Naval Architecture & Ocean Engineering

[2008][04-2]

Engineering Mathematics 2

September, 2008

Prof. Kyu-Yeul Lee

Department of Naval Architecture and Ocean Engineering,
Seoul National University of College of Engineering

Naval Architecture & Ocean Engineering

Laplace Transforms (1)
: Applications

Linear ODE Solution

Linear Control

What is Transform ?

complexity reduced

✓conventional analysis

Problem statement

$$y = \frac{x}{z}$$

Complex analysis

Long-hand division

Problem solution

✓transform analysis

Transform

$$\log y = \log x - \log z$$

Simplified analysis

Table look-up subtraction

Inverse transform

Anti-logarithm Table look-up

What kind of Transform ?

$$y = \frac{x}{z} \xrightarrow{\log} \log y = \log x - \log z$$

$$y = x^2 \xrightarrow{\frac{d}{dx}} \frac{d}{dx} y = 2x$$

$$y = x^2 \xrightarrow{\int dx} \int y dx = \frac{1}{3} x^3 + c$$

Laplace Transform

$$f(t) \xrightarrow{\int_0^\infty e^{-st} \quad dt} \int_0^\infty e^{-st} f(t) dt = \mathscr{L}\{ f(t) \}$$

Fourier Transform

$$s(t) \xrightarrow{\int_{-\infty}^\infty e^{-i\omega t} \quad dt} \int_{-\infty}^\infty e^{-i\omega t} s(t) dt = S(\omega)$$

▷ **Ref. Fourier Transform**

Laplace Transform

Definition

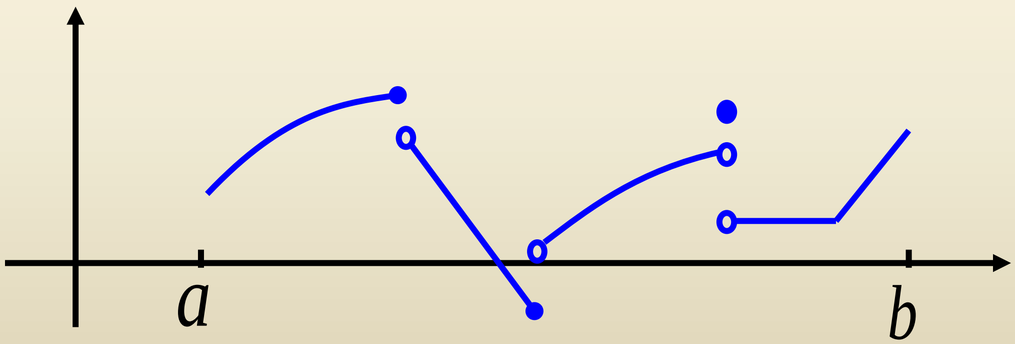
$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$ **,provided the integral converges**

sufficient conditions for existence

$f(t)$ **: be piecewise continuous on** $[0,\infty)$

<u>Piecewise Continuous Function</u>

- **A function has a value on a finite interval [*a*, *b*].**
- **It has finite limits as *t* approaches either endpoints of a interval.**



