Convex Optimization

A supplementary note to Chapter 4 of *Convex Optimization* by S. Boyd and L. Vandenberghe

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Optimization problem:

$$\begin{array}{lll} \min & f_0(x) & \\ \text{s.t.} & f_i(x) \leq 0, & i = 1, \cdots, m \\ & h_j(x) = 0, & j = 1, \cdots, p. \end{array}$$

- Decision variables: $x \in \mathbb{R}^n$
- Objective function: $f_0(x)$
- Constraints (inequality and equality): $f_i(x) \leq 0$ and $h_j(x) = 0$.
- Domain of problem: $\mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{j=1}^{p} \operatorname{dom} h_j$

  $x \in \mathcal{D}$ is feasible if it satisfies all constraints; infeasible, otherwise.

- Optimal value: $p^* = \inf\{f_0(x) | f_i(x) \le 0, \forall i, h_j(x) = 0, \forall j\}$.
- Optimal point: x^* is optimal point if x^* is feasible and $f_0(x^*) = p^*$. Denote by X_{opt} the set of optimal points.
- A feasible x is an ϵ-*suboptimal point* if $f_0(x) \le p^* + \epsilon$.
- A feasible x is locally optimal if $\exists R > 0$ s.t. $f_0(x) = \inf\{f_0(z) | f_i(x) \le 0, \forall i, h_j(x) = 0, \forall j, ||z - x||_2 \le R\}$
- Feasibility problem: "Find x satisfying all the constraints."

Expressing problems in standard form

- Standard form: Min-version

$$
\begin{array}{lll}
\min & f_0(x) & \\
\text{s.t.} & f_i(x) \leq 0, & i = 1, \cdots, m \\
 & h_j(x) = 0, & j = 1, \cdots, p.
\end{array}
$$

- Standard form: Max-version

$$
\begin{array}{lll}
\max & f_0(x) & \\
\text{s.t.} & f_i(x) \geq 0, & i = 1, \cdots, m \\
 & h_j(x) = 0, & j = 1, \cdots, p.
\end{array}
$$

Equivalent problems

We say two problems are *equivalent* if, given a solution of one, we can efficiently find a solution of the other, and vice versa.

Example

Two problems are equivalent if $\alpha_i > 0, \forall i, \beta_j \neq 0, \forall j$:

$$\begin{array}{lll|lll} \min & f_0(x) & & \min & \alpha_0 f_0(x) & \\ \text{s.t.} & f_i(x) \leq 0, & \forall i & \text{s.t.} & \alpha_i f_i(x) \leq 0, & \forall i \\ & h_j(x) = 0, & \forall j & & \beta_j h_j(x) = 0, & \forall j \end{array}$$

Change of variables

Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one mapping. Define $\tilde{f}_i(x) = f_i(\phi(x))$ and $\tilde{h}_j(x) = h_j(\phi(x))$ Then the following two problems are equivalent:

$$\begin{array}{ll} \min & f_0(x) \\ s.t. & f_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & h_i(x) = 0, \;\; i = 1, \ldots, p \end{array}$$

$$\begin{array}{ll} \min & \tilde{f}_0(x) \\ s.t. & \tilde{f}_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & \tilde{h}_i(x) = 0, \;\; i = 1, \ldots, p. \end{array}$$

Transformation of obj. and const. functions

Suppose that $\psi_0 : \mathbb{R} \to \mathbb{R}$ is monotone increasing, $\psi_1, \dots, \psi_m : \mathbb{R} \to \mathbb{R}$ satisfy $\psi_i(u) \leq 0$ if and only if $u \leq 0$, and $\psi_{m+1}, \dots, \psi_{m+p} : \mathbb{R} \to \mathbb{R}$ satisfy $\psi_i(u) = 0$ if and only if $u = 0$. Define

$$\tilde{f}_i(x) = \psi_i(f_i(x)), \ \ i = 0, \dots, m, \ \text{and}$$

$$\tilde{h}_i(x) = \psi_{m+i}(h_i(x)), \ \ i = 1, \dots, p.$$

Then following two are equivalent:

$$\begin{array}{ll} \min & f_0(x) \\ s.t. & f_i(x) \leq 0, \ \ i = 1, \dots, m \\ & h_i(x) = 0, \ \ i = 1, \dots, p, \end{array}$$

$$\begin{array}{ll} \min & \tilde{f}_0(x) \\ s.t. & \tilde{f}_i(x) \leq 0, \ \ i = 1, \dots, m \\ & \tilde{h}_i(x) = 0, \ \ i = 1, \dots, p. \end{array}$$

