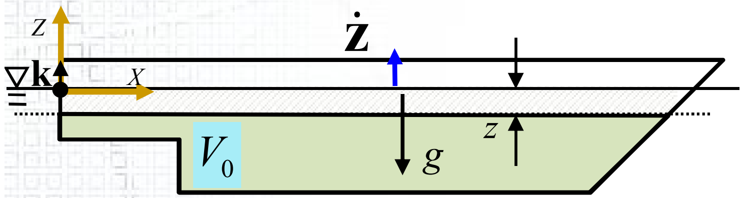
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

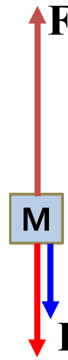
Linearization

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 5



m : mass
ρ : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0 \mathbf{k} - k\mathbf{z}$$

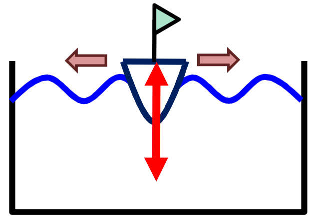
$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}}$$
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}}$$
$$= -k\mathbf{z} - c\dot{\mathbf{z}}$$

opposite to velocity

정수 중 선박의 강제
운동에 의해 발생한 힘



Radiation Force

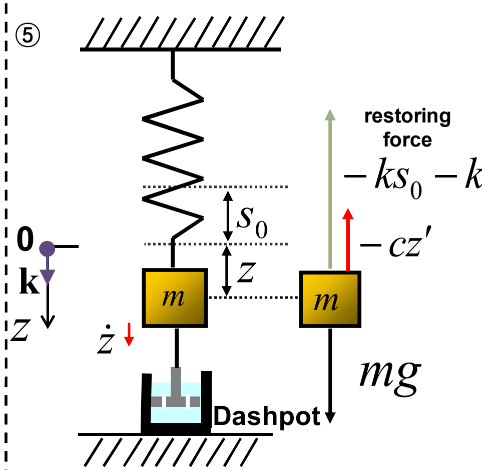
$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS = -c\dot{\mathbf{z}}$$

c : damping coefficient

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system

⑤



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k}$$
$$= -kz\mathbf{k} - cz'\mathbf{k}$$

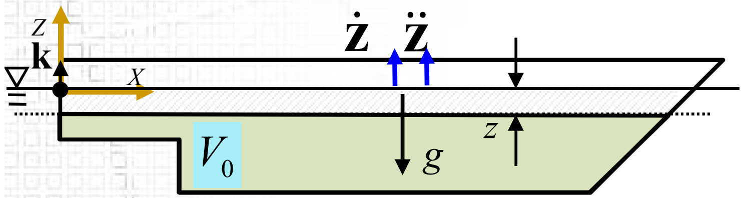
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

Linearization

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 5



m : mass

ρ : density of sea water

V_0: submerged volume

S_B : submerged surface area

Awp : waterplane area

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z}$$
$$= \rho g V_0 \mathbf{k} - k\mathbf{z}$$

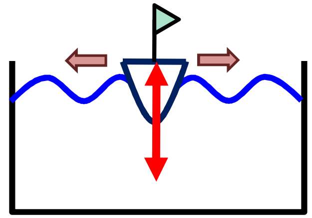
M

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} \ -\rho g A_{wp}\mathbf{z} \ -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$
$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

정수 중 선박의 강제 운동에 의해 발생한 힘



Radiation Force

$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS$$
$$= -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

opposite to velocity

opposite to acceleration

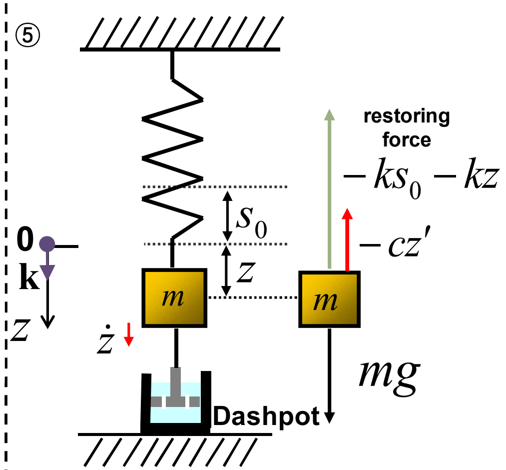
c : damping coefficient

m_a : added mass

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system

⑤



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} \ -cz'\mathbf{k}$$
$$= -kz\mathbf{k} - cz'\mathbf{k}$$

