Naval Architecture & Ocean Engineering

[2008][02-2]

Engineering Mathematics 2

September, 2008

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Ordinary Differential Equations (3)

Basics of Matrices and Vectors
Systems of ODEs as Models
Basic Theory of Systems of ODEs
Constant-Coefficient Systems. Phase Plane Method
Criteria for Critical Points. Stability
Qualitative Methods for Nonlinear Nonhomogeneous Linear Systems of ODEs

Basic of Matrices and Vectors

- System of linear equations is a set of linear equations such as

$$
\begin{aligned}
3x_1 + 2x_2 - x_3 &= 1 \\
2x_1 - 2x_2 + 4x_3 &= -2 \\
-x_1 + \frac{1}{2}x_2 - x_3 &= 0
\end{aligned}
$$

Then, we can make a array with coefficients of system of linear equations above.

$$
\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & 1/2 & -1 \end{bmatrix}
$$

We call this kind of array Matrix.

- A general form of Matrix is

$$
\begin{array}{cc}
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{11} & a_{11} & \cdots & a_{2n} \\
\vdots & \vdots & \cdots & \vdots \\
a_{m1} & a_{m1} & \cdots & a_{mn}
\end{bmatrix} & \begin{array}{l} \\ \text{Row} \\ \\ \\ \end{array} \\
\begin{array}{cccc} & \text{Column} & & \quad \text{entry} \end{array} &
\end{array}
$$

This Matrix has *m rows* and *n Columns*. So It is m x n Matrix.

Basic of Matrices and Vectors

- Vector can be expressed by a matrix form.

- A column vector $\mathbf{x}$ with n components x_1, .., x_n is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ thus if } n = 2, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Similarly, a row vector $\mathbf{v}$ is of the from

$$\mathbf{v} = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$$

$$\text{thus if } n = 2, \ \mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

Basic of Matrices and Vectors
: Calculations with Matrices and Vectors

Equality

- Two n x n matrices are *equal* if and only if corresponding entries are equal.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\text{if } \mathbf{A} = \mathbf{B}, \quad \begin{matrix} a_{11} = b_{11} & a_{12} = b_{12} \\ a_{21} = b_{21} & a_{22} = b_{22} \end{matrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\text{if } \mathbf{x} = \mathbf{v}, \quad \begin{matrix} x_1 = v_1 \\ x_2 = v_2 \end{matrix}$$

Addition

- Addition is performed by adding corresponding entries.

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

$$\mathbf{x} + \mathbf{v} = \begin{bmatrix} x_1 + v_1 \\ x_2 + v_2 \end{bmatrix}$$

Basic of Matrices and Vectors
: Calculations with Matrices and Vectors

Scalar multiplication

- It is performed multiplying each entry by a number.

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\quad c\mathbf{A}=\begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

For example,

$$\mathbf{A}=\begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix},\quad -7\mathbf{A}=\begin{bmatrix} -63 & -21 \\ 14 & 0 \end{bmatrix}$$

$$\mathbf{v}=\begin{bmatrix} 0.4 \\ -13 \end{bmatrix},\quad \text{then}\quad 10\mathbf{v}=\begin{bmatrix} 4 \\ -130 \end{bmatrix}$$

Matrix Multiplication

- The Product C=AB of two n x n matrices $A=[a_{jk}]$ and $B=[b_{jk}]$ is the n x n matrix $C=[c_{jk}]$ with entries

$$c_{jk}=\sum_{m=1}^{n} a_{jm} n_{mk}$$

For example,

$$\begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}=\begin{bmatrix} 9\cdot 1+3\cdot 2 & 9\cdot(-4)+3\cdot 5 \\ (-2)\cdot 1+0\cdot 2 & (-2)\cdot(-4)+0\cdot 5 \end{bmatrix}$$

$$=\begin{bmatrix} 15 & -21 \\ -2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 9 & 3 \\ -2 & 0 \end{bmatrix}=\begin{bmatrix} 1\cdot 9+(-4)\cdot(-2) & 1\cdot 3+(-4)\cdot 0 \\ 2\cdot 9+5\cdot(-2) & 2\cdot 3+5\cdot 0 \end{bmatrix}$$

$$=\begin{bmatrix} 17 & 3 \\ 8 & 9 \end{bmatrix}\qquad AB \neq BA$$

Basic of Matrices and Vectors
: Systems of ODEs as Vector Equations

Differentiation

- The *derivative* of a matrix(or vector) with variable entries(or components) is obtained by differentiating each entry(or component).

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ \sin t \end{bmatrix}, \quad \text{then} \quad \mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2e^{-2t} \\ \cos t \end{bmatrix}$$

$$\left.\begin{aligned} y_1'(t) &= a_{11}y_1(t) + a_{12}y_2(t) \\ y_2'(t) &= a_{21}y_1(t) + a_{22}y_2(t) \\ \mathbf{A} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{aligned}\right\} \mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \mathbf{Ay} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Basic of Matrices and Vectors
: Some Further Operations and Terms

Transposition

- It is the operation of writing columns as rows.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

Inverse of a Matrix

- If for a given n x n matrix A,B such that AB=BA=I, then A is nonsingular and B is called the inverse of A.

- B=A^{-1}(I is the unit matrix of n x n matrix with main diagonal 1, 1, …, 1 and all other entries zero)

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

- For n = 2,

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

If A has no inverse, it is called singular

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Linear Independence

- r given vector v(1), …, v(r) are *linearly independent* if all scalars are zero.

$$c_1\mathbf{v}^{(1)} + \ldots + c_r\mathbf{v}^{(r)} = \mathbf{0}$$

- if it is not, one of them can be expressed as a linear combination of the others. So it is *linearly dependent*

$$c_1 \neq 0, \quad \mathbf{v}^{(1)} = -\frac{1}{c_1}(c_2\mathbf{v}^{(2)} + \ldots + c_r\mathbf{v}^{(r)})$$

Basic of Matrices and Vectors
: Eigenvalues, Eigenvectors

- Let A is an n x n matrix.
- Consider the equation

$$\boxed{\mathbf{A}\mathbf{x} = \lambda \mathbf{x}}$$

λ is a scalar to be determined and x is a a vector to be determined. For every λ, a solution is x = 0.

- A scalar λsuch that the equation holds for some vector $x \neq 0$ is called an *eigenvalue of A*.

At that time, vector x is called eigenvector of A.

Basic of Matrices and Vectors
: Eigenvalues, Eigenvectors

- Consider the equation again

$$\boxed{\begin{aligned} \mathbf{Ax} &= \lambda \mathbf{x} \\ \mathbf{Ax} - \lambda \mathbf{x} &= \mathbf{0} \\ (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} &= \mathbf{0} \end{aligned}}$$

- For these equations to have a solution x ≠ 0, the determinant of the coefficient matrix (A-λx) must be zero.

- if n = 2,

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 &= 0 \end{aligned}$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \underline{\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0} \end{aligned}$$

We can get x easily after calculating λ

Ex.) Eigenvalue Problem(1/2)

$$\mathbf{Ax} = \lambda\mathbf{x}$$
$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{vmatrix}$$
$$= \lambda^2 + 2.8\lambda + 1.6$$
$$= (\lambda+2)(\lambda+0.8) = 0$$

$$\boxed{\lambda_1 = -2, \quad \lambda_2 = -0.8}$$

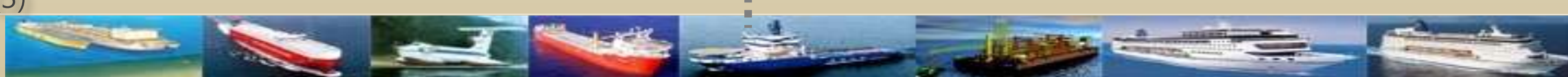
$$\begin{bmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

1) $\lambda = \lambda_1 = -2$

$$\begin{bmatrix} -4-(-2) & 4 \\ -1.6 & 1.2-(-2) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

$$-2x_1 + 4x_2 = 0$$
$$4x_2 = 2x_1$$
$$x_2 = \frac{1}{2}x_1$$

$$x^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



Ex.) Eigenvalue Problem(2/2)

$$\mathbf{Ax} = \lambda \mathbf{x}$$
$$\mathbf{Ax} - \lambda \mathbf{x} = \mathbf{0}$$
$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}$$

2) $\lambda = \lambda_2 = -0.8$

$$\begin{bmatrix} -4-(-0.8) & 4 \\ -1.6 & 1.2-(-0.8) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

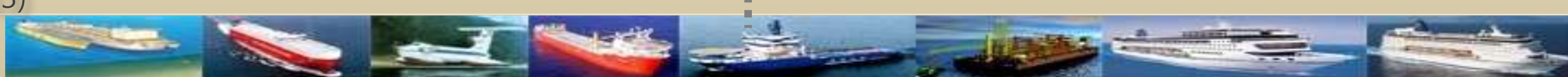
$$-3.2x_1 + 4x_2 = 0$$
$$4x_2 = 3.2x_1$$
$$x_2 = 0.8x_1$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

eigenvalue of $\mathbf{A}$ $\boxed{\lambda_1 = -2, \quad \lambda_2 = -0.8}$

corresponding eigenvector of $\mathbf{A}$

$$\mathbf{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$



Systems of ODEs as Models

☑We first illustrate with a few typical examples that systems of ODEs can serve as models in various applications.

☑We further show that a higher order ODE can be reduced to a first-order system.

Ex.) mixing problem Involving Two Tanks

1) <英> 4.546ℓ, <美> 3.785ℓ, 2) 0.37324kg

Ex.) mixing problem Involving Two Tanks

Tank T1, T2 contain initially 100gal[1] of water each

In T1 the water is pure, whereas 150 lb[2] of salt are dissolved in T2.

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T1

Initial condition :

water 100 gal

T2

Initial condition :

Salt 150 lb mixed in brine 100 gal

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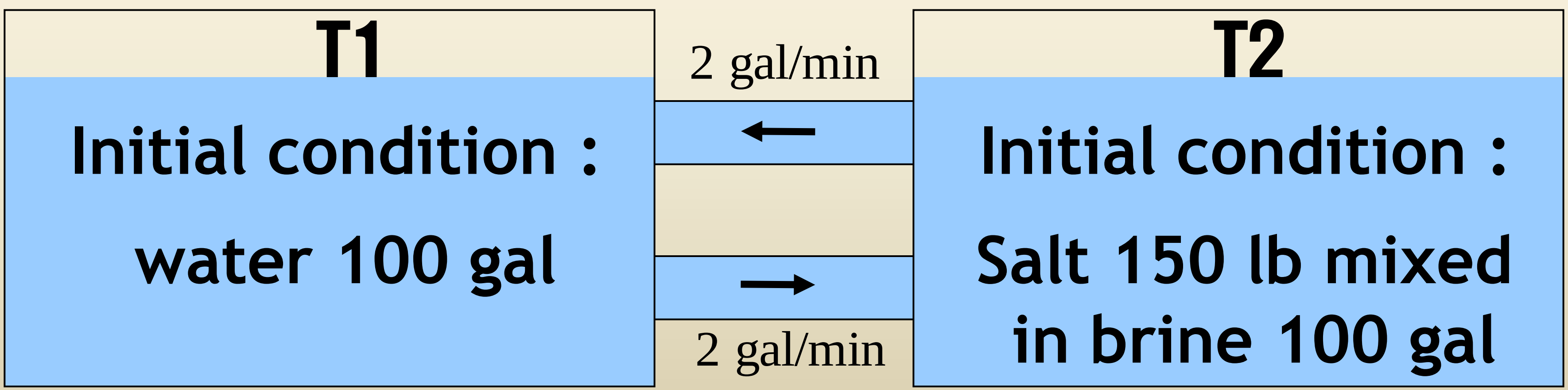
By circulating liquid at a rate of 2 gal/min the amounts of salt $y_1(t)$ in T1 and $y_2(t)$ in T2 change with time t.



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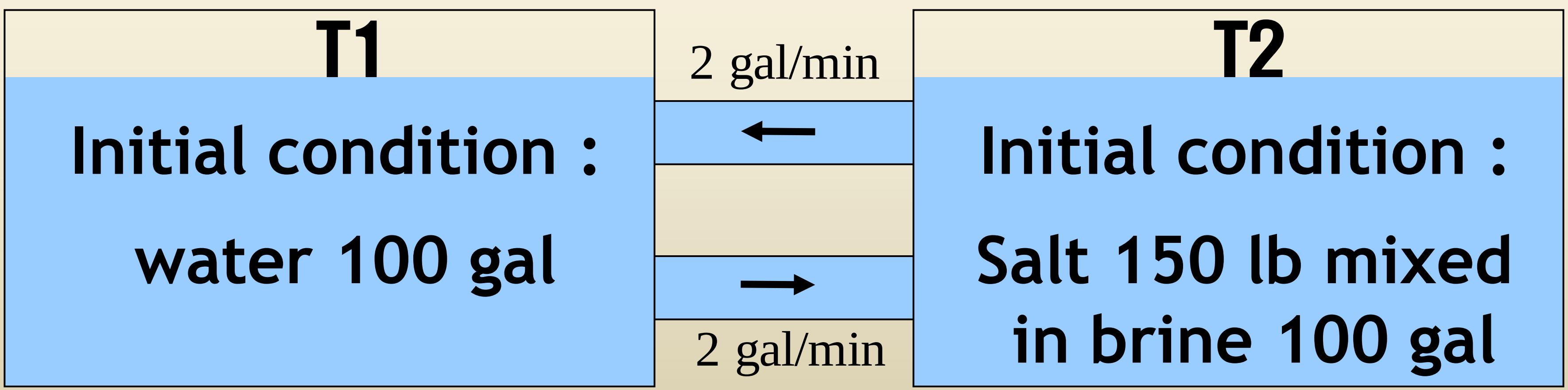
By circulating liquid at a rate of 2 gal/min the amounts of salt $y_1(t)$ in T1 and $y_2(t)$ in T2 change with time t.

Find $y_1(t)$, $y_2(t)$.



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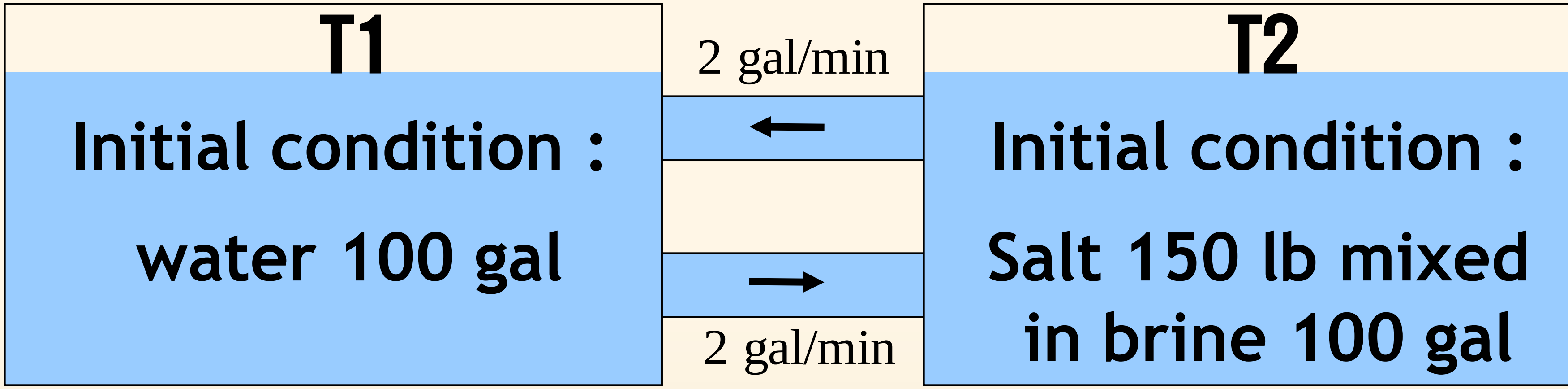

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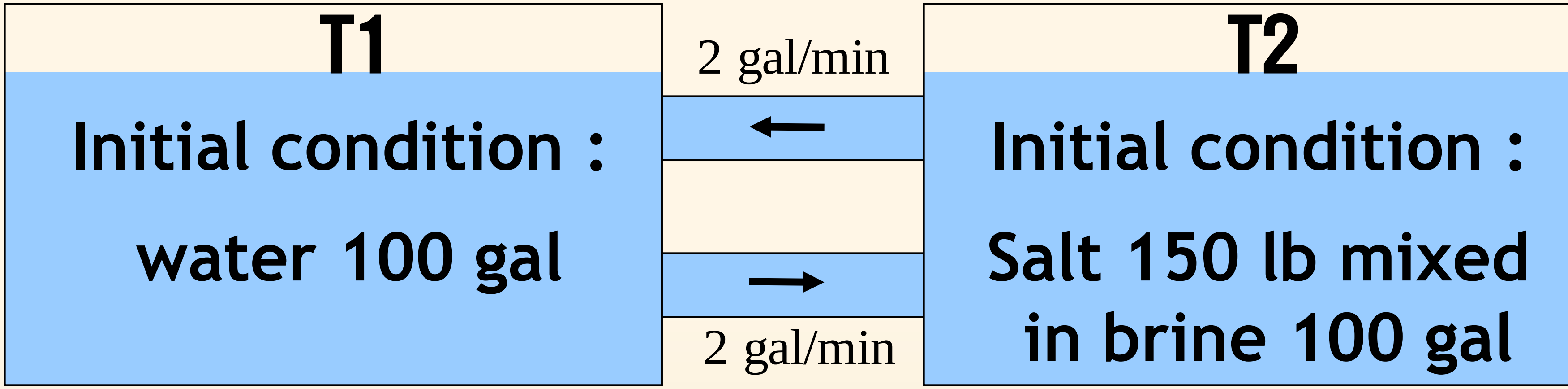

- Assumption : The mixture is kept uniform by stirring.
  (in constant concentration)

t : time [min.]

$y_1(t)$: the amount of salt in T1 [lb]

$y_2(t)$: the amount of salt in T2 [lb]

- Problem : Find the amount of salt in each tank at any time t.

Ex.) mixing problem Involving Two Tanks

Ex.) mixing problem Involving Two Tanks

Salt in the tank (y) = Salt inflow (y_{in}) − Salt outflow (y_{out})

Ex.) mixing problem Involving Two Tanks

$$\text{Salt in the tank}\,(y) = \text{Salt inflow}\,(y_{in}) - \text{Salt outflow}(y_{out})$$

$$\text{time rate of change of salt in the tank}\,(y')$$

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Salt outflow rate of T1, y'_{1_out} **:**

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Salt outflow rate of T1, y'_{1_out} :

The amount of Salt in 2gal of brine of T1 $= y_{1\_2gal}$

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$$100(gal) : y_1(lb) = 2(gal) : y_{1\_2gal}(lb)$$

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$$y_{1\_2gal} = \frac{2}{100} y_1$$

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$$= 0.02 y_1$$

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Outflow per minute is 2 gal

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So, $y'_{1\_out} = y_{1\_2gal} = \dfrac{2}{100} y_1$

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$$y_{1\_2gal} = \frac{2}{100} y_1$$

$$= 0.02 y_1$$

Outflow per minute is 2 gal

So, $y'_{1\_out} = y_{1\_2gal} = \dfrac{2}{100} y_1$

Salt inflow rate of T1, y'_{1_in} :

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Salt outflow rate of T1, y'_{1_out} :

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Outflow per minute is 2 gal

So, $y'_{1\_out} = y_{1\_2gal} = \dfrac{2}{100} y_1$

Salt inflow rate of T1, y'_{1_in} :

The amount of Salt in 2gal of brine of T2 $= y_{2\_2gal}$

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time rate of change of salt in the tank (y')

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Salt outflow rate of T1, y'_{1_out} :

The amount of Salt in 2gal of brine of T1 $= y_{1\_2gal}$

$$100(gal) : y_1(lb) = 2(gal) : y_{1\_2gal}(lb)$$

$$y_{1\_2gal} = \frac{2}{100} y_1$$

$$= 0.02 y_1$$

Outflow per minute is 2 gal

So, $y'_{1\_out} = y_{1\_2gal} = \dfrac{2}{100} y_1$

Salt inflow rate of T1, y'_{1_in} :

The amount of Salt in 2gal of brine of T2 $= y_{2\_2gal}$

$$100(gal) : y_2(lb) = 2(gal) : y_{2\_2gal}(lb)$$

Ex.) mixing problem Involving Two Tanks

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Salt outflow rate of T1, y'_{1_out} **:**

The amount of Salt in 2gal of brine of T1 $= y_{1\_2gal}$

$$100(gal) : y_1(lb) = 2(gal) : y_{1\_2gal}(lb)$$

$$y_{1\_2gal} = \frac{2}{100} y_1$$

$$= 0.02 y_1$$

Outflow per minute is 2 gal

So, $y'_{1\_out} = y_{1\_2gal} = \dfrac{2}{100} y_1$

Salt inflow rate of T1, y'_{1_in} **:**

The amount of Salt in 2gal of brine of T2 $= y_{2\_2gal}$

$$100(gal) : y_2(lb) = 2(gal) : y_{2\_2gal}(lb)$$

$$y_{2\_2gal} = \frac{2}{100} y_2$$

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So, $y'_{1\_out} = y_{1\_2gal} = \dfrac{2}{100} y_1$

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The amount of Salt in 2gal of brine of T2 $= y_{2\_2gal}$

$$100(gal) : y_2(lb) = 2(gal) : y_{2\_2gal}(lb)$$

$$y_{2\_2gal} = \frac{2}{100} y_2$$

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Inflow per minute is 2 gal

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Outflow per minute is 2 gal

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Ex.) mixing problem Involving Two Tanks

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$$y_1' = y_{1\_in}' \text{-} y_{1\_out}' = 0.02 y_2 - 0.02 y_1$$

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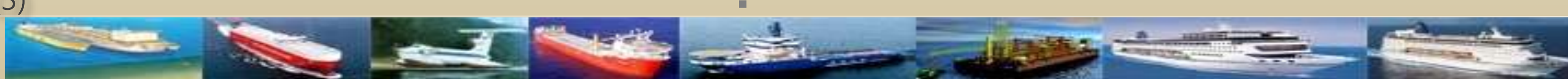

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$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

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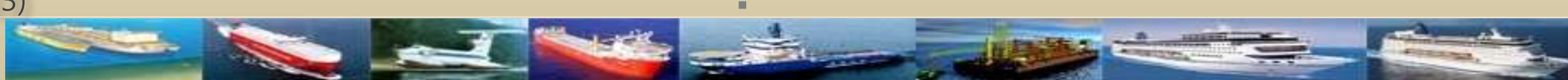
General solution

$$y_1 = x_1 e^{\lambda t}$$

$$y_2 = x_2 e^{\lambda t}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$



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$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

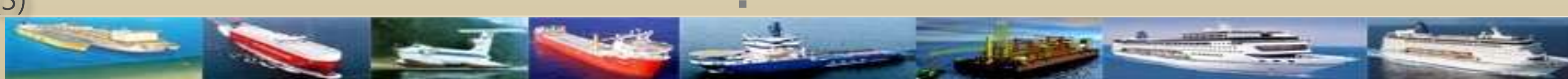
where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1{}' \\ y_2{}' \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

General solution

$$y_1 = x_1 e^{\lambda t}, \quad y_2 = x_2 e^{\lambda t} \quad \Rightarrow \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

where x_1 and x_2 is constant.



Ex.) mixing problem Involving Two Tanks

$$y_1' = y'_{1\_in} - y'_{1\_out} = 0.02y_2 - 0.02y_1$$

$$y_2' = y'_{2\_in} - y'_{2\_out} = 0.02y_1 - 0.02y_2$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1{}' \\ y_2{}' \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

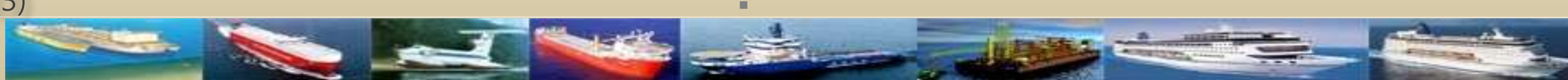
General solution

$$y_1 = x_1 e^{\lambda t}$$
$$y_2 = x_2 e^{\lambda t}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

where x_1 and x_2 is constant.

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} e^{\lambda t}$$



Ex.) mixing problem Involving Two Tanks

$$y_1' = y'_{1\_in} - y'_{1\_out} = 0.02y_2 - 0.02y_1$$

$$y_2' = y'_{2\_in} - y'_{2\_out} = 0.02y_1 - 0.02y_2$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

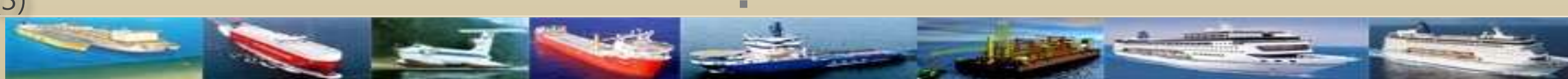
General solution

$$y_1 = x_1 e^{\lambda t}, \quad y_2 = x_2 e^{\lambda t} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} e^{\lambda t}$$

where x_1 and x_2 is constant.

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} e^{\lambda t}$$

$\mathbf{y} = \mathbf{x}e^{\lambda t}$ **then** $\mathbf{y}' = \lambda \mathbf{x} e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$



Ex.) mixing problem Involving Two Tanks

Ex.) mixing problem Involving Two Tanks

$$y_1' = y_{1\_in}' \text{-} y_{1\_out}' = 0.02y_2 - 0.02y_1$$

$$y_2' = y_{2\_in}' \text{-} y_{2\_out}' = 0.02y_1 - 0.02y_2$$

Ex.) mixing problem Involving Two Tanks

$$y_1' = y_{1\_in}' \text{-} y_{1\_out}' = 0.02 y_2 - 0.02 y_1$$

$$y_2' = y_{2\_in}' \text{-} y_{2\_out}' = 0.02 y_1 - 0.02 y_2$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1{}' \\ y_2{}' \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

Ex.) mixing problem Involving Two Tanks

$$y_1' = y_{1\_in}' - y_{1\_out}' = 0.02y_2 - 0.02y_1$$

$$y_2' = y_{2\_in}' - y_{2\_out}' = 0.02y_1 - 0.02y_2$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

$\mathbf{y} = \mathbf{x}e^{\lambda t}$ **then** $\mathbf{y}' = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$

Ex.) mixing problem Involving Two Tanks

$$y_1' = y_{1\_in}' \text{-} y_{1\_out}' = 0.02y_2 - 0.02y_1$$

$$y_2' = y_{2\_in}' \text{-} y_{2\_out}' = 0.02y_1 - 0.02y_2$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where $$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} {y_1}' \\ {y_2}' \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

$\mathbf{y} = \mathbf{x}e^{\lambda t}$ **then** $\mathbf{y}' = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

Ex.) mixing problem Involving Two Tanks

$$y_1' = y_{1\_in}' \text{-} y_{1\_out}' = 0.02y_2 - 0.02y_1$$

$$y_2' = y_{2\_in}' \text{-} y_{2\_out}' = 0.02y_1 - 0.02y_2$$

$$\mathbf{y}' = \mathbf{Ay}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1{}' \\ y_2{}' \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

$\mathbf{y} = \mathbf{x}e^{\lambda t}$ **then** $\mathbf{y}' = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -0.02 - \lambda & 0.02 \\ 0.02 & -0.02 - \lambda \end{vmatrix} = 0$$

Ex.) mixing problem Involving Two Tanks

$$y_1' = y_{1\_in}' - y_{1\_out}' = 0.02y_2 - 0.02y_1$$

$$y_2' = y_{2\_in}' - y_{2\_out}' = 0.02y_1 - 0.02y_2$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1{}' \\ y_2{}' \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

$\mathbf{y} = \mathbf{x}e^{\lambda t}$ **then** $\mathbf{y}' = \lambda \mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -0.02 - \lambda & 0.02 \\ 0.02 & -0.02 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 0.04\lambda + 0.0004 - 0.0004$$
$$= \lambda(\lambda + 0.04) = 0$$

Ex.) mixing problem Involving Two Tanks

$$y_1' = y_{1\_in}' - y_{1\_out}' = 0.02y_2 - 0.02y_1$$

$$y_2' = y_{2\_in}' - y_{2\_out}' = 0.02y_1 - 0.02y_2$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

$\mathbf{y} = \mathbf{x}e^{\lambda t}$ **then** $\mathbf{y}' = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{x}e^{\lambda t}$

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 0.04\lambda + 0.0004 - 0.0004$$

$$= \lambda(\lambda + 0.04) = 0$$

$$\lambda_1 = 0, \; \lambda_2 = -0.04$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

Ex.) mixing problem Involving Two Tanks

$$\boxed{\begin{aligned} \mathbf{A}\mathbf{x} &= \lambda\mathbf{x} \\ \mathbf{A}\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} &= \mathbf{0} \end{aligned}}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\lambda_1 = 0, \quad \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{Ax} = \lambda \mathbf{x}$$

$$\mathbf{Ax} - \lambda \mathbf{x} = \mathbf{0}$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\lambda_1 = 0, \quad \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{array}{r} -0.02x_1 + 0.02x_2 = 0 \\ 0.02x_1 - 0.02x_2 = 0 \end{array}\right\} \; x_1 - x_2 = 0$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{Ax} = \lambda\mathbf{x}$$
$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\lambda_1 = 0, \quad \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} -0.02x_1 + 0.02x_2 &= 0 \\ 0.02x_1 - 0.02x_2 &= 0 \end{aligned}\right\} x_1 - x_2 = 0$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \mathbf{x}^{(1)} = \begin{pmatrix} a \\ a \end{pmatrix}$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{Ax} = \lambda\mathbf{x}$$
$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -0.02 - \lambda & 0.02 \\ 0.02 & -0.02 - \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\lambda_1 = 0, \quad \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} -0.02x_1 + 0.02x_2 &= 0 \\ 0.02x_1 - 0.02x_2 &= 0 \end{aligned}\right\} x_1 - x_2 = 0$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \mathbf{x}^{(1)} = \begin{pmatrix} a \\ a \end{pmatrix}$$

$$\lambda_2 = -0.04, \quad \begin{pmatrix} 0.02 & 0.02 \\ 0.02 & 0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{Ax} = \lambda\mathbf{x}$$
$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\lambda_1 = 0, \quad \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} -0.02x_1 + 0.02x_2 &= 0 \\ 0.02x_1 - 0.02x_2 &= 0 \end{aligned}\right\} x_1 - x_2 = 0$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \mathbf{x}^{(1)} = \begin{pmatrix} a \\ a \end{pmatrix}$$

$$\lambda_2 = -0.04, \quad \begin{pmatrix} 0.02 & 0.02 \\ 0.02 & 0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} 0.02x_1 + 0.02x_2 &= 0 \\ 0.02x_1 + 0.02x_2 &= 0 \end{aligned}\right\} x_1 + x_2 = 0$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{Ax} = \lambda \mathbf{x}$$
$$\mathbf{Ax} - \lambda \mathbf{x} = \mathbf{0}$$
$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\mathbf{y} = \mathbf{x}e^{\lambda t}$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\lambda_1 = 0, \quad \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} -0.02x_1 + 0.02x_2 &= 0 \\ 0.02x_1 - 0.02x_2 &= 0 \end{aligned}\right\} x_1 - x_2 = 0$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{or} \quad \mathbf{x}^{(1)} = \begin{pmatrix} a \\ a \end{pmatrix}$$

$$\lambda_2 = -0.04, \quad \begin{pmatrix} 0.02 & 0.02 \\ 0.02 & 0.02 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} 0.02x_1 + 0.02x_2 &= 0 \\ 0.02x_1 + 0.02x_2 &= 0 \end{aligned}\right\} x_1 + x_2 = 0$$

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{or} \quad \mathbf{x}^{(1)} = \begin{pmatrix} b \\ -b \end{pmatrix}$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{A}\mathbf{x}=\lambda\mathbf{x}$$
$$\mathbf{A}\mathbf{x}-\lambda\mathbf{x}=\mathbf{0}$$
$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$$

$$\mathbf{y}=\mathbf{x}e^{\lambda t}$$

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$$

$$\begin{pmatrix}-0.02-\lambda & 0.02\\ 0.02 & -0.02-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=0$$

$$\lambda_1=0,\quad \begin{pmatrix}-0.02 & 0.02\\ 0.02 & -0.02\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=0$$

$$\left.\begin{aligned}-0.02x_1+0.02x_2&=0\\ 0.02x_1-0.02x_2&=0\end{aligned}\right\}\ x_1-x_2=0$$

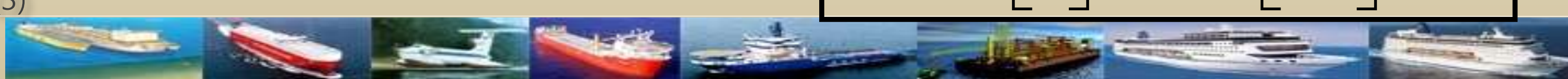
$$\mathbf{x}^{(1)}=\begin{pmatrix}1\\ 1\end{pmatrix}\quad \text{or}\quad \mathbf{x}^{(1)}=\begin{pmatrix}a\\ a\end{pmatrix}$$

$$\lambda_2=-0.04,\quad \begin{pmatrix}0.02 & 0.02\\ 0.02 & 0.02\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=0$$

$$\left.\begin{aligned}0.02x_1+0.02x_2&=0\\ 0.02x_1+0.02x_2&=0\end{aligned}\right\}\ x_1+x_2=0$$

$$\mathbf{x}^{(2)}=\begin{pmatrix}1\\ -1\end{pmatrix}\quad \text{or}\quad \mathbf{x}^{(1)}=\begin{pmatrix}b\\ -b\end{pmatrix}$$

$$\boxed{\begin{aligned}\mathbf{y}&=c_1\mathbf{x}^{(1)}e^{\lambda_1 t}+c_2\mathbf{x}^{(2)}e^{\lambda_2 t}\\ &=c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{0t}+c_2\begin{bmatrix}1\\ -1\end{bmatrix}e^{-0.04t}\\ &\quad or\ c_3\begin{bmatrix}a\\ a\end{bmatrix}e^{0t}+c_4\begin{bmatrix}b\\ -b\end{bmatrix}e^{-0.04t}\end{aligned}}$$



Ex.) mixing problem Involving Two Tanks

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{0t} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$$

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_3\begin{bmatrix}a\\a\end{bmatrix}e^{0t} + c_4\begin{bmatrix}b\\-b\end{bmatrix}e^{-0.04t}$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{0t} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$$

when $y_1(0) = 0,\ y_2(0) = 150$

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_3\begin{bmatrix}a\\a\end{bmatrix}e^{0t} + c_4\begin{bmatrix}b\\-b\end{bmatrix}e^{-0.04t}$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{0t} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$$

when $y_1(0) = 0,\ y_2(0) = 150$

$$\mathbf{y}(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}c_1 + c_2\\c_1 - c_2\end{bmatrix} = \begin{bmatrix}0\\150\end{bmatrix}$$

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_3\begin{bmatrix}a\\a\end{bmatrix}e^{0t} + c_4\begin{bmatrix}b\\-b\end{bmatrix}e^{-0.04t}$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{0t} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$$

when $y_1(0)=0,\ y_2(0)=150$

$$\mathbf{y}(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}c_1+c_2\\c_1-c_2\end{bmatrix} = \begin{bmatrix}0\\150\end{bmatrix}$$

$$c_1 = 75,\ c_2 = -75$$

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_3\begin{bmatrix}a\\a\end{bmatrix}e^{0t} + c_4\begin{bmatrix}b\\-b\end{bmatrix}e^{-0.04t}$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{0t} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$$

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_3\begin{bmatrix}a\\a\end{bmatrix}e^{0t} + c_4\begin{bmatrix}b\\-b\end{bmatrix}e^{-0.04t}$$

when $y_1(0) = 0,\ y_2(0) = 150$

$$\mathbf{y}(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}c_1 + c_2\\c_1 - c_2\end{bmatrix} = \begin{bmatrix}0\\150\end{bmatrix}$$

$$c_1 = 75,\ c_2 = -75$$

$$\mathbf{y}(0) = 75\begin{bmatrix}1\\1\end{bmatrix} - 75\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\150\end{bmatrix}$$

Ex.) mixing problem Involving Two Tanks

$$\mathbf{y} = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{0t} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$$

when $y_1(0)=0,\ y_2(0)=150$

$$\mathbf{y}(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}c_1+c_2\\c_1-c_2\end{bmatrix} = \begin{bmatrix}0\\150\end{bmatrix}$$

$$c_1 = 75,\ c_2 = -75$$

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Ex.) mixing problem Involving Two Tanks

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when $y_1(0) = 0,\ y_2(0) = 150$

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$$=$$

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Ex.) mixing problem Involving Two Tanks

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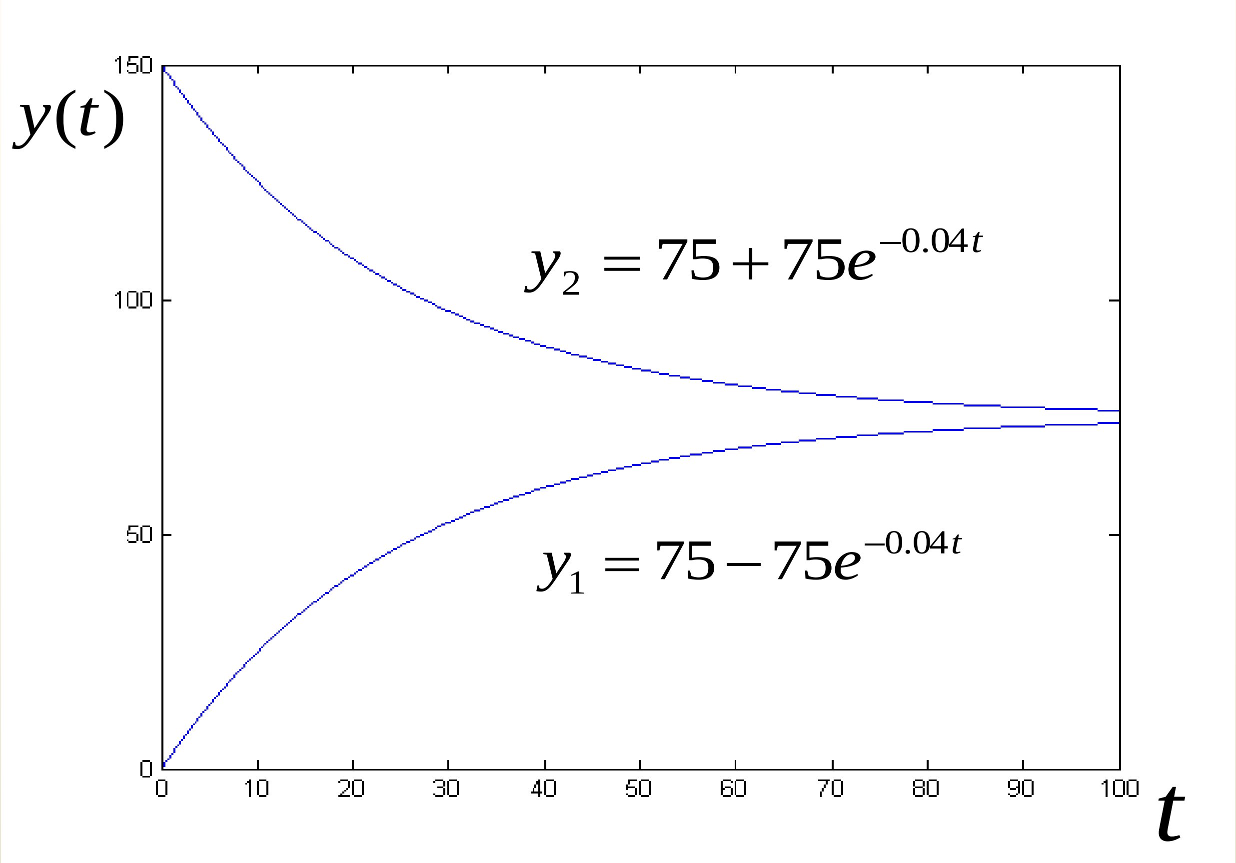
t : time [min.]