Least-norm and least-norm-squared

$$\min \ \|Ax - b\|_2 \text{ versus } \min \ \|Ax - b\|_2^2.$$

Slack variables

- $f_i(x) \leq 0$ if and only if $\exists\ s_i \geq 0$ such that $f_i(x) + s_i = 0$
- s_i is called a slack variable.
- Then following two are equivalent:

$$\begin{array}{ll} \min & f_0(x) \\ s.t. & f_i(x) \leq 0,\ \ i = 1, \ldots, m \\ & h_i(x) = 0,\ \ i = 1, \ldots, p \end{array}$$

$$\begin{array}{ll} \min & f_0(x) \\ s.t. & f_i(x) + s_i = 0,\ \ i = 1, \ldots, m \\ & h_i(x) = 0,\ \ i = 1, \ldots, p \\ & s_i \geq 0,\ \ i = 1, \ldots, m. \end{array}$$

Introducing equality constraints

- The following two problems are equivalent:

$$\begin{array}{lll} \min & f_0(A_0x+b_0) & \\ \text{s.t.} & f_i(A_ix+b_i)\leq 0, & \forall i \\ & h_j(x)=0, & \forall j \end{array} \qquad \begin{array}{lll} \min & f_0(y_0) & \\ \text{s.t.} & f_i(y_i)\leq 0, & \forall i \\ & y_i = Ax_i+b_i & \forall i \\ & h_j(x)=0 & \forall j \end{array}$$

Optimizing over some variables

Since $\inf_{x,y} f(x,y) = \inf_x \tilde{f}(x)$, where $\tilde{f}(x) = \inf_y f(x,y)$, following two are equivalent:

$$\begin{array}{lll} \min & f_0(x_1, x_2) & \\ \text{s.t.} & f_i(x_1) \leq 0, & i = 1, \cdots, m_1 \\ & g_j(x_2) \leq 0, & j = 1, \cdots, m_2 \end{array} \quad \Leftrightarrow \quad \begin{array}{lll} \min & \tilde{f}_0(x_1) & \\ \text{s.t.} & f_i(x_1) \leq 0, & i = 1, \cdots, m_1 \end{array}$$

where $\tilde{f}_0(x_1) = \inf\{f_0(x_1, z) | g_j(z) \leq 0, j = 1, \cdots, m_2\}$.

Example

Consider a strictly convex quadratic program constrained on some variables: $\min x_1^T P_{11} x_1 + 2x_1^T P_{12} x_2 + x_2^T P_{22} x_2$ s.t. $f_i(x_1) \leq 0,\ i = 1, \ldots, m$. Since, $\inf_{x_2} x_1^T P_{11} x_1 + 2x_1^T P_{12} x_2 + x_2^T P_{22} x_2 = x_1^T (P_{11} - P_{12} P_{22}^{-1} P_{12}^T) x_1$, we can obtain equivalent problem:

$$\min\ x_1^T (P_{11} - P_{12} P_{22}^{-1} P_{12}^T) x_1, \text{s.t. } f_i(x_1) \leq 0,\ i = 1, \ldots, m.$$

Epigraph problem form

- The following two problems are equivalent:

$$\begin{array}{lll} \min & f_0(x) & \\ \text{s.t.} & f_i(x) \leq 0, & i = 1, \cdots, m \\ & h_j(x) = 0, & j = 1, \cdots, p \end{array}$$

$$\begin{array}{lll} \min & t & \\ \text{s.t.} & f_0(x) - t \leq 0, & i = 1, \cdots, m_1 \\ & f_i(x) \leq 0, & i = 1, \cdots, m \\ & h_j(x) = 0, & j = 1, \cdots, p \end{array}$$

Convex optimization problems in standard form

- Convex minimization

$$\begin{array}{llll} \min & f_0(x) & & f_0 \text{ is convex.} \\ \text{s.t.} & f_i(x) \leq 0, & i = 1, \dots, m, & f_i \text{ are convex.} \\ & a_j^T x = b_j, & j = 1, \dots, p. & \end{array}$$

  If f_0 is quasiconvex instead of convex, then problem is a quasiconvex minimization problem.