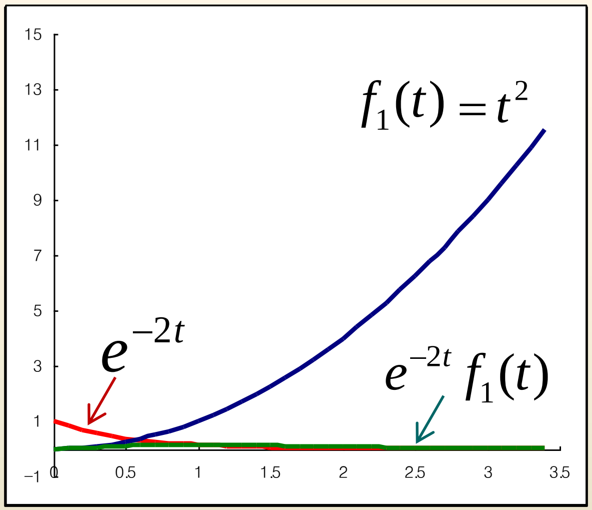
Laplace Transform

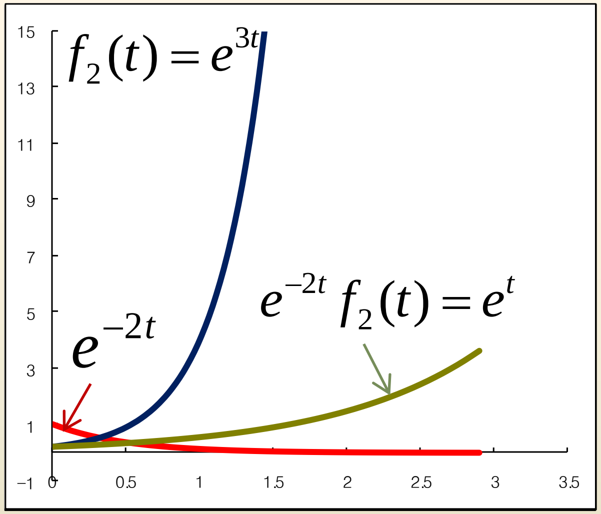
Definition

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$$

sufficient conditions for existence

$f(t)$ **: be of exponential order for** $t > T$





t^2 **Increase as** $t \to \infty$

$e^{-st}t^2$ **Converge to 0 as** $t \to \infty$

$\int_0^{\infty} e^{-st}t^2\,dt$ ➔ **The integral exists**

e^{3t} **Increase as** $t \to \infty$

$e^{-st}t^2$ **Diverge to infinite as** $t \to \infty$

$\int_0^{\infty} e^{-st}t^2\,dt$ ➔ **The integral does not exist**

Laplace Transform

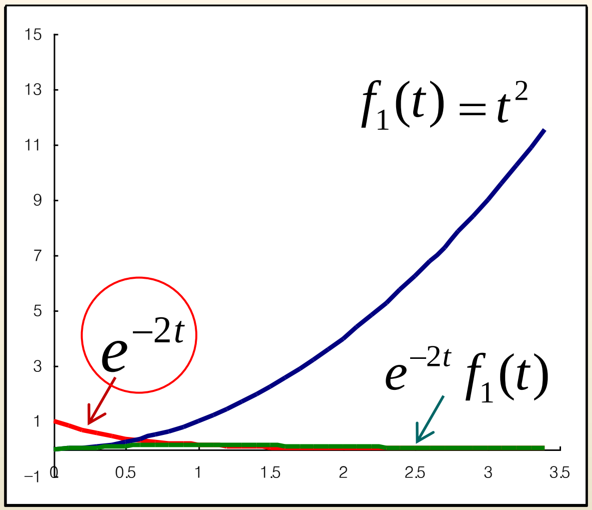
Definition

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Playing an important role in cooling $f(t)$ down to be convergent on [0,∞)

sufficient conditions for existence

$f(t)$ **: be of exponential order for** $t > T$



t^2 **Increase as** $t \to \infty$

$e^{-st}t^2$ **Converge to 0 as** $t \to \infty$

$\int_0^\infty e^{-st}t^2 dt$ **➔ The integral exists**

Let $g(t)=Me^{kt}$ **and prove the existence of Laplace transform**

$$\int_0^\infty e^{-st} g(t)dt = \int_0^\infty e^{-st} Me^{kt} dt = \int_0^\infty Me^{-(s-k)t} dt$$