$y_1(t)$: the amount of salt in T1 [lb]

$y_2(t)$: the amount of salt in T2 [lb]



Ex.) Electrical Network

$$E_R = RI$$
$$E_L = LI'$$
$$E_C = \frac{1}{C}\int I dt$$

inductor

capacitor

capacitor

resistor

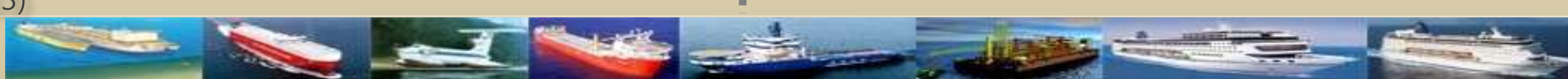
volts

henry

farad

ohmn

ohmn



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Ex.) Electrical Network

Find the current I_1(t), I_2(t) in the network. Assume all currents and charges to be zero at t=0, the instant when the switch is closed

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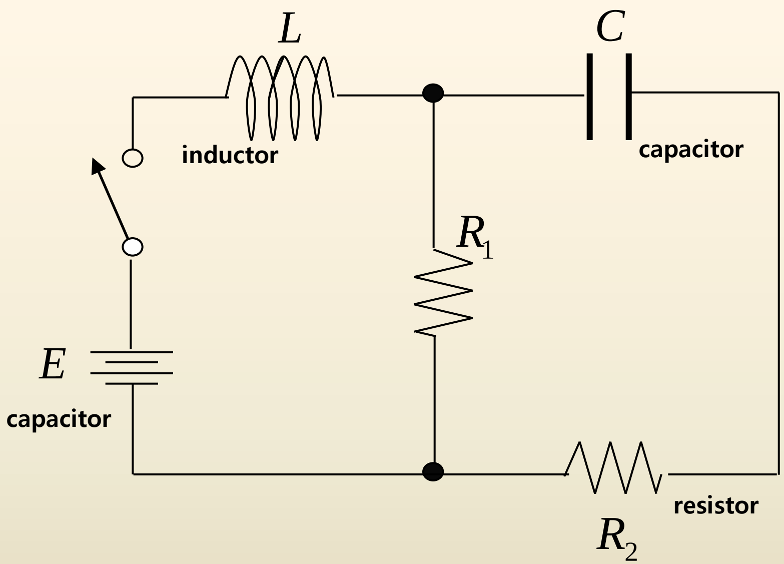

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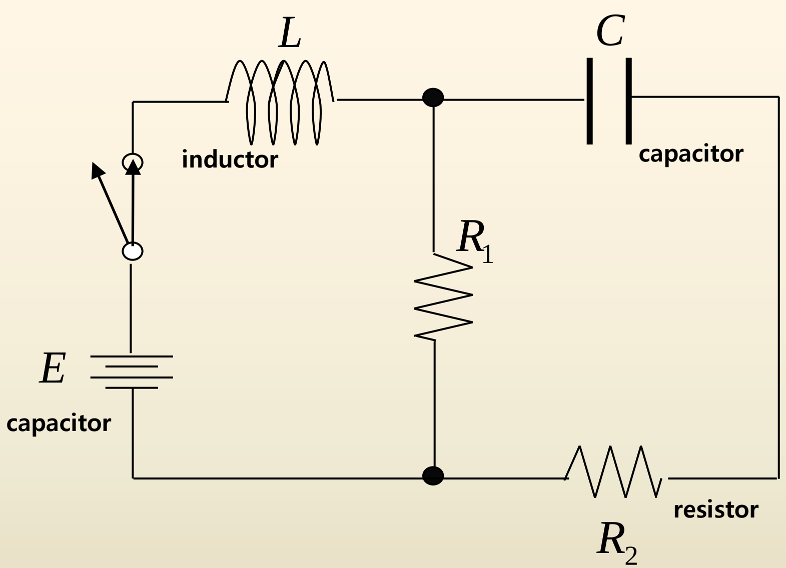

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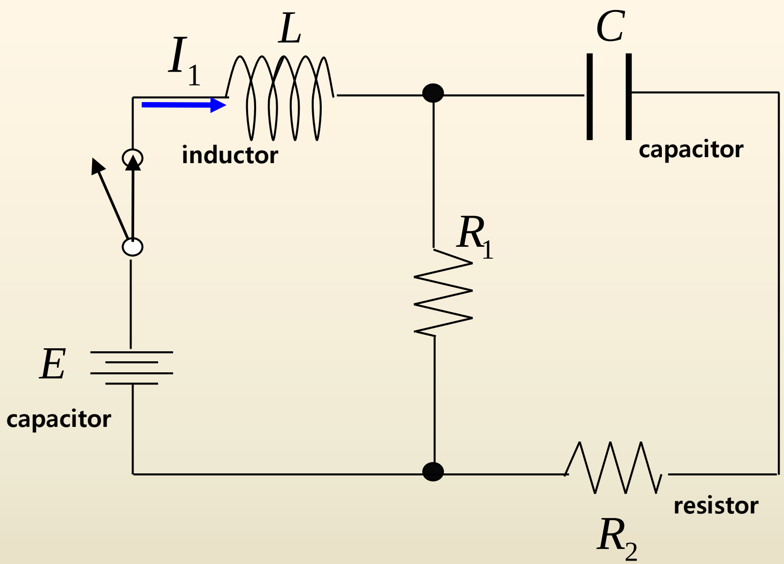

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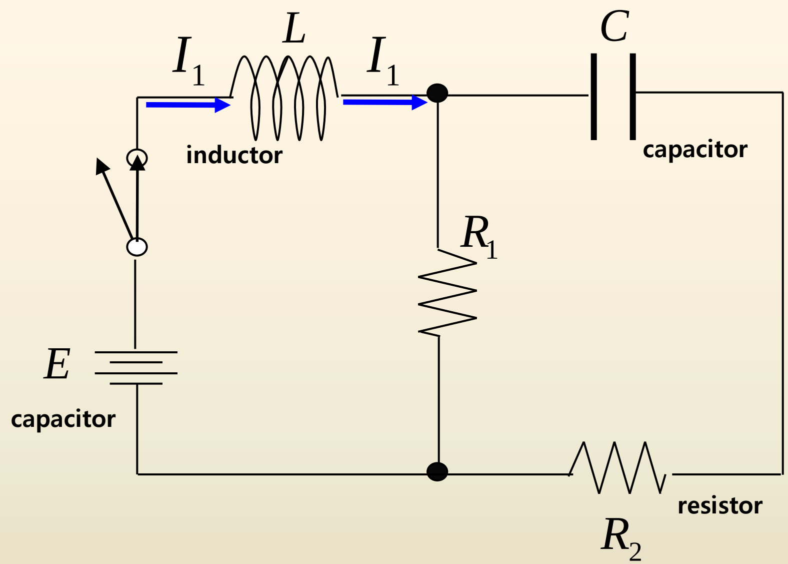

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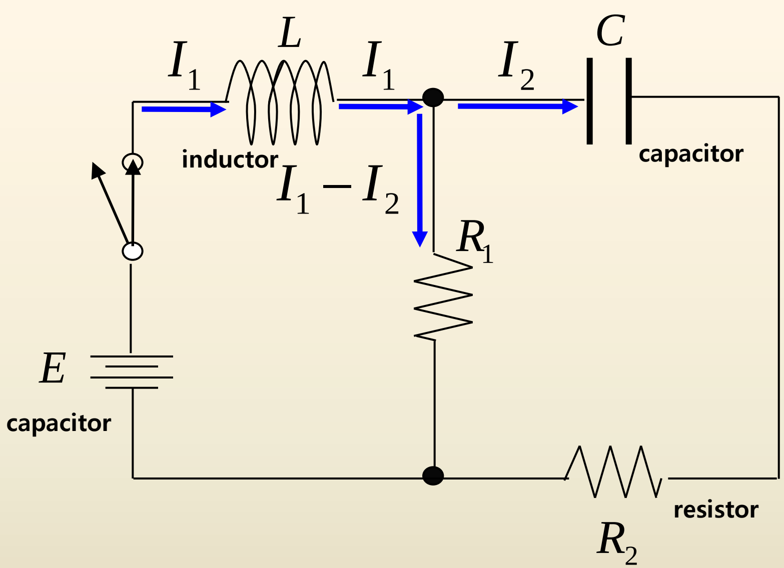

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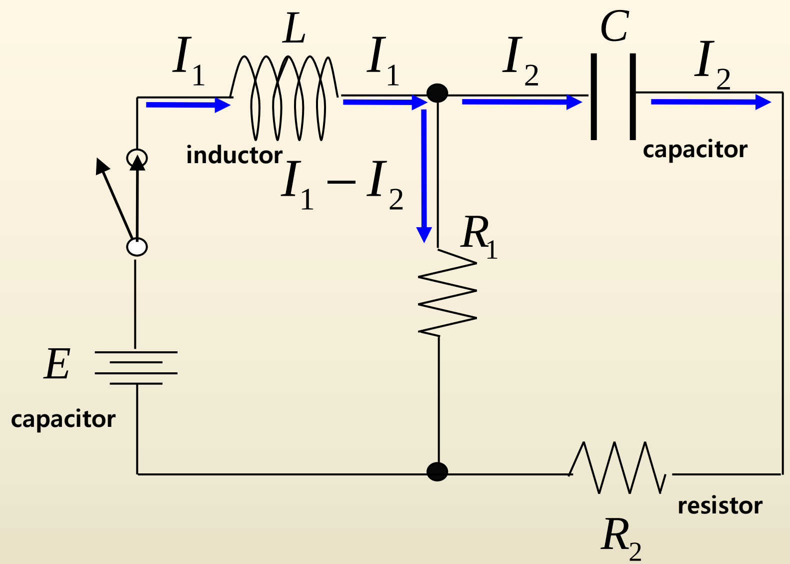

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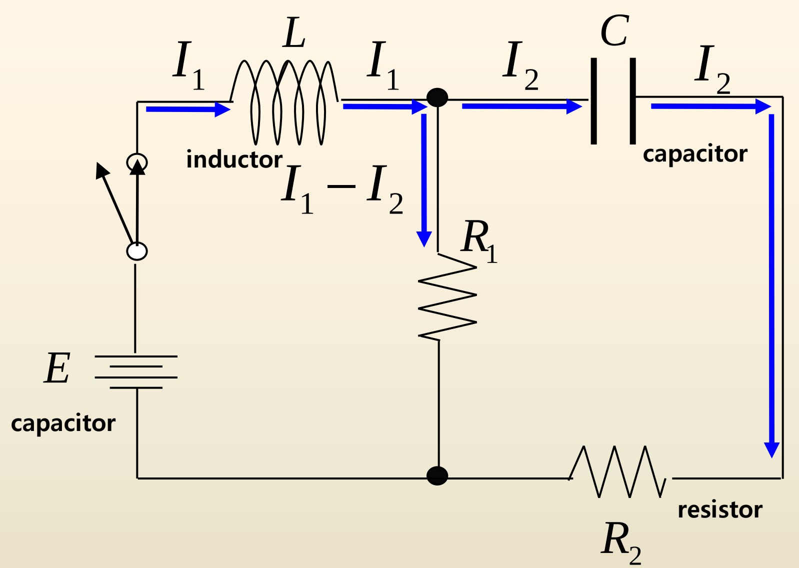

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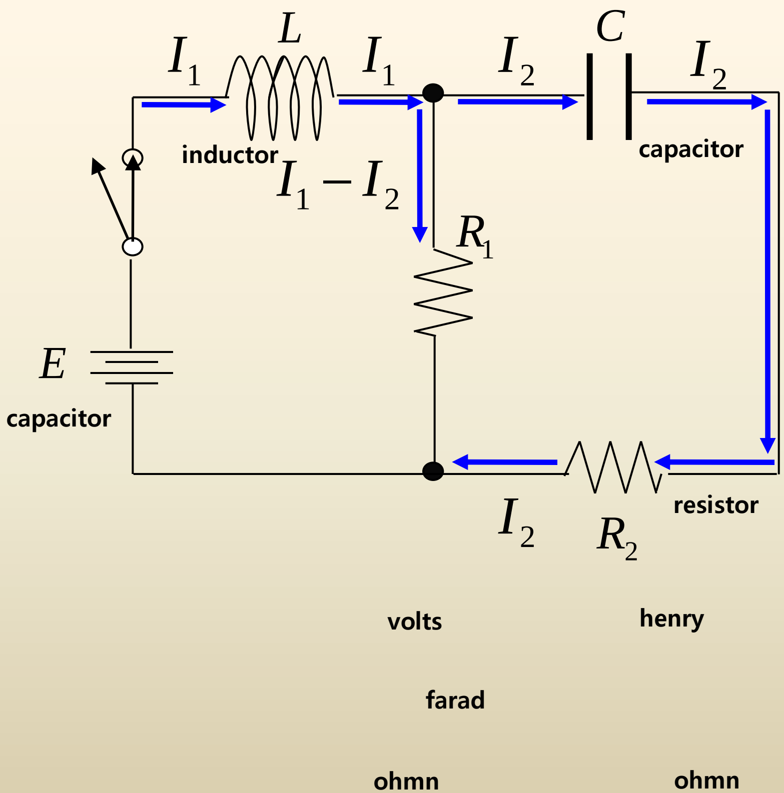

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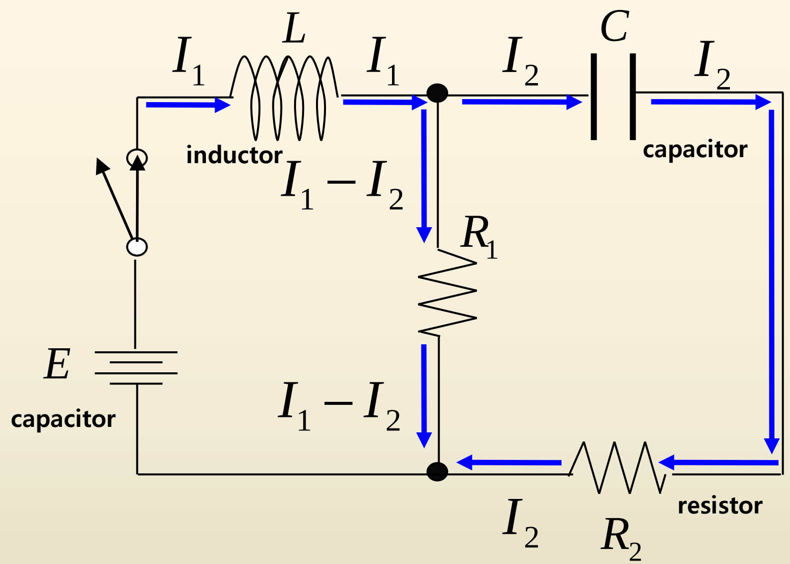

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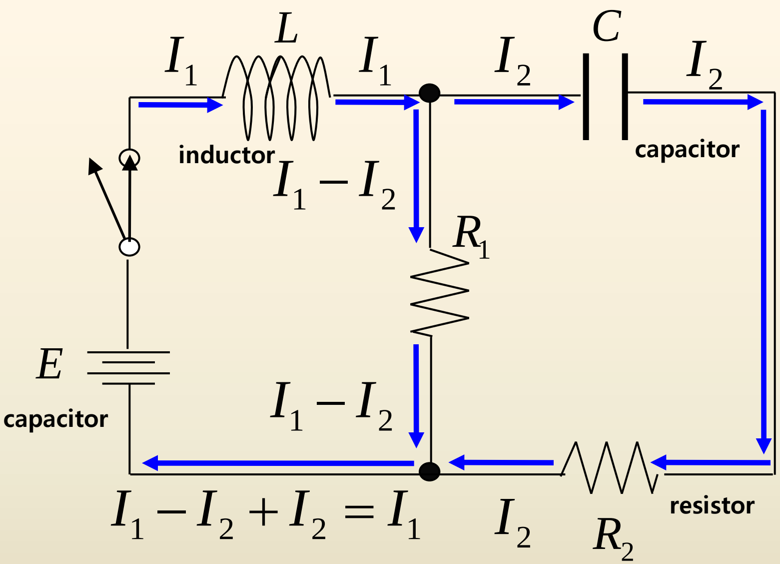

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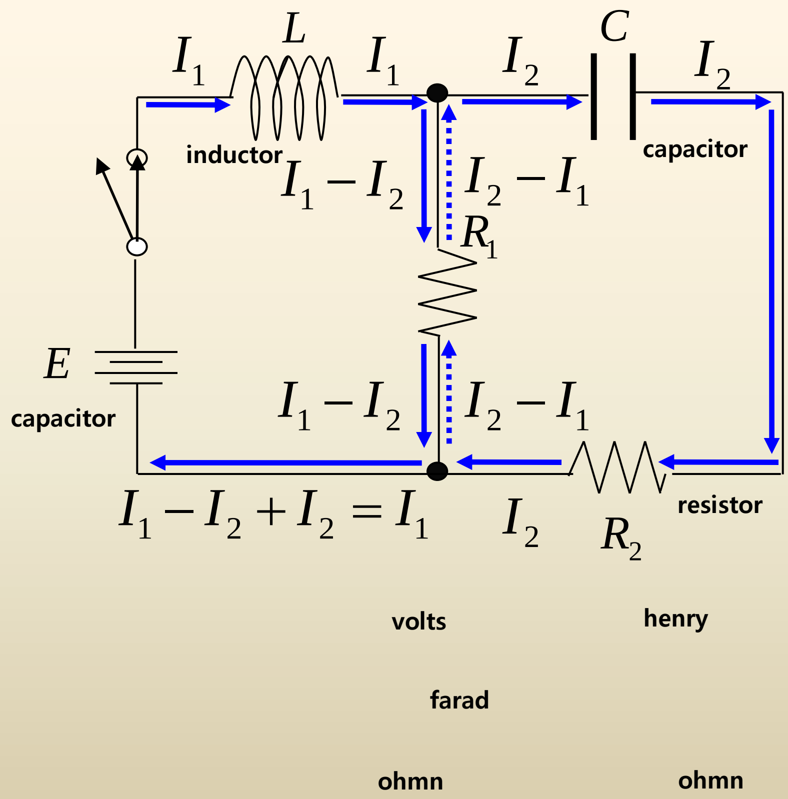

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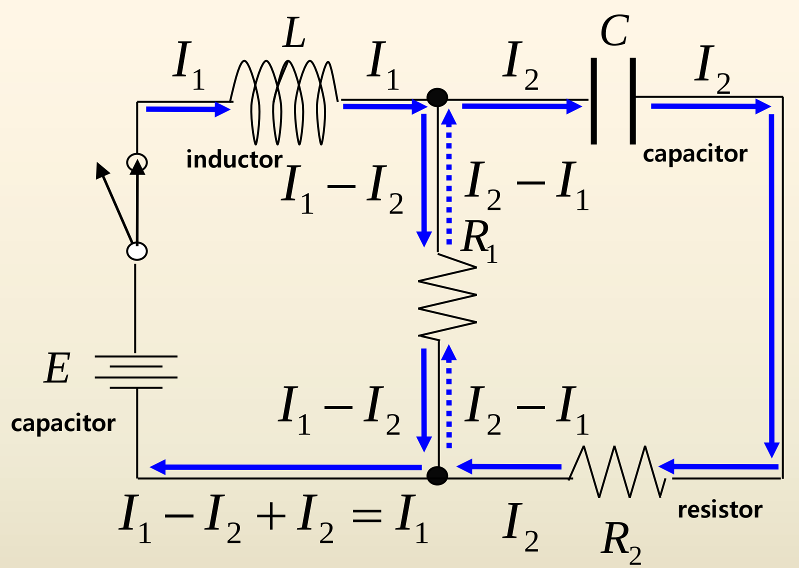

$E = 12V$ volts , $L = 1H$ henry

$C = 0.25F$ farad

$R_1 = 4\Omega$ ohmn , $R_2 = 6\Omega$ ohmn

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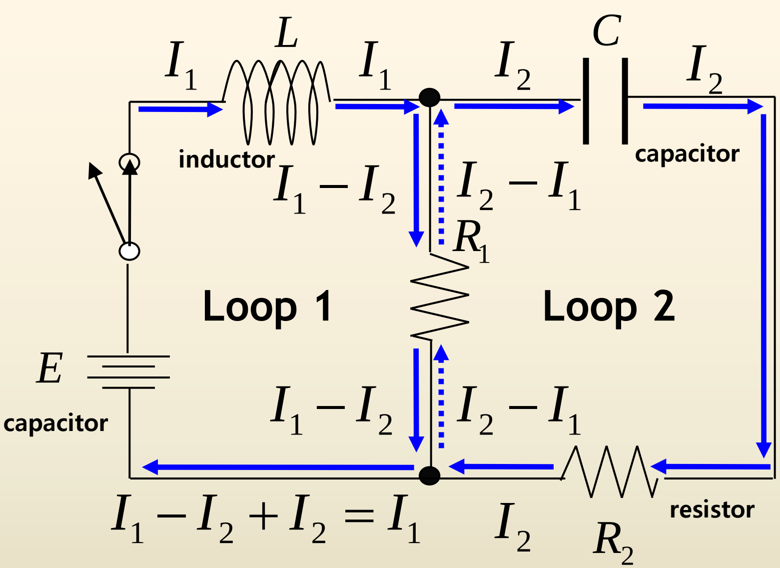

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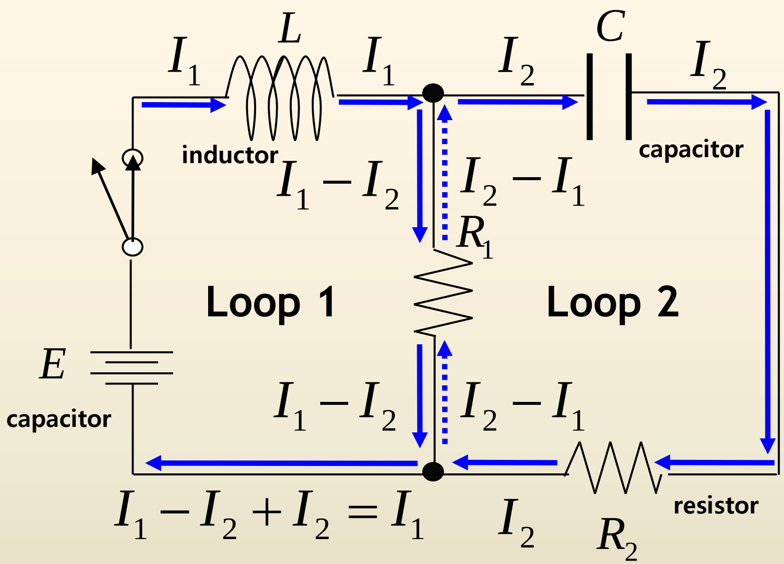

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- **By KVL(Kirchhoff's Voltage Law)**

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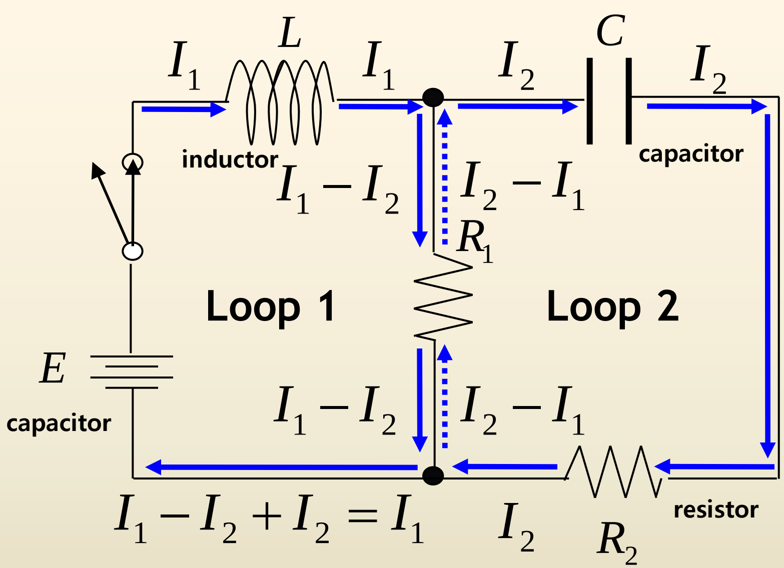
Find the current $I_1(t)$, $I_2(t)$ in the network. Assume all currents and charges to be zero at $t=0$, the instant when the switch is closed



$E = 12V$ volts , $L = 1H$ henry

$C = 0.25F$ farad

$R_1 = 4\Omega$ ohmn , $R_2 = 6\Omega$ ohmn

- **By KVL(Kirchhoff's Voltage Law)**

At Loop 1

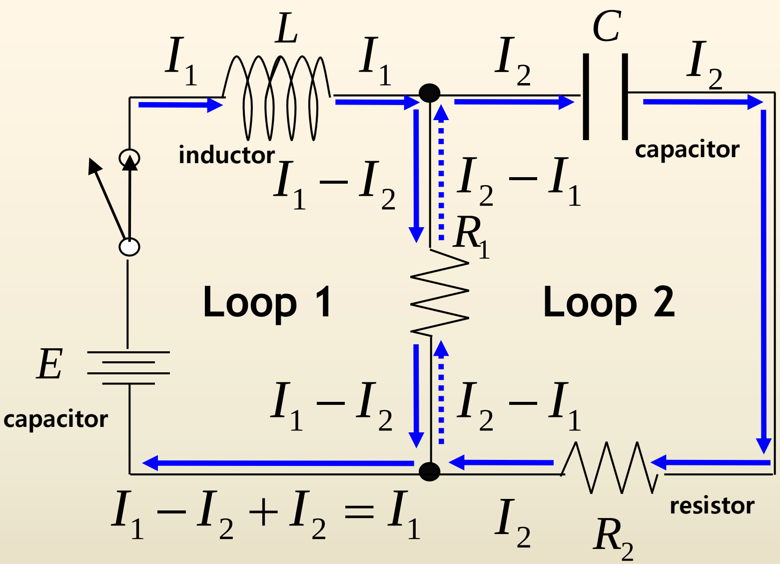
$$E_L + E_R + E_C = E$$

$$LI_1' + R_1(I_1 - I_2) + 0 = E$$

$$E_R = RI$$
$$E_L = LI'$$
$$E_C = \frac{1}{C}\int I\,dt$$

Ex.) Electrical Network

Find the current $I_1(t)$, $I_2(t)$ in the network. Assume all currents and charges to be zero at $t=0$, the instant when the switch is closed



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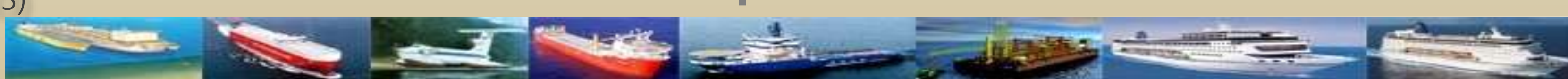
At Loop 2

$$E_L + E_R + E_C = E$$

$$0 + R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2\,dt = 0$$



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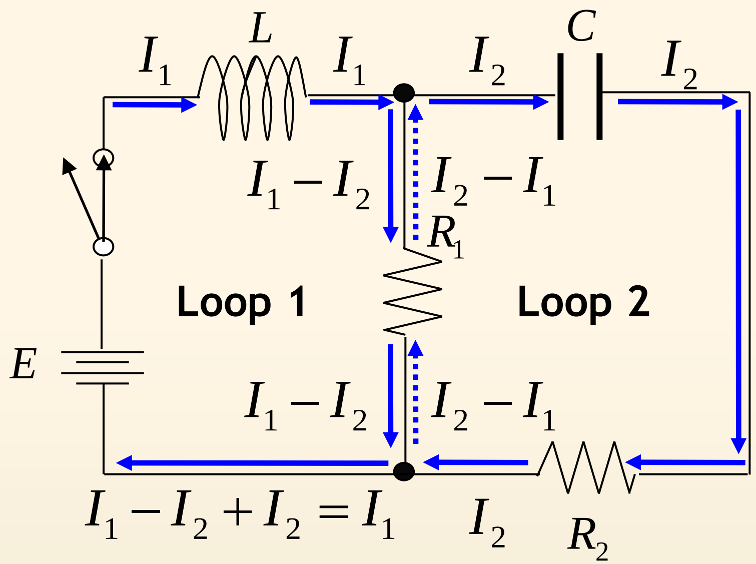

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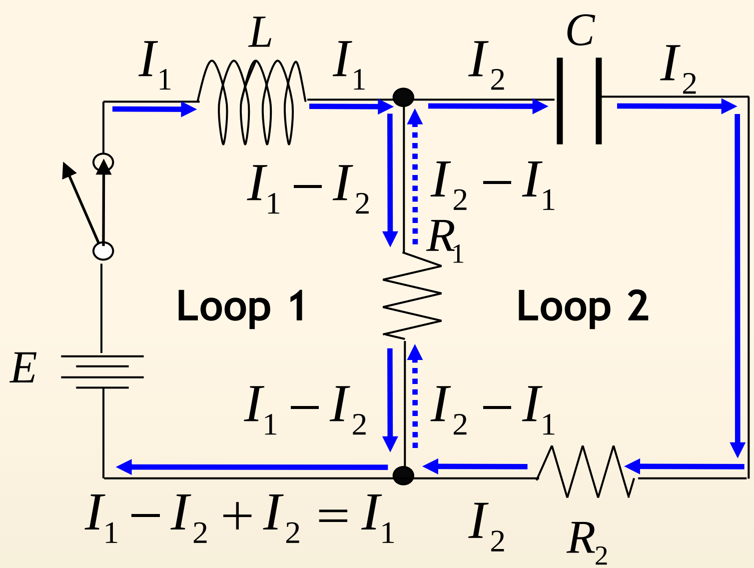

At Loop 1

$$LI_1' + R_1(I_1 - I_2) = E$$

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Ex.) Electrical Network

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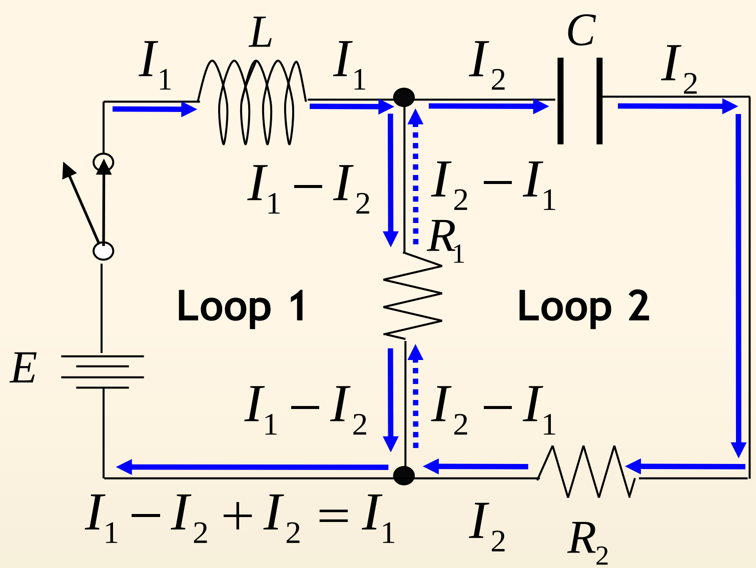

At Loop 1

$$LI_1' + R_1(I_1 - I_2) = E$$

$$LI_1' + R_1 I_1 - R_1 I_2 = E$$

$$LI_1' = -R_1 I_1 + R_1 I_2 + E$$

Ex.) Electrical Network

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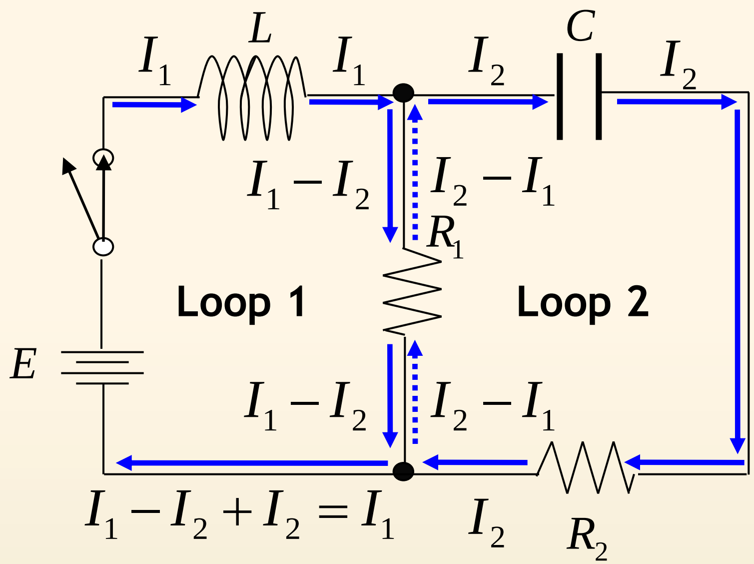

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$$LI_1' + R_1(I_1 - I_2) = E$$

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$$I_1' = -\frac{R_1}{L} I_1 + \frac{R_1}{L} I_2 + \frac{E}{L}$$

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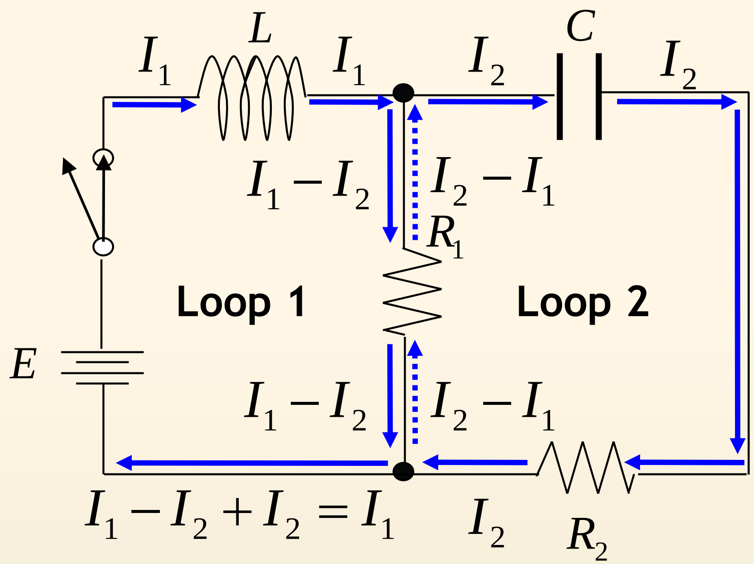

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$$I_1' = -\frac{R_1}{L} I_1 + \frac{R_1}{L} I_2 + \frac{E}{L}$$

$$I_1' = -\frac{4}{1} I_1 + \frac{4}{1} I_2 + \frac{12}{1}$$

Ex.) Electrical Network

$$E_L + E_R + E_C = E$$

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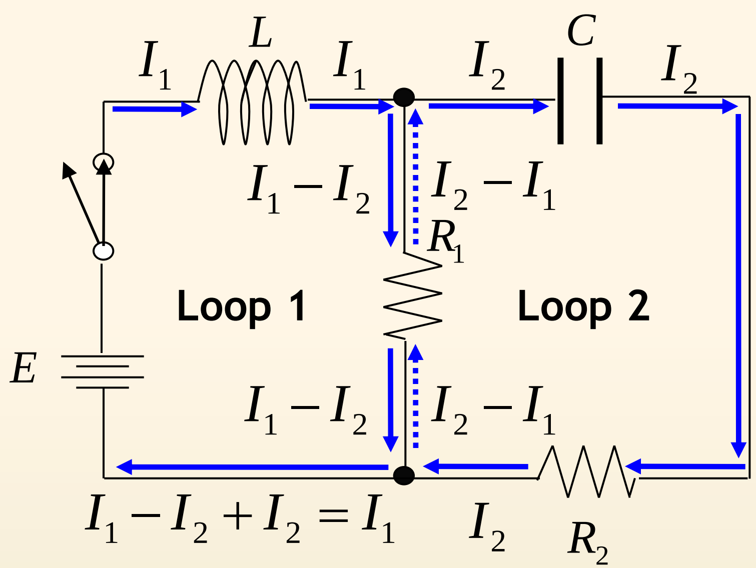

$$LI_1' = -R_1 I_1 + R_1 I_2 + E$$

$$I_1' = -\frac{R_1}{L} I_1 + \frac{R_1}{L} I_2 + \frac{E}{L}$$

$$I_1' = -\frac{4}{1} I_1 + \frac{4}{1} I_2 + \frac{12}{1}$$

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 1

$$LI_1' + R_1(I_1 - I_2) = E$$

$$LI_1' + R_1 I_1 - R_1 I_2 = E$$

Ex.) Electrical Network

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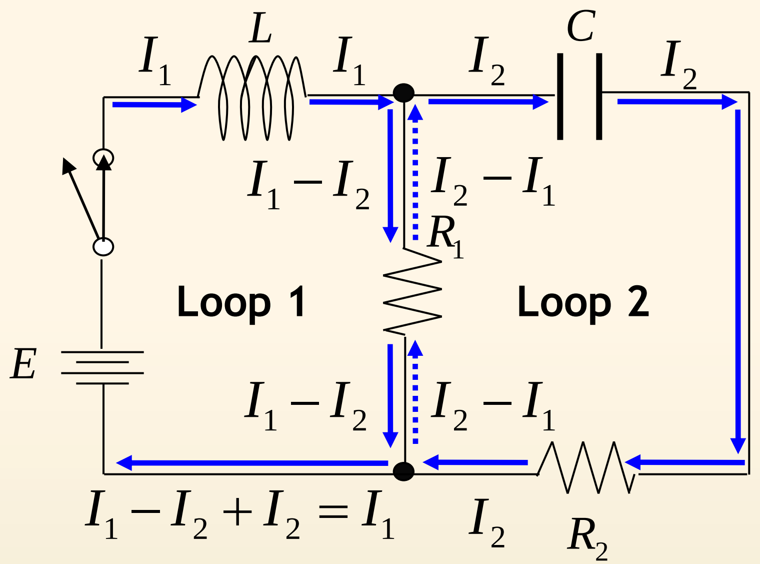

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

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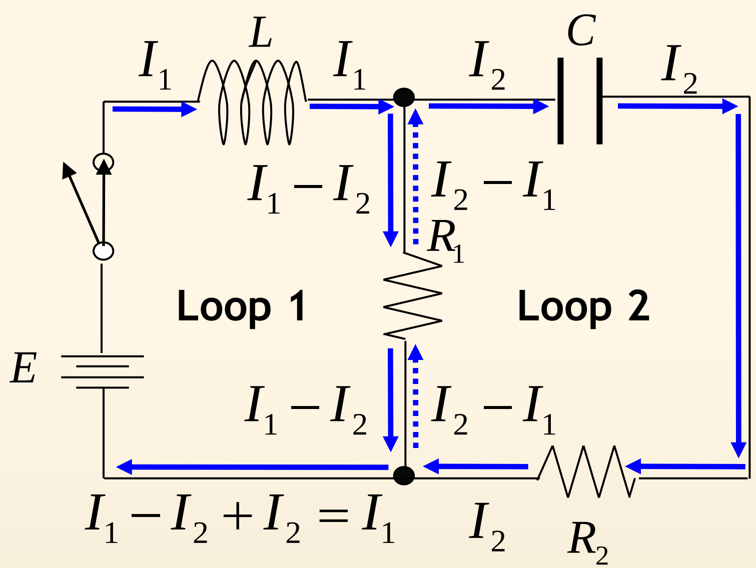

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

Ex.) Electrical Network

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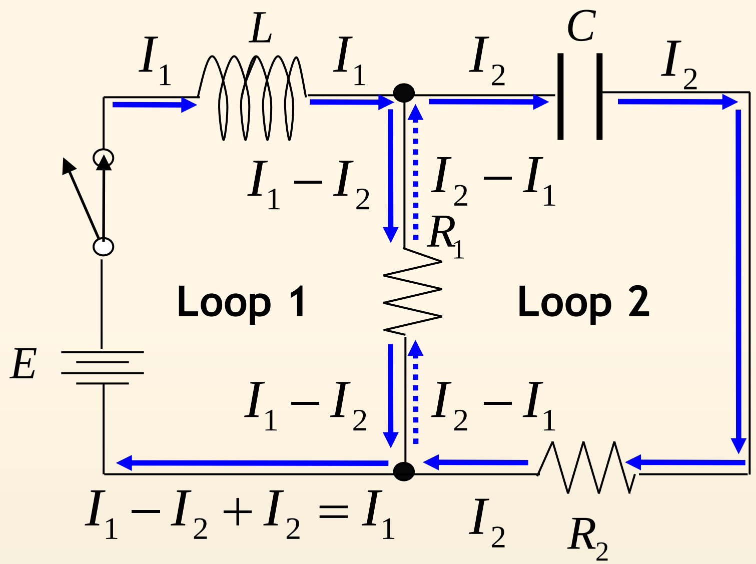

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$(R_1 + R_2)I_2 - R_1 I_1 + \frac{1}{C}\int I_2 dt = 0$$

Ex.) Electrical Network

$$E_L + E_R + E_C = E$$

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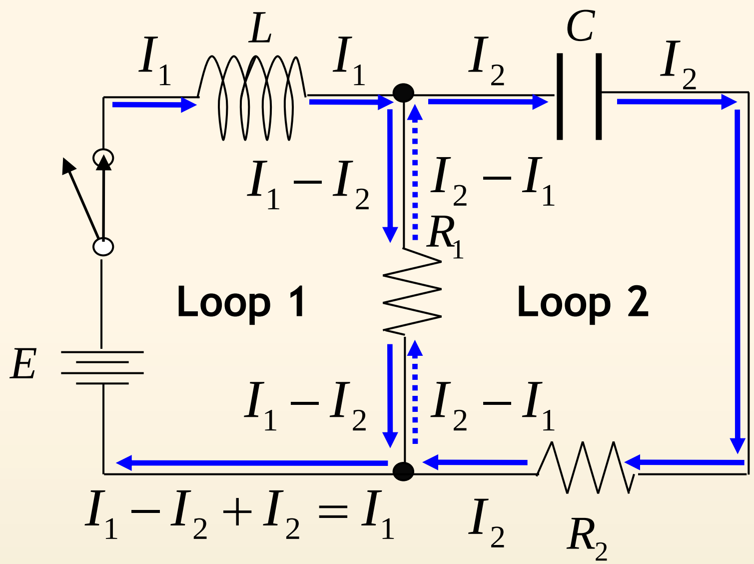

$$I_1 - I_2 + I_2 = I_1$$

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$(R_1 + R_2) I_2 - R_1 I_1 + \frac{1}{C}\int I_2 dt = 0$$

Differentiate

Ex.) Electrical Network

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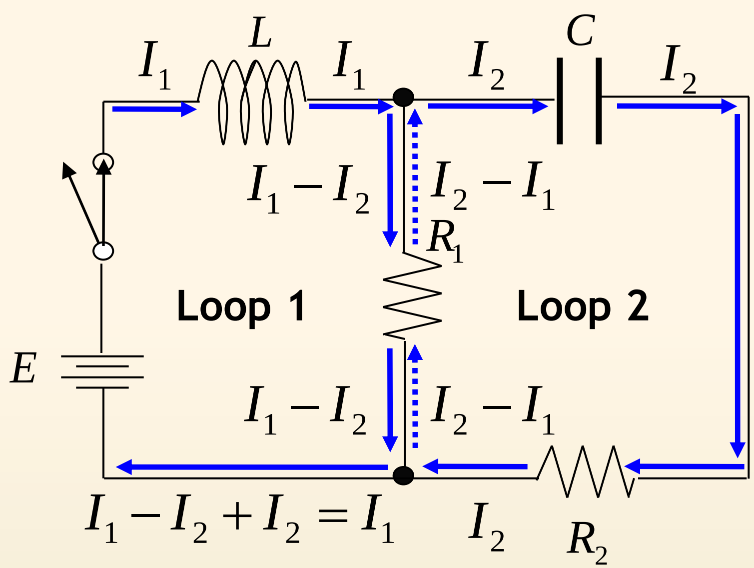

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$(R_1 + R_2)I_2 - R_1 I_1 + \frac{1}{C}\int I_2 dt = 0$$

Differentiate

$$(R_1 + R_2)I_2' - R_1 I_1' + \frac{1}{C} I_2 = 0$$

Ex.) Electrical Network

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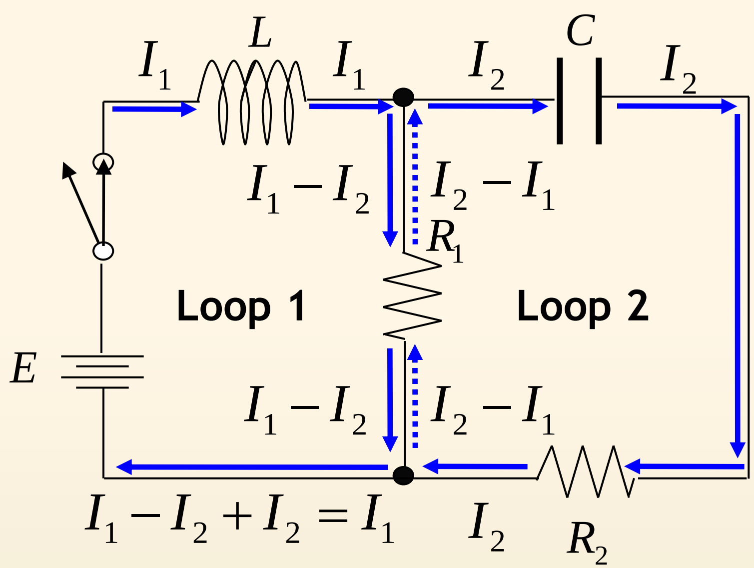