- Concave maximization

$$\begin{array}{llll} \max & f_0(x) & & f_0 \text{ is concave.} \\ \text{s.t.} & f_i(x) \geq 0, & i = 1, \dots, m, & f_i \text{ are concave.} \\ & a_j^T x = b_j, & j = 1, \dots, p. & \end{array}$$

Local and global optima

Theorem

Any local optimum of convex optimization problems is also a global optimum.

Proof Let x be a local optimum: $\exists\ R > 0$ s.t. $f_0(x) = \inf\{f_0(z) |\ z,$ feasible, $\|z - x\|_2 \leq R\}$. Suppose, on the contrary, $\exists\ z \in \mathcal{D}$ such that $f(z) < f(x)$. Then, $\exists y$ such that $\|y - x\|_2 < R$ and $y = \lambda x + (1 - \lambda)z$ for some $0 < \lambda < 1$. Since $f(y) \geq f(x) > f(z)$, $f(y) > \lambda f(x) + (1 - \lambda)f(z)$. A contradiction to convexity of f_0. $\square$

Remark

Not necess. true for quasiconvex minimization.

Theorem

For convex minimization with differentiable f_0, feasible x is optimal iff $\nabla f_0(x)^T(y-x) \geq 0$ for any feasible y.

Proof "If" part. For any feasible y we have $f_0(y) \geq f_0(x) + \nabla f_0(x)^T(y-x) \geq f_0(x)$.
"Only if" part. Suppose not: $\exists y \in X$ such that $\nabla f_0(x)^T(y-x) < 0$.
For $\lambda \in [0,1]$, let $g(\lambda) = f_0(\lambda y + (1-\lambda)x)$. Then,

$$\frac{d}{d\lambda} g(\lambda)\Big|_{\lambda=0} = \nabla f_0(x)^T(y-x) < 0,$$

which implies that for small enough $\lambda > 0$, we have $g(\lambda) < g(0)$. A contradiction to optimality of x. □

Corollary

For unconstrained convex minimization, x is optimal iff x is feasible and $\nabla f_0(x) = 0$.

Corollary

For convex minimization with equality constraints only, x is optimal iff $\exists \lambda$ s.t. $A^T\lambda = \nabla f_0(x)$, where $\lambda \in \mathbb{R}^p$.

Proof

$$\begin{aligned} x \text{ optimal} &\iff \forall y \text{ s.t. } Ay = b, \nabla f_0(x)^T(y - x) \geq 0 \\ &\iff \nabla f_0(x)^T z \geq 0, \forall z \in \mathcal{N}(A) \iff \nabla f_0(x)^T z = 0, \forall z \in \mathcal{N}(A) \\ &\iff \nabla f_0(x) \perp \mathcal{N}(A) \iff \nabla f_0(x) \in \mathcal{R}(A^T) \\ &\iff \exists \lambda \text{ s.t. } A^T\lambda = \nabla f_0(x), \lambda \in \mathbb{R}^p. \end{aligned}$$

Example

Consider the problem

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & x \geq 0 \end{array}$$

- Optimality condition : $x \geq 0$, $\nabla f_0(x)^T(y - x) \geq 0$ for all $y \geq 0$.
- $\nabla f_0(x)^T y$ is unbounded below on $y \geq 0$ unless $\nabla f_0(x) \geq 0$, which implies $-\nabla f_0(x)^T x \geq 0$.
- But, as $x \geq 0$ and $\nabla f_0(x) \geq 0$, we get $\nabla f_0(x)^T x = 0$.
- Thus, optimality condition becomes

$$x \geq 0, \quad \nabla f_0(x) \geq 0, \; x_i(\nabla f_0(x))_i = 0, \; \text{complementary condition.}$$

$$
\begin{array}{lll}
\min & f_0(x) & f_0 \text{ is quasiconvex,} \\
\text{s.t.} & f_i(x) \le 0,\ i = 1, \ldots, m, & f_i\text{'s are convex,} \\
 & Ax = b. &
\end{array}
$$