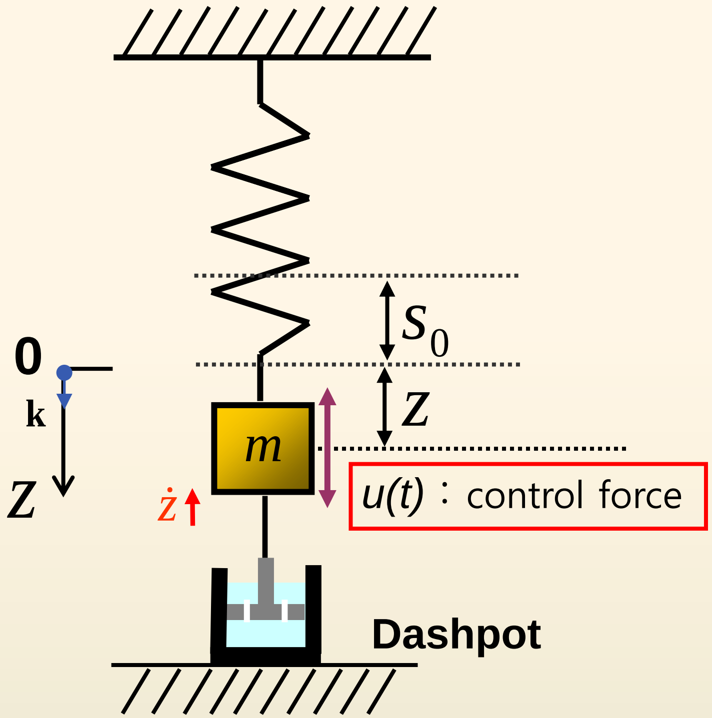
$$= \left[-\frac{M}{s-k} e^{-(s-k)t} \right]_0^\infty = \begin{cases} \dfrac{M}{s-k}, & (s > k) \\ \infty \quad , & (s \le k) \end{cases}$$

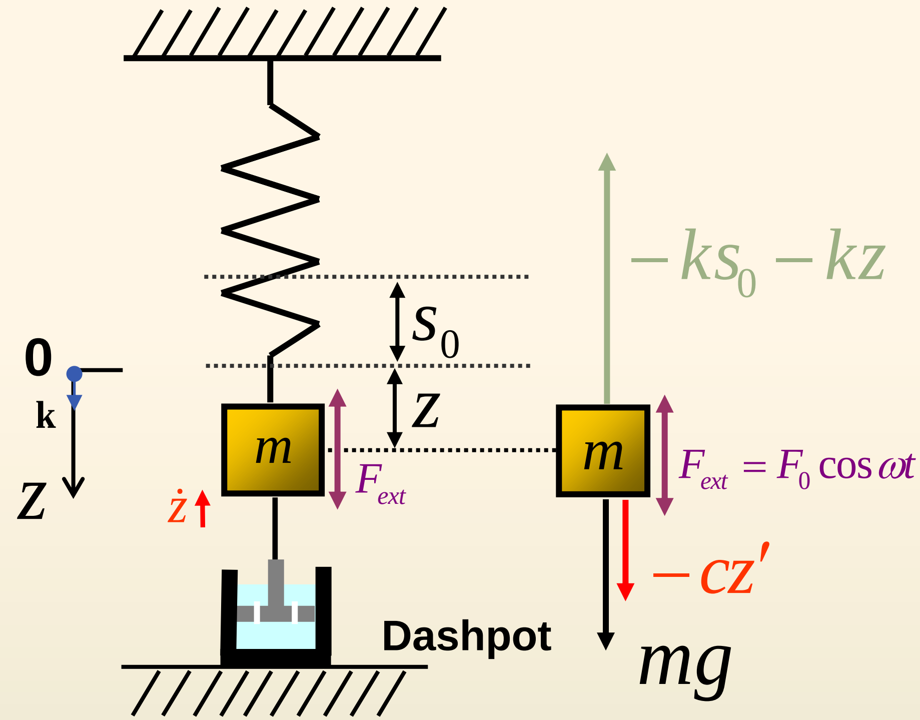
A function of which absolute value is smaller than exponential function $g(t)$ **has Laplace transform.**