$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

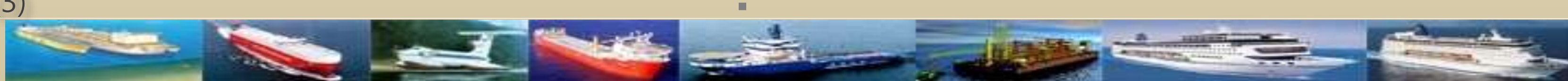
$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$(R_1 + R_2)I_2 - R_1 I_1 + \frac{1}{C}\int I_2 dt = 0$$

Differentiate

$$(R_1 + R_2)I_2' - R_1 I_1' + \frac{1}{C} I_2 = 0$$

$$(4+6)I_2' - 4I_1' + \frac{1}{0.25} I_2 = 0$$



Ex.) Electrical Network

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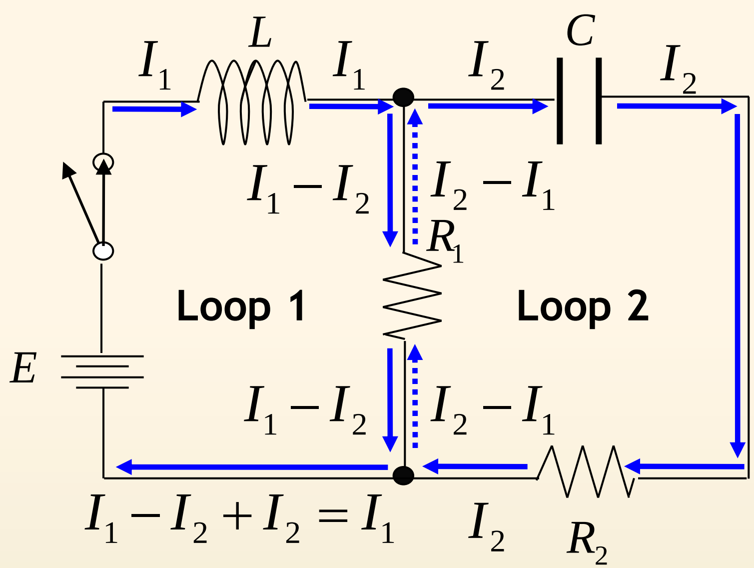


$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

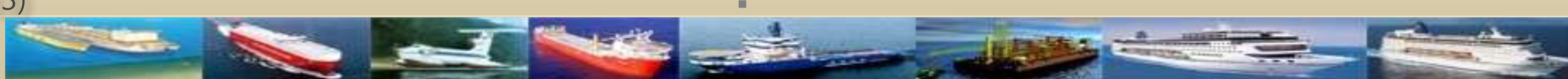
$$(R_1 + R_2)I_2 - R_1 I_1 + \frac{1}{C}\int I_2 dt = 0$$

Differentiate

$$(R_1 + R_2)I_2' - R_1 I_1' + \frac{1}{C} I_2 = 0$$

$$(4+6)I_2' - 4I_1' + \frac{1}{0.25} I_2 = 0$$

$$10I_2' - 4I_1' + 4I_2 = 0$$



Ex.) Electrical Network

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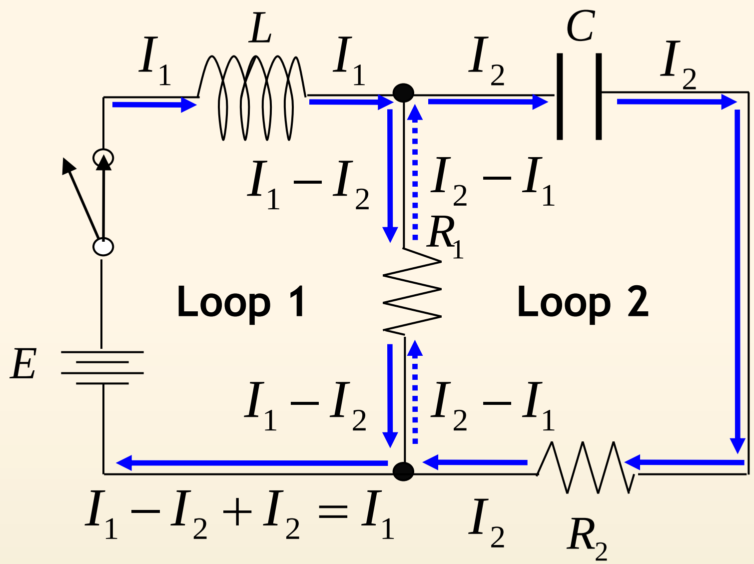

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$(R_1 + R_2)I_2 - R_1 I_1 + \frac{1}{C}\int I_2 dt = 0$$

Differentiate

$$(R_1 + R_2)I_2' - R_1 I_1' + \frac{1}{C} I_2 = 0$$

$$(4+6)I_2' - 4I_1' + \frac{1}{0.25} I_2 = 0$$

$$10I_2' - 4I_1' + 4I_2 = 0$$

$$10I_2' - 4(-4I_1 + 4I_2 + 12) + 4I_2 = 0$$

Ex.) Electrical Network

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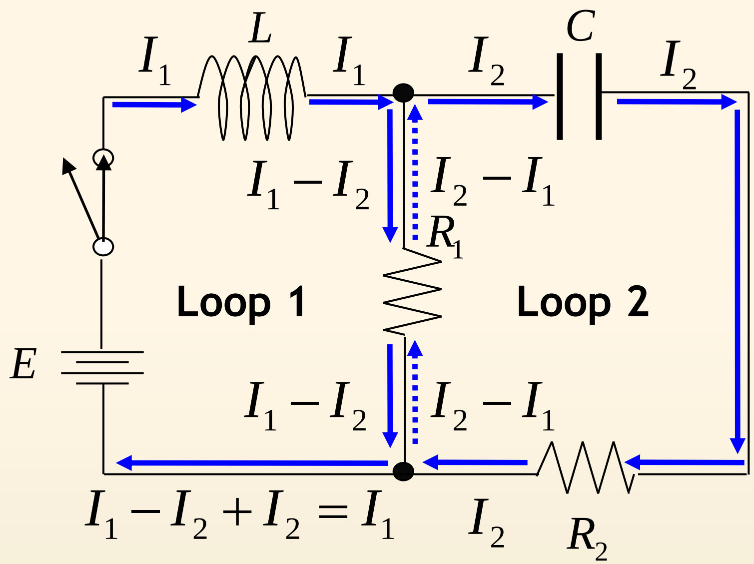

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

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$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$(R_1 + R_2)I_2 - R_1 I_1 + \frac{1}{C}\int I_2 dt = 0$$

Differentiate

$$(R_1 + R_2)I_2' - R_1 I_1' + \frac{1}{C} I_2 = 0$$

$$(4+6)I_2' - 4I_1' + \frac{1}{0.25} I_2 = 0$$

$$10I_2' - 4I_1' + 4I_2 = 0$$

$$10I_2' - 4(-4I_1 + 4I_2 + 12) + 4I_2 = 0$$

$$10I_2' = 4(-4I_1 + 4I_2 + 12) - 4I_2$$

Ex.) Electrical Network

$$E_L + E_R + E_C = E$$

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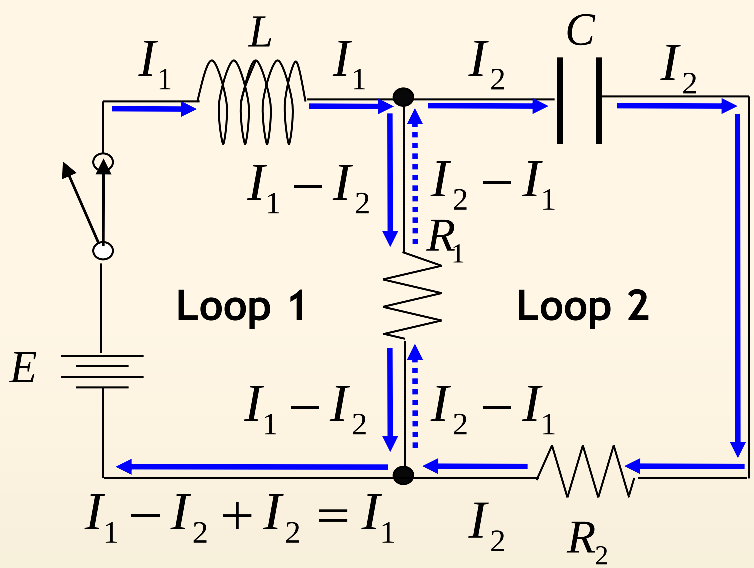

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$(R_1 + R_2)I_2 - R_1 I_1 + \frac{1}{C}\int I_2 dt = 0$$

Differentiate

$$(R_1 + R_2)I_2' - R_1 I_1' + \frac{1}{C} I_2 = 0$$

$$(4+6)I_2' - 4I_1' + \frac{1}{0.25} I_2 = 0$$

$$10I_2' - 4I_1' + 4I_2 = 0$$

$$10I_2' - 4(-4I_1 + 4I_2 + 12) + 4I_2 = 0$$

$$10I_2' = 4(-4I_1 + 4I_2 + 12) - 4I_2$$

$$10I_2' = -16I_2 + 16I_2 + 48 - 4I_2$$

Ex.) Electrical Network

$$E_L + E_R + E_C = E$$

$E = 12V,\quad L = 1H$

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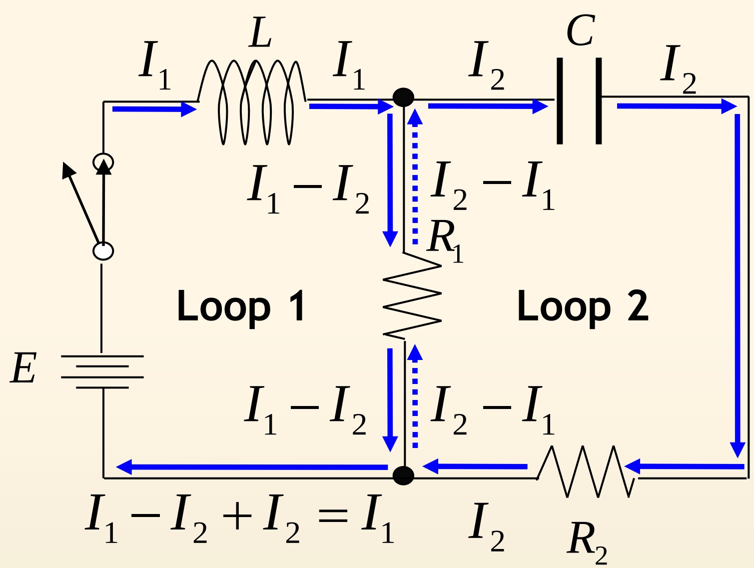

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2\,dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2\,dt = 0$$

$$(R_1 + R_2)I_2 - R_1 I_1 + \frac{1}{C}\int I_2\,dt = 0$$

Differentiate

$$(R_1 + R_2)I_2' - R_1 I_1' + \frac{1}{C} I_2 = 0$$

$$(4+6)I_2' - 4I_1' + \frac{1}{0.25} I_2 = 0$$

$$10I_2' - 4I_1' + 4I_2 = 0$$

$$10I_2' - 4(-4I_1 + 4I_2 + 12) + 4I_2 = 0$$

$$10I_2' = 4(-4I_1 + 4I_2 + 12) - 4I_2$$

$$10I_2' = -16I_2 + 16I_2 + 48 - 4I_2$$

$$10I_2' = -16I_2 + 12I_2 + 48$$

Ex.) Electrical Network

$$E_L + E_R + E_C = E$$

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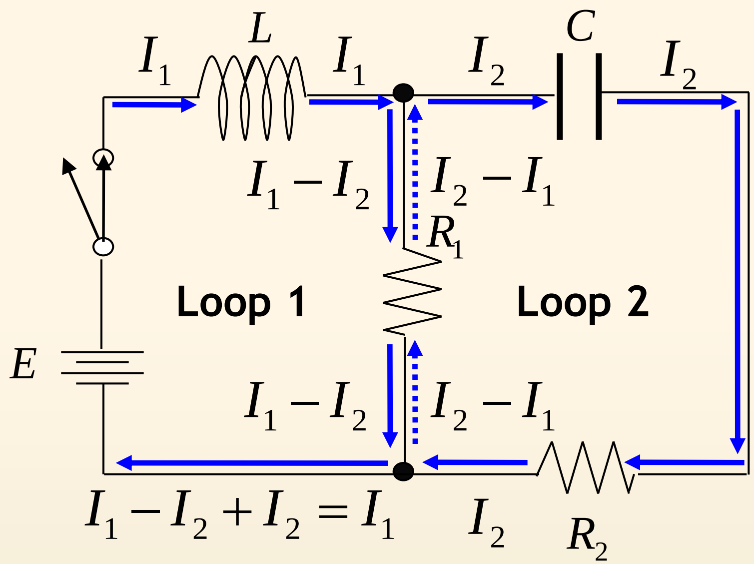

$$\boxed{I_1' = -4I_1 + 4I_2 + 12}$$

At Loop 2

$$R_1(I_2 - I_1) + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$R_1 I_2 - R_1 I_1 + R_2 I_2 + \frac{1}{C}\int I_2 dt = 0$$

$$(R_1 + R_2)I_2 - R_1 I_1 + \frac{1}{C}\int I_2 dt = 0$$

Differentiate

$$(R_1 + R_2)I_2' - R_1 I_1' + \frac{1}{C} I_2 = 0$$

$$(4+6)I_2' - 4I_1' + \frac{1}{0.25} I_2 = 0$$

$$10I_2' - 4I_1' + 4I_2 = 0$$

$$10I_2' - 4(-4I_1 + 4I_2 + 12) + 4I_2 = 0$$

$$10I_2' = 4(-4I_1 + 4I_2 + 12) - 4I_2$$

$$10I_2' = -16I_2 + 16I_2 + 48 - 4I_2$$

$$10I_2' = -16I_2 + 12I_2 + 48$$

$$\boxed{I_2' = -1.6I_2 + 1.2I_2 + 4.8}$$

Ex.) Electrical Network

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$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

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$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

$\longrightarrow$ **System of nonhomogeneous ODEs**

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

$\longrightarrow$ System of nonhomogeneous ODEs

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

$\longrightarrow$ System of nonhomogeneous ODEs

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

→ **System of nonhomogeneous ODEs**

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

$\longrightarrow$ System of nonhomogeneous ODEs

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

Ex.) Electrical Network

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$\longrightarrow$ System of nonhomogeneous ODEs

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I}$$

Assume that $\mathbf{I} = \mathbf{x}e^{\lambda t}$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

→ System of nonhomogeneous ODEs

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I}$$

Assume that $\mathbf{I} = \mathbf{x}e^{\lambda t}$

$$\mathbf{I}' = \lambda\mathbf{x}e^{\lambda t} = \mathbf{A}\mathbf{I} = \mathbf{A}\mathbf{x}e^{\lambda t}$$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

$\longrightarrow$ System of nonhomogeneous ODEs

$$\mathbf{I}' = \mathbf{AI} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\mathbf{I}' = \mathbf{AI}$$

Assume that $\mathbf{I} = \mathbf{x}e^{\lambda t}$

$$\mathbf{I}' = \underline{\lambda \mathbf{x} e^{\lambda t}} = \mathbf{AI} = \underline{\mathbf{A}\mathbf{x}e^{\lambda t}}$$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

$\longrightarrow$ System of nonhomogeneous ODEs

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I}$$

Assume that $\mathbf{I} = \mathbf{x}e^{\lambda t}$

$$\mathbf{I}' = \underline{\lambda \mathbf{x} e^{\lambda t}} = \mathbf{A}\mathbf{I} = \underline{\mathbf{A}\mathbf{x}e^{\lambda t}}$$

$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$: Eigen Value Problem

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

$\longrightarrow$ System of nonhomogeneous ODEs

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I}$$

Assume that $\mathbf{I} = \mathbf{x}e^{\lambda t}$

$$\mathbf{I}' = \underline{\lambda \mathbf{x} e^{\lambda t}} = \mathbf{A}\mathbf{I} = \underline{\mathbf{A}\mathbf{x}e^{\lambda t}}$$

$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$: **Eigen Value Problem**

$$(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

$\longrightarrow$ **System of nonhomogeneous ODEs**

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I}$$

Assume that $\mathbf{I} = \mathbf{x}e^{\lambda t}$

$$\mathbf{I}' = \underline{\lambda \mathbf{x} e^{\lambda t}} = \mathbf{A}\mathbf{I} = \underline{\mathbf{A}\mathbf{x}e^{\lambda t}}$$

$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$: **Eigen Value Problem**

$$(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$$

$$\det(\mathbf{A} - \lambda I) = 0$$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

$\longrightarrow$ System of nonhomogeneous ODEs

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I}$$

Assume that $\mathbf{I} = \mathbf{x}e^{\lambda t}$

$$\mathbf{I}' = \underline{\lambda \mathbf{x} e^{\lambda t}} = \mathbf{A}\mathbf{I} = \underline{\mathbf{A}\mathbf{x} e^{\lambda t}}$$

$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$: Eigen Value Problem

$$(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$$

$$\det(\mathbf{A} - \lambda I) = 0$$

$$\begin{vmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2.8\lambda + 1.6 = 0$$

Ex.) Electrical Network

$$\begin{cases} I_1' = -4I_1 + 4I_2 + 12 \\ I_2' = -1.6I_1 + 1.2I_2 + 4.8 \end{cases}$$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \boxed{\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}}$$

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$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\boxed{\mathbf{I} = \mathbf{I}_h + \mathbf{I}_p}$$

To calculate $\mathbf{I}_h$, we assume $\mathbf{g} = \mathbf{0}$

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I}$$

Assume that $\mathbf{I} = \mathbf{x}e^{\lambda t}$

$$\mathbf{I}' = \underline{\lambda \mathbf{x} e^{\lambda t}} = \mathbf{A}\mathbf{I} = \underline{\mathbf{A}\mathbf{x}e^{\lambda t}}$$

$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$: Eigen Value Problem

$$(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$$

$$\det(\mathbf{A} - \lambda I) = 0$$

$$\begin{vmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2.8\lambda + 1.6 = 0$$

$$\lambda = \frac{-2.8 \pm \sqrt{2.8^2 - 6.4}}{2}, \qquad \lambda_1 = -2, \quad \lambda_2 = -0.8$$

Ex.) Electrical Network

$$\mathbf{A} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}$$

$$(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$$

Ex.) Electrical Network

$$\mathbf{A} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}$$

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$$\begin{pmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Ex.) Electrical Network

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$$(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

① $\lambda_1 = -2, \begin{pmatrix} -2 & 4 \\ -1.6 & 3.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Ex.) Electrical Network

$$\mathbf{A} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}$$

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$$\begin{pmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

① $\lambda_1 = -2, \begin{pmatrix} -2 & 4 \\ -1.6 & 3.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} -2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 3.2x_2 &= 0 \end{aligned}\right\}$$

Ex.) Electrical Network

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$$\begin{pmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\left.\begin{aligned} -2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 3.2x_2 &= 0 \end{aligned}\right\} x_1 - 2x_2 = 0$$

Ex.) Electrical Network

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$$\left.\begin{aligned} -2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 3.2x_2 &= 0 \end{aligned}\right\} x_1 - 2x_2 = 0$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Ex.) Electrical Network

$$\mathbf{A}=\begin{pmatrix}-4 & 4\\ -1.6 & 1.2\end{pmatrix}$$

$$(\mathbf{A}-\lambda I)\mathbf{x}=\mathbf{0}$$

$$\begin{pmatrix}-4-\lambda & 4\\ -1.6 & 1.2-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

① $\lambda_1=-2,\ \begin{pmatrix}-2 & 4\\ -1.6 & 3.2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\left.\begin{aligned}-2x_1+4x_2&=0\\ -1.6x_1+3.2x_2&=0\end{aligned}\right\}\ x_1-2x_2=0$$

$$\mathbf{x}^{(1)}=\begin{pmatrix}2\\ 1\end{pmatrix}$$

② $\lambda_2=-0.8,\ \begin{pmatrix}-3.2 & 4\\ -1.6 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

Ex.) Electrical Network

$$\mathbf{A}=\begin{pmatrix}-4 & 4\\ -1.6 & 1.2\end{pmatrix}$$

$$(\mathbf{A}-\lambda I)\mathbf{x}=\mathbf{0}$$

$$\begin{pmatrix}-4-\lambda & 4\\ -1.6 & 1.2-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

① $\lambda_1=-2,\ \begin{pmatrix}-2 & 4\\ -1.6 & 3.2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\left.\begin{aligned}-2x_1+4x_2&=0\\ -1.6x_1+3.2x_2&=0\end{aligned}\right\}\ x_1-2x_2=0$$

$$\mathbf{x}^{(1)}=\begin{pmatrix}2\\ 1\end{pmatrix}$$

② $\lambda_2=-0.8,\ \begin{pmatrix}-3.2 & 4\\ -1.6 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\left.\begin{aligned}-3.2x_1+4x_2&=0\\ -1.6x_1+2x_2&=0\end{aligned}\right\}$$

Ex.) Electrical Network

$$\mathbf{A} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}$$

$$(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

① $\lambda_1 = -2,\ \begin{pmatrix} -2 & 4 \\ -1.6 & 3.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} -2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 3.2x_2 &= 0 \end{aligned}\right\} x_1 - 2x_2 = 0$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

② $\lambda_2 = -0.8,\ \begin{pmatrix} -3.2 & 4 \\ -1.6 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} -3.2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 2x_2 &= 0 \end{aligned}\right\} -0.8x_1 + x_2 = 0$$

Ex.) Electrical Network

$$\mathbf{A} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}$$

$$(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

① $\lambda_1 = -2$, $\begin{pmatrix} -2 & 4 \\ -1.6 & 3.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} -2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 3.2x_2 &= 0 \end{aligned}\right\} x_1 - 2x_2 = 0$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

② $\lambda_2 = -0.8$, $\begin{pmatrix} -3.2 & 4 \\ -1.6 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} -3.2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 2x_2 &= 0 \end{aligned}\right\} -0.8x_1 + x_2 = 0$$

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0.8 \end{pmatrix}$$

Ex.) Electrical Network

$$\mathbf{A} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix}$$

$$(\mathbf{A} - \lambda I)\mathbf{x} = \mathbf{0}$$

$$\begin{pmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

① $\lambda_1 = -2,\ \begin{pmatrix} -2 & 4 \\ -1.6 & 3.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} -2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 3.2x_2 &= 0 \end{aligned}\right\} x_1 - 2x_2 = 0$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

② $\lambda_2 = -0.8,\ \begin{pmatrix} -3.2 & 4 \\ -1.6 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\left.\begin{aligned} -3.2x_1 + 4x_2 &= 0 \\ -1.6x_1 + 2x_2 &= 0 \end{aligned}\right\} -0.8x_1 + x_2 = 0$$

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0.8 \end{pmatrix}$$

$$I_h = c_1\mathbf{x}^{(1)}e^{\lambda_1 t} + c_2\mathbf{x}^{(2)}e^{\lambda_2 t}$$

Ex.) Electrical Network

$$\mathbf{A}=\begin{pmatrix}-4 & 4\\ -1.6 & 1.2\end{pmatrix}$$

$$(\mathbf{A}-\lambda I)\mathbf{x}=\mathbf{0}$$

$$\begin{pmatrix}-4-\lambda & 4\\ -1.6 & 1.2-\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

① $\lambda_1=-2,\ \begin{pmatrix}-2 & 4\\ -1.6 & 3.2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\left.\begin{aligned}-2x_1+4x_2&=0\\ -1.6x_1+3.2x_2&=0\end{aligned}\right\}\ x_1-2x_2=0$$

$$\mathbf{x}^{(1)}=\begin{pmatrix}2\\ 1\end{pmatrix}$$

② $\lambda_2=-0.8,\ \begin{pmatrix}-3.2 & 4\\ -1.6 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\left.\begin{aligned}-3.2x_1+4x_2&=0\\ -1.6x_1+2x_2&=0\end{aligned}\right\}\ -0.8x_1+x_2=0$$

$$\mathbf{x}^{(2)}=\begin{pmatrix}1\\ 0.8\end{pmatrix}$$

$$I_h=c_1\mathbf{x}^{(1)}e^{\lambda_1 t}+c_2\mathbf{x}^{(2)}e^{\lambda_2 t}$$

$$\boxed{I_h=c_1\begin{pmatrix}2\\ 1\end{pmatrix}e^{\lambda_1 t}+c_2\begin{pmatrix}1\\ 0.8\end{pmatrix}e^{\lambda_2 t}}$$

Ex.) Electrical Network

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$$

$$I_h = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{\lambda_2 t}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$\text{Paricular Solution} : \mathbf{I}_p ,$

Ex.) Electrical Network

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$$

$$I_h = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{\lambda_2 t}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\text{Paricular Solution} : \mathbf{I}_p ,$$

Nonhomogeneous term :

Ex.) Electrical Network

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$$

$$I_h = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{\lambda_2 t}$$

$$\mathbf{I}' = \mathbf{A}\mathbf{I} + \mathbf{g}$$

$$\text{Paricular Solution} : \mathbf{I}_p ,$$

Nonhomogeneous term : $\begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$

Ex.) Electrical Network

$$\begin{pmatrix} I_1' \\ I_2' \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$$

$$I_h = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{\lambda_2 t}$$

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General Solution

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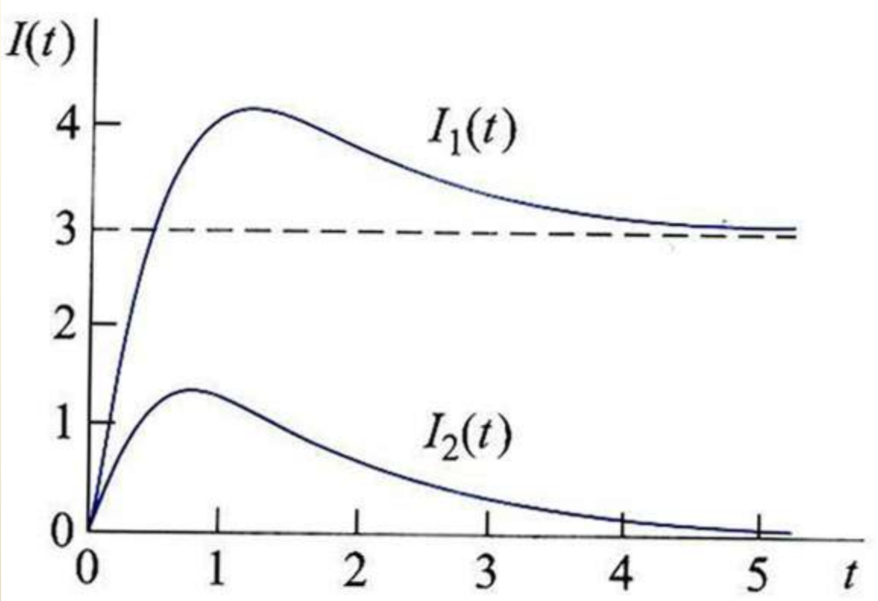

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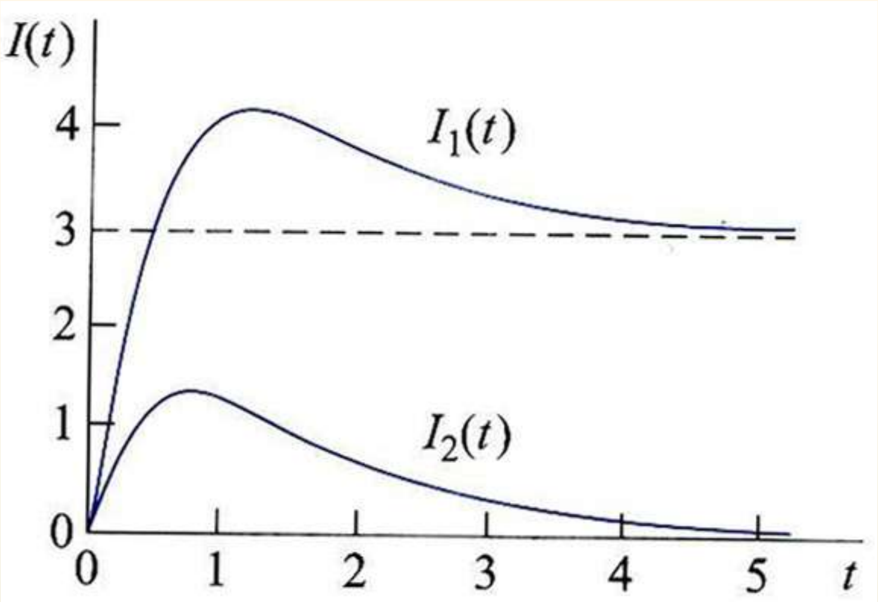

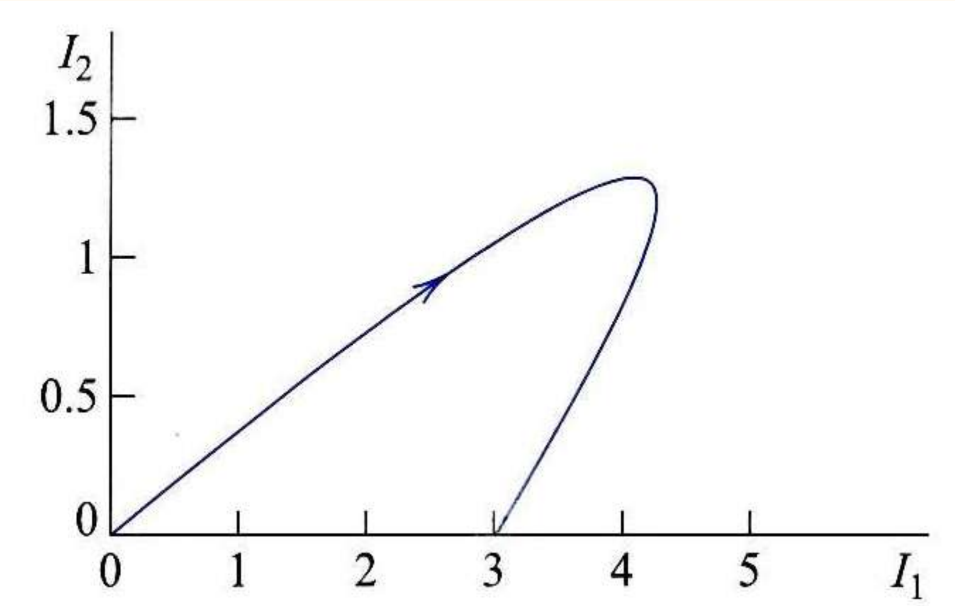

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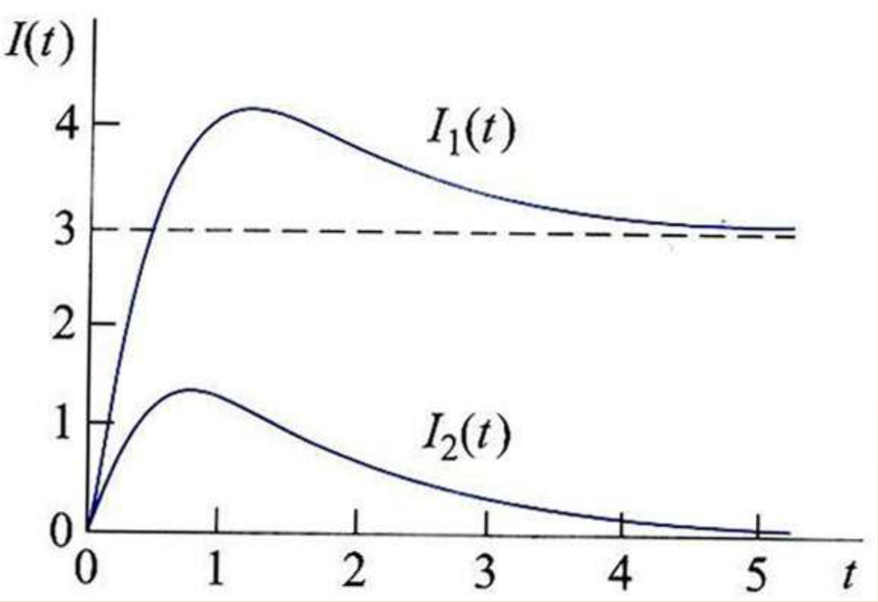
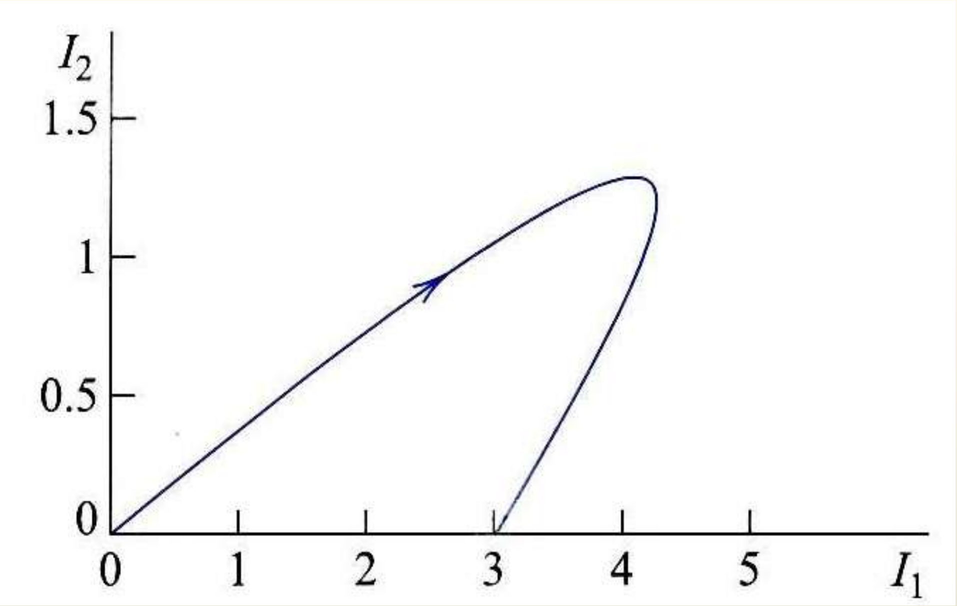


Graph at I-t plane



Graph at I_1-I_2 plane

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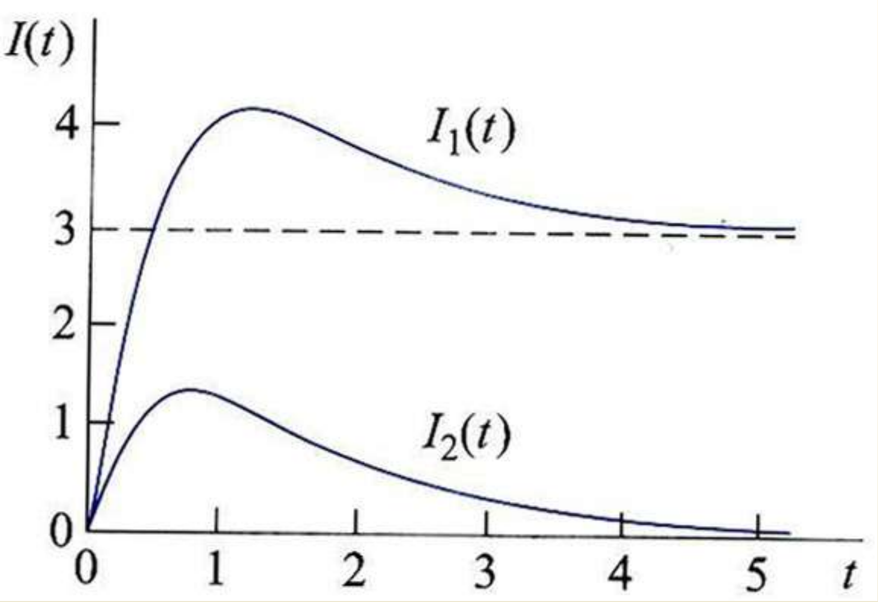

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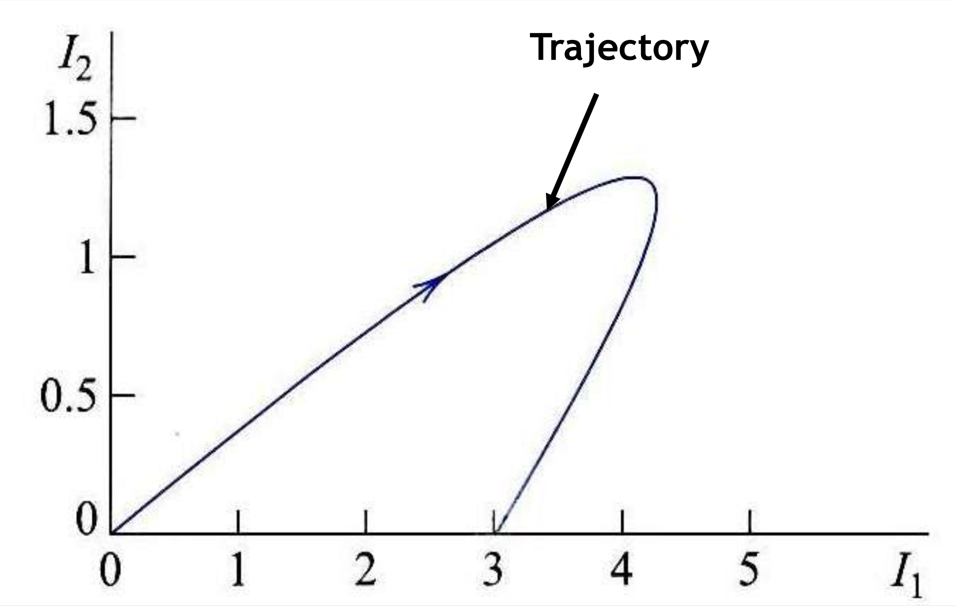

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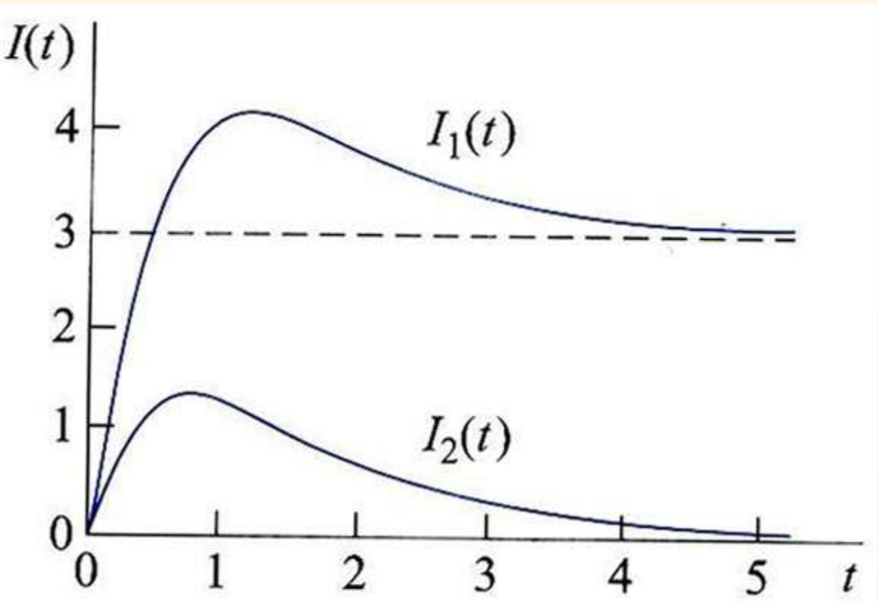

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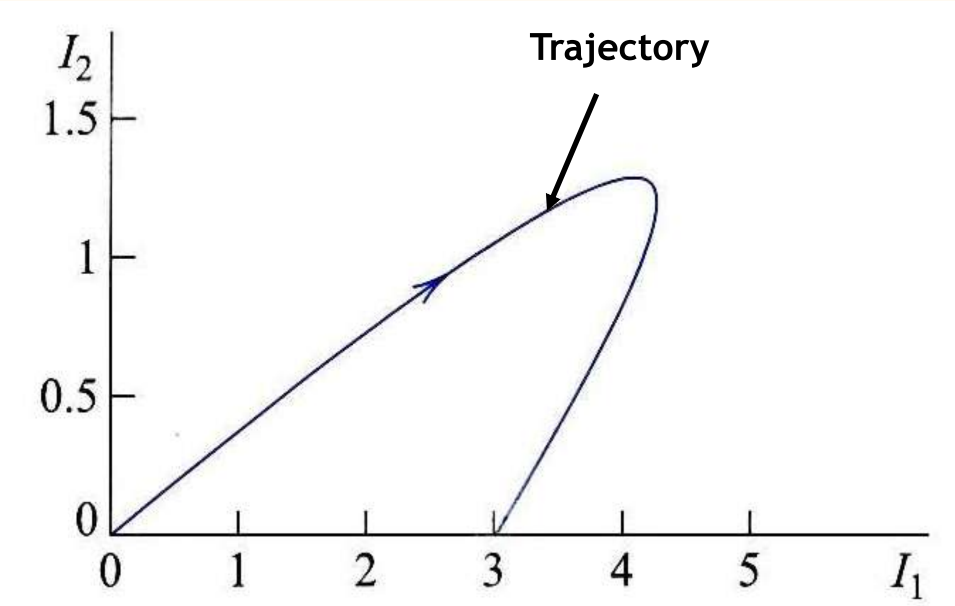

Graph at I_1-I_2 plane

We can represent graph at Graph at I_1-I_2 plane,

we call I_1-I_2 plane as *Phase Plane*.