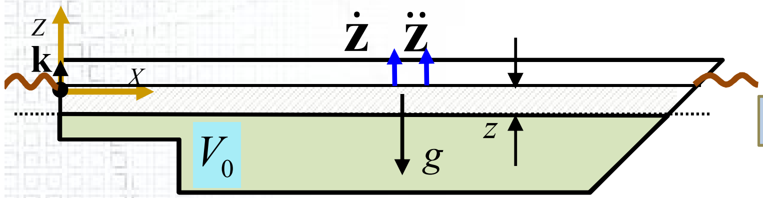
Nonlinearity

Nonlinearity of the nature → **Nonlinear Mathematical Model** → **Numerical Method**

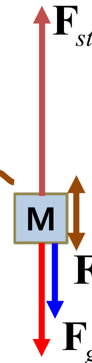
Linearization ↓

Linear Mathematical Model → **Analytic Solution**

Ex) Heave Motion of a Ship – step 6



m : mass ρ : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0\mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0\mathbf{k} - k\mathbf{z}$$

$$\mathbf{F}_{exciting}$$

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

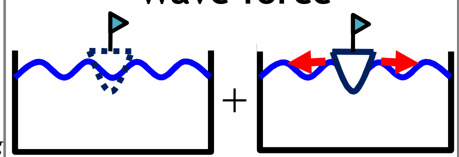
$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0\mathbf{k}$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} + \mathbf{F}_{exciting}$$
$$= -mg\mathbf{k} + \rho g V_0\mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

c : damping coefficient
m_a : added mass

Wave force

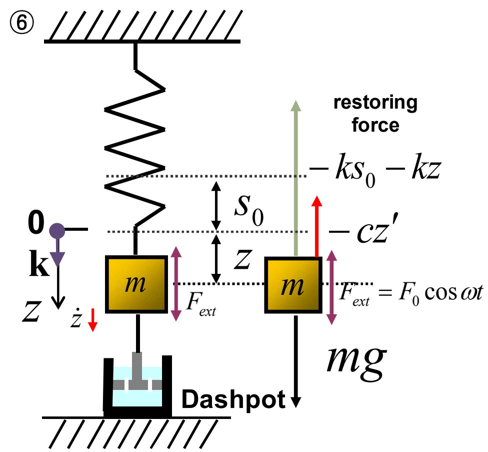


Froude-Kriloff Force Diffraction Force

$$\mathbf{F}_{wave\ exciting} = \iint_{S_B} P_{wave\ exciting}\mathbf{n}dS \quad (= \mathbf{F}_{exciting})$$

✓ Mass-Spring-Damper system

⑥



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0\cos\omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0\cos\omega t$$

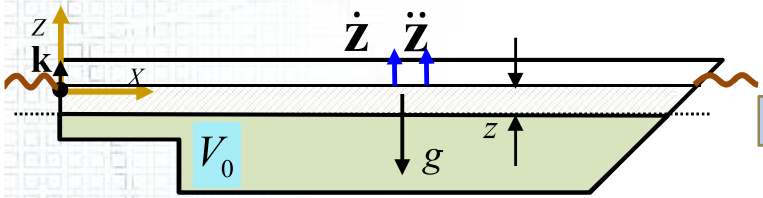
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

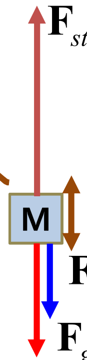
↓ **Linearization**

Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 6



m: mass
ρ: density of sea water
V_0: submerged volume
S_B: submerged surface area
Awp: waterplane area



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0\mathbf{k} - \rho g A_{WP}\mathbf{z}$$
$$= \rho g V_0\mathbf{k} - k\mathbf{z}$$

$\mathbf{F}_{exciting}$

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} + \mathbf{F}_{exciting}$$
$$= -mg\mathbf{k} + \rho g V_0\mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