$$|f(t)| \le Me^{kt}$$

The Laplace transform exists for all $s>k$**.**

Dynamic System Modeling : Linear ODE





$$mz'' = F$$

$$= mg - k(s_0 + z) - cz' + \boxed{u(t)}$$

$$= -kz - cz' + \boxed{u(t)}$$

$$mz'' + cz' + kz = \boxed{u(t)}$$

$$\mathbf{mz'' = F}$$

Physical Phenomenon

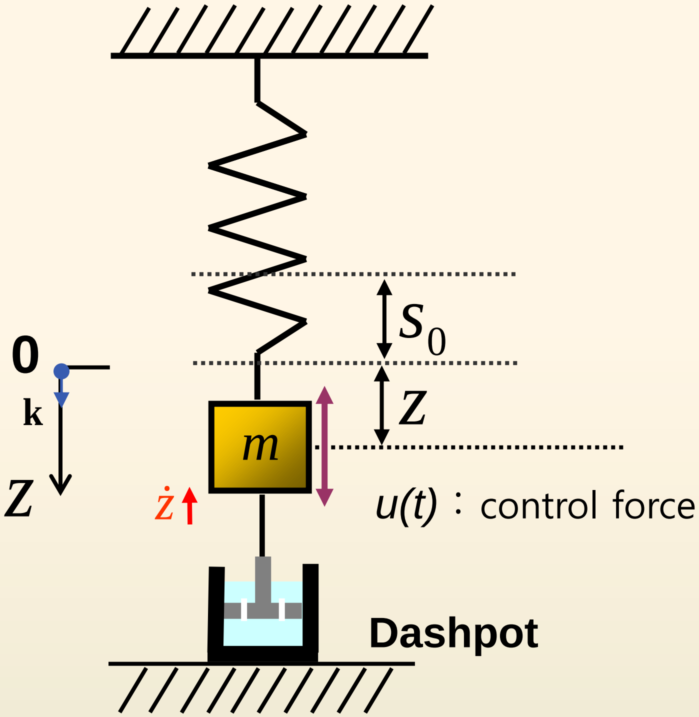
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0} \cos \omega t$$