Systems of ODEs as Models
: Conversion of an nth-Order ODE to a System

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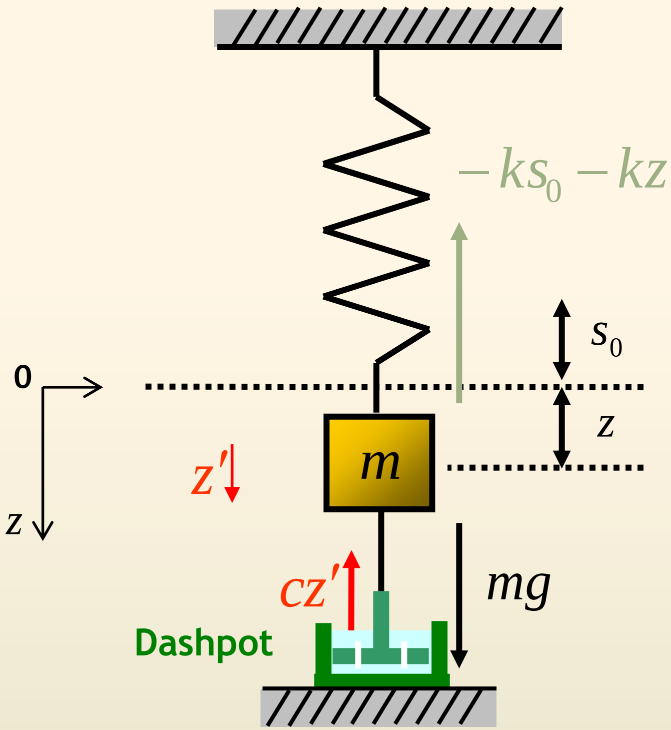
$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \ldots, y_n) \end{aligned}}$$

Ex.) Mass on a Spring

$$\mathbf{Ax} = \lambda \mathbf{x}$$
$$\mathbf{Ax} - \lambda \mathbf{x} = 0$$
$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

- Free, Damped vibration



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$$= mg - k(s_0 + z) - cz'$$
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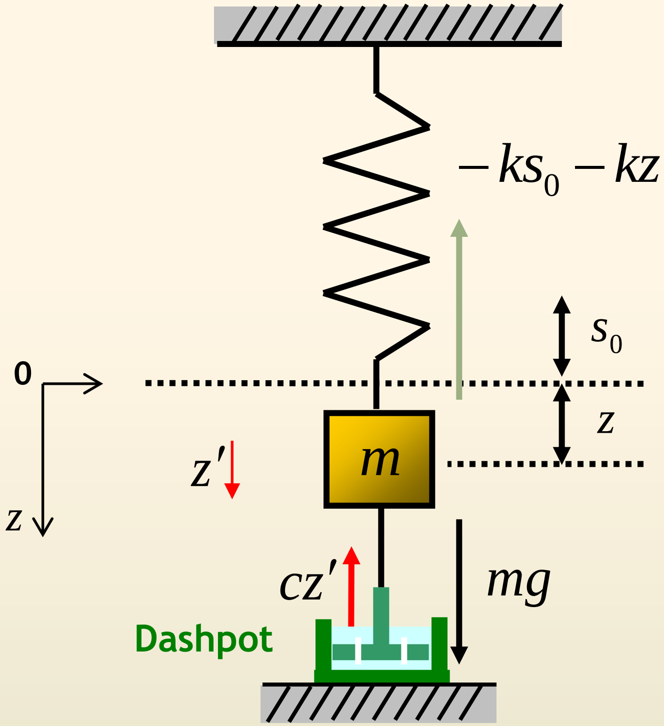

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$$mz'' + cz' + kz = 0$$

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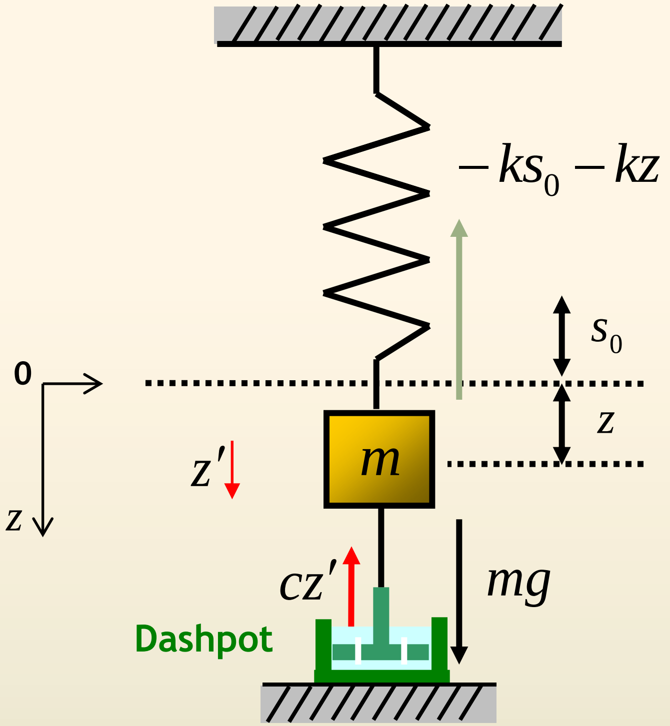
2008_O.D.E(3)

Ex.) Mass on a Spring

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$
$$\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = 0$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

- Free, Damped vibration



$$mz'' = F$$
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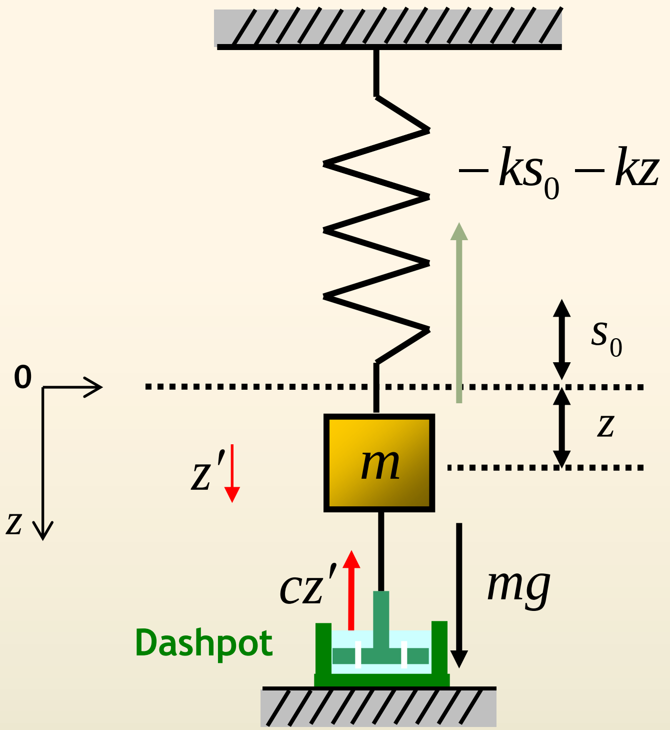

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$$z = z_1$$

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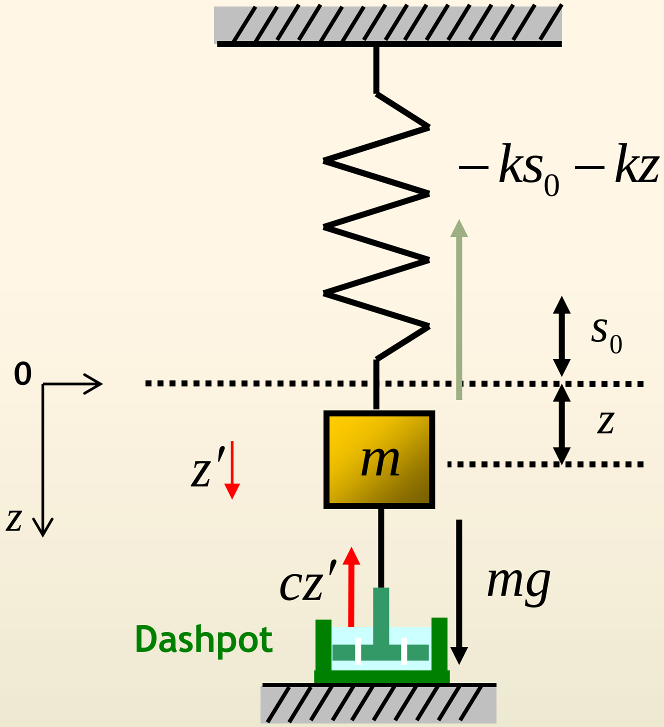

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$z = z_1$ ← This is what we want to find

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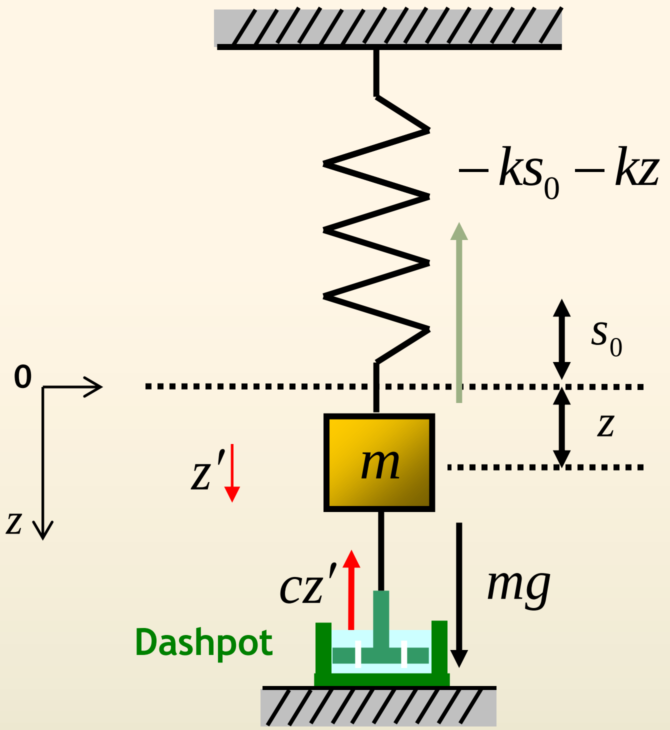

$$\begin{aligned} mz'' &= F \\ &= mg - k(s_0 + z) - cz' \\ &= -kz - cz' \end{aligned}$$

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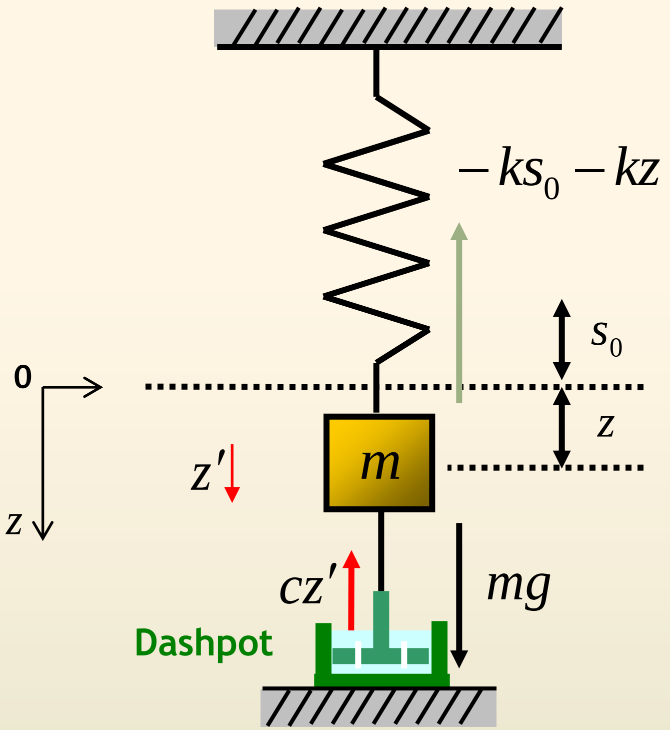

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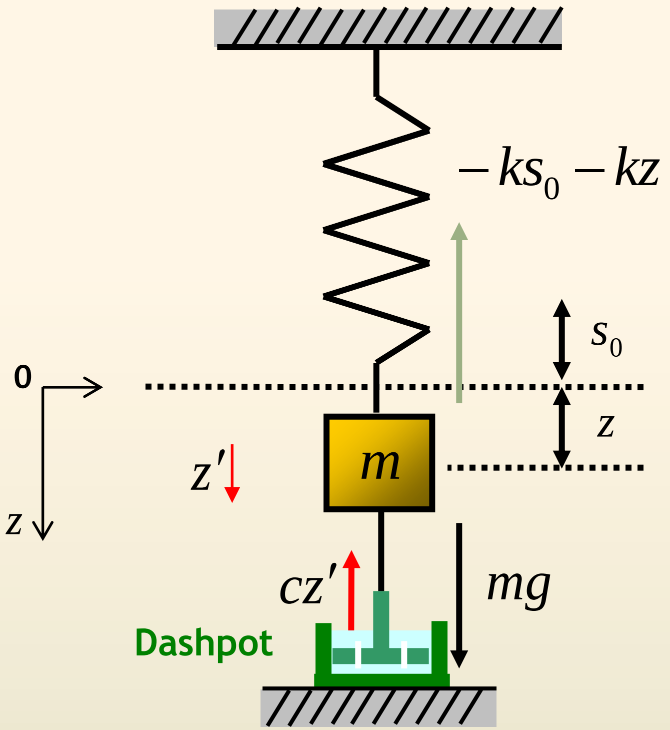

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$$\boxed{\begin{aligned} z_1' &= z_2 \\ z_2' &= -\frac{k}{m}z_1 - \frac{c}{m}z_2 \end{aligned}}$$

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$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\mathbf{z}' = \mathbf{A}\mathbf{z}$$

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$$\mathbf{z}' = \mathbf{Az}$$

Characteristic equation $\mathbf{z} = \mathbf{x} e^{\lambda t}$

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Characteristic equation $\mathbf{z} = \mathbf{x} e^{\lambda t}$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} - \lambda \end{vmatrix}$$

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$$= \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

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If we consider

$m = 1, c = 2, k = 0.75$

$$\lambda^2 + 2\lambda + 0.75 = 0$$

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$\boxed{\text{Characteristic equation}}$ $\mathbf{z} = \mathbf{x}e^{\lambda t}$

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$$m = 1, c = 2, k = 0.75$$

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$$\lambda = \frac{-2 \pm \sqrt{2^2 - 3}}{2}, \qquad \lambda_1 = -0.5, \quad \lambda_2 = -1.5$$

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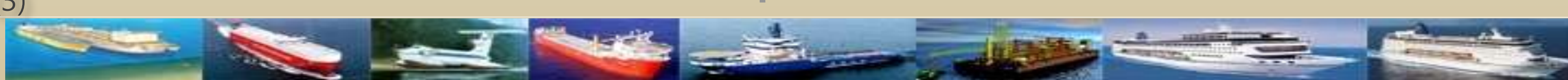

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$$\begin{pmatrix} -\lambda & 1 \\ -0.75 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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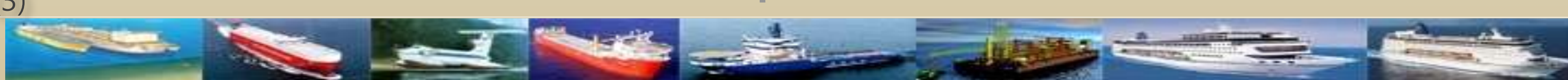

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$$0.5x_1 + x_2 = 0, \qquad \mathbf{x}^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

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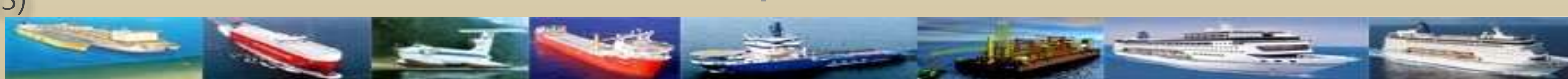

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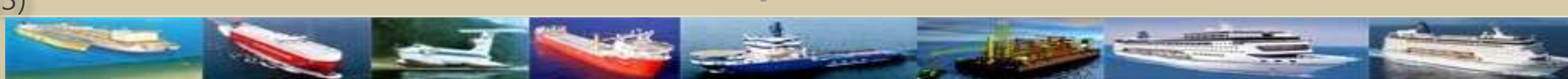

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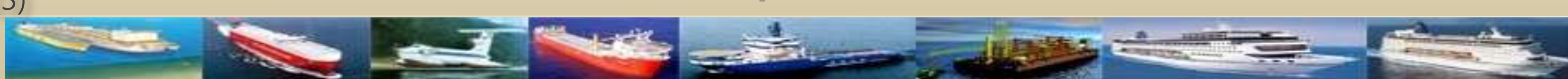

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$$\mathbf{z} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t}$$

$$\mathbf{z} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} e^{\lambda_2 t}$$

$$\boxed{z = z_1 = -2c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y}$$

in components,

$$\begin{aligned} y_1' &= -3y_1 + y_2 \\ y_2' &= y_1 - 3y_2 \end{aligned}$$

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$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = (\lambda+4)(\lambda+2) = 0$$

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$$\lambda_1 = -4,\ \lambda_2 = -2, \quad \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

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$$\lambda_1 = -4,\ \lambda_2 = -2, \quad \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\lambda_1 = -4, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = -3y_1 + y_2$$
$$y_2' = y_1 - 3y_2$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = (\lambda+4)(\lambda+2) = 0$$

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$$\lambda_2 = -2, \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}$$

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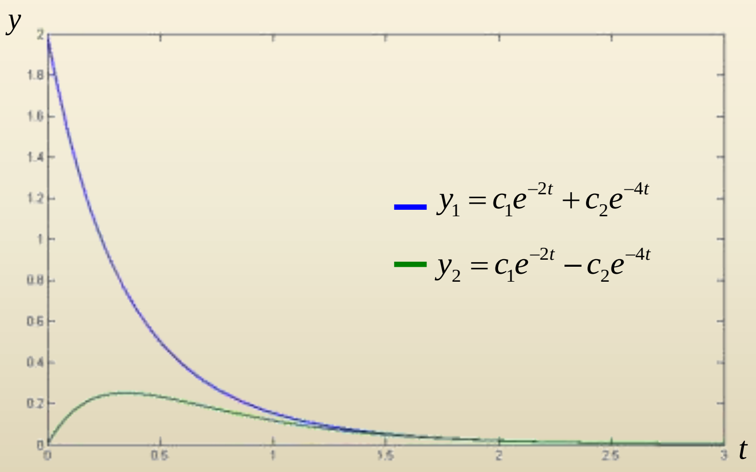
$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} \quad \Rightarrow \quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

Constant-Coefficient Systems. Phase Plane Method

$$y_1 = c_1 e^{-2t} + c_2 e^{-4t} \qquad y_2 = c_1 e^{-2t} - c_2 e^{-4t}$$

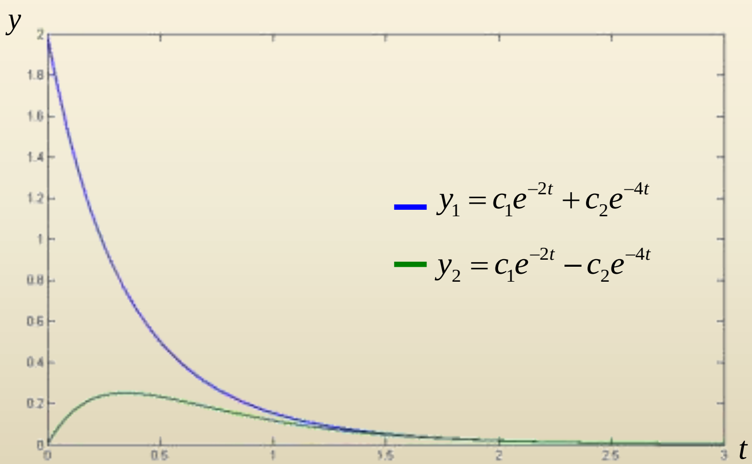
Constant-Coefficient Systems. Phase Plane Method

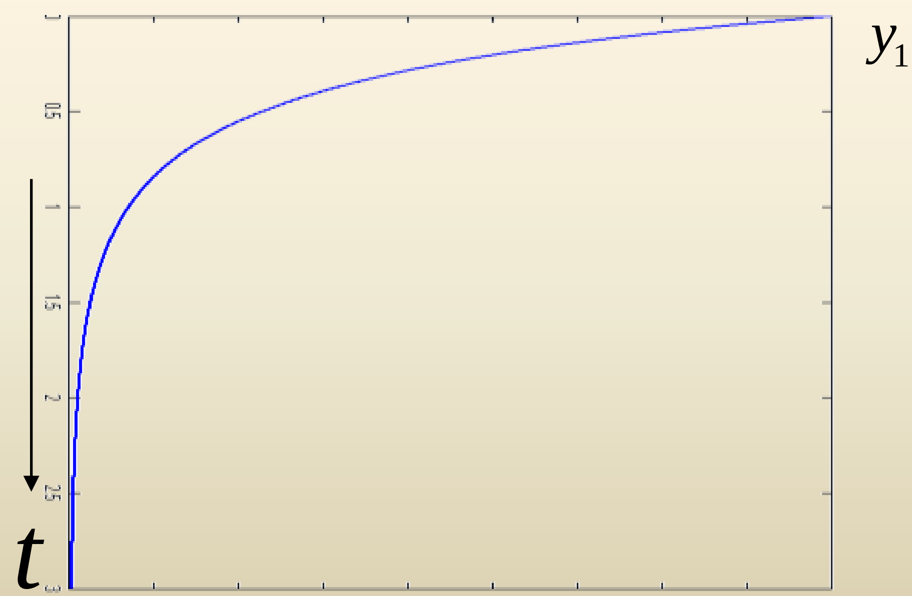
$$y_1 = c_1 e^{-2t} + c_2 e^{-4t} \qquad y_2 = c_1 e^{-2t} - c_2 e^{-4t}$$



Constant-Coefficient Systems. Phase Plane Method

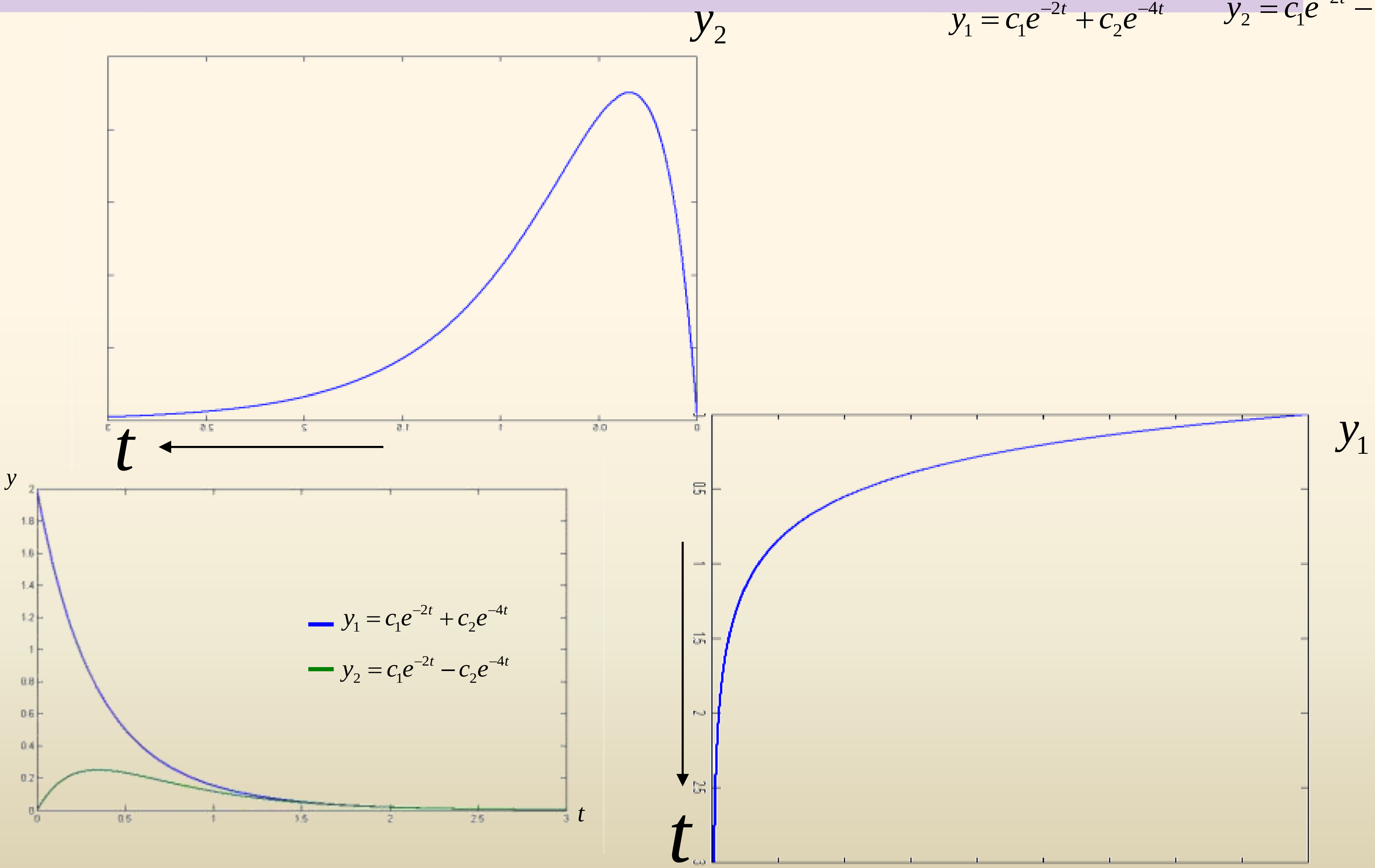
$y_1 = c_1 e^{-2t} + c_2 e^{-4t}$ $\qquad y_2 = c_1 e^{-2t} - c_2 e^{-4t}$





Constant-Coefficient Systems. Phase Plane Method

$y_1 = c_1 e^{-2t} + c_2 e^{-4t}$ $\quad$ $y_2 = c_1 e^{-2t} - c_2 e^{-4t}$



Constant-Coefficient Systems. Phase Plane Method

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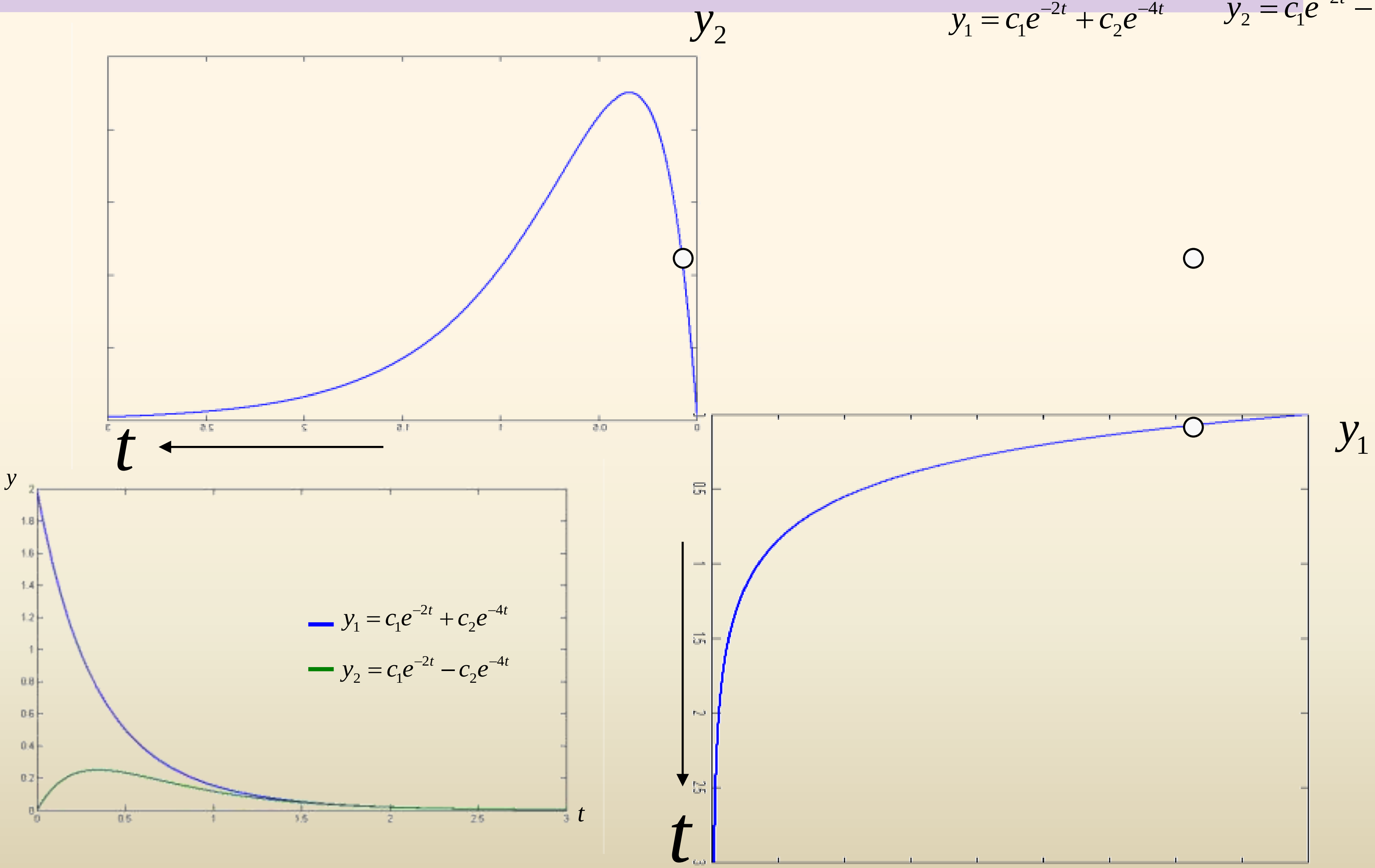

Constant-Coefficient Systems. Phase Plane Method

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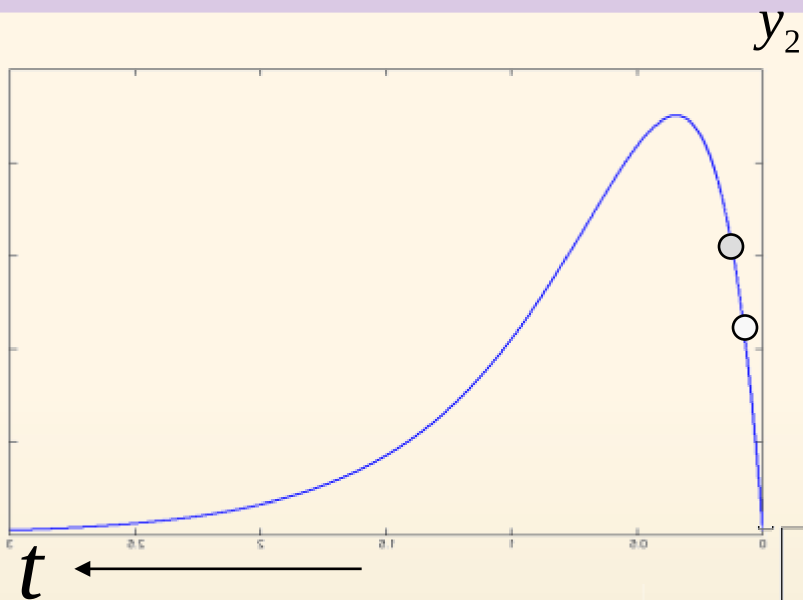
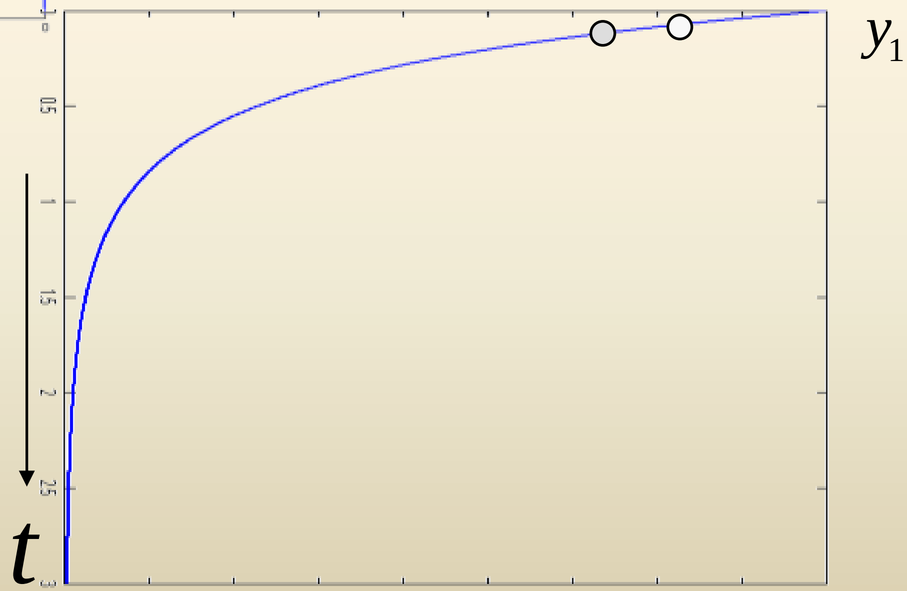
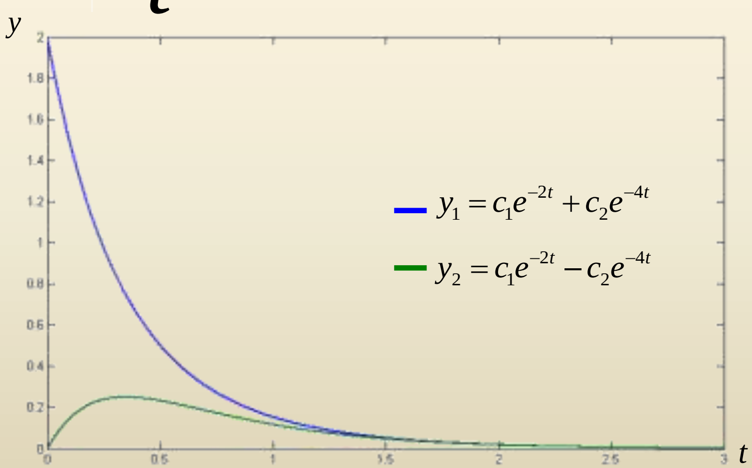
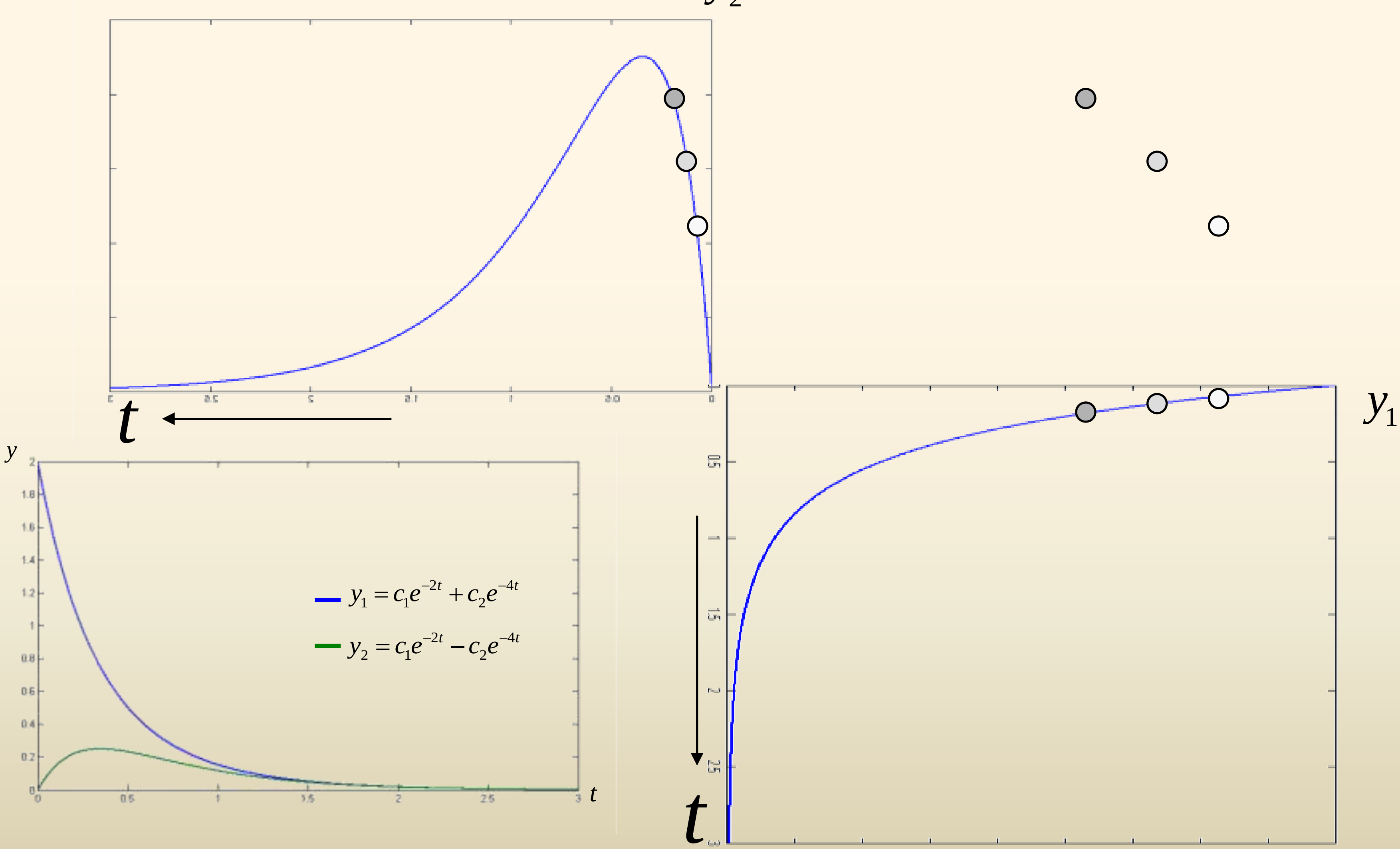




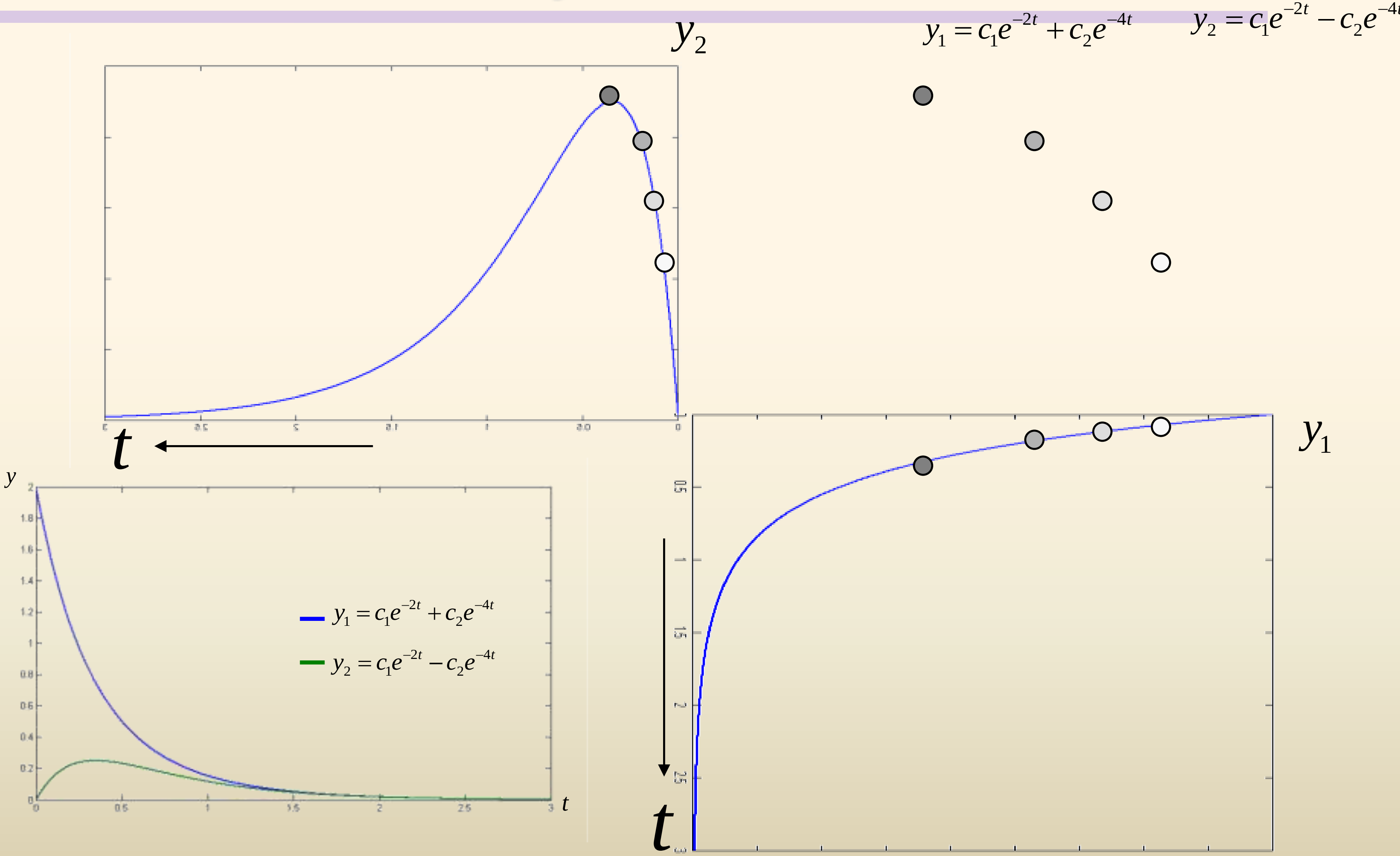


Constant-Coefficient Systems. Phase Plane Method

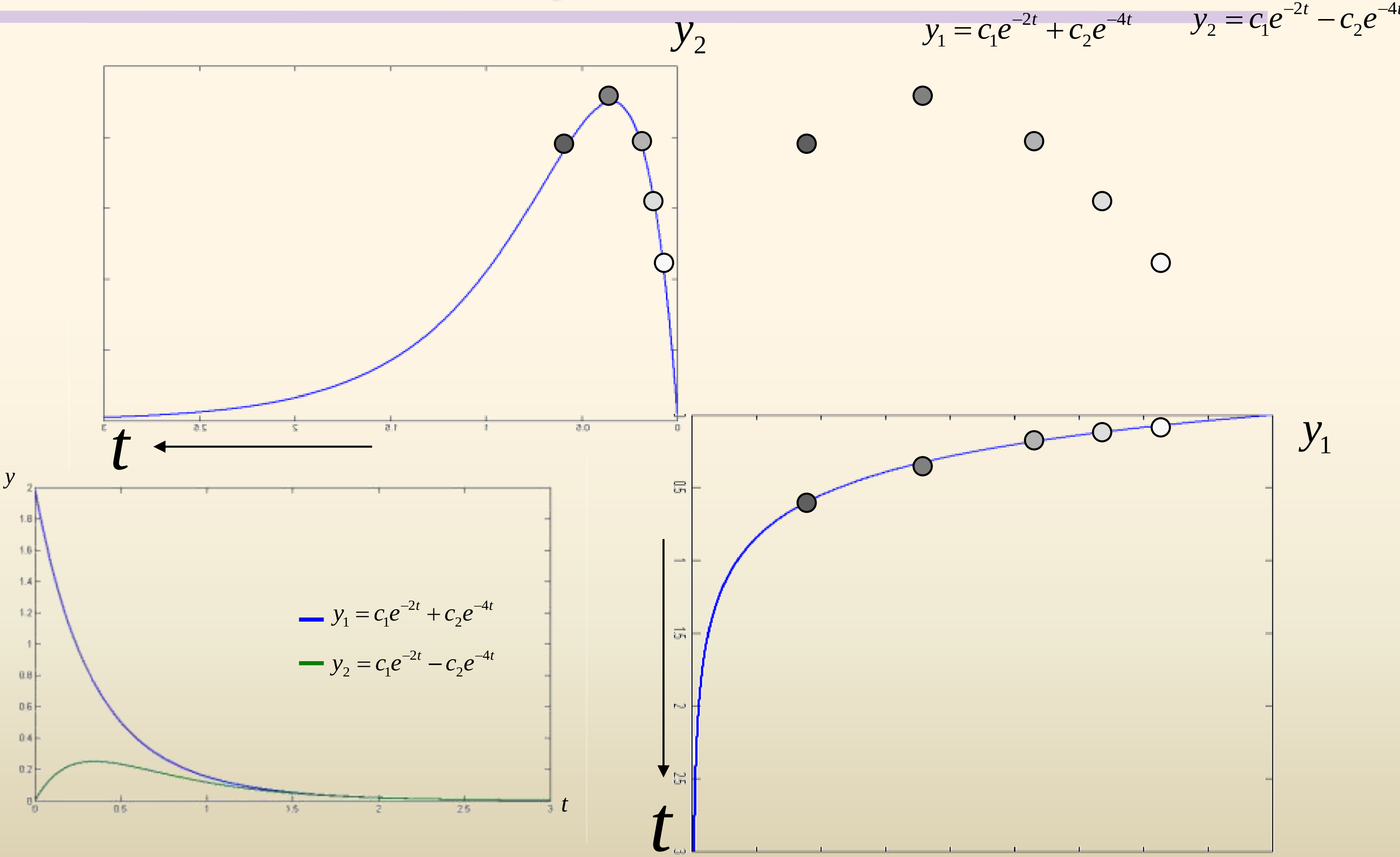
$y_1 = c_1 e^{-2t} + c_2 e^{-4t}$ $\quad y_2 = c_1 e^{-2t} - c_2 e^{-4t}$