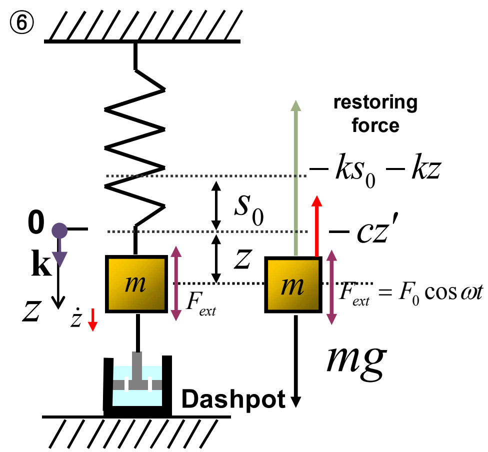
$$\boxed{(m + m_a)\ddot{\mathbf{z}} + c\dot{\mathbf{z}} + k\mathbf{z} = \mathbf{F}_{exciting}}$$

c: damping coefficient
m_a: added mass

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0\mathbf{k}$

✓ **Mass-Spring-Damper system**

⑥



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0\cos\omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0\cos\omega t$$

$$\boxed{m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0\cos\omega t}$$

Linked slide
: Homogeneous / Particular solution & Zero Input / Zero State

Comparison : example

$y(t) = y_h(t) + y_p(t)$

example*

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t \quad y(0) = 1, \dot{y}(0) = 5$$

solution

1) Homogeneous Solution

Homogeneous

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0$$

Try : $y_h(t) = e^{mt}$

$$(\ddot{m} + 3\dot{m} + 2)e^{mt} = 0$$

$$(m+1)(m+2) = 0$$

$$\therefore m = -1, m = -2$$

$$y_h(t) = e^{mt}$$

e^{-t} and e^{-2t} : linearly independent

$$\therefore y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$$

! Notation changed in the example

$z(t)$	➡	$y(t)$
$z'(t), z''(t)$	➡	$y'(t), y''(t)$
$z_{\text{transient}}, z_{\text{steady}}$	➡	$y_h(t), y_p(t)$
$z_{\text{zero-input}}, z_{\text{zero-state}}$	➡	$y_0(t), y_1(t)$

*Zill & Cullen, Advanced Engineering Mathematics, 3rd Edition p204, example 5 modified

$y(t) = y_h(t) + y_p(t)$

example

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t \quad y(0) = 1, \dot{y}(0) = 5$$

Solution

2) Particular Solution

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t$$

Try : $y_p(t) = A\sin 2t + B\cos 2t$

$$\ddot{y}_p = -4A\sin 2t - 4B\cos 2t$$

$$\dot{y}_p = 2A\cos 2t - 2B\sin 2t$$

L.H.S.: $-4(A\sin 2t + B\cos 2t) + 6(A\cos 2t - B\sin 2t) + 2(A\sin 2t + B\cos 2t) = (-2A - 6B)\sin 2t + (6A - 2B)\cos 2t$

R.H.S.: $\sin 2t$

L.H.S. = R.H.S.

$$\begin{aligned} -2A - 6B &= 1 \\ 6A - 2B &= 0 \end{aligned} \Rightarrow A = -\frac{1}{20}, B = -\frac{3}{20}$$

$$\therefore y_p = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : example

$y(t) = y_h(t) + y_p(t)$

example

$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t \quad y(0) = 1, \dot{y}(0) = 5$

solution

3) General Solution

$$y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$$

$$y_p = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Initial condition : $y(0) = 1, \dot{y}(0) = 5$

$$y(0): c_1 + c_2 - \frac{3}{20} = 1$$

$$\dot{y}(0): -c_1 - 2c_2 - \frac{2}{20} = 5$$

$$\Rightarrow \quad c_1 = \frac{37}{5}, \quad c_2 = -\frac{25}{4}$$

$$y_h(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t}, \quad y_p(t) = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$