1. It can have local optima that are not global optima.
2. A sufficient optimality condition: If x is feasible and $\nabla f_0(x)^T(y - x) > 0$, $\forall$ feasible y, then x is globally optimal.
3. Solvable via a sequence of convex feasibility problems.

Let $\phi_t : \mathbb{R}^n \to \mathbb{R}, t \in \mathbb{R}$ such that

$$f_0(x) \le t \iff \phi_t(x) \le 0 \text{ and } \phi_s(x) \le \phi_t(x) \text{ for } s \ge t.$$

Then, optimal value p^* can be computed as follows:

$\begin{array}{lll} \text{find} & x & \\ \text{s.t.} & \phi_t(x) \le 0 & \\ & f_i(x) \le 0, & \forall i \\ & Ax = b & \end{array}$	If the problem is feasible, then $p^* \le t = UB$. Otherwise, $p^* \ge t = LB$. $t \leftarrow (UB + LB)/2$ Repeat until $UB - LB \le \epsilon$.

Note that it requires $\lceil \log_2((UB - LB)/\epsilon) \rceil$ iterations for an ϵ-suboptimal solution.

- Quadratic program (QP) minimizes convex quadratic over polyhedron.

$$\begin{array}{ll} \min & \frac{1}{2}x^T P x + q^T x + r \\ \text{s.t.} & Gx \leq h \\ & Ax = b, \end{array}$$

  where $P \in \mathbb{S}_+^n$, $G \in \mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{p \times n}$.

- Quadratically constrained quadratic program (QCQP) minimizes convex quadratic over intersection of ellipsoids.

$$\begin{array}{ll} \min & \frac{1}{2}x^T P_0 x + q_0^T x + r_0 \\ \text{s.t.} & \frac{1}{2}x^T P_i x + q_i^T x + r_0 \leq 0, \ i = 1, \cdots, m \\ & Ax = b, \end{array}$$

  where $P_i \in \mathbb{S}_+^n$, $i = 0, 1, \cdots, m$.

- QCQP $\supseteq$ QP $\supseteq$ LP.

Minimizing Euclidean norm of affine functions

- Least-squares and regression

$$\|Ax - b\|_2^2 = x^T A^T Ax - 2b^T Ax + b^T b.$$

Analytical solution is $x = A^\dagger b$. But, if linear inequality constraints are added, e.g.

$$\begin{array}{ll} \min & \|Ax - b\|_2^2 \\ \text{s.t.} & l_i \le x_i \le u_i, i = 1, \cdots, n, \end{array}$$

problem does not have analytical solution and solvable via QP.

- Distance between polyhedra For $\mathcal{P}_1 = \{x | A_1 x \le b_1\}, \mathcal{P}_2 = \{x | A_2 x \le b_2\}$,

$$\mathrm{dist}(\mathcal{P}_1, \mathcal{P}_2) = \inf\{\|x_1 - x_2\|_2 | x_1 \in \mathcal{P}_1, x_2 \in \mathcal{P}_2\},$$

can be computed via following QP:

$$\begin{array}{ll} \min & \|x_1 - x_2\|_2^2 \\ \text{s.t.} & A_1 x \le b_1 (x_1 \in \mathcal{P}_1) \\ & A_2 x \le b_2 (x_2 \in \mathcal{P}_2). \end{array}$$

Linear program with random cost

Suppose that $c \in \mathbb{R}^n$ is random with mean $\bar{c}$ and covariance Σ. Then,

$$\mathrm{E}(c^T x) = \bar{c}^T x, \ \mathrm{Var}(c^T x) = \mathrm{E}(c^T x - \bar{c}^T x)^2 = x^T \Sigma x$$