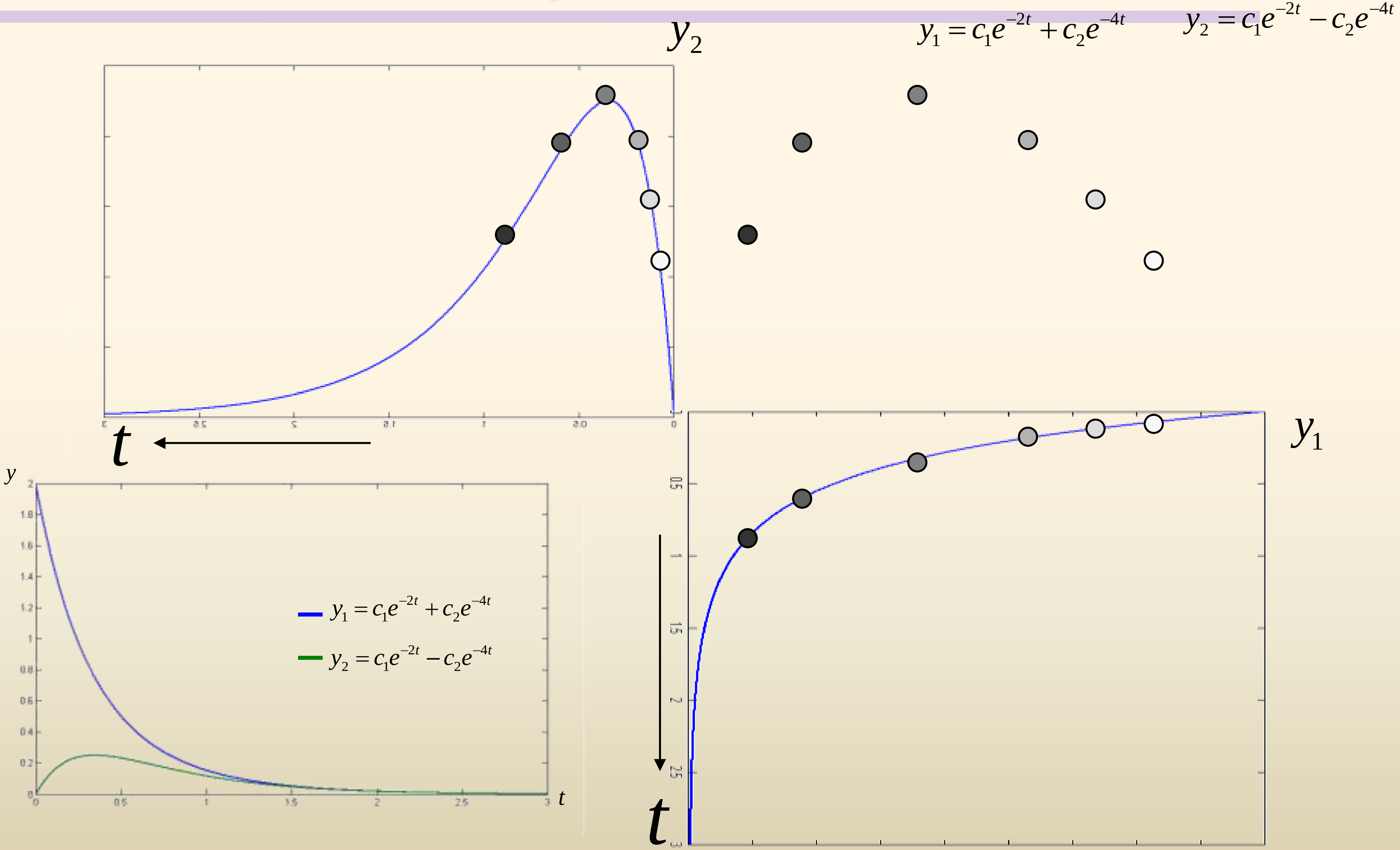
Constant-Coefficient Systems. Phase Plane Method



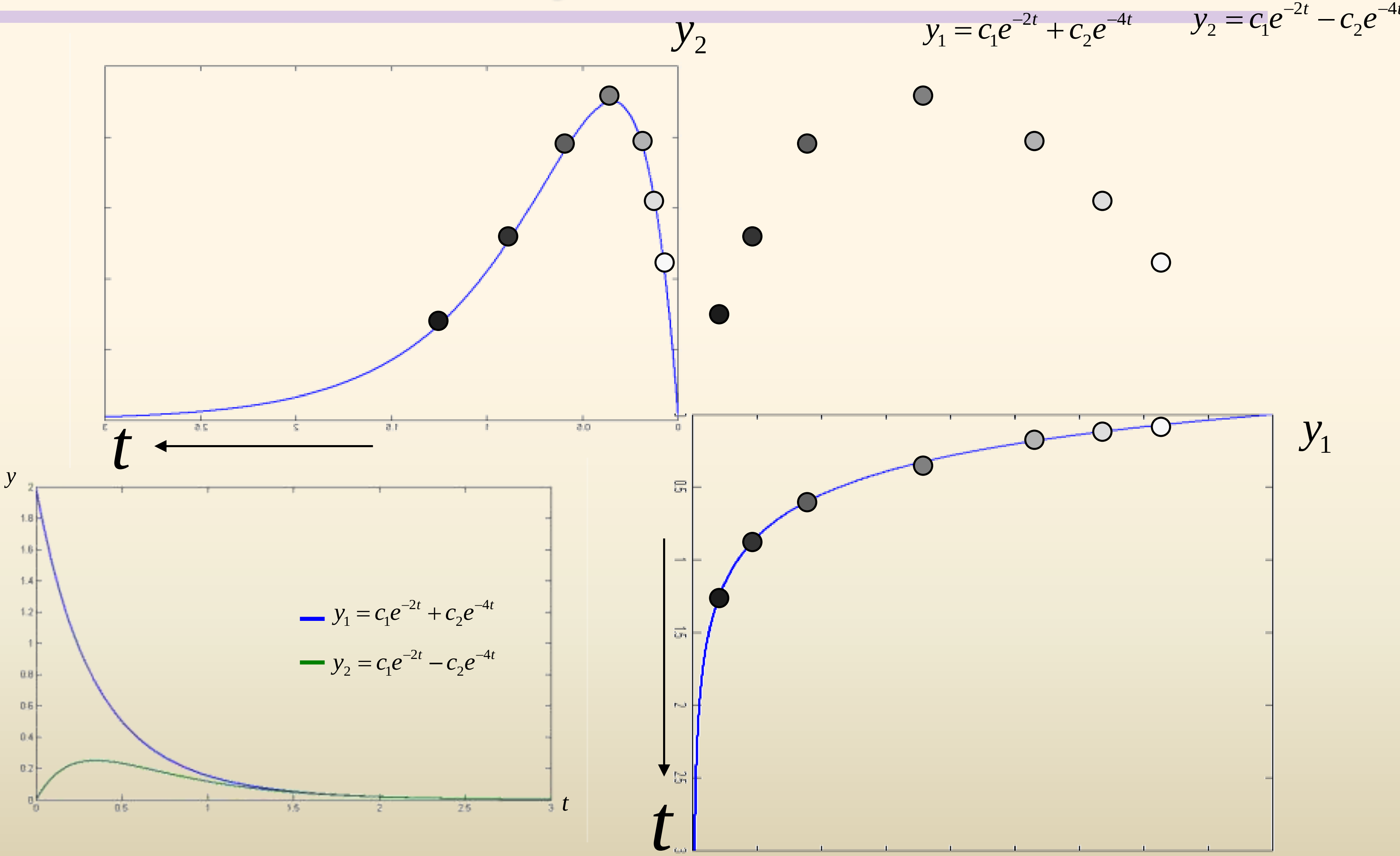
Constant-Coefficient Systems. Phase Plane Method



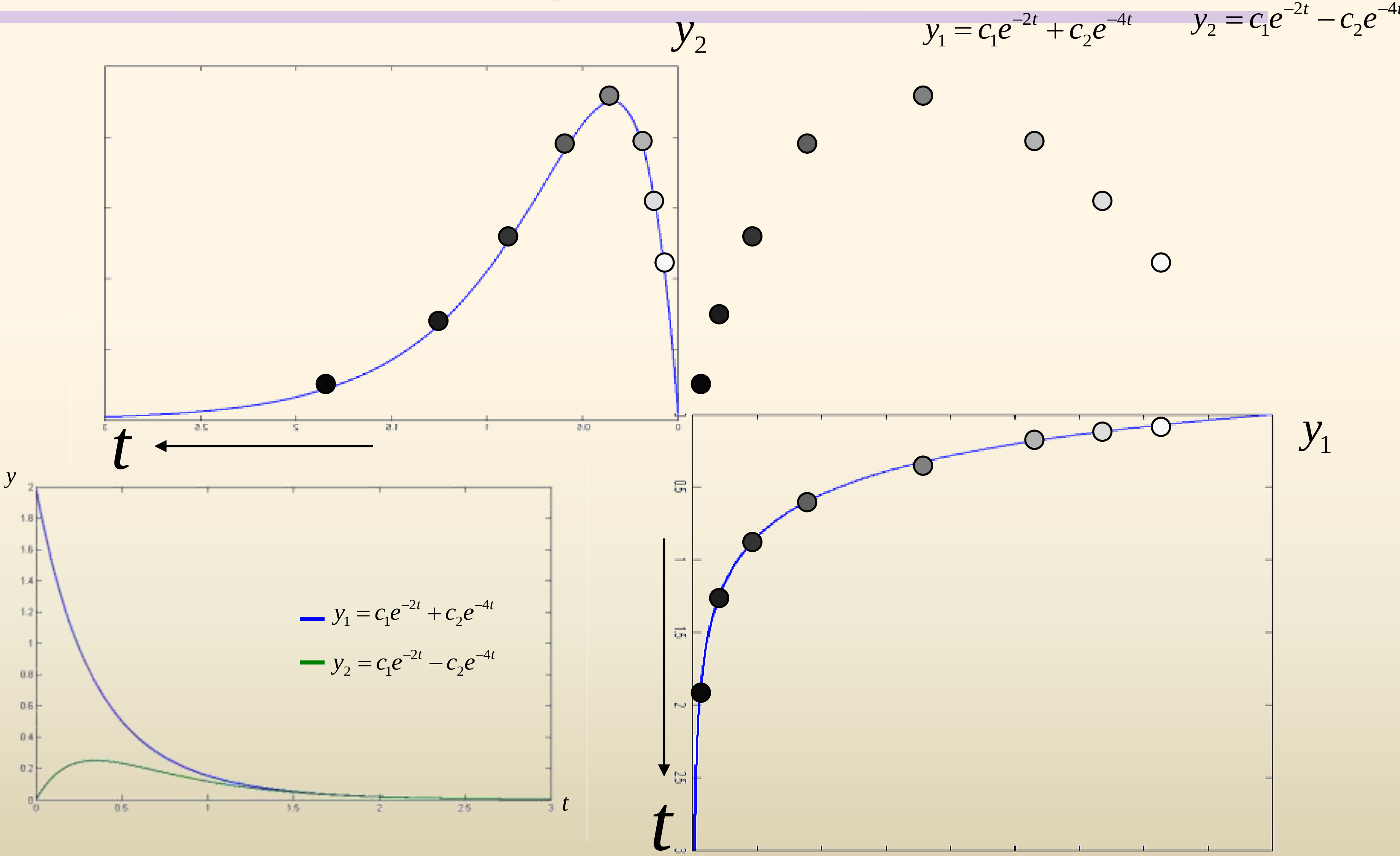
Constant-Coefficient Systems. Phase Plane Method



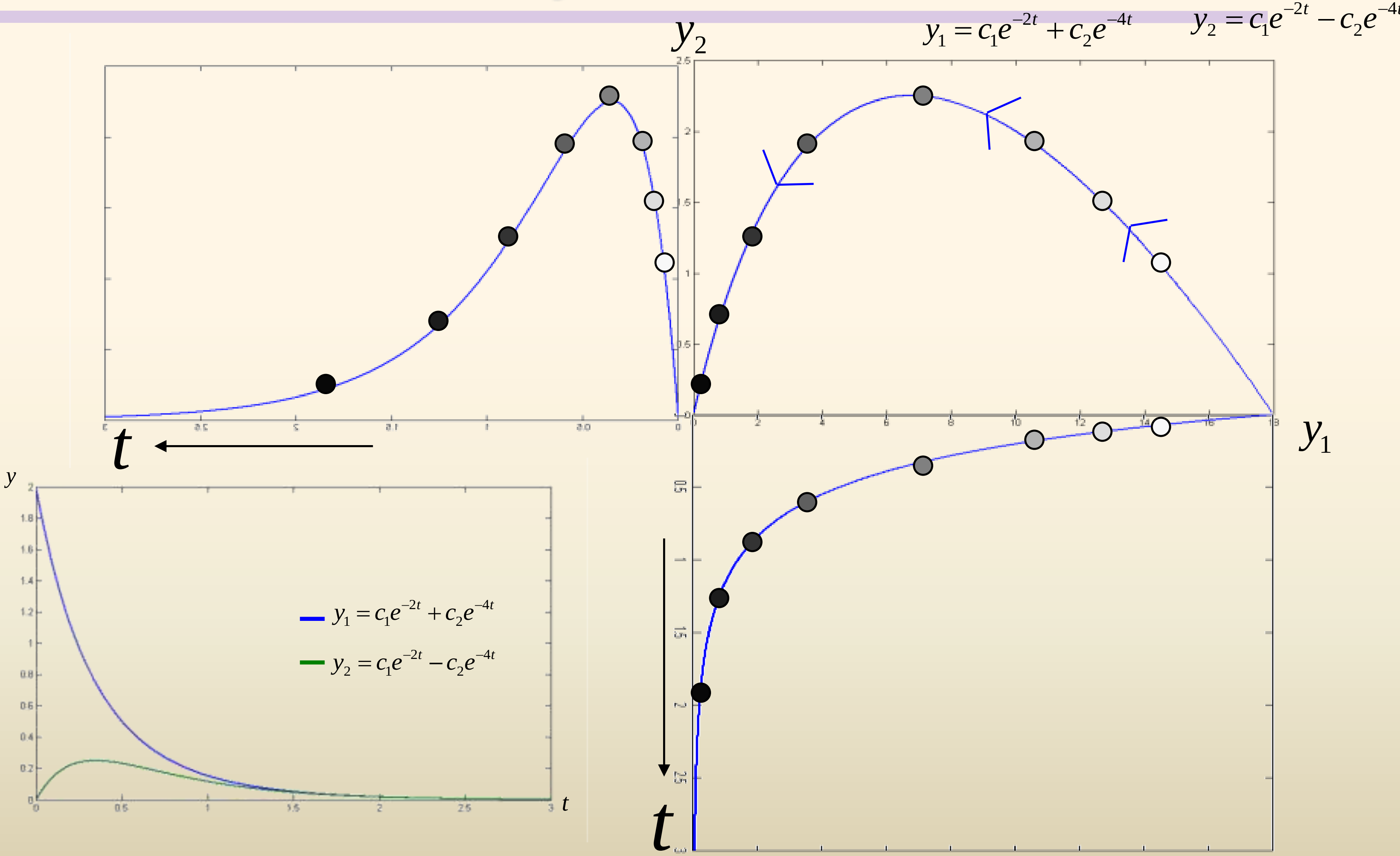
Constant-Coefficient Systems. Phase Plane Method



Constant-Coefficient Systems. Phase Plane Method



Constant-Coefficient Systems. Phase Plane Method



Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{Ax} = \lambda\mathbf{x}$$
$$\mathbf{Ax} - \lambda\mathbf{x} = 0$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

$$y_1' = -3y_1 + y_2$$
$$y_2' = y_1 - 3y_2$$

$$\Rightarrow \quad \mathbf{y} = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-4t}$$

$$y_1 = c_1e^{-2t} + c_2e^{-4t}$$
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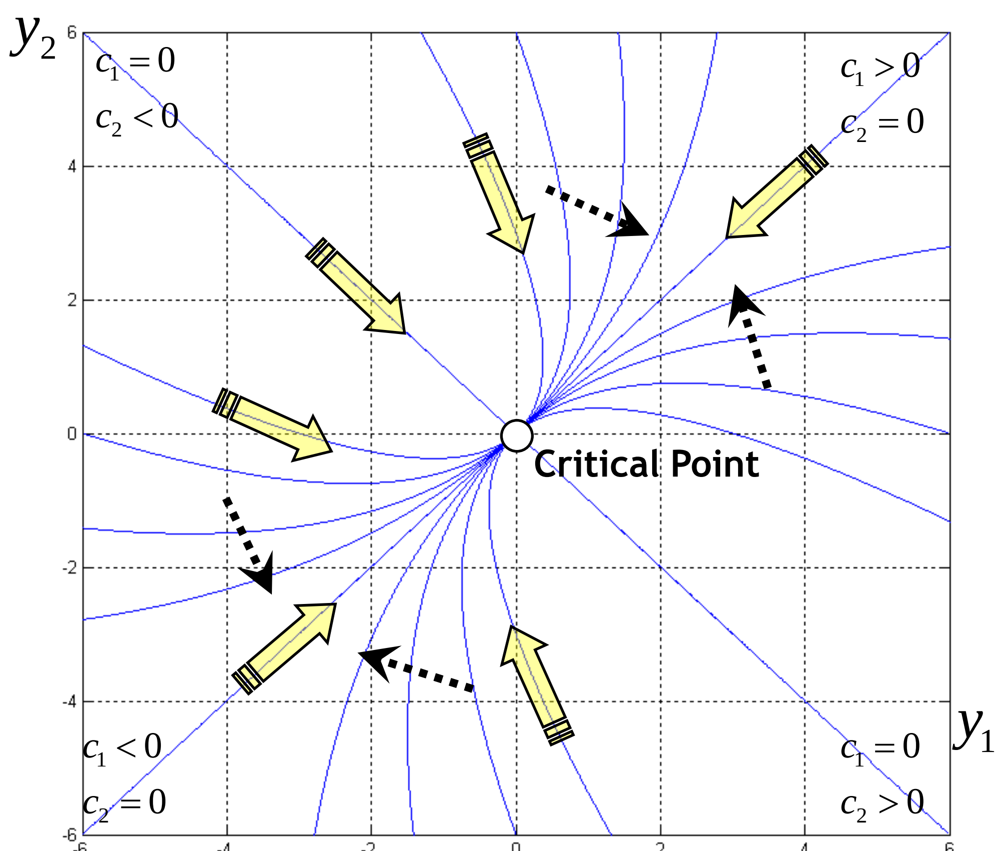

$\left|\frac{c_1}{c_2}\right|$ 가 증가할 수록 ⇢ 방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{Ax} = \lambda \mathbf{x}$$
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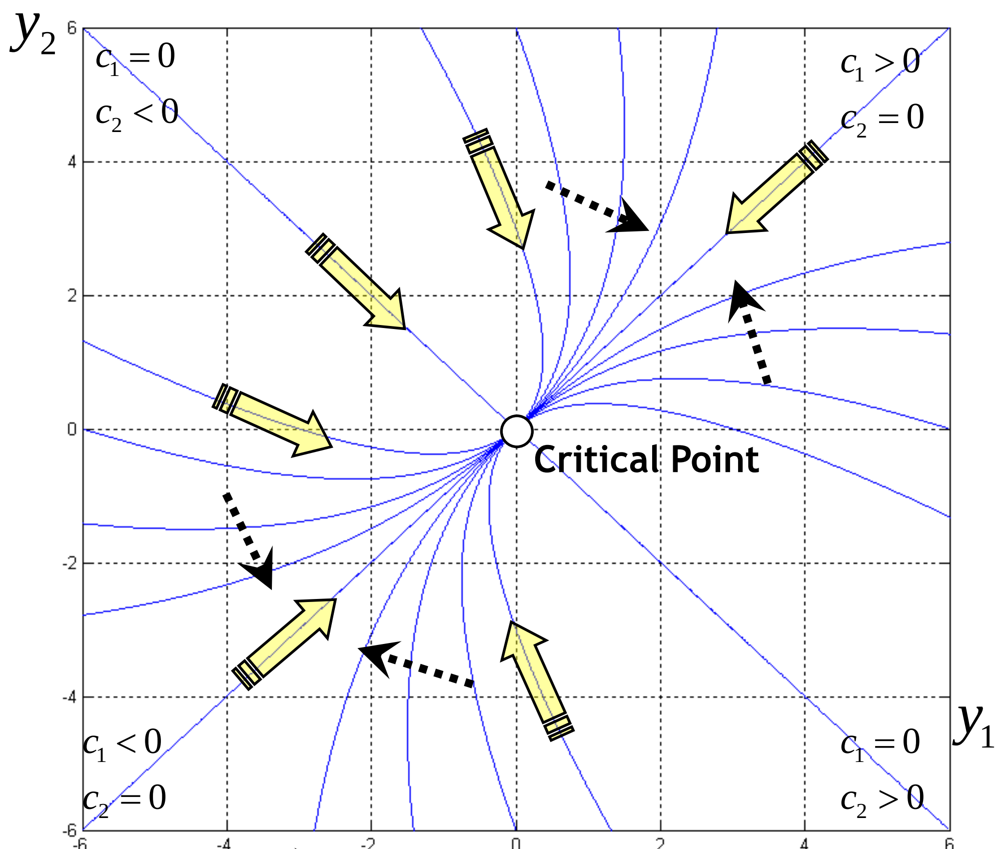

$\left|\dfrac{c_1}{c_2}\right|$ 가 증가할 수록 ⇢ 방향으로 변함

1. Critical Point :

   point(y_1, y_2)at which both y_1' and y_2' are 0

2. Five types of critical points depending on eigen values

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{Ax} = \lambda \mathbf{x}$$
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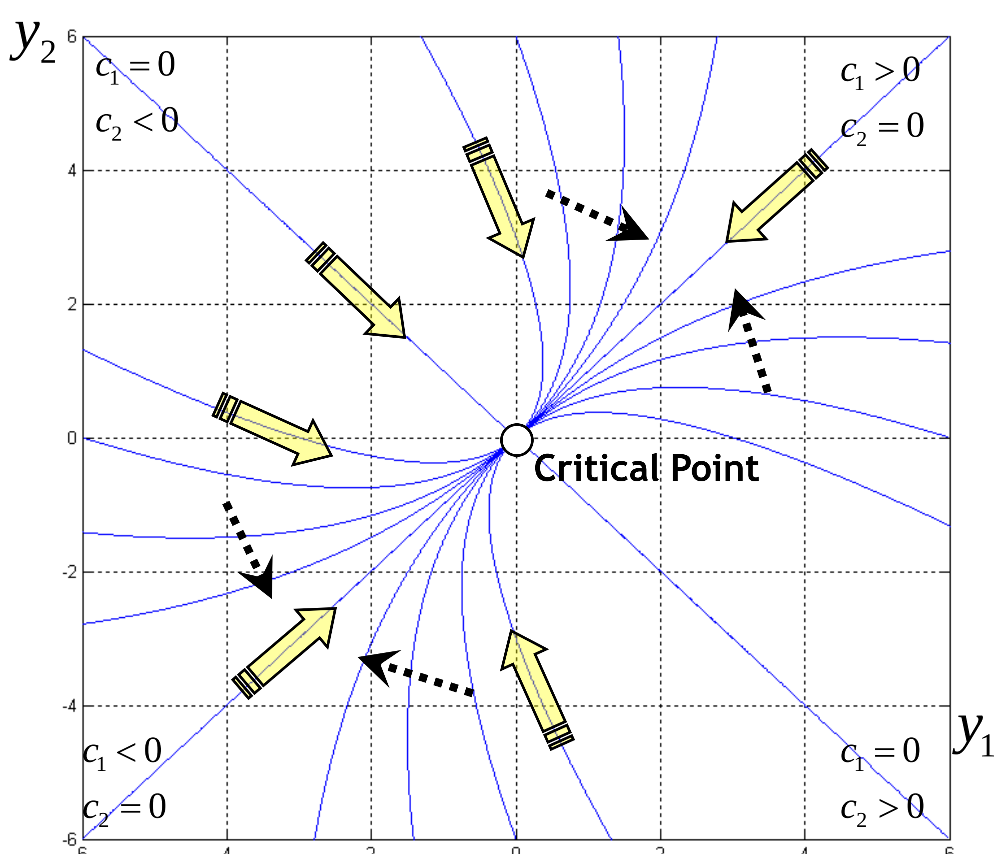

$\left|\dfrac{c_1}{c_2}\right|$ 가 증가할 수록 ⋯➤ 방향으로 변함

1. Critical Point :

point(y_1, y_2)at which both y_1' and y_2' are 0

2. Five types of critical points depending on eigen values

Critical Point: (0, 0)

Improper node(**비고유마디점**)

λ_1 and λ_2 are real numbers(ex:-2, -4)

$\lambda_1 \neq \lambda_2$

$\lambda_1 \lambda_2 > 0$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{Ax} = \lambda \mathbf{x}$$
$$\mathbf{Ax} - \lambda \mathbf{x} = 0$$
$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$$

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t}$$

$$y_1' = -3y_1 + y_2$$
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$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

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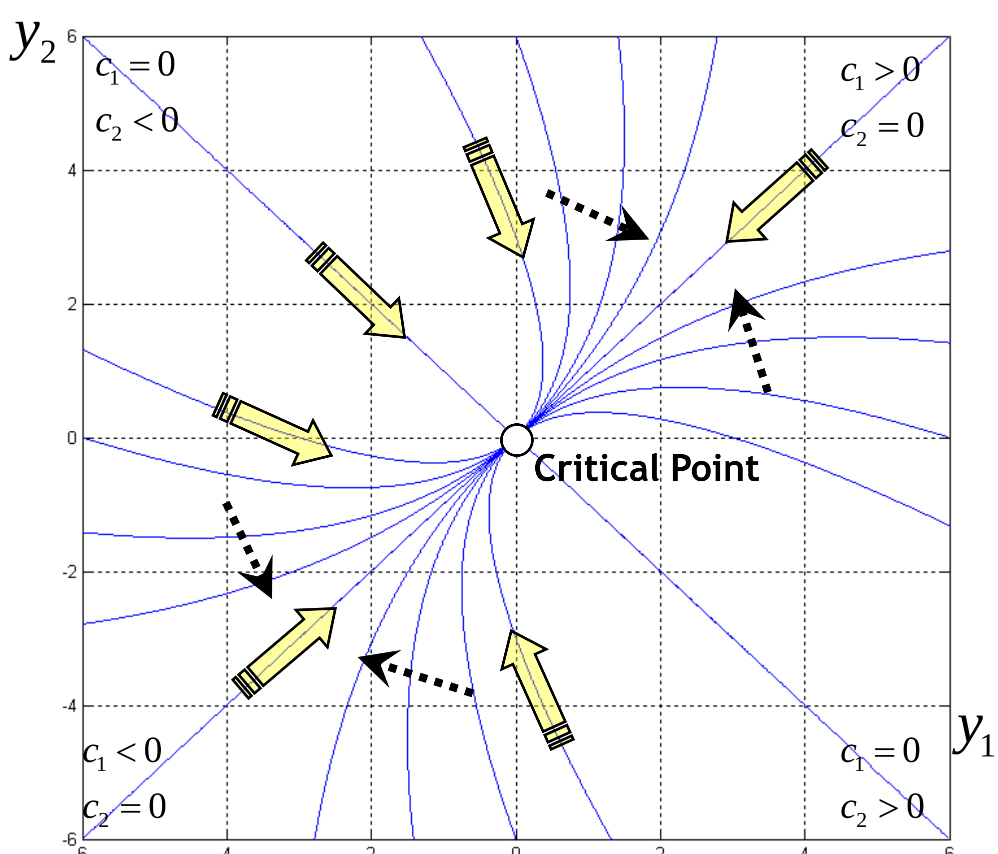

$\left|\frac{c_1}{c_2}\right|$ 가 증가할 수록 ···➤방향으로 변함

1. Critical Point :

point(y_1, y_2)at which both y_1' and y_2' are 0

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Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = y_1$$
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Constant-Coefficient Systems. Phase Plane Method

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in components,

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$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$$

Constant-Coefficient Systems. Phase Plane Method

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$$\lambda_1,\ \lambda_2 = 1, \qquad \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}$$

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$$\lambda_1,\ \lambda_2 = 1, \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \mathbf{x}^{(2)} = \begin{bmatrix} c \\ d \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}$$

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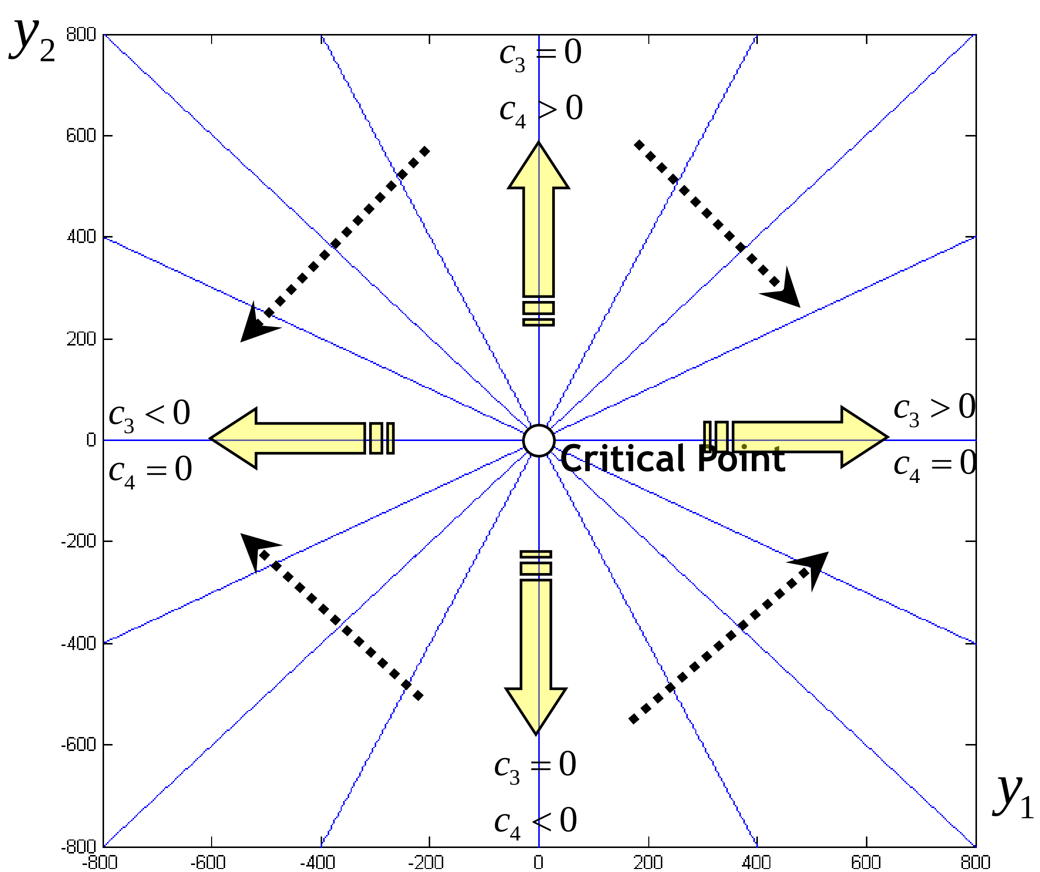
$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} \quad \Rightarrow \quad \mathbf{y} = c_1 \begin{bmatrix} a \\ b \end{bmatrix} e^{t} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} e^{t}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} a \\ b \end{bmatrix} e^t + c_2 \begin{bmatrix} c \\ d \end{bmatrix} e^t$$

$$y_1 = (c_1 a + c_2 c)e^t = c_3 e^t$$
$$y_2 = (c_1 b + c_2 d)e^t = c_4 e^t$$



$\left|\frac{c_3}{c_4}\right|$ 가 증가할 수록 ⋯▶ 방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} a \\ b \end{bmatrix} e^t + c_2 \begin{bmatrix} c \\ d \end{bmatrix} e^t$$

$$y_1 = (c_1 a + c_2 c)e^t = c_3 e^t$$
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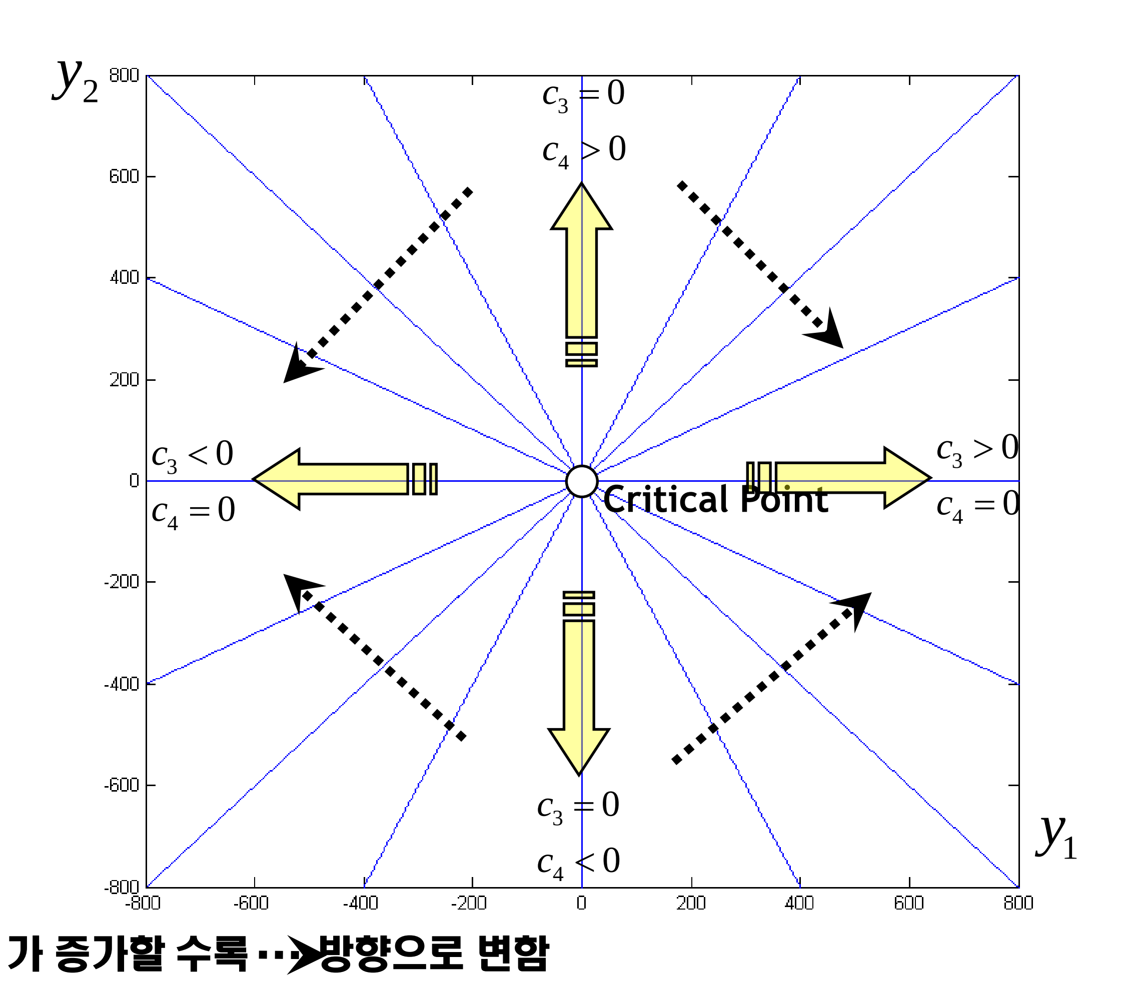
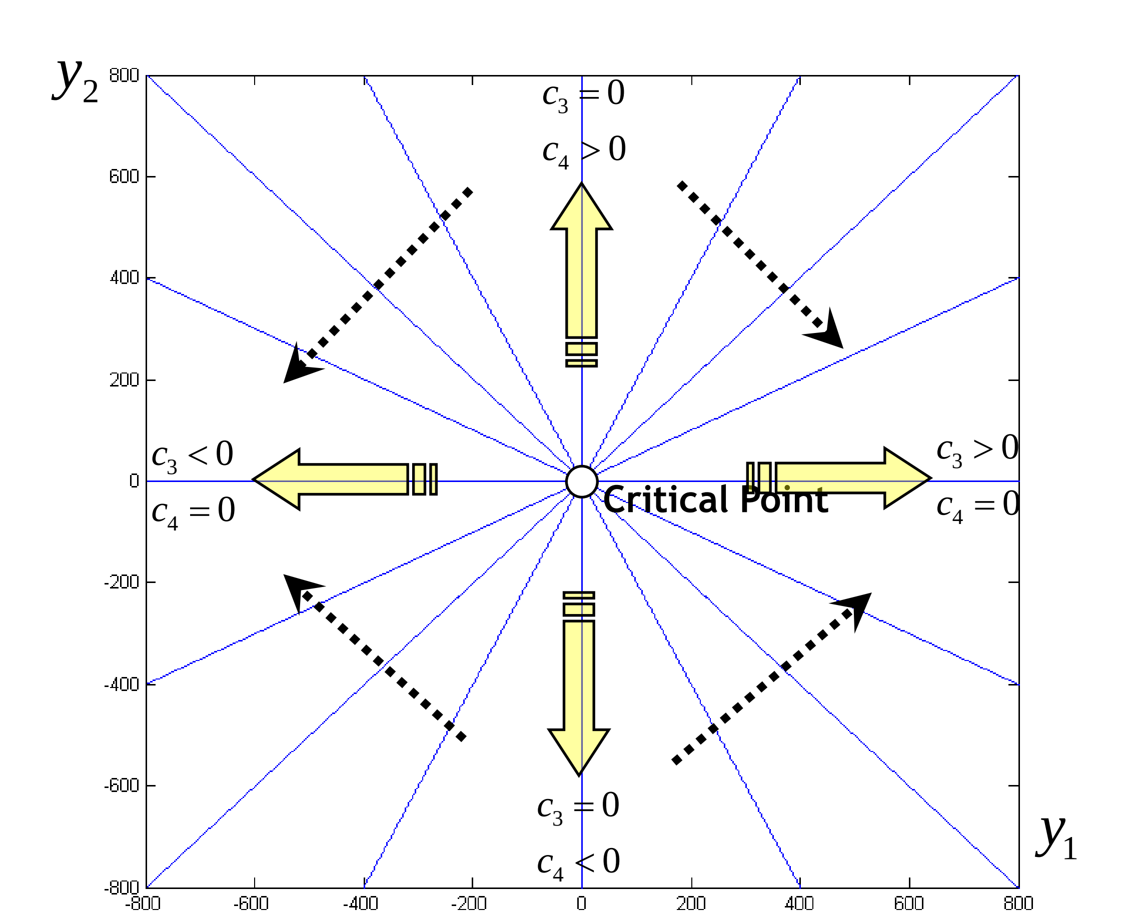


Critical Point: (0, 0)

$\left|\dfrac{c_3}{c_4}\right|$ 가 증가할 수록 ··➤방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1\begin{bmatrix} a \\ b \end{bmatrix}e^t + c_2\begin{bmatrix} c \\ d \end{bmatrix}e^t \qquad \begin{aligned} y_1 &= (c_1 a + c_2 c)e^t = c_3 e^t \\ y_2 &= (c_1 b + c_2 d)e^t = c_4 e^t \end{aligned}$$



$\left|\dfrac{c_3}{c_4}\right|$ 가 증가할 수록 ···➤방향으로 변함

Critical Point: (0, 0)

proper node(고유마디점)

λ_1 and λ_2 are real numbers

$\lambda_1 = \lambda_2$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = y_1$$
$$y_2' = -y_2$$

Constant-Coefficient Systems. Phase Plane Method

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$$\lambda_1 = 1,\ \lambda_2 = -1 \qquad \begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

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$$\lambda_1 = 1 \qquad \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

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in components,

$$y_1' = y_1$$
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$$\lambda_1 = 1 \qquad \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1 \qquad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = y_1$$
$$y_2' = -y_2$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) = 0$$

$$\lambda_1 = 1,\ \lambda_2 = -1 \qquad \begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\lambda_1 = 1 \qquad \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