$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$

$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0 \qquad \ddot{y}(t)-3\dot{y}(t)+2y(t)=0$$

$$y_{0\_h}(t)=e^{mt} \qquad y_{1\_h}(t)=e^{mt}$$

$$(m+1)(m+2)e^{mt}=0 \qquad (m+1)(m+2)e^{mt}=0$$

$$y_{0\_h}(t)=c_1e^{-t}+c_2e^{-2t} \qquad y_{1\_h}(t)=c_1e^{-t}+c_2e^{-2t}$$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$

$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$

$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0 \qquad \ddot{y}(t)-3\dot{y}(t)+2y(t)=0$$

$$y_{0\_h}(t)=c_1e^{-t}+c_2e^{-2t} \qquad y_{1\_h}(t)=c_1e^{-t}+c_2e^{-2t}$$

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2) Particular Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0 \qquad \ddot{y}(t)-3\dot{y}(t)+2y(t)=\sin 2t$$

$$y_{1\_p}=A\sin 2t+B\cos 2t$$

L.H.S.: $(-2A-6B)\sin 2t+(6A-2B)\cos 2t$

R.H.S.: $\sin 2t$

L.H.S. = R.H.S. $\quad A=-\frac{1}{20}, B=-\frac{3}{20}$

$$y_{0\_p}(t)=0 \qquad y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Comparison : example

예제

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

	Zero Input	Zero State
	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$ $y(0)=1, \dot{y}(0)=5$	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$ $y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$
1) Homogeneous Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$ $y_{0\_h}(t)=c_1e^{-t}+c_2e^{-2t}$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=0$ $y_{1\_h}(t)=c_1e^{-t}+c_2e^{-2t}$
2) Particular Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$ $y_{0\_p}(t)=0$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=\sin 2t$ $y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
3) General Solution	$y_{0\_g}(t)=c_1e^{-t}+c_2e^{-2t}$	$y_{1\_g}(t)=c_1e^{-t}+c_2e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
Initial condition	$y(0)=1, \dot{y}(0)=5$	$y(0)=0\ , \dot{y}(0)=0$
	$c_1+c_2=1$ $-c_1-2c_2=5$ ➡ $c_1=7$ $c_2=-6$	$y(0): c_1+c_2-\frac{3}{20}=1$ $\dot{y}(0): -c_1-2c_2-\frac{2}{20}=5$ ➡ $c_1=\frac{2}{5}$ $c_2=-\frac{1}{4}$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

	Zero Input	Zero State
	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$ $y(0)=1, \dot{y}(0)=5$	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$ $y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$
1) Homogeneous Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$ $y_{0\_h}(t)=c_1e^{-t}+c_2e^{-2t}$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=0$ $y_{1\_h}(t)=c_1e^{-t}+c_2e^{-2t}$
2) Particular Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$ $y_{0\_p}(t)=0$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=\sin 2t$ $y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
3) General Solution	$y_{0\_g}(t)=c_1e^{-t}+c_2e^{-2t}$	$y_{1\_g}(t)=c_1e^{-t}+c_2e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
Initial condition	$y(0)=1, \dot{y}(0)=5$	$y(0)=0\ , \dot{y}(0)=0$
General Solution	$y_{0\_g}(t)=7e^{-t}-6e^{-2t}$	$y_{1\_g}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$

$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$

$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

$$y_{0\_h}(t)=7e^{-t}-6e^{-2t}$$

$$y_{1\_h}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}$$

2) Particular Solution

$$y_{0\_p}(t)=0$$

$$y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

3)General Solution

$$y_{0\_g}(t)=y_0(t)=7e^{-t}-6e^{-2t}$$

$$y_{1\_g}(t)=y_1(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=(\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t})-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t)$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$

$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$

$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

$$y_{0\_h}(t)=7e^{-t}-6e^{-2t} \qquad y_{1\_h}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}$$