A possible problem is to minimize a weighted sum of expected cost and uncertainty cost. To do so, we can consider γ, *risk-sensitive* cost so that the problem is formulated as follows:

$$\begin{array}{ll} \min & \bar{c}^T x + \gamma x^T \Sigma x \\ \text{s.t.} & Gx \leq h \\ & Ax = b. \end{array}$$

Markowitz portfolio optimization

- Let x_i be amount of asset or stock $i = 1, \ldots, n$; $x_i > 0 \leftrightarrow$ long position in asset i, $x_i < 0 \leftrightarrow$ short position in asset i.
- Let p_i be increase of price of i during a period; we assume p has mean vector $\bar{p}$ and covariance matrix Σ.
- Classical model minimizes risk guaranteeing a return with no shorting:

$$\begin{array}{ll} \min & x^T \Sigma x \\ \text{s.t.} & \bar{p}^T x \geq r_{\min}, \\ & \mathbf{1}^T x = 1, \;\; x \geq 0. \end{array}$$

- Various extensions are possible.

For $x \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n_i \times n}$, SOCP has the form

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, i = 1, \cdots, m \\ & Fx = g. \end{array}$$

- The first constraints require affine transforms $(A_i x + b_i, c_i^T x + d_i)$ to be in second-order cones in $\mathbb{R}^{n_i+1}$.
- Note that SOCP $\supseteq$ QCQP (by squaring soc-constraints) and SOCP $\supseteq$ LP (when $A_i = 0$, $\forall i$).

Robust linear programming

Let a_i's be uncertain and lie in given ellipsoids $\mathcal{E}_i = \{\bar{a}_i + P_i u \,|\, \|u\|_2 \le 1\}$.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & a_i^T x \le b_i, i = 1, \cdots, m \end{array} \xrightarrow{\textit{robust version}} \begin{array}{ll} \min & c^T x \\ \text{s.t.} & a_i^T x \le b_i, \forall a_i \in \mathcal{E}_i, \forall i \end{array}$$

$a_i^T x \le b_i, \forall a_i \in \mathcal{E}_i \Leftrightarrow \sup\{a_i^T x | a_i \in \mathcal{E}_i\} \le b_i$, and

$$\sup\{a_i^T x | a_i \in \mathcal{E}_i\} = \bar{a}_i^T x + \sup\{u^T P_i^T x \,|\, \|u\| \le 1\} = \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i$$

Thus, we have the following SOCP:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i, \forall i. \end{array}$$

Linear programming with random constraints

Suppose that a_i's are indep. Gaussian random vectors with mean $\bar{a}_i$ and covariance Σ_i. Also suppose $\mathrm{P}(a_i^T x \le b_i) \ge \eta \ge 1/2$. Let $u = a_i^T x$ and σ^2 be its variance. Normalizing we have

$$\mathrm{P}\left(\frac{u-\bar{u}}{\sigma} \le \frac{b_i - \bar{u}}{\sigma}\right) \ge \eta \iff \frac{b_i - \bar{u}}{\sigma} \ge \Phi^{-1}(\eta) \iff \bar{u} + \Phi^{-1}(\eta)\sigma \le b_i.$$

From $\sigma = (x^T \Sigma_i x)^{1/2}$, we have $\bar{a}_i^T x + \Phi^{-1}(\eta)\|\Sigma_i^{1/2} x\|_2 \le b_i$. Hence

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & \mathrm{P}(a_i^T x \le b_i) \ge \eta, \forall i \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & \bar{a}_i^T x + \Phi^{-1}(\eta)\|\Sigma_i^{1/2} x\|_2 \le b_i, \forall i \end{array}$$

Since $\eta \ge 1/2$, $\Phi^{-1}(\eta) \ge 0$, and problem is thus an SOCP.

- Monomials

$$f(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, c > 0, a_i \in \mathbb{R}, \mathrm{dom} f = \mathbb{R}^n_{++}$$

Closed under multiplication and division.

- Posi(tively weighted sum of mo)nomials

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}, c_k > 0$$

Closed under addition, multiplication, and nonnegative scaling.