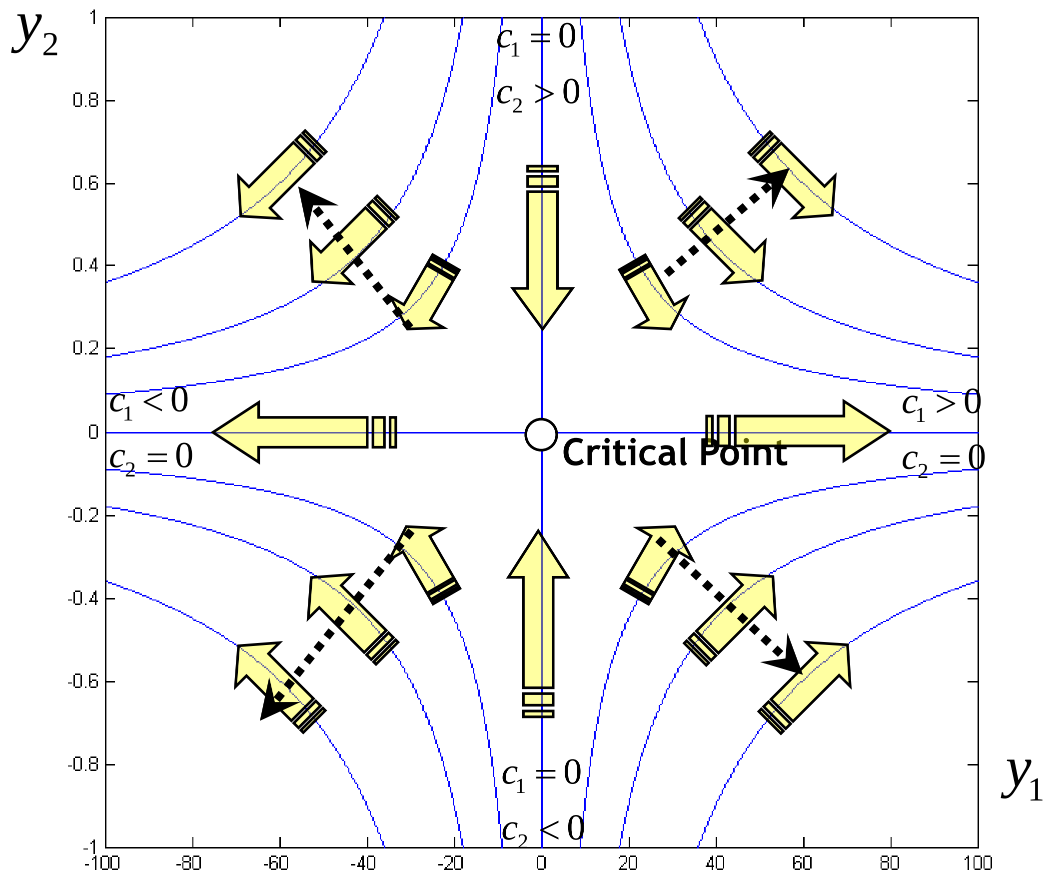
$$\lambda_2 = -1 \qquad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} \quad \Rightarrow \quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$$

$$y_1 = c_1 e^{t}$$
$$y_2 = c_2 e^{-t}$$



$\left|\dfrac{c_1}{c_2}\right|$ 가 증가할 수록 ···➤ 방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$$

$$y_1 = c_1 e^{t}$$
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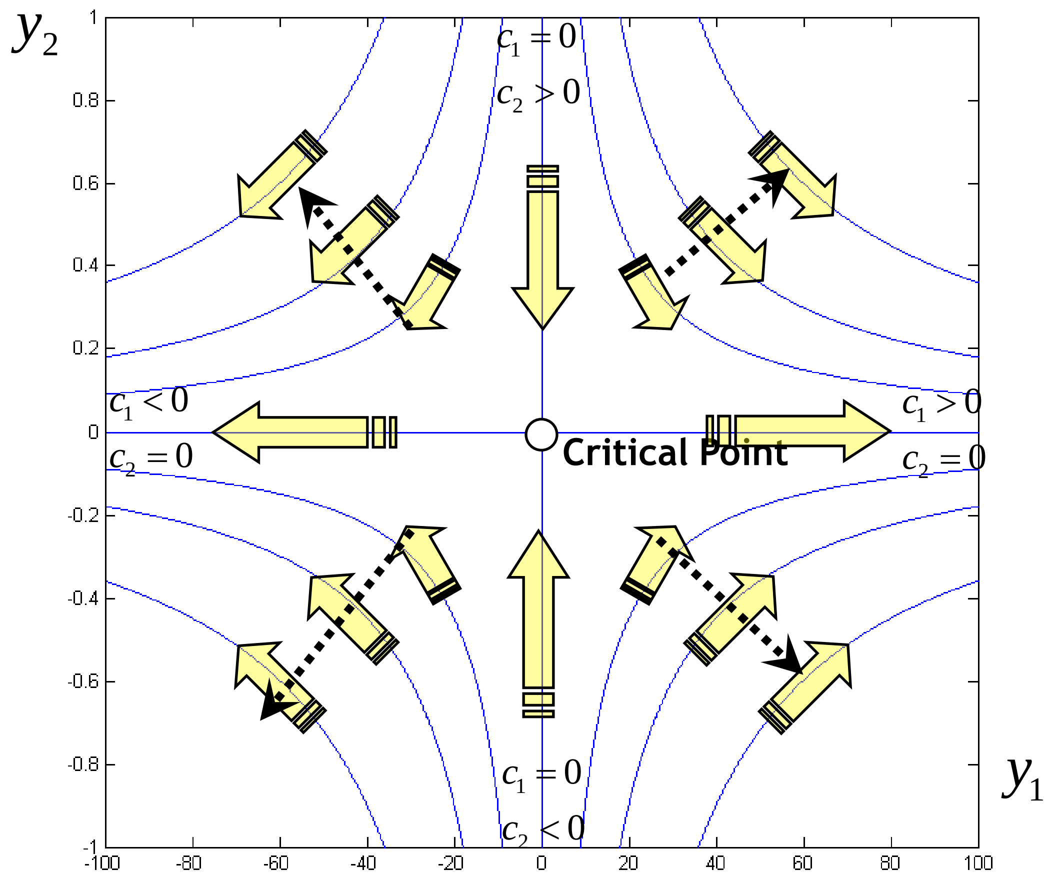


Critical Point: (0, 0)

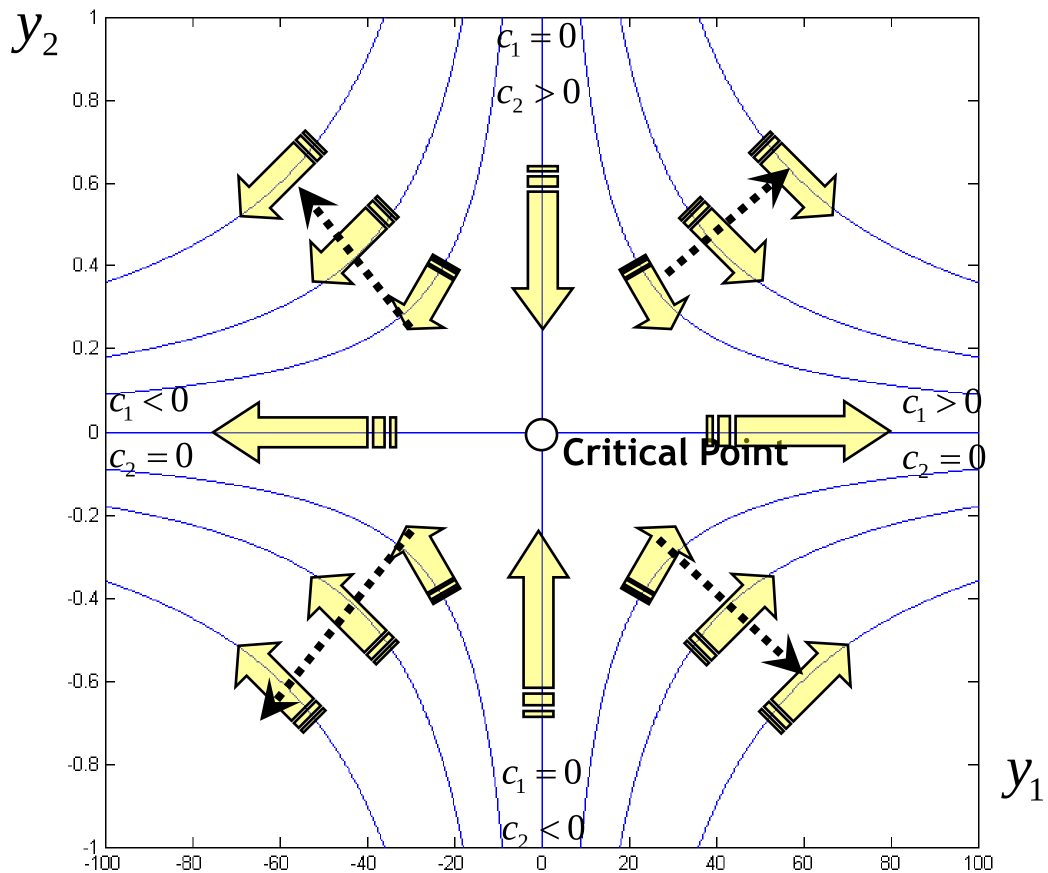
$\left|\dfrac{c_1}{c_2}\right|$ 가 증가할 수록 ···➤방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$$

$$y_1 = c_1 e^{t}$$

$$y_2 = c_2 e^{-t}$$



$\left|\frac{c_1}{c_2}\right|$ 가 증가할 수록 ···▶ 방향으로 변함

Critical Point: (0, 0)

Saddle point(안장점)

λ_1 and λ_2 are real numbers

$\lambda_1 \lambda_2 < 0$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\mathbf{y}$$

in components,

$$y_1' = y_2$$
$$y_2' = -4y_1$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = y_2$$
$$y_2' = -4y_1$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\mathbf{y}$$

in components,

$$y_1' = y_2$$
$$y_2' = -4y_1$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

$$\lambda_1 = 2i,\ \lambda_2 = -2i \qquad \begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = y_2$$
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$$\lambda_1 = 2i,\ \lambda_2 = -2i \qquad \begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\lambda_1 = 2i \qquad \begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = y_2$$
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$$\lambda_2 = -2i \qquad \begin{bmatrix} 2i & 1 \\ -4 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = y_2$$
$$y_2' = -4y_1$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

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$$\lambda_1 = 2i \qquad \begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

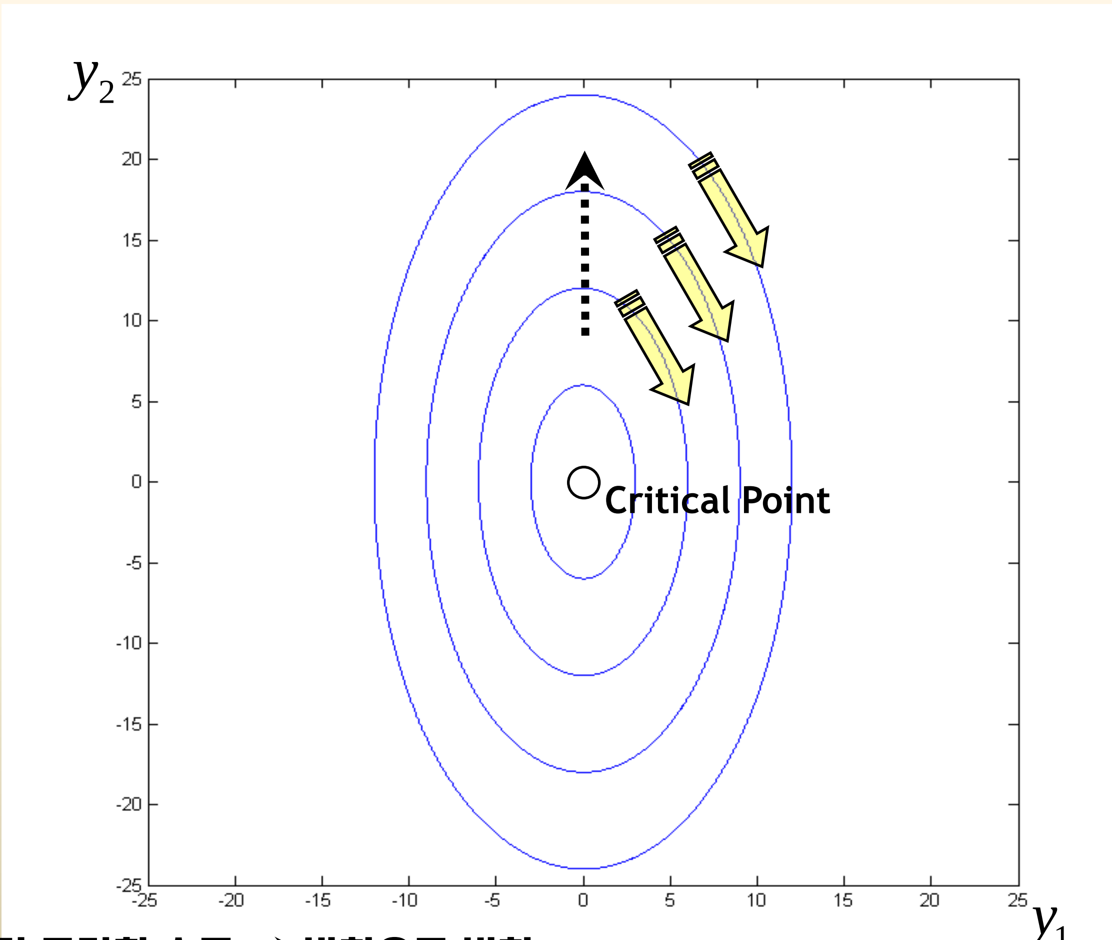
$$\lambda_2 = -2i \qquad \begin{bmatrix} 2i & 1 \\ -4 & 2i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} \quad \Rightarrow \quad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}$$

$$y_1 = c_1 e^{2it} + c_2 e^{-2it}$$
$$y_2 = c_1 2ie^{2it} - c_2 2ie^{-2it}$$



$\left|\dfrac{c_1}{c_2}\right|$ 가 증가할 수록 ···➤방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}$$

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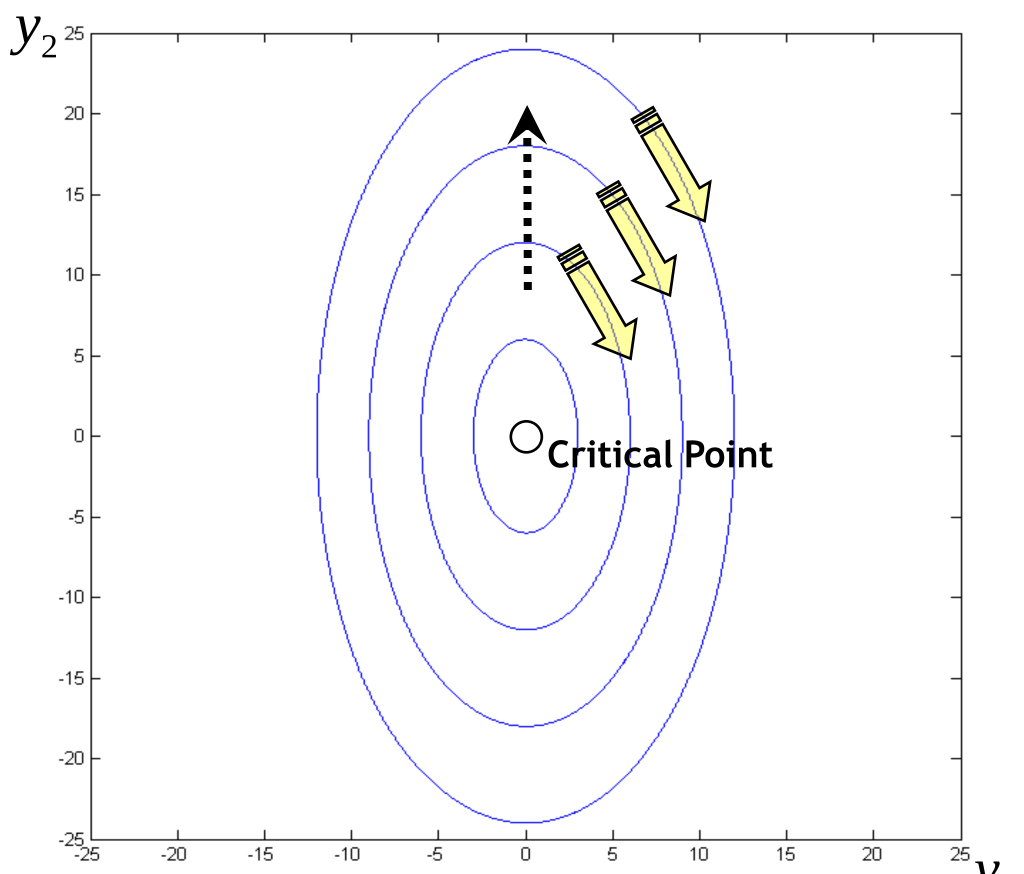

Critical Point: (0, 0)

$\left|\frac{c_1}{c_2}\right|$ 가 증가할 수록 ··➤ 방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}$$

$$y_1 = c_1 e^{2it} + c_2 e^{-2it}$$
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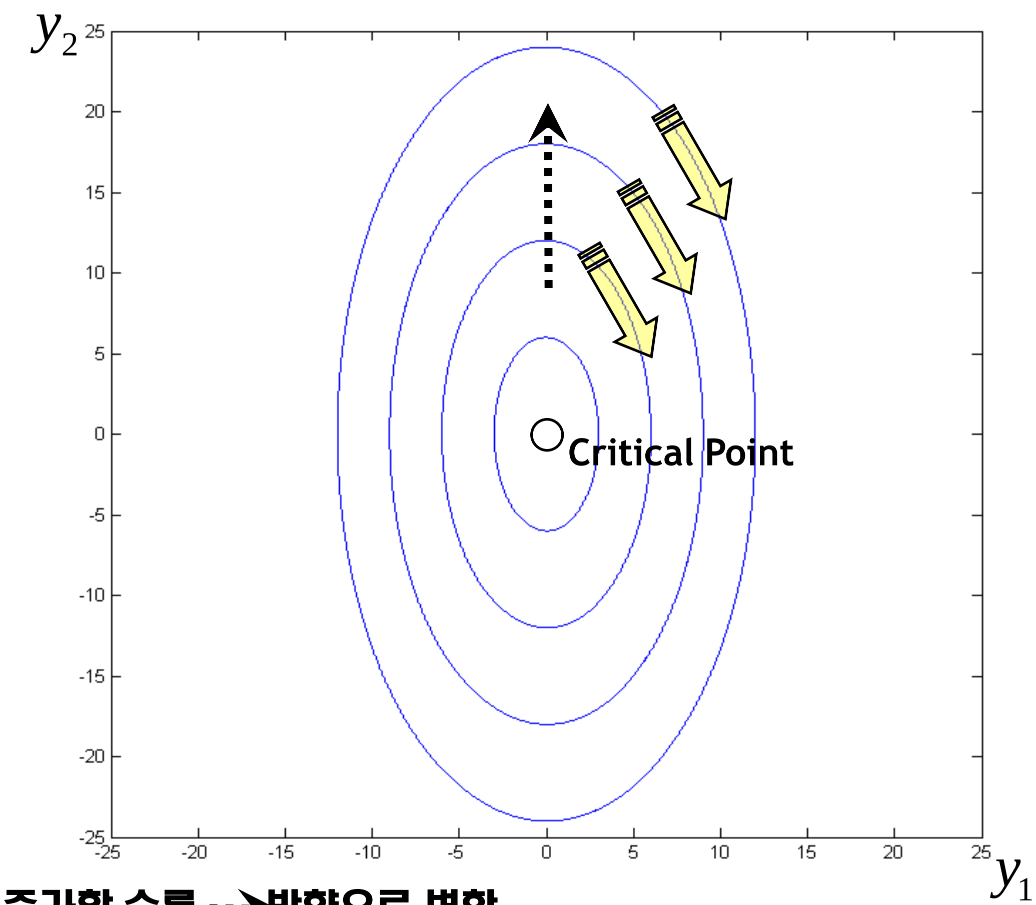

Critical Point: (0, 0)

centers(중심)

λ_1 and λ_2 are pure imaginaries

$\left|\frac{c_1}{c_2}\right|$ 가 증가할 수록 ···➤방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}$$

in components,

$$\begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = -y_1 + y_2$$
$$y_2' = -y_1 - y_2$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = -y_1 + y_2$$
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$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\lambda_1 = -1 + i,\ \lambda_2 = -1 - i$$

$$\begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = -y_1 + y_2$$
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$$\lambda_1 = -1 + i,\ \lambda_2 = -1 - i \qquad \begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\lambda_1 = -1 + i \qquad \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = -y_1 + y_2$$
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$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0$$

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$$\lambda_1 = -1+i,\ \lambda_2 = -1-i \qquad \begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\lambda_1 = -1+i \qquad \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda_2 = -1-i \qquad \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = -y_1 + y_2$$
$$y_2' = -y_1 - y_2$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0 \qquad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\lambda_1 = -1 + i,\ \lambda_2 = -1 - i \qquad \begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\lambda_1 = -1 + i \qquad \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

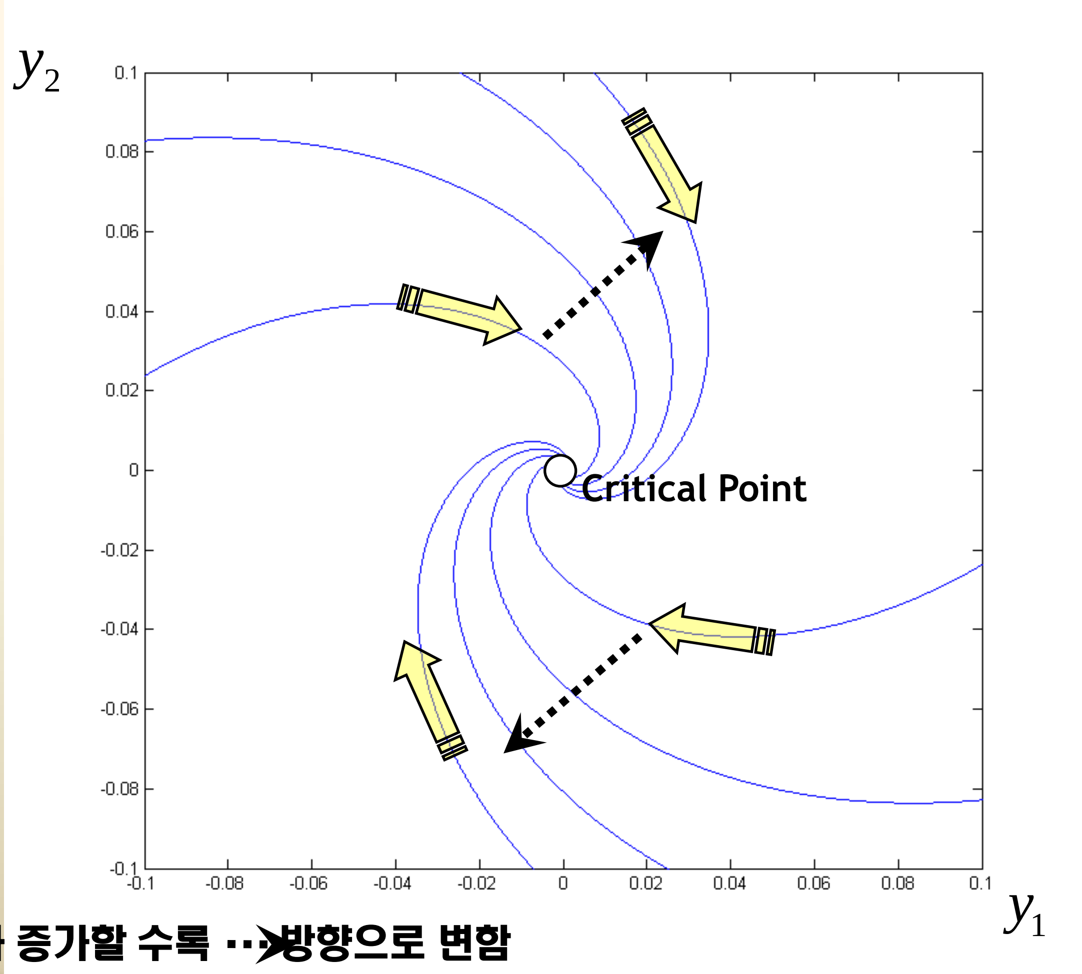
$$\lambda_2 = -1 - i \qquad \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\mathbf{y} = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} \Rightarrow \mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

$$y_1 = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}$$
$$y_2 = c_1 i e^{(-1+i)t} - c_2 i e^{(-1-i)t}$$

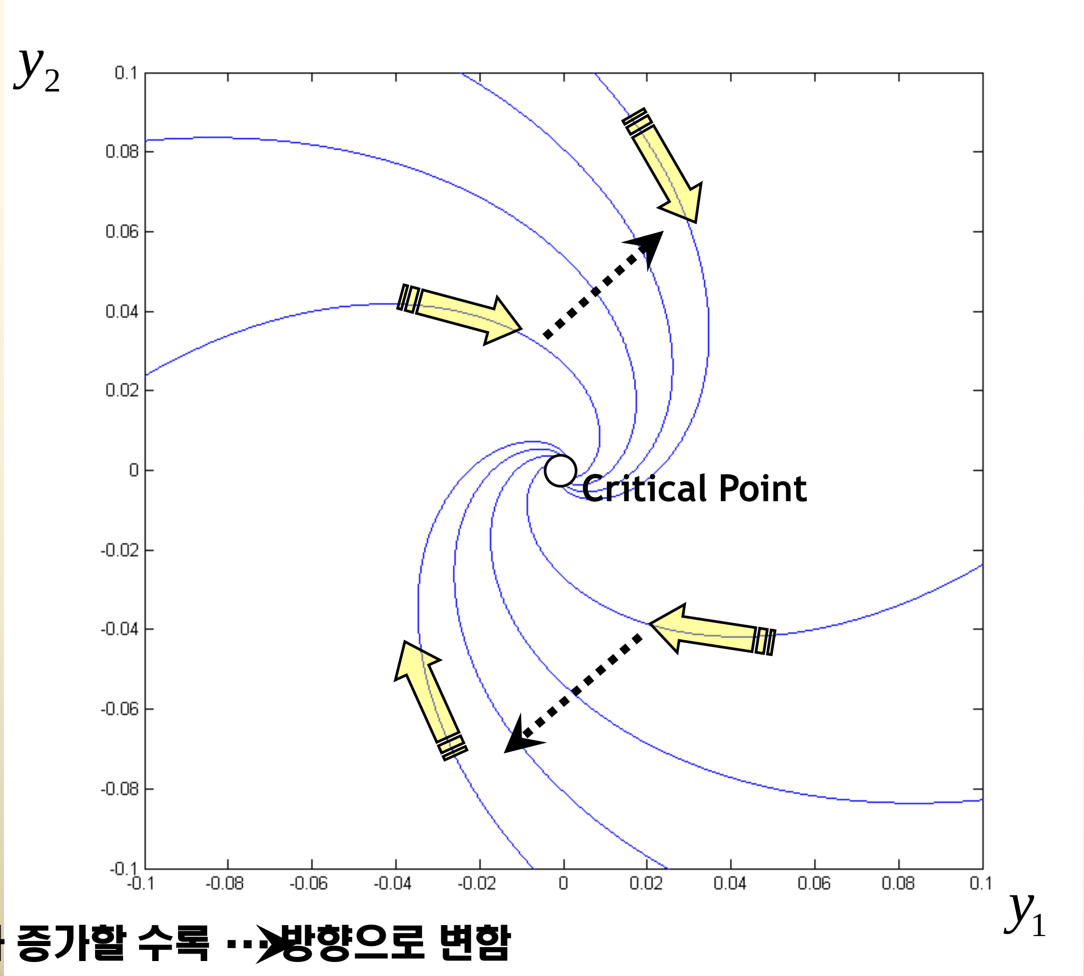


$\left|\frac{c_1}{c_2}\right|$ 가 증가할 수록 ···➤방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

$$y_1 = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}$$
$$y_2 = c_1 i e^{(-1+i)t} - c_2 i e^{(-1-i)t}$$



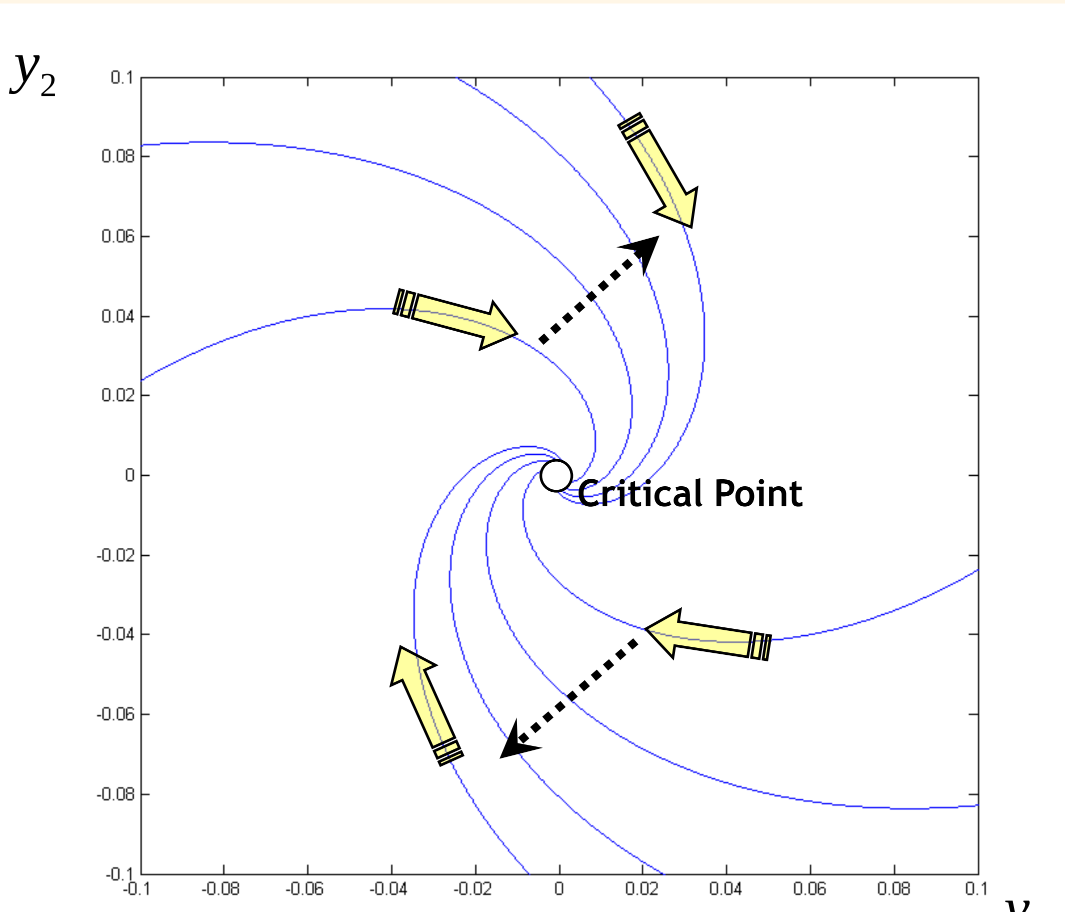
Critical Point: (0, 0)

$\left|\frac{c_1}{c_2}\right|$ 가 증가할 수록 ···➤방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}$$

$$y_1 = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}$$
$$y_2 = c_1 i e^{(-1+i)t} - c_2 i e^{(-1-i)t}$$



Critical Point: (0, 0)

Spiral point(**나선점**)

λ_1 and λ_2 are complex not pure imaginary

$\left|\dfrac{c_1}{c_2}\right|$ **가 증가할 수록 ···> 방향으로 변함**

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}\mathbf{y}$$

in components,

$$y_1' = 4y_1 + y_2$$
$$y_2' = -y_1 + 2y_2$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = 4y_1 + y_2$$
$$y_2' = -y_1 + 2y_2$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = 4y_1 + y_2$$
$$y_2' = -y_1 + 2y_2$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$$

$$\lambda_1 = \lambda_2 = 3 \qquad \begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = 4y_1 + y_2$$
$$y_2' = -y_1 + 2y_2$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$$

$$\lambda_1 = \lambda_2 = 3 \qquad \begin{bmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Suppose that $\mathbf{y}^{(2)} = \mathbf{x} t e^{\lambda t} + \mathbf{u} e^{\lambda t}$

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}\mathbf{y}$$

in components,

$$y_1' = 4y_1 + y_2$$
$$y_2' = -y_1 + 2y_2$$

$$\lambda = 3 \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Suppose that $\mathbf{y}^{(2)} = \mathbf{x}te^{\lambda t} + \mathbf{u}e^{\lambda t}$

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

$$\mathbf{y}'^{(2)} = \mathbf{x}e^{\lambda t} + \lambda\mathbf{x}te^{\lambda t} + \lambda\mathbf{u}e^{\lambda t} = \mathbf{A}\mathbf{y}^{(2)} = \mathbf{A}\mathbf{x}te^{\lambda t} + \mathbf{A}\mathbf{u}e^{\lambda t}$$

$$\mathbf{x}e^{\lambda t} + \cancel{\mathbf{A}\mathbf{x}te^{\lambda t}} + \lambda\mathbf{u}e^{\lambda t} = \cancel{\mathbf{A}\mathbf{x}te^{\lambda t}} + \mathbf{A}\mathbf{u}e^{\lambda t}$$

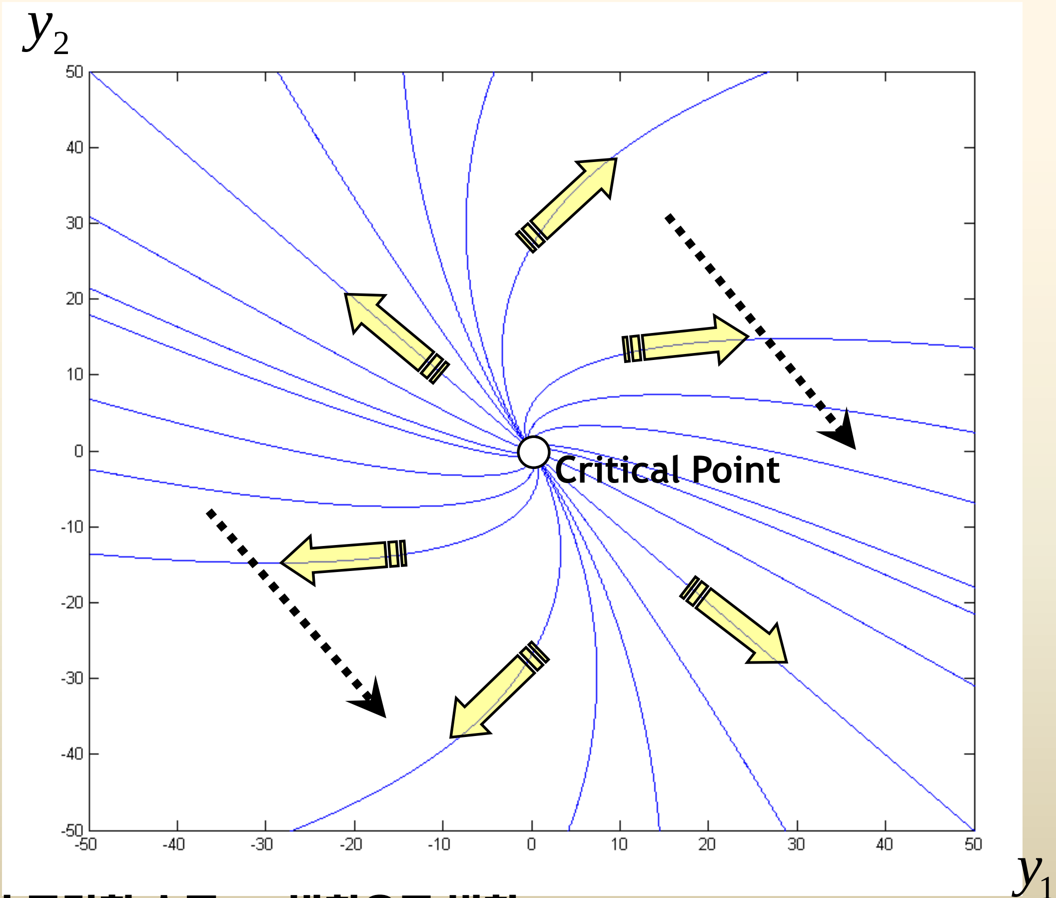
$$\mathbf{x} + \lambda\mathbf{u} = \mathbf{A}\mathbf{u}, \quad So \;\; (\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{x}$$

$$(\mathbf{A} - 3\mathbf{I})\mathbf{u} = \begin{bmatrix} 4-3 & 1 \\ -1 & 2-3 \end{bmatrix}\mathbf{u} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{array}{l} u_1 + u_2 = 1 \\ -u_1 - u_2 = -1 \end{array} \Rightarrow \mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{3t} + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)e^{3t}$$

Constant-Coefficient Systems. Phase Plane Method

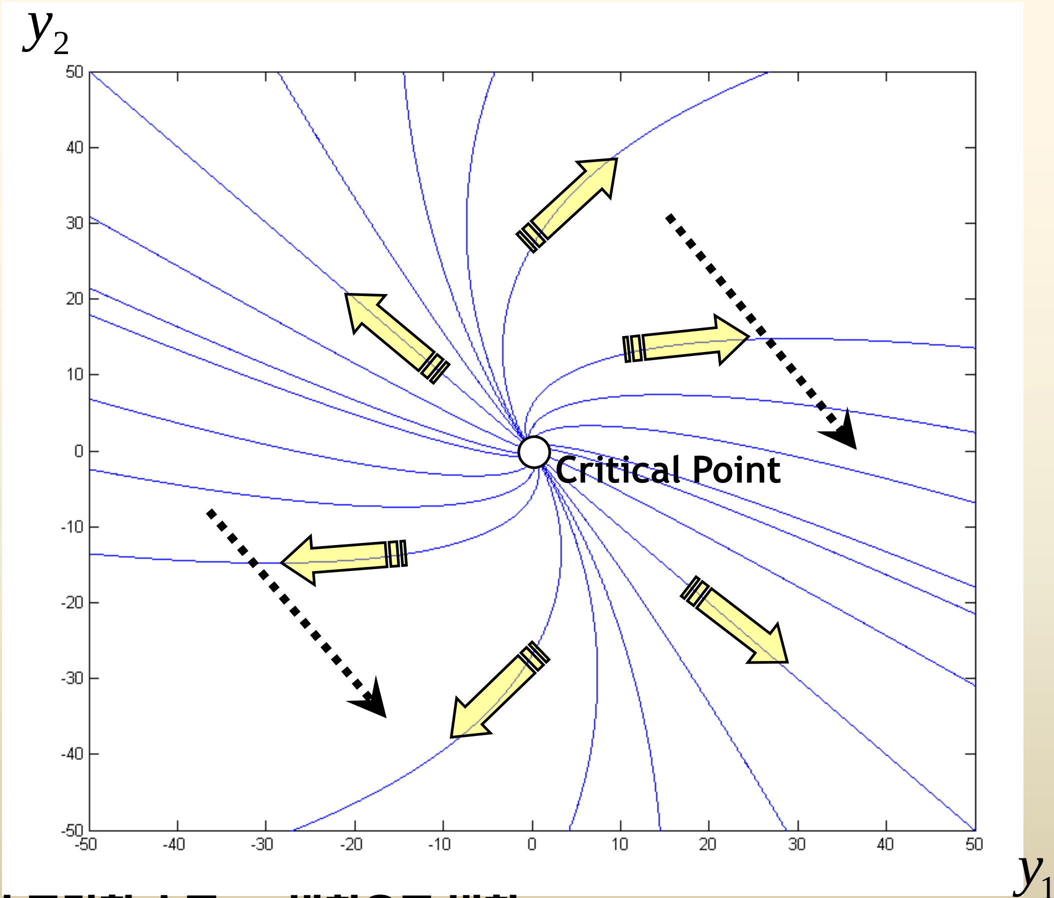
$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{3t} + c_2\left(\begin{bmatrix}1\\-1\end{bmatrix}t + \begin{bmatrix}0\\1\end{bmatrix}\right)e^{3t}$$



c_1 이 증가할 수록…➤방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1\begin{bmatrix}1\\-1\end{bmatrix}e^{3t} + c_2\left(\begin{bmatrix}1\\-1\end{bmatrix}t + \begin{bmatrix}0\\1\end{bmatrix}\right)e^{3t}$$

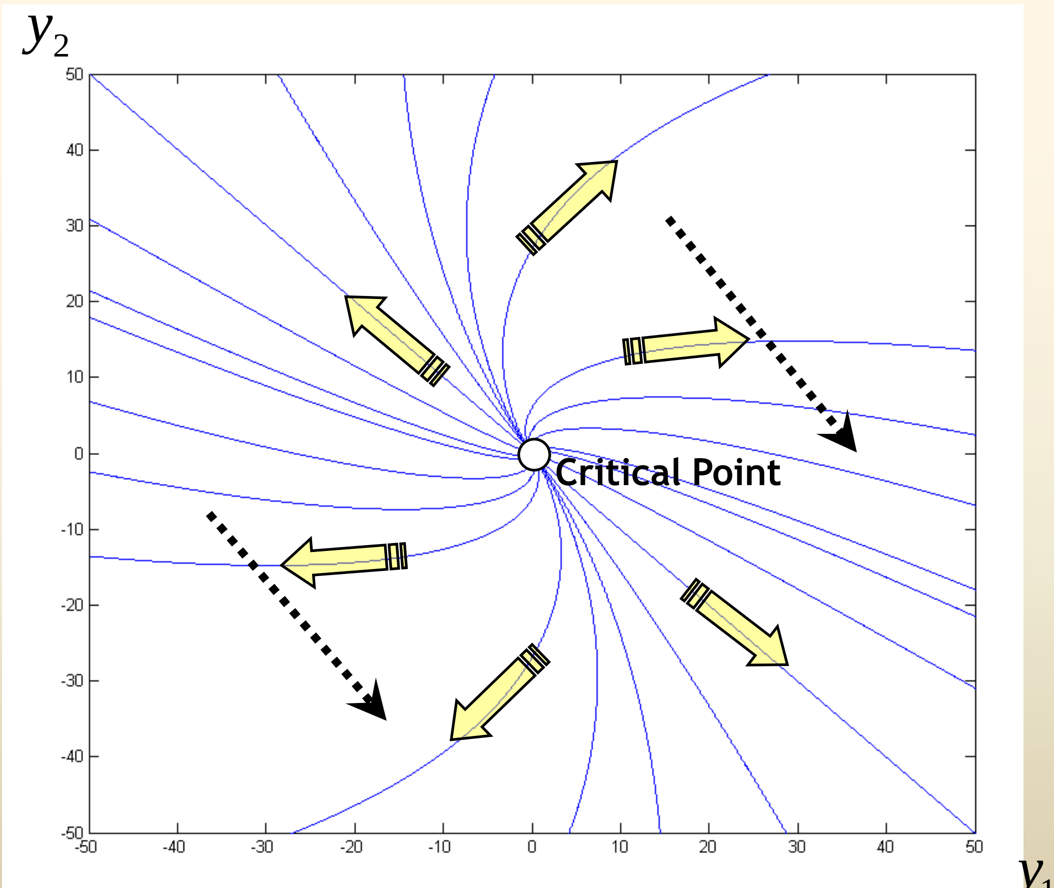


Critical Point: (0, 0)

c_1 이 증가할 수록… 방향으로 변함

Constant-Coefficient Systems. Phase Plane Method

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^{3t}$$



Critical Point: (0, 0)

Degenerate point(퇴화마디점)

λ_1 and λ_2 are complex not pure imaginary

c_1 이 증가할 수록 ···➤ 방향으로 변함

Criteria for Critical Points. Stability

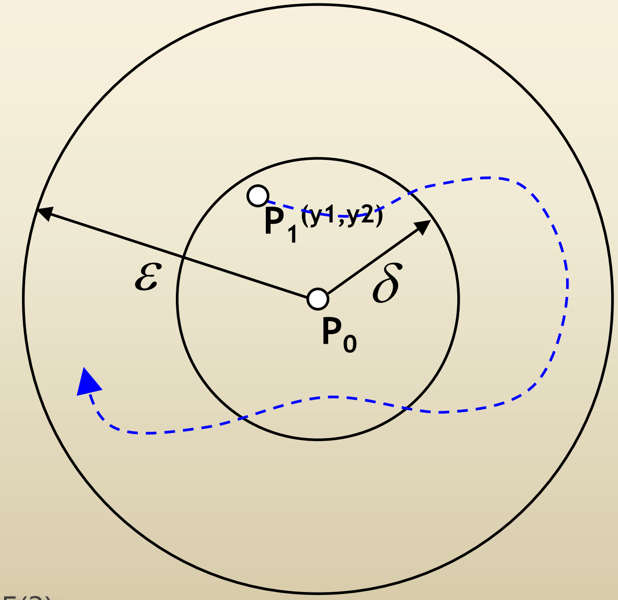
$$(1) \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y}$$

Criteria for Critical Points. Stability

$$(1) \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y}$$

Stable Critical Point, P_0

- P_0 is critical point of (1)
- All trajectories of (1) remain close to P0 at all future times



Criteria for Critical Points. Stability

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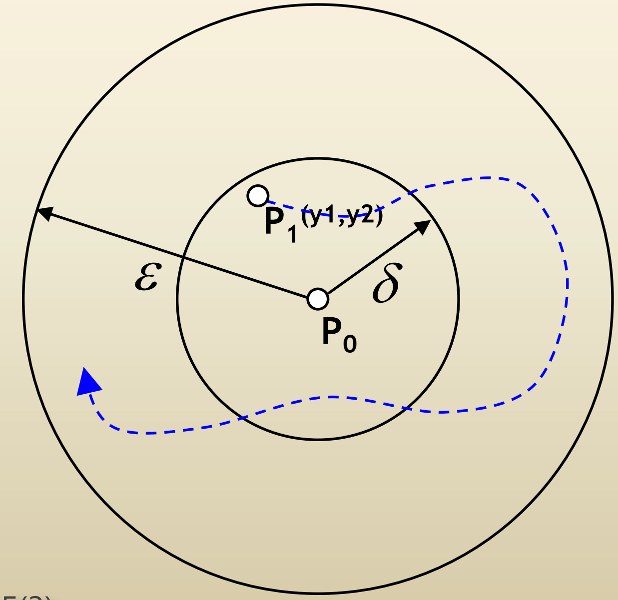
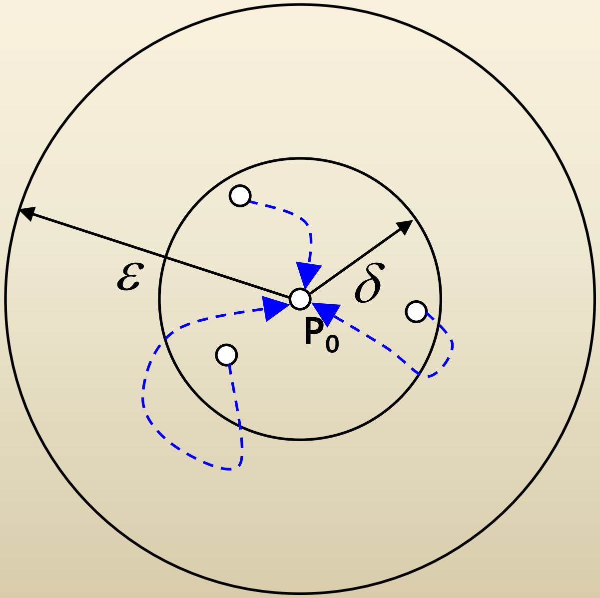


Stable and attractive Critical Point, P_0

- P_0 is stable critical point of (1)
- Every trajectory of (1) approaches P_0 as t → ∞.