$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0} \cos \omega t$$

Mathematical Equation

$$mz'' + cz' + kz = \mathbf{F}_0 \cos \omega t$$

Dynamic System Modeling : Linear ODE



$$mz'' = F$$

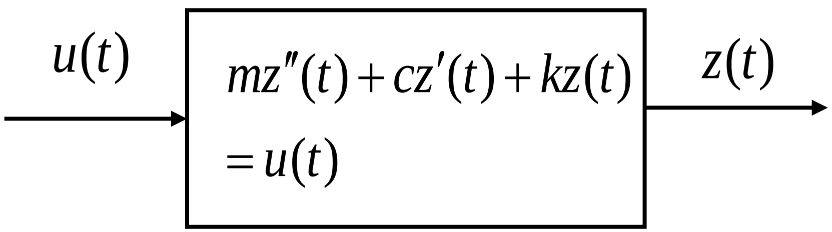
$$= mg - k(s_0 + z) - cz' + u(t)$$

$$= -kz - cz' + u(t)$$

$$\boxed{mz'' + cz' + kz = u(t)}$$

$$mz''(t) + cz'(t) + kz(t) = u(t)$$

Input(control force): *u(t)*   Output: *z(t)*

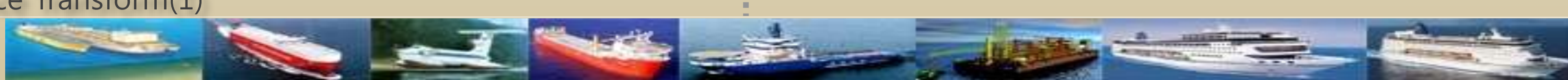


Method to find the solution of the 2nd-order O.D.E

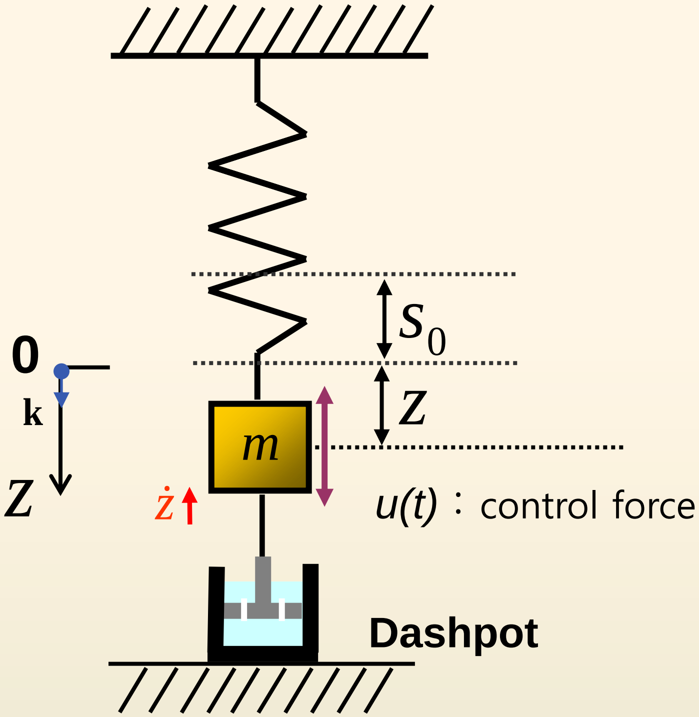
Laplace Transform

$$\mathcal{L}\{z(t)\}$$

$$z(t) = e^{\lambda t}$$



Solution of Linear ODE by Laplace Transform



$$
\begin{aligned}
mz'' &= F \\
&= mg - k(s_0 + z) - cz' + u(t) \\
&= -kz - cz' + u(t)
\end{aligned}
$$

$$\boxed{mz'' + cz' + kz = u(t)}$$

Laplace Transform

Find unknown z(t) that satisfies a D.E. and initial condition

↓ Laplace transform $\mathscr{L}$

Transformed DE becomes an algebraic equation in Z(s)

↓

Solve transformed algebraic equation for Z(s)

↓ inverse transform $\mathscr{L}^{-1}$

Solution z(t) of original D.E

Solution of Linear ODE by Laplace Transform

$$mz''(t)+cz'(t)+kz(t)=u(t)$$

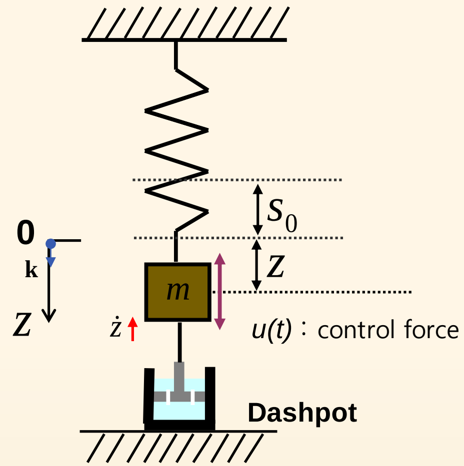
Laplace Transform

$$\mathscr{L}(mz''(t)+cz'(t)+kz(t))=\mathscr{L}(u(t))$$

Linearity

$$\mathscr{L}(mz''(t))+\mathscr{L}(cz'(t))+\mathscr{L}(kz(t))=\mathscr{L}(u(t))$$

$$m[s^2Z(s)-sz(0)-z'(0)]+c[sZ(s)-z(0)]+kZ(s)=U(s)$$



$$mz''=F$$
$$=mg-k(s_0+z)-cz'+u(t)$$
$$=-kz-cz'+u(t)$$

$$mz''+cz'+kz=u(t)$$

$$\mathscr{L}\{u(t)\}=U(s)$$

$$\mathscr{L}\{z(t)\}=Z(s)$$

Integrating by Part

$$\int_a^b u(x)v'(x)dx=[u(x)v(x)]_a^b-\int_a^b u'(x)v(x)dx$$

$$\mathscr{L}\{z'(t)\}=\int_0^\infty e^{-st}z'(t)dt=e^{-st}z(t)\Big|_0^\infty-\int_0^\infty(-s)e^{-st}z(t)dt=-z(0)+s\mathscr{L}\{z(t)\}$$

$$\therefore \mathscr{L}\{z'(t)\}=sZ(s)-z(0)$$

$$\mathscr{L}\{z''(t)\}=\int_0^\infty e^{-st}z''(t)dt=e^{-st}z'(t)\Big|_0^\infty+s\int_0^\infty e^{-st}z'(t)dt=-z'(0)+s\mathscr{L}\{z'(t)\}$$

$$=s[sZ(s)-z(0)]-z'(0)$$

$$\therefore \mathscr{L}\{z''(t)\}=s^2Z(s)-sz(0)-z'(0)$$

Solution of Linear ODE by Laplace Transform

$$mz''(t)+cz'(t)+kz(t)=u(t)$$

D.E.