Definition

- Geometric programming

$$\begin{array}{llll} \min & f_0(x) & & f_0 \text{ posynomial} \\ \text{s.t.} & f_i(x) \leq 1 & i = 1, \cdots, m & f_i \text{ posynomials} \\ & h_j(x) = 1 & j = 1, \cdots, p & h_j \text{ monomials} \\ & x > 0. & & \text{implicit constraints} \end{array}$$

- Extensions of geometric programming

$$\begin{array}{ll} \max & x/y \\ \text{s.t.} & 2 \leq x \leq 3 \\ & x^2 + 3y/z \leq \sqrt{y} \\ & x/y = z^2 \end{array} \Rightarrow \begin{array}{ll} \min & x^{-1}y \\ \text{s.t.} & 2x^{-1} \leq 1, (1/3)x \leq 1 \\ & x^2y^{-1/2} + 3y^{1/2}z^{-1} \leq 1 \\ & xy^{-1}z^{-2} = 1 \end{array}$$

- Monomials and posynomials can be converted into convex form:

$$
\begin{aligned}
f_{mono}(x) &= cx_1^{a_1}\cdots x_n^{a_n} = c(e^{y_1})^{a_1}\cdots(e^{y_n})^{a_n} = e^{a^Ty+b} \\
f_{posy}(x) &= \sum_{k=1}^{K} c_k x_1^{a_{1k}}\cdots x_n^{a_{nk}} = \sum_{k=1}^{K} c_k(e^{y_1})^{a_{1k}}\cdots(e^{y_n})^{a_{nk}} = \sum_{k=1}^{K} e^{a_k^Ty+b_k},
\end{aligned}
$$

where $y_i = \log x_i$, $b = \log c$, $b_k = \log c_k$.

- Using above conversion, we can transform GP into a convex optimization:

$$
\begin{aligned}
\min\quad & \sum_{k=1}^{K_0} e^{a_{0k}^Ty+b_{0k}} & \Rightarrow\quad \min\quad & \tilde{f}_0(y) = \log\left(\sum_{k=1}^{K_0} e^{a_{0k}^Ty+b_{0k}}\right) \\
\text{s.t.}\quad & \sum_{k=1}^{K_i} e^{a_{ik}^Ty+b_{ik}} \le 1 & \text{s.t.}\quad & \tilde{f}_i(y) = \log\left(\sum_{k=1}^{K_i} e^{a_{ik}^Ty+b_{ik}}\right) \le 0 \\
& e^{g_j^Ty+h_j} = 1 & & \tilde{h}_j(y) = g_j^Ty + h_j = 0.
\end{aligned}
$$

Since $\tilde{f}_i$'s are convex and $\tilde{h}_j$'s are affine, this is convex optimization.

- Frobenius diagonal scaling

$$y = Mu \xrightarrow{\text{scaling coordinates}} \tilde{y} = DMD^{-1}\tilde{u} \text{ with } \tilde{u} = Du, \tilde{y} = Dy.$$

$$\|DMD^{-1}\|_F^2 = \mathrm{tr}\big((DMD^{-1})^T(DMD^{-1})\big) = \sum_{i,j=1}^{n} (DMD^{-1})_{ij}^2 = \sum_{i,j=1}^{n} M_{ij}^2 d_i^2/d_j^2$$

Thus, minimizing Frobenius norm is an unconstrained GP.

Design of cantilever beam

Minimize total volume of beam, $w_1 h_1 + \cdots + w_n h_n$ subject to some constraints.