2) Particular Solution

$$y_{0\_p}(t)=(0) \qquad y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

3)General Solution

$$y_{0\_g}(t)=y_0(t)=(7e^{-t}-6e^{-2t}) \qquad y_{1\_g}(t)=y_1(t)=(\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t})-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t)$$

$$y(t) \;=\; \qquad + \qquad$$

$$\| \qquad \| \qquad \|$$

$$y_0(t) \;=\; \qquad + \qquad$$

$$+ \qquad + \qquad +$$

$$y_1(t) \;=\; \qquad + \qquad$$

Comparison : example-proof

$y(t)=y_0(t)+y_1(t)$

$$m\ddot{y}(t)+c\dot{y}(t)+ky(t)=u(t) \quad ,u(t)\neq 0 \quad \cdots(1)$$

$$y(0)=a,\dot{y}(0)=b \qquad ,a\neq 0,b\neq 0 \quad \cdots(2)$$

Zero Input solution : $y_0(t)$

$\boxed{m\ddot{y}_0(t)+c\dot{y}_0(t)+ky_0(t)=0}$

$\boxed{y_0(0)=a},\boxed{\dot{y}_0(0)=b}$

Zero state solution : $y_1(t)$

$\boxed{m\ddot{y}_1(t)+c\dot{y}_1(t)+ky_1(t)=u(t)}$

$\boxed{y_1(0)=0},\boxed{\dot{y}_1(0)=0}$

assum. : $y(t)=y_0(t)+y_1(t)$

$y_t \rightarrow$ (1) :

L.H.S.:

$$m\ddot{y}(t)+c\dot{y}(t)+ky(t)$$
$$=m(\ddot{y}_0(t)+\ddot{y}_1(t))+c(\dot{y}_0(t)+\dot{y}_1(t))+k(y_0(t)+y_1(t))$$
$$=\boxed{[m\ddot{y}_0(t)+c\dot{y}_0(t)+ky_0(t)]}+\boxed{[m\ddot{y}_1(t)+c\dot{y}_1(t)+ky_1(t)]}$$
$$=\boxed{0}+\boxed{u(t)}$$

R.H.S.: $u(t)$

∴ L.H.S.=R.H.S "(1) satisfied"

$\therefore y(t)=y_0(t)+y_1(t)$

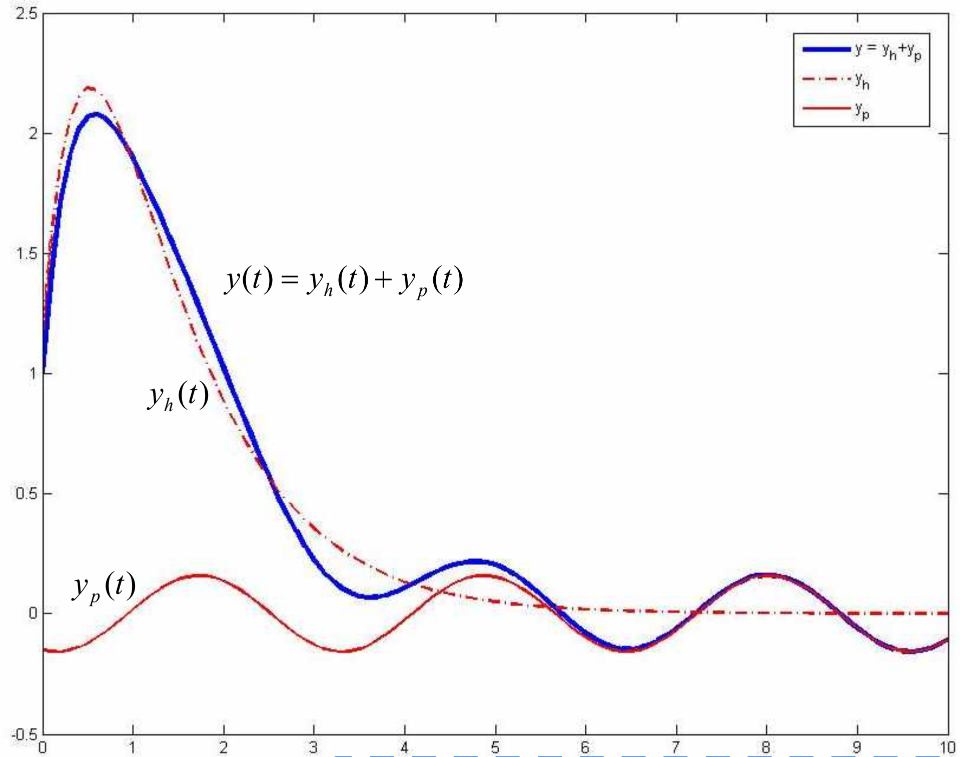
$y_t \rightarrow$ (2) :

$$y(0)=\boxed{y_0(0)}+\boxed{y_1(0)}=\boxed{a}+\boxed{0}$$
$$\dot{y}(0)=\boxed{\dot{y}_0(0)}+\boxed{\dot{y}_1(0)}=\boxed{b}+\boxed{0}$$

$$\therefore y(0)=a$$
$$\dot{y}(0)=b$$

"(2) satisfied"