Laplace Transform

$$\mathcal{L}(mz''(t)+cz'(t)+kz(t))=\mathcal{L}(u(t))$$

Linearity

$$\mathcal{L}(mz''(t))+\mathcal{L}(cz'(t))+\mathcal{L}(kz(t))=\mathcal{L}(u(t))$$

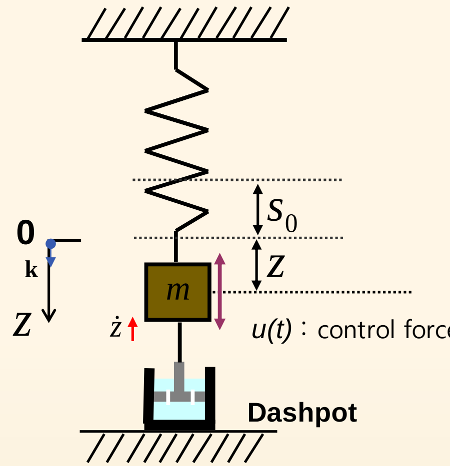
$$m[s^2Z(s)-sz(0)-z'(0)]+c[sZ(s)-z(0)]+kZ(s)=U(s)$$

$$ms^2Z(s)-msz(0)-mz'(0)+csZ(s)-cz(0)+kZ(s)=U(s)$$

$$(ms^2+cs+k)Z(s)=msz(0)+mz'(0)+cz(0)+U(s)$$

$$Z(s)=\frac{1}{(ms^2+cs+k)}[\overline{Q(s)}+U(s)]$$

algebraic equation



$$mz''=F$$
$$=mg-k(s_0+z)-cz'+u(t)$$
$$=-kz-cz'+u(t)$$

$$mz''+cz'+kz=u(t)$$

Solution of Linear ODE by Laplace Transform

$$mz''(t)+cz'(t)+kz(t)=u(t)$$

Laplace Transform

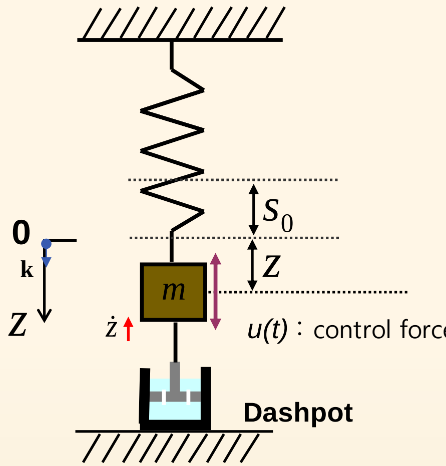
$$\mathscr{L}(mz''(t)+cz'(t)+kz(t))=\mathscr{L}(u(t))$$

$$\left(ms^2+cs+k\right)Z(s)=msz(0)+mz'(0)+cz(0)+U(s)$$

$$Z(s)=\frac{1}{\left(ms^2+cs+k\right)}\left[Q(s)+U(s)\right]$$

$$Z(s)=W(s)\left[Q(s)+U(s)\right]=W(s)Q(s)+W(s)U(s)$$

$$\boxed{Z(s)=W(s)Q(s)+W(s)U(s)}$$



$$mz''=F$$
$$=mg-k(s_0+z)-cz'+u(t)$$
$$=-kz-cz'+u(t)$$

$$\boxed{mz''+cz'+kz=u(t)}$$

$W(s)$ **: transfer function**

$W(s)Q(s)$ **: the effect on the response that are due to the initial condition**