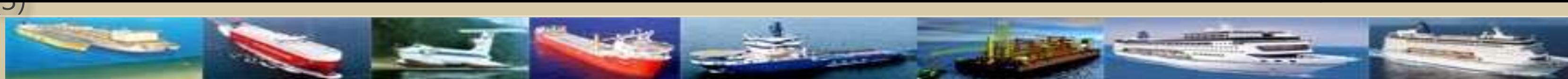
Criteria for Critical Points. Stability

$$(1) \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + \det \mathbf{A} = 0$$

when $p = a_{11} + a_{22} = \lambda_1 + \lambda_2, \quad q = \det \mathbf{A} = \lambda_1 \lambda_2, \quad \Delta = p^2 - 4q = (\lambda_1 - \lambda_2)^2$

		Stable Attractive
		Stable
		Unstable
		Unstable



Criteria for Critical Points. Stability

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$q = \lambda_1\lambda_2 > 0$ $\lambda_1, \lambda_2 > 0$		**Stable Attractive**
$\lambda_1, \lambda_2 < 0$		**Stable**
$\lambda_1, \lambda_2 = \alpha \pm \beta i$		**Unstable**
		Unstable

Criteria for Critical Points. Stability

(1) $$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y}$$

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$q = \lambda_1\lambda_2 > 0$ ⇒ $\lambda_1, \lambda_2 > 0$; $\lambda_1, \lambda_2 < 0$; $\lambda_1, \lambda_2 = \alpha \pm \beta i$	$p = \lambda_1 + \lambda_2 < 0$ ⇒ $\lambda_1, \lambda_2 < 0$ *real number of* λ_1, λ_2 $\alpha < 0$	**Stable Attractive**
		Stable
		Unstable
		Unstable

Criteria for Critical Points. Stability

(1) $$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\mathbf{y}$$

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$q = \lambda_1\lambda_2 > 0$ $\Rightarrow$ $\lambda_1,\lambda_2 > 0$ $\lambda_1,\lambda_2 < 0$ $\lambda_1,\lambda_2 = \alpha \pm \beta i$	$p = \lambda_1 + \lambda_2 < 0$ $\Rightarrow$ $\lambda_1,\lambda_2 < 0$ *real number of* λ_1,λ_2 $\alpha < 0$	**Stable Attractive**
	$p = \lambda_1 + \lambda_2 = 0$ $\Rightarrow$ λ_1,λ_2 *both are pure imaginaries*	**Stable**
		Unstable
		Unstable

Criteria for Critical Points. Stability

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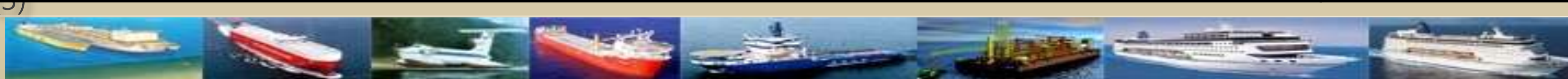
$q = \lambda_1 \lambda_2 > 0 \Rightarrow$ $\lambda_1, \lambda_2 > 0$; $\lambda_1, \lambda_2 < 0$; $\lambda_1, \lambda_2 = \alpha \pm \beta i$	$p = \lambda_1 + \lambda_2 < 0 \Rightarrow \lambda_1, \lambda_2 < 0$ *real number of* λ_1, λ_2; $\alpha < 0$	**Stable Attractive**
	$p = \lambda_1 + \lambda_2 = 0 \Rightarrow \lambda_1, \lambda_2$ *both are pure imaginaries*	**Stable**
	$p = \lambda_1 + \lambda_2 > 0 \Rightarrow \lambda_1, \lambda_2 > 0$ *real number of* λ_1, λ_2; $\alpha > 0$	**Unstable**
		Unstable



Criteria for Critical Points. Stability

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	$p = \lambda_1 + \lambda_2 = 0$ ⇨	λ_1, λ_2 *both are pure imaginaries*	**Stable**
	$p = \lambda_1 + \lambda_2 > 0$ ⇨	$\lambda_1, \lambda_2 > 0$ *real number of* λ_1, λ_2; $\alpha > 0$	**Unstable**
$q = \lambda_1\lambda_2 < 0$ ⇨	$\lambda_1 > 0,\ \lambda_2 < 0$ **or** $\lambda_1 < 0,\ \lambda_2 > 0$		**Unstable**

Criteria for Critical Points. Stability

(1) $\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\mathbf{y}$ **when** $p = a_{11} + a_{22} = \lambda_1 + \lambda_2, \quad q = \det \mathbf{A} = \lambda_1\lambda_2, \quad \Delta = p^2 - 4q = (\lambda_1 - \lambda_2)^2$

$q = \lambda_1 \lambda_2$	$p = \lambda_1 + \lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	**Explanation of** λ_1, λ_2	
$q > 0$ ⇒ $\lambda_1, \lambda_2 > 0$ $\lambda_1, \lambda_2 < 0$ $\lambda_1, \lambda_2 = \alpha \pm \beta i$		$\Delta \geq 0$	λ_1, λ_2 *both are real numbers*	**마디점**
	$p = 0$		λ_1, λ_2 *both are pure imaginaries*	**중심**
$q < 0$			$\lambda_1 < 0,\ \lambda_2 > 0$ **or** $\lambda_1 > 0,\ \lambda_2 < 0$	**안장점**
	$p \neq 0$	$\Delta < 0$	λ_1, λ_2 *both are complex not pure imaginaries*	**나선점**

Ex.) Application of the Criteria in Tables

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y}$$

in components,

$$y_1' = -3y_1 + y_2$$
$$y_2' = y_1 - 3y_2$$

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$$p = -6, \quad q = 8, \quad \Delta = 4$$

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	$p = \lambda_1 + \lambda_2 = 0$	Stable
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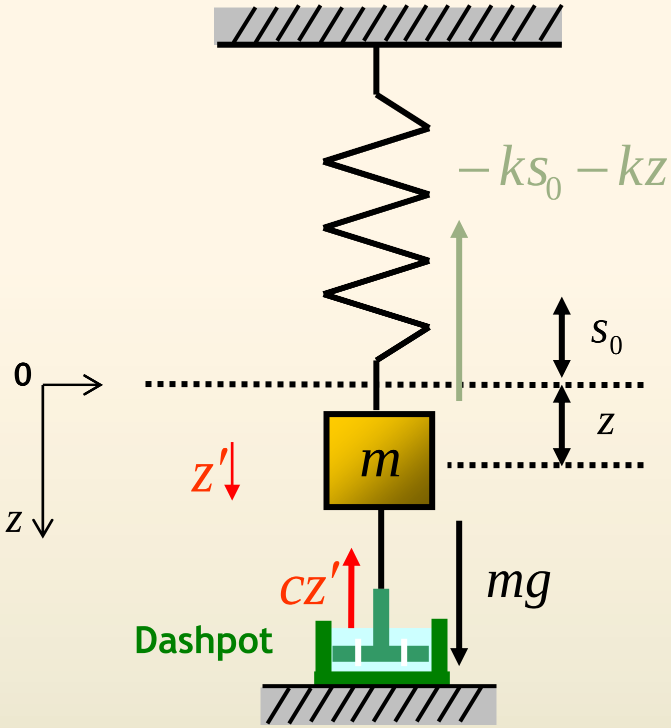
Ex.) Free Motions of a Mass on a Spring

- **Forced, Damped vibration**



$$mz'' = F$$
$$= mg - k(s_0 + z) - cz'$$
$$= -kz - cz'$$

$$\boxed{mz'' + cz' + kz = 0}$$

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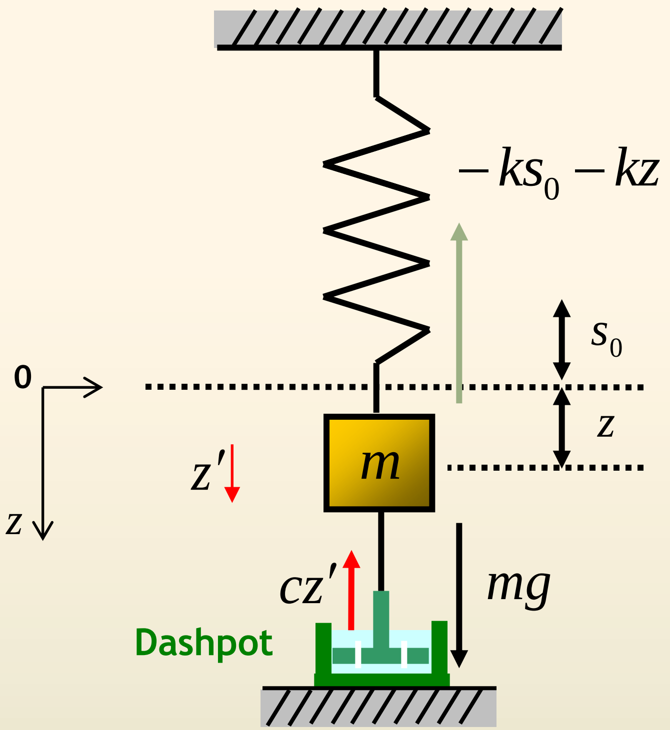

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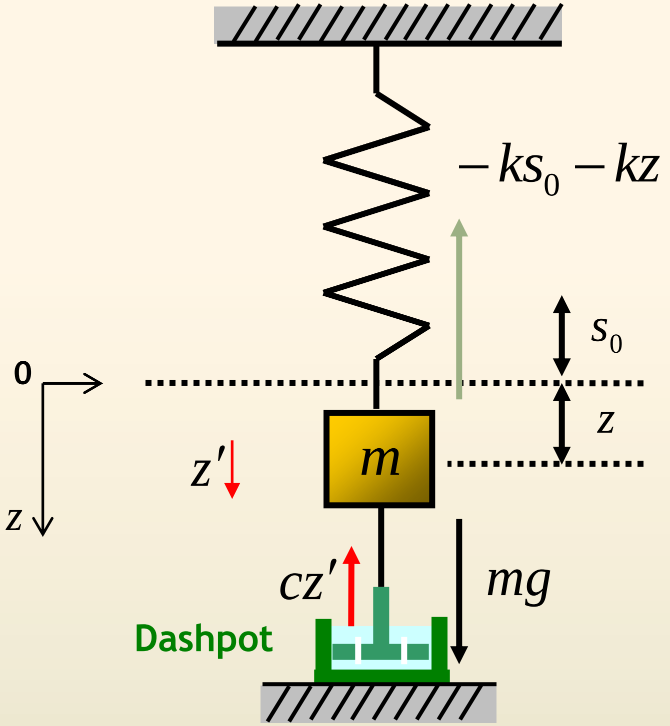

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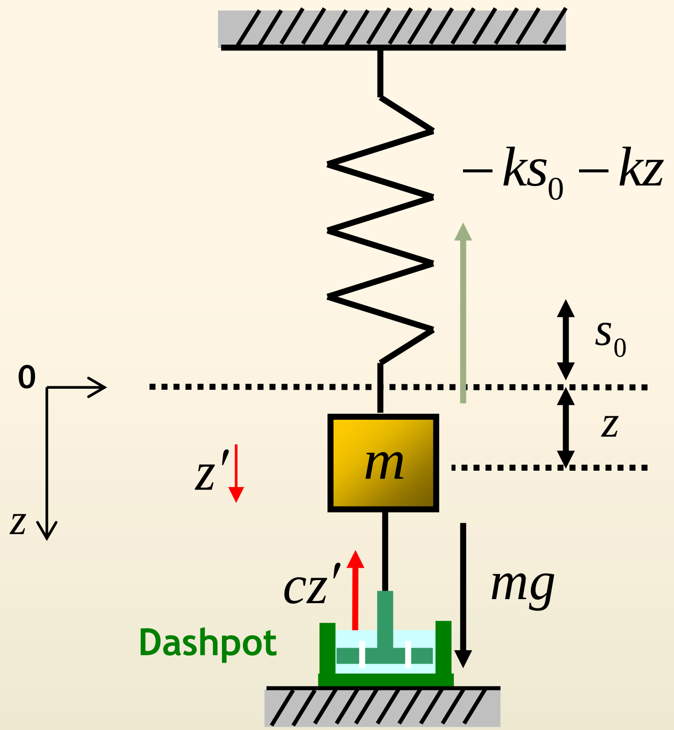

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Ex.) Free Motions of a Mass on a Spring

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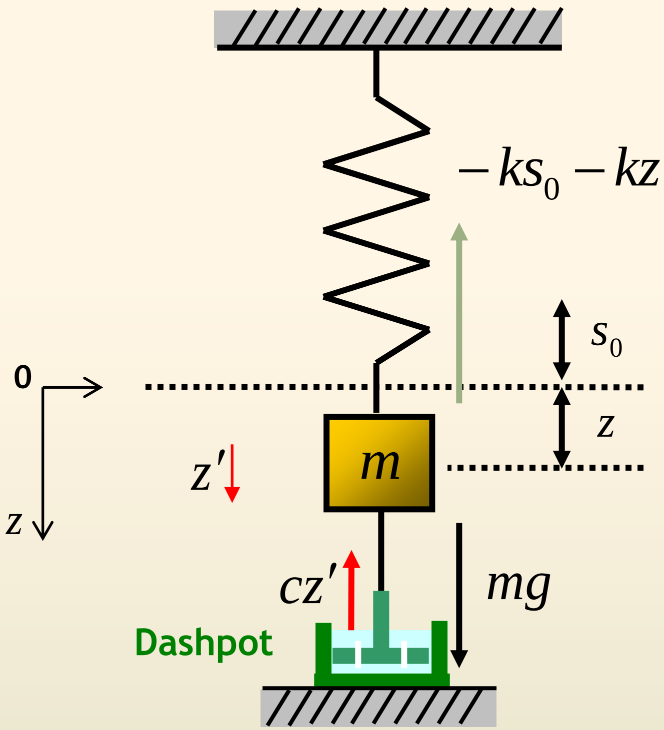

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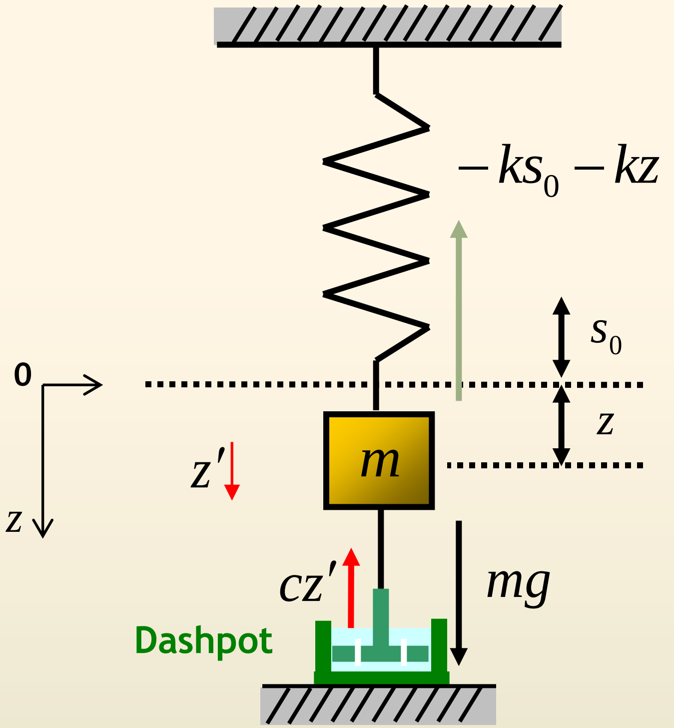
$$\boxed{\begin{aligned} z_1' &= z_2 \\ z_2' &= -\frac{k}{m}z_1 - \frac{c}{m}z_2 \end{aligned}}$$

Ex.) Free Motions of a Mass on a Spring

- Forced, Damped vibration



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Ex.) Free Motions of a Mass on a Spring

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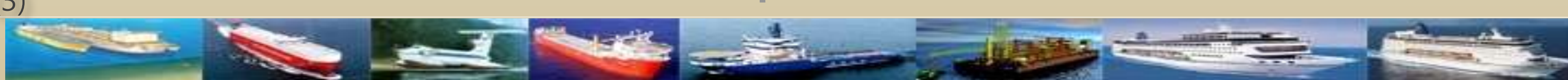

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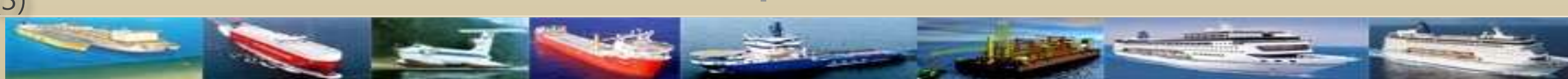

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① No Damping

② Under Damping

③ Critical Damping

④ Over Damping

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$$\boxed{mz'' + cz' + kz = 0}$$

① **No Damping**

$$c = 0, \ p = 0, \ q > 0$$

② **Under Damping**

③ **Critical Damping**

④ **Over Damping**

Ex.) Free Motions of a Mass on a Spring

$$p = -\frac{c}{m}, \quad q = \frac{k}{m},$$

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① **No Damping**

$$c = 0, \ p = 0, \ q > 0$$ **Center**

② **Under Damping**

③ **Critical Damping**

④ **Over Damping**

Ex.) Free Motions of a Mass on a Spring

$$p = -\frac{c}{m}, \quad q = \frac{k}{m},$$
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① No Damping

$$c = 0,\ p = 0,\ q > 0$$ **Center**

② Under Damping

$$c^2 < 4mk,\ p < 0,\ q > 0,\ \Delta < 0$$

③ Critical Damping

④ Over Damping

Ex.) Free Motions of a Mass on a Spring

$$p = -\frac{c}{m}, \quad q = \frac{k}{m},$$
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$$mz'' + cz' + kz = 0$$

① No Damping

$$c = 0,\ p = 0,\ q > 0$$

Center

② Under Damping

$$c^2 < 4mk,\ p < 0,\ q > 0,\ \Delta < 0$$

Stable Attractive Spiral Point

③ Critical Damping

④ Over Damping

Ex.) Free Motions of a Mass on a Spring

$$p = -\frac{c}{m}, \quad q = \frac{k}{m},$$

$$\Delta = \left(\frac{c}{m}\right)^2 - \frac{4k}{m} = \frac{c^2 - 4mk}{m^2}$$

$$mz'' + cz' + kz = 0$$

① **No Damping**

$$c = 0, \; p = 0, \; q > 0$$

Center

② **Under Damping**

$$c^2 < 4mk, \; p < 0, \; q > 0, \; \Delta < 0$$

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Stable Attractive Node

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- Critical Point of (1) is a point $P_0(a,b)$ that $y_1' = 0 \quad y_2' = 0$

- $P_0(a,b)$ can be translated to *(0,0)* which is easy to handle.

$$\tilde{y}_1 = y_1 - a$$

$$\tilde{y}_2 = y_2 - b$$

- We use y_1, y_2 instead of $\tilde{y}_1, \tilde{y}_2$

Qualitative Methods for Nonlinear Systems

: Linearization of Nonlinear System

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) \quad \text{thus} \quad \begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2) \end{aligned} \qquad (1)$$

We can express it again like below by using Taylor Expansion at critical point $P_0(0,0)$.

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$$f_1(0,0) = 0$$
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Qualitative Methods for Nonlinear Systems

: Linearization of Nonlinear System

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) \quad \text{thus} \quad \begin{matrix} y_1' = f_1(y_1, y_2) \\ y_2' = f_2(y_1, y_2) \end{matrix} \qquad (1)$$

We can express it again like below by using Taylor Expansion at critical point $P_0(0,0)$.

$$f_1(y_1, y_2) = f_1(0,0) + \frac{\partial f_1}{\partial y_1}(y_1 - 0) + \frac{\partial f_1}{\partial y_2}(y_2 - 0)$$

$$f_1(0,0) = 0$$

$$f_2(0,0) = 0$$

$$+ \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial y_1^2}(y_1 - 0)^2 + 2\frac{\partial^2 f_1}{\partial y_1 \partial y_2}(y_1 - 0)(y_2 - 0) + \frac{\partial^2 f_1}{\partial x_2^2}(y_2 - 0)^2 \right) + R_1$$

$$= \frac{\partial f_1}{\partial y_1} y_1 + \frac{\partial f_1}{\partial y_2} y_2 + \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial y_1^2} y_1^{\ 2} + 2\frac{\partial^2 f_1}{\partial y_1 \partial y_2} y_1 y_2 + \frac{\partial^2 f_1}{\partial x_2^2} y_2^{\ 2} \right) + R_1$$

$$= \frac{\partial f_1}{\partial y_1} y_1 + \frac{\partial f_1}{\partial y_2} y_2 + h_1(y_1, y_2)$$

Qualitative Methods for Nonlinear Systems

: Linearization of Nonlinear System

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) \quad \text{thus} \quad \begin{aligned} y_1' &= f_1(y_1, y_2) \\ y_2' &= f_2(y_1, y_2) \end{aligned} \qquad (1)$$

We can express it again like below by using Taylor Expansion at critical point $P_0(0,0)$.

$$f_1(0,0) = 0$$

$$f_2(0,0) = 0$$

$$f_1(y_1, y_2) = f_1(0,0) + \frac{\partial f_1}{\partial y_1}(y_1 - 0) + \frac{\partial f_1}{\partial y_2}(y_2 - 0)$$

$$+ \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial y_1^2}(y_1 - 0)^2 + 2\frac{\partial^2 f_1}{\partial y_1 \partial y_2}(y_1 - 0)(y_2 - 0) + \frac{\partial^2 f_1}{\partial x_2^2}(y_2 - 0)^2 \right) + R_1$$

$$= \frac{\partial f_1}{\partial y_1} y_1 + \frac{\partial f_1}{\partial y_2} y_2 + \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial y_1^2} y_1^{\,2} + 2\frac{\partial^2 f_1}{\partial y_1 \partial y_2} y_1 y_2 + \frac{\partial^2 f_1}{\partial x_2^2} y_2^{\,2} \right) + R_1$$

$$= \frac{\partial f_1}{\partial y_1} y_1 + \frac{\partial f_1}{\partial y_2} y_2 + h_1(y_1, y_2)$$

$$f_2(y_1, y_2) = \frac{\partial f_2}{\partial y_1} y_1 + \frac{\partial f_2}{\partial y_2} y_2 + \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y_1^2} y_1^{\,2} + 2\frac{\partial^2 f_2}{\partial y_1 \partial y_2} y_1 y_2 + \frac{\partial^2 f_2}{\partial x_2^2} y_2^{\,2} \right) + R_2$$

$$= \frac{\partial f_2}{\partial y_1} y_1 + \frac{\partial f_2}{\partial y_2} y_2 + h_2(y_1, y_2)$$

Qualitative Methods for Nonlinear Systems
: Linearization of Nonlinear System

$$
\begin{aligned}
y_1' &= f_1(y_1, y_2) = \boxed{\frac{\partial f_1}{\partial y_1} y_1 + \frac{\partial f_1}{\partial y_2} y_2} + h_1(y_1, y_2) \\
y_2' &= f_2(y_1, y_2) = \boxed{\frac{\partial f_2}{\partial y_1} y_1 + \frac{\partial f_2}{\partial y_2} y_2} + h_2(y_1, y_2)
\end{aligned}
\quad \text{thus} \quad
\begin{aligned}
y_1' &= \boxed{a_{11} y_1 + a_{12} y_2} + h_1(y_1, y_2) \\
y_2' &= \boxed{a_{21} y_1 + a_{22} y_2} + h_2(y_1, y_2)
\end{aligned}
$$

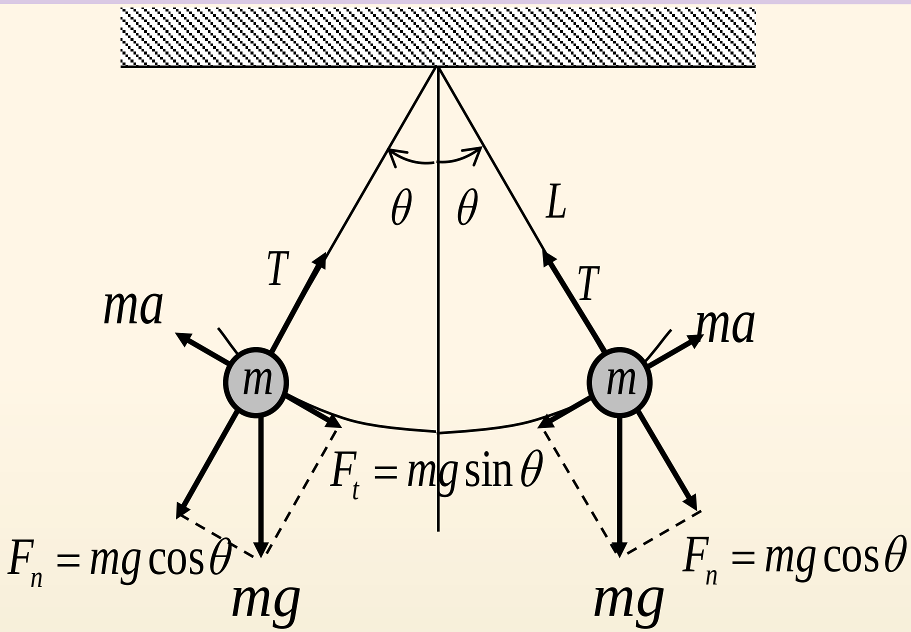
Linear Terms

- **To linearize, we only choose linear terms.**

  So, it can be written like below.

$$
\mathbf{y}' = \mathbf{A}\mathbf{y} \quad \text{thus} \quad
\begin{aligned}
y_1' &= a_{11} y_1 + a_{12} y_2 \\
y_2' &= a_{21} y_1 + a_{22} y_2
\end{aligned}
$$

Ex.) Free Undamped Pendulum. Linearization



- **Acceleration at time *t* is** $\dfrac{d^2R}{dt^2}$ **where *R* is arc length of pendulum**

$$R = L\theta, \qquad \frac{dR}{dt} = L\frac{d\theta}{dt},$$

$$\frac{d^2R}{dt^2} = L\frac{d^2\theta}{dt^2} = L\theta''$$

- **By Newton's 2nd law**

$$F = ma$$

$$ma = m\frac{d^2R}{dt^2} = mL\frac{d^2\theta}{dt^2}$$
$$= mL\theta''$$

$$F = -F_t = -mg\sin\theta$$

$$mL\theta'' = -mg\sin\theta$$

$$mL\theta'' + mg\sin\theta = 0$$

$$\boxed{\theta'' + k\sin\theta = 0(\text{where } k = \frac{g}{L})}$$

Ex.) Free Undamped Pendulum. Linearization

$$\theta'' + k\sin\theta = 0$$

- **If we assume** $\theta = y_1,\ \theta' = y_2$

Then,

$$y_1 = \theta$$

$$y_1' = \theta' = y_2$$

$$\theta'' + k\sin\theta = 0$$

And

$$y_2' + k\sin y_1 = 0$$

$$y_2' = -k\sin y_1$$

$$y_1' = y_2$$

$$y_2' = -k\sin y_1$$

- **Critical point of above eq.**

$$y_1' = 0 \qquad y_2' = 0$$

$$y_1' = y_2 = 0$$

$$y_1' = -k\sin y_1 = 0$$

$$\text{then } y_1 = n\pi (n = 0, \pm 1, \pm 2, \ldots.)$$

- **Critical point P_0 is** $P_0 : (n\pi, 0)$

Ex.) Free Undamped Pendulum. Linearization

1. Critical Point (0,0), ±(2π,0), ±(4π,0),... Linearization

$$y_1' = y_2$$
$$y_2' = -k \sin y_1$$

$$P_0 : (n\pi, 0)$$

- **We should linearize** $\sin y_1$ at P_0

- **First, we think about n=0,** $P_0 : (0,0)$

$$\sin y_1 = \sin(0 + y_1)$$

$$\boxed{= \sin(0) + \left.\frac{d \sin t}{dt}\right|_{t=0} y_1} + \frac{1}{2!}\left.\frac{d^2 \sin t}{dt^2}\right|_{t=0} y_1^{\,2} + \frac{1}{3!}\left.\frac{d^3 \sin t}{dt^3}\right|_{t=0} y_1^{\,3}$$

$$\boxed{= \cos(0)\, y_1} - \frac{1}{6}\cos(0)\, y_1^{\,3}$$

$$\boxed{= y_1} + \frac{1}{6} y_1^{\,3}$$

$$y_1' = y_2$$
$$y_2' = -k y_1$$

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$$P_0 : (n\pi, 0)$$

▪ We only need blue boxed one. So, from now on we will use first and second term of Taylor expansion to linearize.

▪ We should linearize $\sin y_1$ at P_0

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$$y_1' = y_2$$
$$y_2' = -k y_1$$

Ex.) Free Undamped Pendulum. Linearization

$$y_1' = y_2$$

$$y_2' = -ky_1$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}\mathbf{y}$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\mathbf{y}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + \det\mathbf{A} = 0$$

$$p = a_{11} + a_{22} = \lambda_1 + \lambda_2,$$

$$q = \det\mathbf{A} = \lambda_1\lambda_2,$$

$$\Delta = p^2 - 4q = (\lambda_1 - \lambda_2)^2$$

$q = \lambda_1 \lambda_2$	$p = \lambda_1 + \lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	Explanation of λ_1, λ_2	
$q > 0$: $\lambda_1, \lambda_2 > 0$; $\lambda_1, \lambda_2 < 0$		$\Delta \ge 0$	λ_1, λ_2 both are real numbers	마디점
$\lambda_1, \lambda_2 = \alpha \pm \beta i$	$p = 0$		λ_1, λ_2 both are pure imaginaries	중심
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	$p \ne 0$	$\Delta < 0$	λ_1, λ_2 both are complex not pure imaginaries	나선점

$$p = a_{11} + a_{22} = 0$$

$$q = \det\mathbf{A} = k = g / L$$

$$\Delta = p^2 - 4q = -4k$$

It is *center* because *p = 0, q > 0*

And every critical points (nπ,0), n=0, ±2, ±4, ±6 *Is center because the function sinθ is periodic with period 2π*

Ex.) Free Undamped Pendulum. Linearization

2. Critical Point ±(π,0), ±(3π,0), … Linearization

$$y_1' = y_2$$
$$y_2' = -k\sin y_1$$

$$P_0 : (n\pi, 0)$$

$$\sin y_1 = \sin(0) + \left.\frac{d\sin t}{dt}\right|_{t=0} y_1 + \frac{1}{2!}\left.\frac{d^2\sin t}{dt^2}\right|_{t=0} y_1^{\,2} + \frac{1}{3!}\left.\frac{d^3\sin t}{dt^3}\right|_{t=0} y_1^{\,3}$$

$$= \cos(0)y_1 - \frac{1}{6}\cos(0)y_1^{\,3}$$

$$= y_1 + \frac{1}{6}y_1^{\,3}$$

$$\sin(y_1 \pm \pi) = -\sin y_1$$

$$= \boxed{-y_1} - \frac{1}{6}y_1^{\,3}$$

$$\Rightarrow$$

$$y_1' = y_2$$
$$y_2' = ky_1$$

Ex.) Free Undamped Pendulum. Linearization

$$y_1' = y_2$$

$$y_2' = ky_1$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & 0 \end{bmatrix}\mathbf{y}$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\mathbf{y}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + \det \mathbf{A} = 0$$

$$p = a_{11} + a_{22} = \lambda_1 + \lambda_2,$$

$$q = \det \mathbf{A} = \lambda_1\lambda_2,$$

$$\Delta = p^2 - 4q = (\lambda_1 - \lambda_2)^2$$

$q = \lambda_1 \lambda_2$	$p = \lambda_1 + \lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	Explanation of λ_1, λ_2	
$q > 0$: $\lambda_1, \lambda_2 > 0$; $\lambda_1, \lambda_2 < 0$; $\lambda_1, \lambda_2 = \alpha \pm \beta i$		$\Delta \geq 0$	λ_1, λ_2 both are real numbers	마디점
	$p = 0$		λ_1, λ_2 both are pure imaginaries	중심
$q < 0$			$\lambda_1 < 0,\ \lambda_2 > 0$ or $\lambda_1 > 0,\ \lambda_2 < 0$	안장점
	$p \neq 0$	$\Delta < 0$	λ_1, λ_2 both are complex not pure imaginaries	나선점

$$p = a_{11} + a_{22} = 0$$

$$q = \det \mathbf{A} = -k = -(g / L)$$

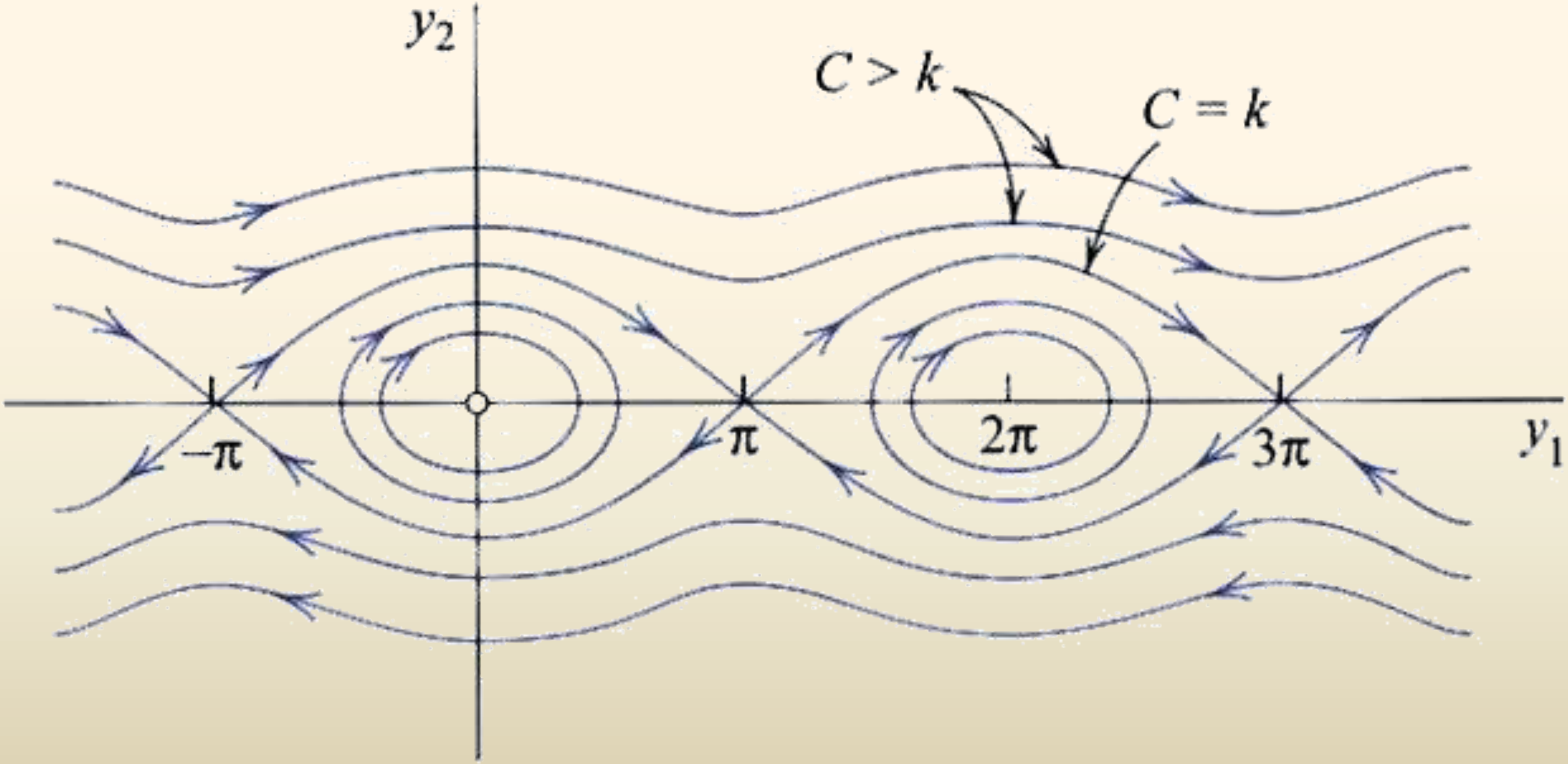
$$\Delta = p^2 - 4q = 4k$$

It is *Saddle Point*

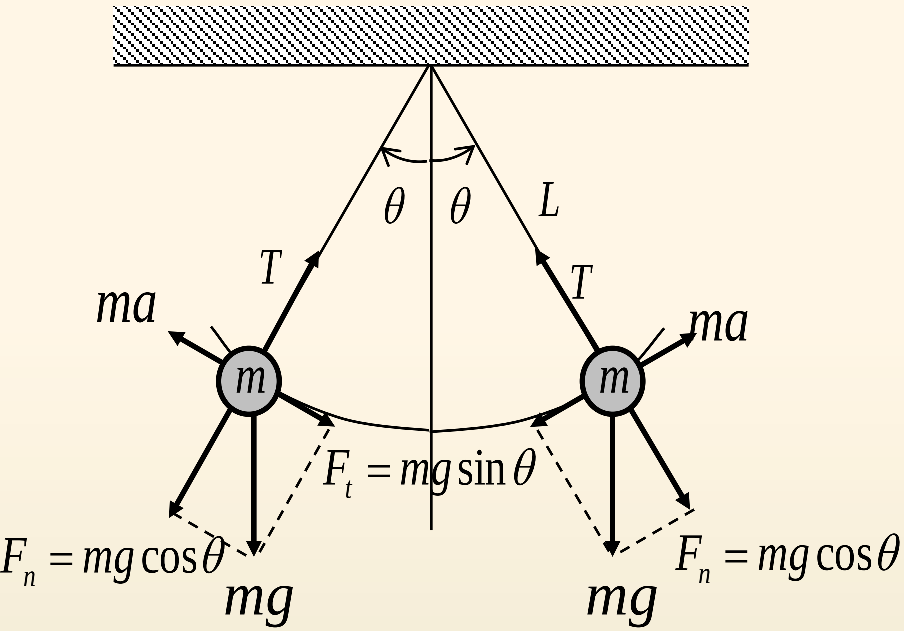
because *p = 0, q < 0*

And every critical points (nπ,0), n=±1, ±3, ±5 *is saddle point because the function sinθ is periodic with period 2π*

Ex.) Free Undamped Pendulum. Linearization



Ex.) Linearization of the Damped Pendulum Equation



No Damping

$$\theta'' + k\sin\theta = 0 \text{(where } k = \frac{g}{L}\text{)}$$

Damping : $c\theta', \quad (k > 0,\ c \geq 0)$

$$\theta'' + c\theta' + k\sin\theta = 0$$

$$\theta'' + c\theta' + k\sin\theta = 0$$

- If we assume $\theta = y_1, \ \theta' = y_2$

Then,

$$y_1 = \theta$$

$$y_1' = \theta' = y_2$$

$$\theta'' + c\theta' + k\sin\theta = 0$$

And $y_2' + cy_2 + k\sin y_1 = 0$

$$y_2' = -k\sin y_1 - cy_2$$

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= -k\sin y_1 - cy_2 \end{aligned}}$$