Design of cantilever beam(*cont'd*)

1. Upper and lower bounds on w_i, h_i

$$w_{min} \leq w_i \leq w_{max}, h_{min} \leq h_i \leq h_{max}, i = 1, \cdots, N.$$

2. Upper and lower bounds on aspect ratios h_i/w_i

$$S_{min} \leq h_i/w_i \leq S_{max}, i = 1, \cdots, N.$$

3. Upper bound on stress in each segment

$$\frac{6iF}{w_i h_i^2} \leq \sigma_{max}, i = 1, \cdots, N.$$

4. Upper bound on vertical deflection at end of beam, $y_1 \leq y_{max}$
   From deflection and slope of beam segments, y_1 can be obtained:

$$v_i = 12(i - 1/2)\frac{F}{Ew_i h_i^3} + v_{i+1}, y_i = 6(i - 1/3)\frac{F}{Ew_i h_i^3} + v_{i+1} + y_{i+1}.$$

- Convex optimization problem with generalized inequality constraints

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \preceq_{K_i} 0, i = 1, \cdots, m \\ & Ax = b \end{array}$$

  where $f_0 : \mathbb{R}^n \to \mathbb{R}$, $K_i \subseteq \mathbb{R}^{k_i}$, and $f_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$ are K_i-convex.

- Many properties of ordinary convex optimization are extended to this problem
  1. Feasible set, any sublevel set, and optimal set are convex.
  2. Any local optimum is global optimum.
  3. Feasible x is globally optimal iff $\nabla f_0(x)^T(y - x) \geq 0 \; \forall$ feasible y.

Conic form problems (conic programs, or cone-LPs). When K is nonnegative orthant, reduces to LP.

1.	2.	3.
$\min \; c^T x$	$\min \; c^T x$	$\min \; c^T x$
s.t. $Fx + g \preceq_K 0$	s.t. $x \succeq_K 0$	s.t. $Fx + g \preceq_K 0$
$Ax = b$	$Ax = b$	

1. General form.
2. Standard form.
3. Inequality form.

$$\begin{array}{ll} \min & f^T x \\ \text{s.t.} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \forall i \\ & Fx = g \end{array} \quad \Rightarrow \quad \begin{array}{ll} \min & f^T x \\ \text{s.t.} & -(A_i x + b_i, c_i^T x + d_i) \preceq_{K_i} 0, \forall i \\ & Fx = g \end{array}$$

where $K_i = \{(y,t) \in \mathbb{R}^{k_i+1} | \|y\|_2 \le t\}$

Here, $K = \mathbb{S}_+^k$. Then conic form is

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & x_1 F_1 + \cdots x_n F_n + G \preceq 0 \quad (LMI) \\ & Ax = b, \end{array}$$

where $G, F_1, \ldots, F_n \in \mathbb{S}^k$, $A \in \mathbb{R}^{p \times n}$.
Note that if $G, F_1, \cdots, F_n$ are all diagonal, then SDP reduces to LP.

1.

$$\begin{array}{ll} \min & \mathrm{tr}(CX) \\ \text{s.t.} & \mathrm{tr}(A_iX) = b_i, i = 1, \cdots, p \\ & X \succeq 0. \end{array}$$

2.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & x_1A_1 + \cdots + x_nA_n \preceq B \end{array}$$

1. Standard form: $C, A_1, \cdots, A_n \in \mathbb{S}^n$.
2. Inequality form: $B, A_1, \cdots, A_n \in \mathbb{S}^k, c \in \mathbb{R}^n$.

Matrix norm (or maximum eigenvalue) minimization

Let $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$, where $A_i \in \mathbb{R}^{p\times q}$. How to minimize maximum eigenvalue $\|A(x)\|_2$ or $\lambda_{max}(A(x))$? Observe that

$$\begin{aligned} \lambda_{max(B)} \le \mu \quad &\iff \quad B = U\lambda U^T, \lambda I - \Lambda \succeq 0 \\ &\iff \quad U(\mu I - \Lambda)U^T \succeq 0 \iff \mu I - U\Lambda U^T \succeq 0 \\ &\iff \quad \mu I - B \succeq 0. \end{aligned}$$

Therefore, we have

$$\lambda_{max}(A(x)^T A(x)) \le s^2 \iff s^2 I - A^T(x)A(x) \succeq 0 \iff \begin{bmatrix} sI & A(x) \\ A(x)^T & xI \end{bmatrix} \succeq 0,$$

an LMI!
In sum, $\min \|A(x)\|_2$ is equivalent to

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \begin{bmatrix} sI & A(x) \\ A(x)^T & xI \end{bmatrix} \succeq 0. \end{aligned}$$