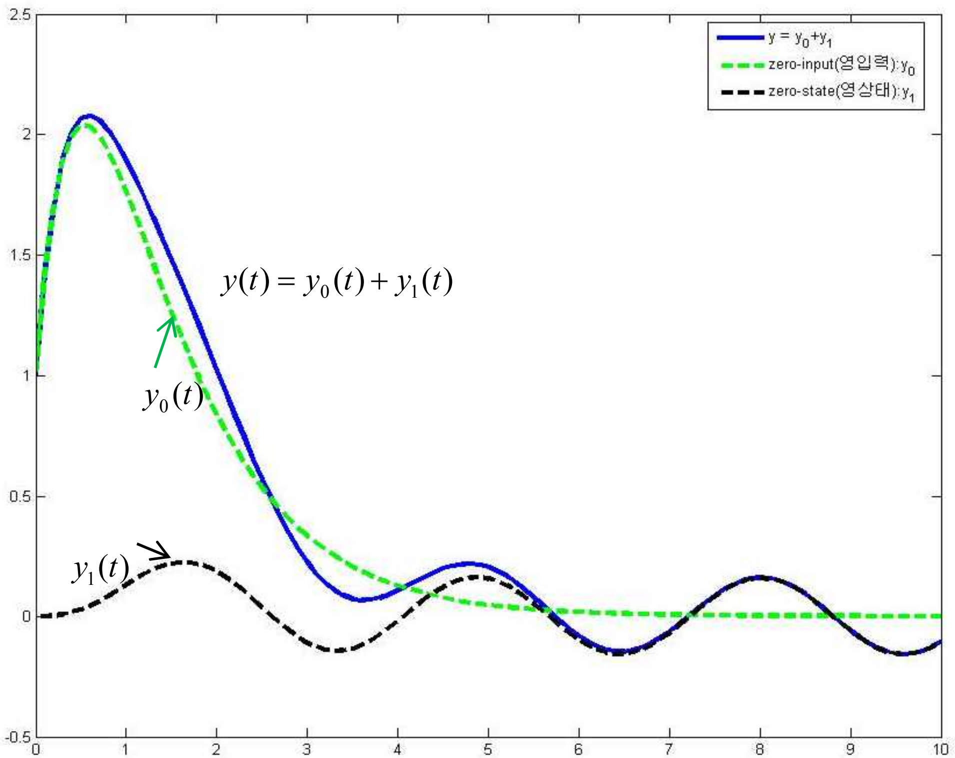
Comparison : graph



$$\boxed{y(t) = y_h(t) + y_p(t)}$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