Ex.) Linearization of the Damped Pendulum Equation

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= -k\sin y_1 - cy_2 \end{aligned}}$$

- **Critical point of above eq.**

$$y_1' = 0 \qquad y_2' = 0$$

$$y_1' = y_2 = 0$$

$$y_1' = -k\sin y_1 - cy_2 = 0$$

$$\text{then } y_1 = n\pi (n = 0, \pm 1, \pm 2, \ldots .)$$

- **Critical point *P₀* is** $P_0 : (n\pi, 0)$

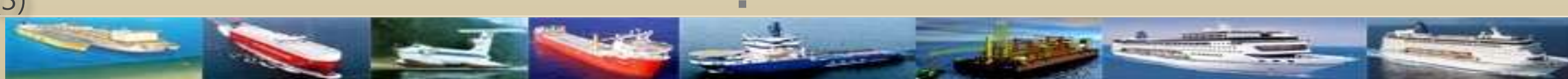
1. Critical Point (0,0)

$$\sin y_1 \approx y_1$$

$$y_1' = y_2 = 0$$

$$y_1' = -ky_1 - cy_2 = 0$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}\mathbf{y}$$



Ex.) Linearization of the Damped Pendulum Equation

$q=\lambda_1\lambda_2$	$p=\lambda_1+\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Explanation of λ_1,λ_2	
$q>0$ $\lambda_1,\lambda_2>0$ $\lambda_1,\lambda_2<0$ $\lambda_1,\lambda_2=\alpha\pm\beta i$		$\Delta\geq 0$	λ_1,λ_2 both are real numbers	마디점
	$p=0$		λ_1,λ_2 both are pure imaginaries	중심
$q<0$			$\lambda_1<0,\ \lambda_2>0$ or $\lambda_1>0,\ \lambda_2<0$	안장점
	$p\neq 0$	$\Delta<0$	λ_1,λ_2 both are complex not pure imaginaries	나선점

$$\mathbf{y}'=\mathbf{A}\mathbf{y}=\begin{bmatrix}0 & 1\\ -k & -c\end{bmatrix}\mathbf{y}$$

$$\mathbf{y}'=\mathbf{A}\mathbf{y}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\mathbf{y}$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix}$$

$$=\lambda^2-(a_{11}+a_{22})\lambda+\det\mathbf{A}=0$$

$$p=a_{11}+a_{22}=\lambda_1+\lambda_2,$$

$$q=\det\mathbf{A}=\lambda_1\lambda_2,$$

$$\Delta=p^2-4q=(\lambda_1-\lambda_2)^2$$

$$p=a_{11}+a_{22}=-c$$

$$q=\det\mathbf{A}=k=g/L$$

$$\Delta=p^2-4q=c^2-4k$$

If $c^2<-4k$ It is *Spiral* **Stable Attractive Spiral Point**

because $p<0$, $q>0$, $\Delta<0$

And every critical points (nπ,0), n=0, ±2, ±4, ±6 *is Spiral Point* because the function sinθ is periodic with period 2π

Ex.) Linearization of the Damped Pendulum Equation

2. Critical Point(π,0)

$$\sin(y_1 \pm \pi) = -\sin y_1 \approx -y_1$$

$$y_1' = y_2 = 0$$

$$y_1' = ky_1 - cy_2 = 0$$

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 0 & 1 \\ k & -c \end{bmatrix}\mathbf{y}$$

$$p = a_{11} + a_{22} = -c$$

$$q = \det \mathbf{A} = -k = -g/L$$

$$\Delta = p^2 - 4q = c^2 + 4k$$

$q=\lambda_1\lambda_2$	$p=\lambda_1+\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Explanation of λ_1,λ_2	
$q>0$: $\lambda_1,\lambda_2>0$; $\lambda_1,\lambda_2<0$		$\Delta\geq 0$	λ_1,λ_2 both are real numbers	마디점
$\lambda_1,\lambda_2=\alpha\pm\beta i$	$p=0$		λ_1,λ_2 both are pure imaginaries	중심
$q<0$			$\lambda_1<0,\ \lambda_2>0$ or $\lambda_1>0,\ \lambda_2<0$	안장점
	$p\neq 0$	$\Delta<0$	λ_1,λ_2 both are complex not pure imaginaries	나선점

No Damping

$$c=0,\ p=0,\ q<0,\ \Delta>0$$

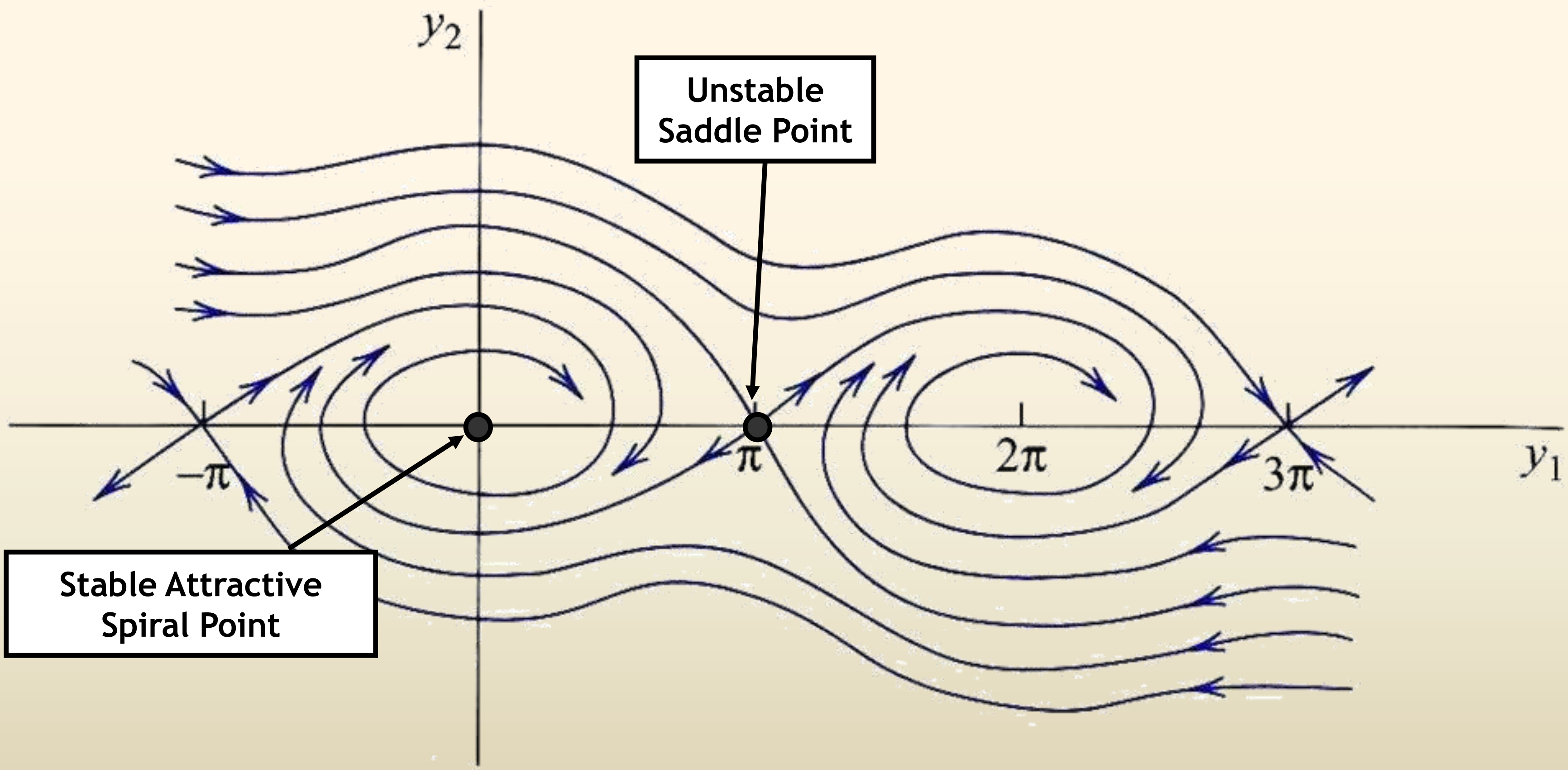
Unstable Saddle Point

Damping

$$c>0,\ p<0,\ q<0,\ \Delta>0$$

Unstable Saddle Point

Ex.) Linearization of the Damped Pendulum Equation



Ex.) Predator-Prey Population Model

This model concerns two species, say, rabbits(y_1) and foxes(y_2), and the foxes prey on the rabbits.

Assumption

1. Rabbit have unlimited food supply. There number $y_1(t)$ would grow exponentially.

$$y_1' = ay_1$$

2. Actually, $y_1(t)$ is decreased by a rate proportional to the number of encounters between predator and prey.

$$\boxed{y_1' = ay_1 - by_1y_2, (a > 0, b > 0)}$$

3. If there were no rabbits, the $y_2(t)$ would exponentially decrease to zero.

$$y_2' = -ly_2$$

4. However $y_2(t)$ is increased by a rate proportional to the number of encounters between predator and prey

$$\boxed{y_2' = -ly_2 + ky_1y_2, (k > 0, l > 0)}$$

Ex.) Predator-Prey Population Model

$$
\begin{aligned}
y_1' &= f_1(y_1, y_2) = ay_1 - by_1y_2 \\
y_2' &= f_2(y_1, y_2) = -ly_2 + ky_1y_2
\end{aligned}
$$

- **Critical point of above eq.**

$$y_1' = 0 \qquad y_2' = 0$$

$$ay_1(1 - \frac{b}{a}y_2) = 0,$$

$$\Rightarrow y_1 = 0,\ y_1 = \frac{a}{b}$$

$$ky_2(y_1 - \frac{l}{k}) = 0,$$

$$\Rightarrow y_2 = 0,\ y_1 = \frac{l}{k}$$

- **Critical point *P₀* is** $P_0 : (0,0)$

$$P_0 : \left(\frac{l}{k}, \frac{a}{b}\right)$$

Ex.) Predator-Prey Population Model

1. Critical Point (0,0)

$$\boxed{\begin{aligned} y_1' &= f_1(y_1, y_2) = ay_1 - by_1y_2 \\ y_2' &= f_2(y_1, y_2) = -ly_2 + ky_1y_2 \end{aligned}}$$

Linearization at P_0

$$y_1' = ay_1 - by_1y_2$$
$$y_2' = -ly_2 + ky_1y_2$$

$$\Downarrow$$

$$y_1' = ay_1$$
$$y_2' = -ly_2$$

$q = \lambda_1 \lambda_2$	$p = \lambda_1 + \lambda_2$	$\Delta = (\lambda_1 - \lambda_2)^2$	Explanation of λ_1, λ_2	
$q > 0$: $\lambda_1, \lambda_2 > 0$, $\lambda_1, \lambda_2 < 0$		$\Delta \ge 0$	λ_1, λ_2 both are real numbers	마디점
$\lambda_1, \lambda_2 = \alpha \pm \beta i$	$p = 0$		λ_1, λ_2 both are pure imaginaries	중심
$q < 0$			$\lambda_1 < 0,\ \lambda_2 > 0$ or $\lambda_1 > 0,\ \lambda_2 < 0$	안장점
	$p \ne 0$	$\Delta < 0$	λ_1, λ_2 both are complex not pure imaginaries	나선점

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a & 0 \\ 0 & -l \end{bmatrix} \mathbf{y}$$

$$p = a_{11} + a_{22} = a - l$$

$$q = \det \mathbf{A} = -al < 0$$

$$\Delta = p^2 - 4q = (a-l)^2 + 4al$$
$$= (a+l)^2$$

At Critical Point(0,0) it is *Saddle Point* because *q<0*

Ex.) Predator-Prey Population Model

2. Critical Point (*a/b,l/k*)

$$y_1' = f_1(y_1, y_2) = ay_1 - by_1y_2$$
$$y_2' = f_2(y_1, y_2) = -ly_2 + ky_1y_2$$

$y_1y_2 - plane$ ⇨ $\tilde{y}_1\tilde{y}_2 - plane$

$P_0 : \left(\frac{l}{k}, \frac{a}{b}\right)$ ⇨ $P_0 : (0,0)$

$$y_1 = \tilde{y}_1 + l/k$$
$$y_2 = \tilde{y}_2 + a/b$$

$$y_1 = \tilde{y}_1 + l/k$$
$$y_2 = \tilde{y}_2 + a/b$$

$$y_1' = ay_1(1 - \frac{b}{a}y_2)$$

$$y_2' = ky_2(y_1 - \frac{l}{k})$$

$$\tilde{y}_1' = a(\tilde{y}_1 + \frac{l}{k})[1 - \frac{b}{a}(\tilde{y}_2 + \frac{a}{b})]$$

$$\tilde{y}_2' = k(\tilde{y}_2 + \frac{a}{b})(\tilde{y}_1 + \frac{l}{k} - \frac{l}{k})$$

$$\tilde{y}_1' = -b\tilde{y}_1\tilde{y}_2 - \frac{lb}{k}\tilde{y}_2$$

$$\tilde{y}_2' = k\tilde{y}_1\tilde{y}_2 + \frac{ak}{b}\tilde{y}_1$$

Ex.) Predator-Prey Population Model

Linearization at P_0 $\tilde{y}_1\tilde{y}_2 - plane$

$$\tilde{y}_1' = -b\tilde{y}_1\tilde{y}_2 - \frac{lb}{k}\tilde{y}_2$$

$$\tilde{y}_2' = k\tilde{y}_1\tilde{y}_2 + \frac{ak}{b}\tilde{y}_1$$

$$\Downarrow$$

$$\tilde{y}_1' = -\frac{lb}{k}\tilde{y}_2$$

$$\tilde{y}_2' = \frac{ak}{b}\tilde{y}_1$$

$$\frac{\tilde{y}_1'}{\tilde{y}_2'} = \frac{-\dfrac{lb}{k}\tilde{y}_2}{\dfrac{ak}{b}\tilde{y}_1}$$

$$\frac{ak}{b}\tilde{y}_1'\tilde{y}_1 = -\frac{lb}{k}\tilde{y}_2'\tilde{y}_2$$

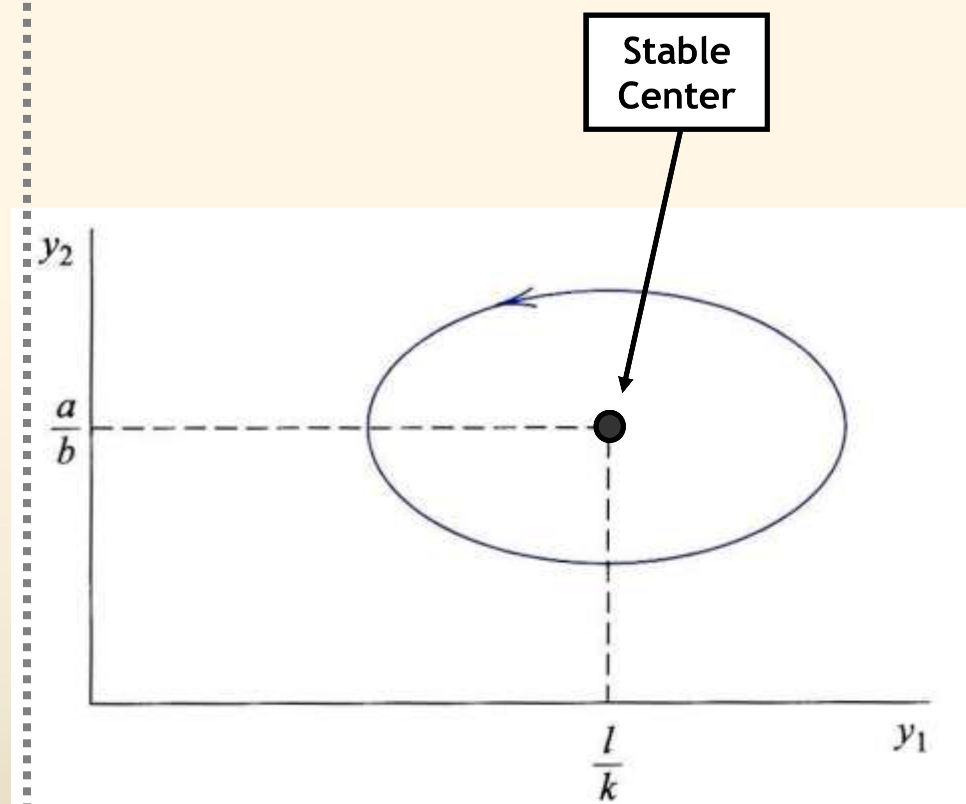
$$\int \frac{ak}{b}\tilde{y}_1'\tilde{y}_1 = \int -\frac{lb}{k}\tilde{y}_2'\tilde{y}_2$$

Ex.) Predator-Prey Population Model

$$\frac{1}{2}\frac{ak}{b}\tilde{y}_1^{\,2}+c_1=-\frac{1}{2}\frac{lb}{k}\tilde{y}_2^{\,2}+c_2$$

$$\frac{ak}{b}\tilde{y}_1^{\,2}=-\frac{lb}{k}\tilde{y}_2^{\,2}+C$$

$$\frac{ak}{b}\tilde{y}_1^{\,2}+\frac{lb}{k}\tilde{y}_2^{\,2}=C$$ **: Ellipse**



Qualitative Methods for Nonlinear Systems

Transformation to a First-Order Equation in the Phase Plane

Another phase plane method

$$F(y, y', y'') = 0$$

We can think about eq above. It's second-order autonomous ODE(an ODE in which t does not occur explicitly)

Setting like below, we can gain much more insight into the behavior of solutions.

$$y = y_1$$

$$y' = y_1' = y_2$$

$$y'' = y_2'$$

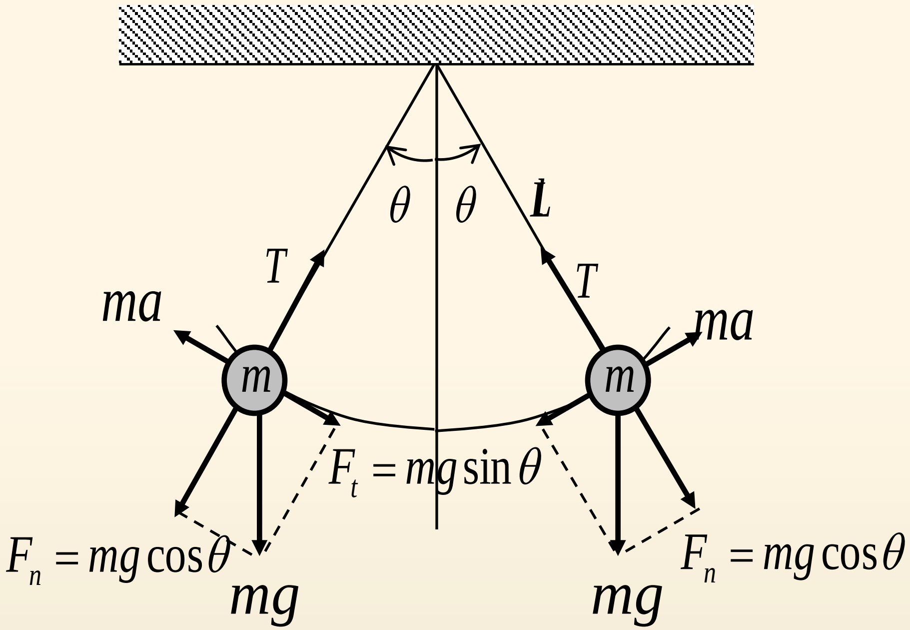
$$= \frac{dy_2}{dt}$$

$$= \frac{dy_2}{dy_1}\frac{dy_1}{dt}$$

$$= \frac{dy_2}{dy_1} y_1' = \frac{dy_2}{dy_1} y_2$$

$$\boxed{F\left(y_1, y_2, \frac{dy_2}{dy_1} y_2\right) = 0}$$

Ex.) Another Phase Plane Method for the Free Undamped Pendulum)



$$mL\theta'' + mg\sin\theta = 0$$

$$\theta'' + k\sin\theta = 0 (\text{where } k = \frac{g}{L})$$

$$\theta'' = -k\sin\theta$$

$$\theta = y_1, \quad \theta' = y_1' = y_2$$

Angular velocity

$$\theta'' = \frac{dy_2}{dt} = \frac{dy_2}{dy_1}\frac{dy_1}{dt} = \frac{dy_2}{dy_1}y_2$$

$$\frac{dy_2}{dy_1}y_2 = -k\sin y_1$$

$$y_2 dy_2 = -k\sin y_1 dy_1$$

$$\int y_2 dy_2 = \int -k\sin y_1 dy_1$$

$$\frac{1}{2}{y_2}^2 + c_1 = k\cos y_1 + c_2$$

$$\frac{1}{2}{y_2}^2 = k\cos y_1 + C$$

Ex.) Another Phase Plane Method for the Free Undamped Pendulum

$$\theta = y_1, \quad \theta' = y_1' = y_2$$

$$\frac{1}{2}{y_2}^2 = k\cos y_1 + C$$

Multiplying this by $\underline{mL^2}$

Mass moment of inertia

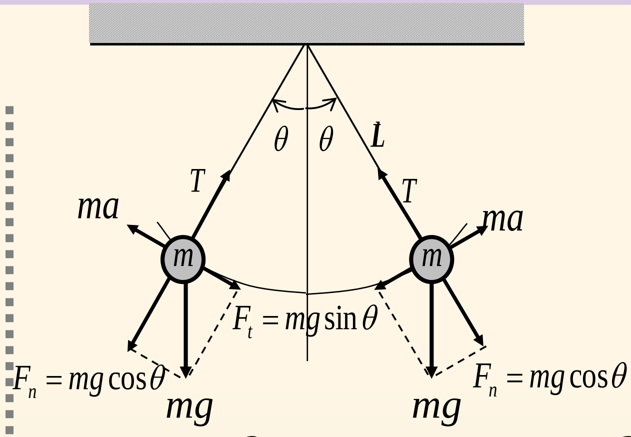
$$\frac{1}{2}{y_2}^2 \times mL^2 = k\cos y_1 \times mL^2 + C \times mL^2$$

$$\underbrace{\frac{1}{2}m(Ly_2)^2}_{①} \underbrace{- mL^2 k\cos y_1}_{②} = \underbrace{mL^2 C}_{③}$$

① $\frac{1}{2}m(Ly_2)^2$ $\quad Ly_2$ is velocity

y_2:angular velocity

It means 1st term is kinetic energy



$$② \; -mL^2 k\cos y_1 = -mL^2\left(\frac{g}{L}\right)\cos y_1$$

$$= -mgL\cos y_1$$

$L\cos y_1$ is perpendicular length from ceiling to pendulum

It means 2nd term is potential energy

③ $mL^2 C$ is constant

① + ② = ③

It shows us

Preservation of Mechanical Energy

Ex.) Another Phase Plane Method for the Free Undamped Pendulum

$\theta = y_1, \quad \theta' = y_1' = y_2$

$$\frac{1}{2}{y_2}^2 = k\cos y_1 + C, \quad \left(k = \frac{g}{L}\right)$$

Pendulum will change its direction of motion if there are points at which *y*$_2$=0

at point $y_2 = 0$,

① If $C = k$

$$k\cos y_1 + k = 0$$
$$k\cos y_1 = -k$$
$$\cos y_1 = -1$$
$$y_1 = \pi$$

② If $C = -k$

$$k\cos y_1 - k = 0$$
$$k\cos y_1 = k$$
$$\cos y_1 = 1$$
$$y_1 = 0$$

③ If $-k < C < k$

$$k\cos y_1 + C = 0$$
$$\cos y_1 = -\frac{C}{k}\left(-1 < -\frac{C}{k} < 1\right)$$

y_1 has value, $-\pi < y_1 < \pi$

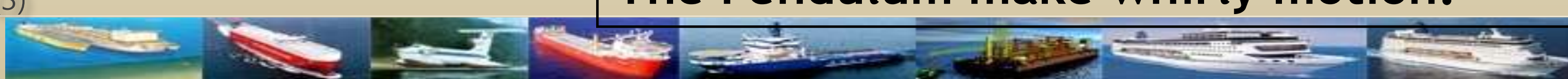
The Pendulum oscillates.

④ If $C > k$

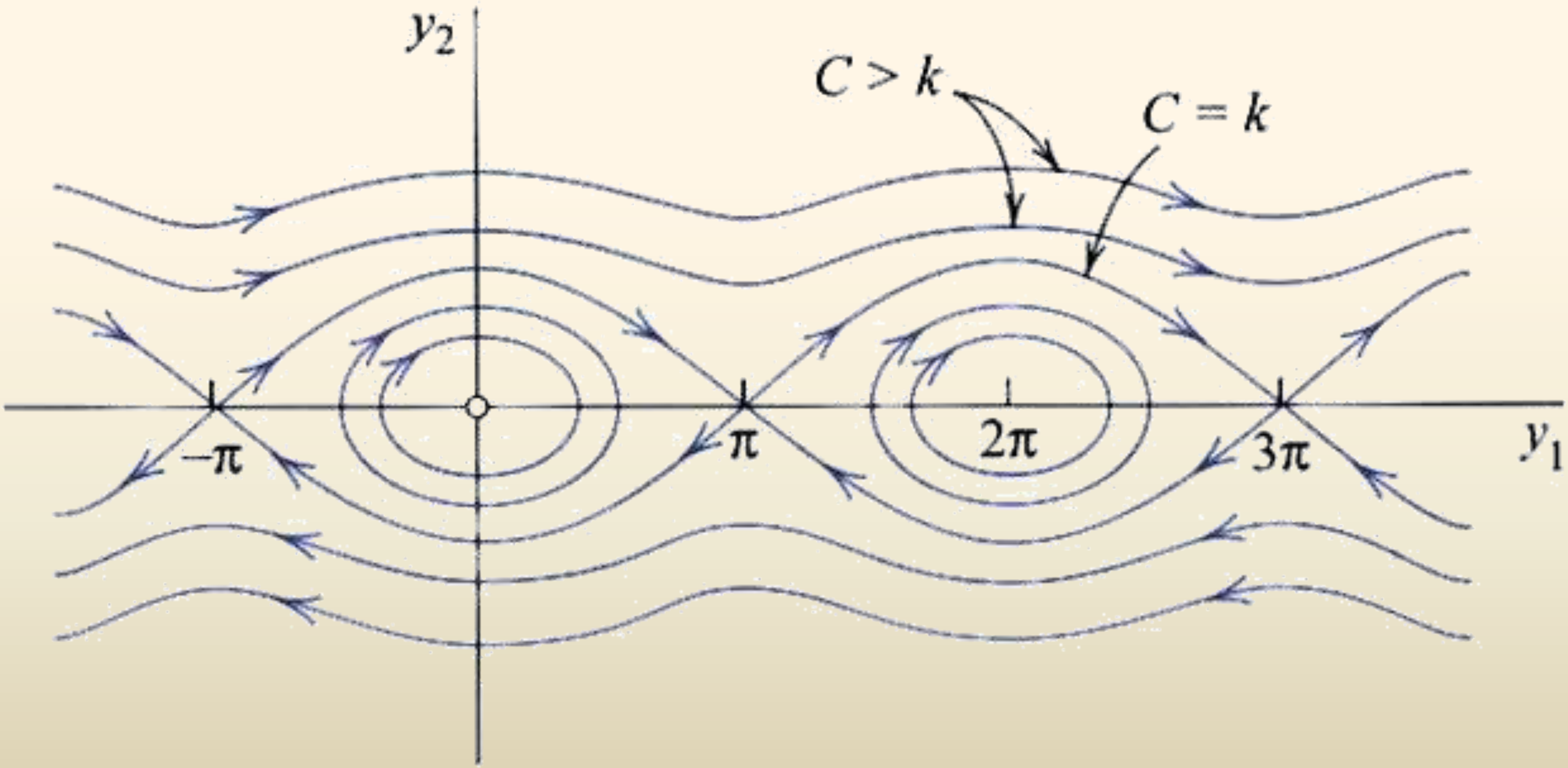
$$k\cos y_1 + C = 0$$
$$\cos y_1 = -\frac{C}{k}, \quad \left(-\frac{C}{k} < -1\right)$$

y_1 doesn't have value

The Pendulum make whirly motion.



Ex.) Another Phase Plane Method for the Free Undamped Pendulum



Ex. Self-Sustained Oscillations. Van der Pol Equation

$$\boxed{y'' - \mu(1 - y^2)y' + y = 0 \qquad (\mu > 0, const)}$$

If $\mu = 0, \quad y'' + y = 0$

It's harmonic oscillation

CASE I : $y^2 < 1,$ **: Negative Damping**

CASE II : $y^2 = 1,$ **: No Damping**

CASE II : $y^2 > 1,$ **: Positive Damping**

If μ is small, **We expect a limit cycle that is almost a circle.**

Because this equation differs but little from $y'' + y = 0$

Ex.) Self-Sustained Oscillations. Van der Pol Equation

$$y'' - \mu(1-y^2)y' + y = 0 \quad (\mu > 0, const)$$

Setting,

$$y = y_1, \quad y' = y_2, \quad y'' = \frac{dy_2}{dy_1} y_2$$

Then

$$\frac{dy_2}{dy_1} y_2 - \mu(1-y_1^{\,2})y_2 + y_1 = 0$$

Isoclines in the phase plane are curve

$$\frac{dy_2}{dy_1} = K$$

$$\frac{dy_2}{dy_1} y_2 = \mu(1-y_1^{\,2})y_2 - y_1$$

$$\frac{dy_2}{dy_1} = \mu(1-y_1^{\,2}) - \frac{y_1}{y_2}$$

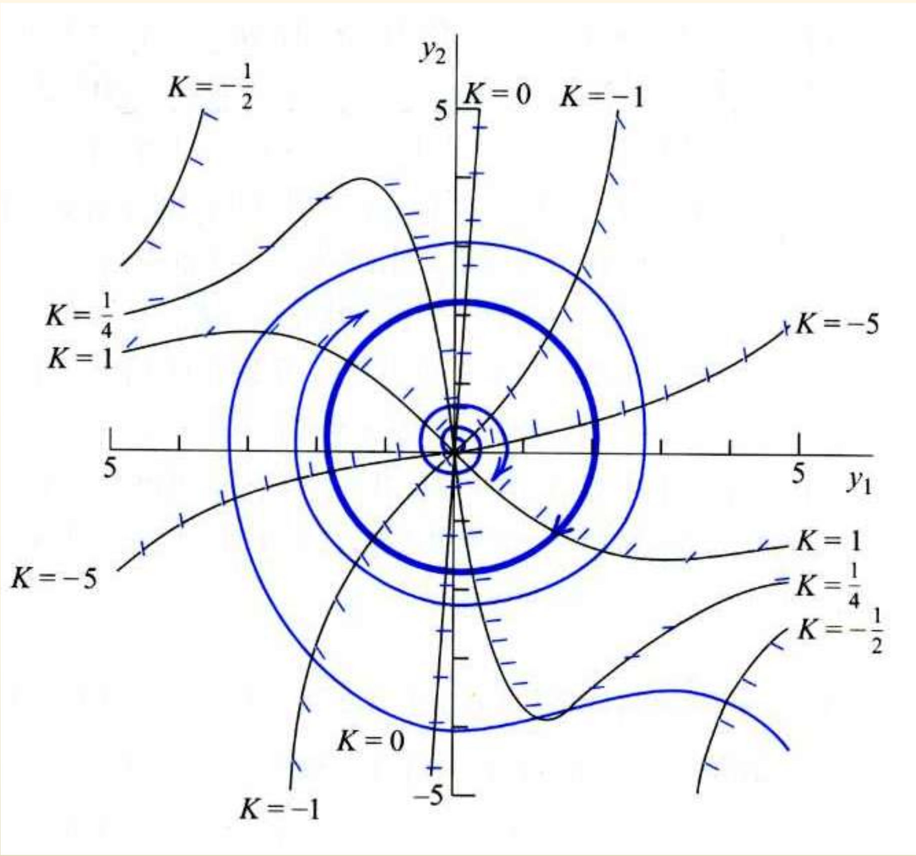
$$\frac{dy_2}{dy_1} = \mu(1-y_1^{\,2}) - \frac{y_1}{y_2} = K$$

$$\frac{y_1}{y_2} = \mu(1-y_1^{\,2}) - K$$

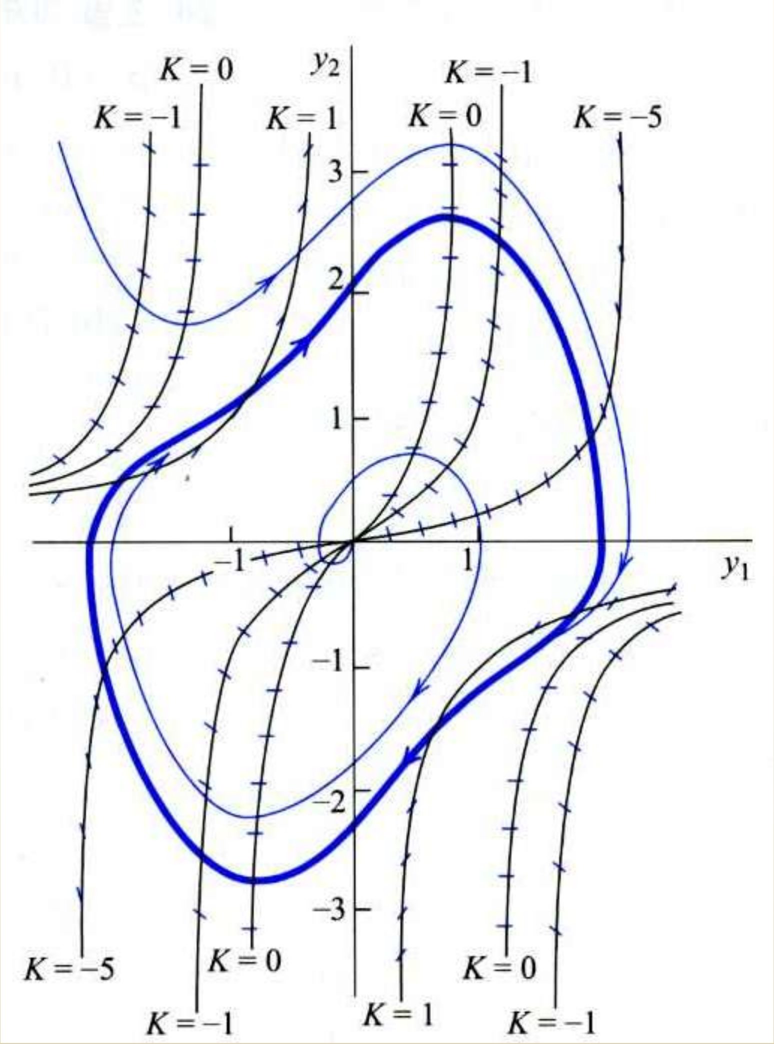
$$\frac{y_2}{y_1} = \frac{1}{\mu(1-y_1^{\,2}) - K}$$

$$\boxed{y_2 = \frac{y_1}{\mu(1-y_1^{\,2}) - K}}$$

Ex.) Self-Sustained Oscillations. Van der Pol Equation



$\mu = 0.1$



$\mu = 1$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = (\lambda+2)(\lambda+4) = 0$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

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$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } \lambda_1 = -2$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

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Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = (\lambda+2)(\lambda+4) = 0$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } \lambda_1 = -2$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \begin{bmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ where } \lambda_1 = -4$$

$$\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\Rightarrow \quad \mathbf{y}^{(p)} = \mathbf{u}\, t\, e^{-2t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\Rightarrow \quad \mathbf{y}^{(p)} = \mathbf{u}\, t\, e^{-2t}$$

$$\mathbf{y}^{(p)\prime} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{g}$$

$$-2\mathbf{u}te^{-2t} + \mathbf{u}e^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\Rightarrow \quad \mathbf{y}^{(p)} = \mathbf{u}\, t\, e^{-2t}$$

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$$\boxed{-2\mathbf{u}}te^{-2t} + \mathbf{u}e^{-2t} = \boxed{\mathbf{A}\mathbf{u}}te^{-2t} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\Rightarrow \quad \mathbf{y}^{(p)} = \mathbf{u}\, t\, e^{-2t}$$

$$\mathbf{y}^{(p)\prime} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{g}$$

$$\boxed{-2\mathbf{u}}te^{-2t} + \mathbf{u}e^{-2t} = \boxed{\mathbf{A}\mathbf{u}}te^{-2t} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$-2\mathbf{u} = \mathbf{A}\mathbf{u}$$

$$(\mathbf{A} + 2\mathbf{I})\mathbf{u} = 0 \quad \Rightarrow \quad (\mathbf{A} + 2\mathbf{I})\mathbf{u} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\Rightarrow \quad \mathbf{y}^{(p)} = \mathbf{u}\,t\,e^{-2t}$$

$$\mathbf{y}^{(p)\prime} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{g}$$

$$\boxed{-2\mathbf{u}}te^{-2t} + \boxed{\mathbf{u}}e^{-2t} = \boxed{\mathbf{A}\mathbf{u}}te^{-2t} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix}}e^{-2t}$$

$$-2\mathbf{u} = \mathbf{A}\mathbf{u}$$

$$(\mathbf{A} + 2\mathbf{I})\mathbf{u} = 0 \quad \Rightarrow \quad (\mathbf{A} + 2\mathbf{I})\mathbf{u} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$\mathbf{y}^{(p)} = \mathbf{u}\, t\, e^{-2t}$$

$$\mathbf{y}^{(p)\prime} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{g}$$

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$$-2\mathbf{u} = \mathbf{A}\mathbf{u}$$

$$(\mathbf{A} + 2\mathbf{I})\mathbf{u} = 0$$

$$(\mathbf{A} + 2\mathbf{I})\mathbf{u} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\Rightarrow \quad \mathbf{y}^{(p)} = \mathbf{u}\, t\, e^{-2t}$$

$$\mathbf{y}^{(p)\prime} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{g}$$

$$\boxed{-2\mathbf{u}}te^{-2t} + \boxed{\mathbf{u}}e^{-2t} = \boxed{\mathbf{A}\mathbf{u}}te^{-2t} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix}}e^{-2t}$$

$$-2\mathbf{u} = \mathbf{A}\mathbf{u}$$

$$(\mathbf{A} + 2\mathbf{I})\mathbf{u} = 0 \quad \Rightarrow \quad (\mathbf{A} + 2\mathbf{I})\mathbf{u} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \quad \Rightarrow \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

Impossible!!!

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

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$$\Rightarrow \quad \mathbf{y}^{(p)} = \mathbf{u}\,t\,e^{-2t} + \mathbf{v}e^{-2t} \qquad \text{not} \qquad \mathbf{y}^{(p)} = \mathbf{u}\,t\,e^{-2t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$\Rightarrow$ $\mathbf{y}^{(p)} = \mathbf{u}te^{-2t} + \mathbf{v}e^{-2t}$ **not** $\mathbf{y}^{(p)} = \mathbf{u}te^{-2t}$

$$\mathbf{y}^{(p)\prime} = \mathbf{u}e^{-2t} - 2\mathbf{u}te^{-2t} - 2\mathbf{v}e^{-2t} = \mathbf{A}(\mathbf{u}te^{-2t} + \mathbf{v}e^{-2t}) + \mathbf{g} = \mathbf{A}\mathbf{u}te^{-2t} + \mathbf{A}\mathbf{v}e^{-2t} + \mathbf{g}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

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Ex.) Method of Undetermined Coefficients. Modification Rule

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Ex.) Method of Undetermined Coefficients. Modification Rule

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$$-2\mathbf{u} = \mathbf{A}\mathbf{u}$$

$$(\mathbf{A} + 2\mathbf{I})\mathbf{u} = 0$$

$\Rightarrow$ $(\mathbf{A} + 2\mathbf{I})\mathbf{u} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$ $\Rightarrow$ $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}$$

$$\Rightarrow \quad \mathbf{y}^{(p)} = \mathbf{u}\,t\,e^{-2t} + \mathbf{v}e^{-2t} \qquad \text{not} \qquad \mathbf{y}^{(p)} = \mathbf{u}\,t\,e^{-2t}$$

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$$\mathbf{u} - 2\mathbf{v} = \mathbf{A}\mathbf{v} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a \\ a \end{bmatrix} - \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \begin{bmatrix} -3v_1 + v_2 - 6 \\ v_1 - 3v_2 + 2 \end{bmatrix}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

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$$\begin{aligned} a - 2v_1 &= -3v_1 + v_2 - 6 \\ a - 2v_2 &= v_1 - 3v_2 + 2 \end{aligned} \Rightarrow \begin{aligned} v_1 - v_2 &= -a - 6 \\ -v_1 + v_2 &= -a + 2 \end{aligned} \Rightarrow \begin{aligned} 0 &= -2a - 4 \\ a &= -2 \end{aligned} \Rightarrow v_2 = v_1 + 4$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{y}^{(p)} = \mathbf{u}\,t\,e^{-2t} + \mathbf{v}e^{-2t}$$

$$\mathbf{u} - 2\mathbf{v} = \mathbf{A}\mathbf{v} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ a \end{bmatrix} - \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \begin{bmatrix} -3v_1 + v_2 - 6 \\ v_1 - 3v_2 + 2 \end{bmatrix}$$

$$\begin{aligned} a - 2v_1 &= -3v_1 + v_2 - 6 \\ a - 2v_2 &= v_1 - 3v_2 + 2 \end{aligned} \Rightarrow \begin{aligned} v_1 - v_2 &= -a - 6 \\ -v_1 + v_2 &= -a + 2 \end{aligned} \Rightarrow \begin{aligned} 0 &= -2a - 4 \\ a &= -2 \end{aligned} \Rightarrow v_2 = v_1 + 4$$

So $$\mathbf{y}^{(p)} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} t\,e^{-2t} + \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} e^{-2t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$\mathbf{y}^{(p)} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} t\, e^{-2t} + \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} e^{-2t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \qquad \mathbf{y}^{(p)} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} e^{-2t}$$

So $$\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} e^{-2t}$$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

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So $$\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} e^{-2t}$$

Let $v_1 = -2$

Ex.) Method of Undetermined Coefficients. Modification Rule

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{y} + \boxed{\begin{bmatrix} -6 \\ 2 \end{bmatrix} e^{-2t}}$$

$$\mathbf{y}^{(h)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} \qquad \mathbf{y}^{(p)} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} e^{-2t}$$

So $\mathbf{y} = \mathbf{y}^{(h)} + \mathbf{y}^{(p)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} v_1 \\ v_1 + 4 \end{bmatrix} e^{-2t}$

Let $v_1 = -2$ **Then** $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} t e^{-2t} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$