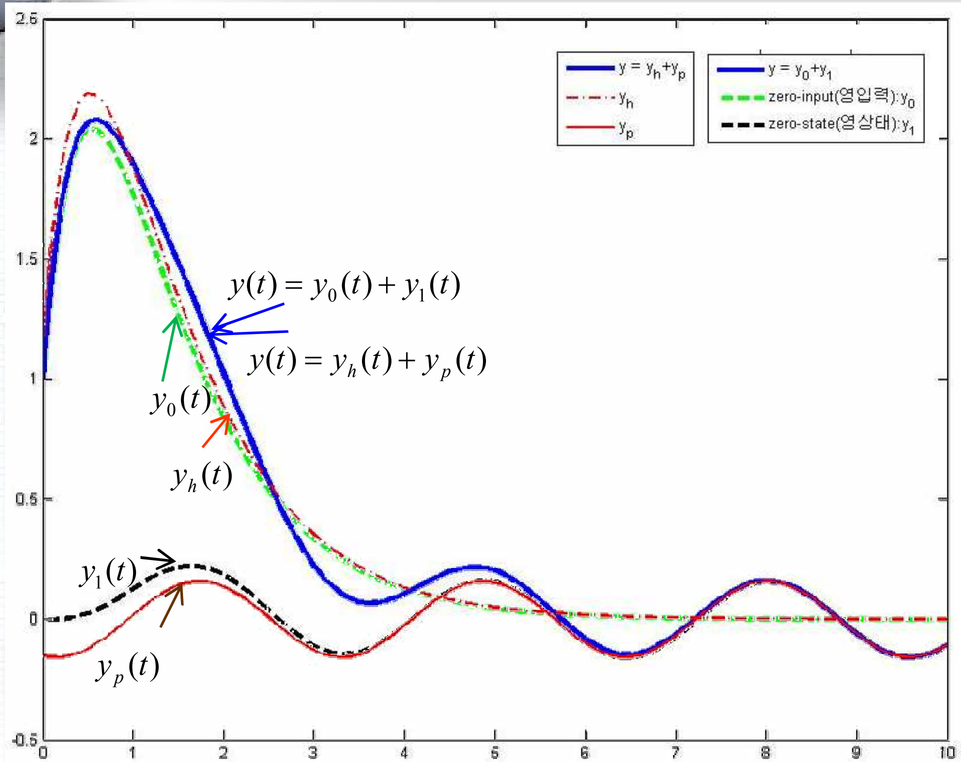
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

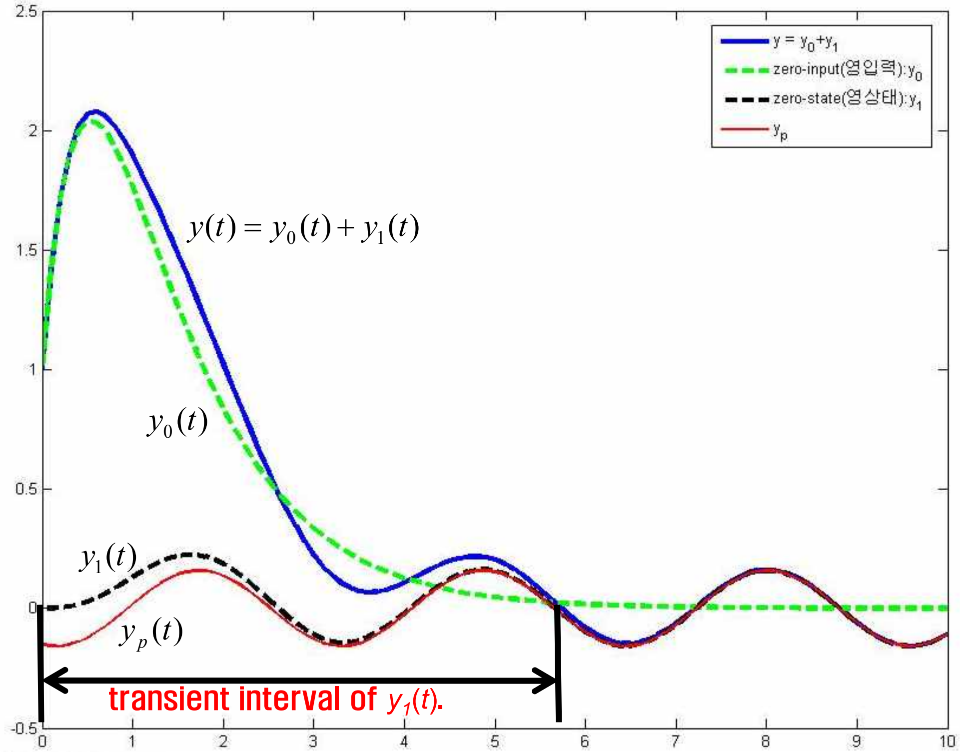
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\| \qquad \| \qquad \|$$

Zero Input solution
$u(t)=0$

$$y_0(t) = y_{0\_h}(t) + y_{0\_p}(t)$$

$$+ \qquad + \qquad +$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t) = \boxed{y_{1\_h}(t)} + y_{1\_p}(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \boxed{\frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t}} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$