$W(s)U(s)$ **: the effect on the response that are due to the input function**

Solution of Linear ODE by Laplace Transform

$$mz''(t)+cz'(t)+kz(t)=u(t)$$

Laplace Transform

$$\mathscr{L}(mz''(t)+cz'(t)+kz(t))=\mathscr{L}(u(t))$$

$$Z(s)=W(s)Q(s)+W(s)U(s)$$

$$\therefore z(t)=\mathscr{L}^{-1}\{W(s)Q(s)\}+\mathscr{L}^{-1}\{W(s)U(s)\}=z_0(t)+z_1(t)$$

If the input $u(t)=0$

$$Z(s)=W(s)Q(s)$$

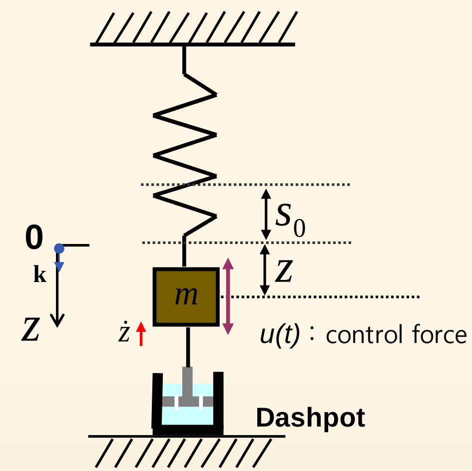
$$\therefore z(t)=z_0(t)=\mathscr{L}^{-1}\{W(s)Q(s)\}$$ **Zero-input response**

If all the initial conditions are zero, $Q(s)=msz(0)+mz'(0)+cz(0)=0$

$$Z(s)=W(s)U(s)$$

$$\therefore z(t)=z_1(t)=\mathscr{L}^{-1}\{W(s)U(s)\}$$ **Zero-state response**

To see the response due to control force only



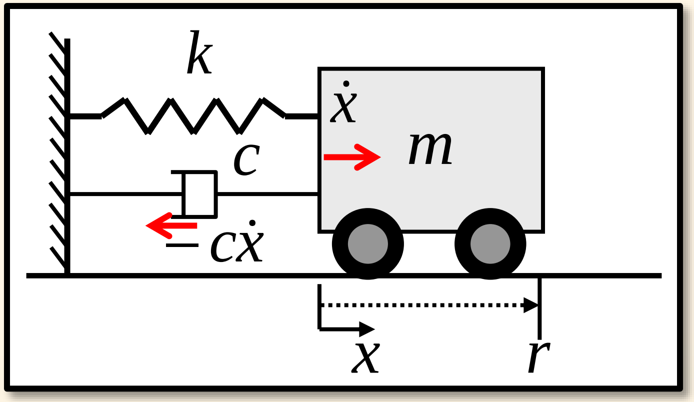
$$\begin{aligned} mz'' &= F \\ &= mg-k(s_0+z)-cz'+u(t) \\ &= -kz-cz'+u(t) \end{aligned}$$

$$mz''+cz'+kz=u(t)$$

Table of Laplace Transform

$f(t)$	$L(f)$	$f(t)$	$L(f)$
1	$\dfrac{1}{s}$	$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$
t	$\dfrac{1}{s^2}$	$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$
t^2	$\dfrac{2!}{s^3}$	$\cosh at$	$\dfrac{s}{s^2-a^2}$
t^n	$\dfrac{n!}{s^{n+1}}$	$\sinh at$	$\dfrac{a}{s^2-a^2}$
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
e^{at}	$\dfrac{1}{s-a}$	$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

Mass-Spring-Damper Linear Mechanical System*



$$u(t) \rightarrow \boxed{\begin{array}{l} mx''(t)+cx'(t)+kx(t) \\ =u(t) \end{array}} \rightarrow x(t)$$

☑ 수레와 지면의 사이의 마찰을 무시함

☑ 초기 상태

- 수레의 초기 위치 : $x=0$
- 수레의 초기 속도 : $x'=0$
- 수레의 초기 가속도 : $x''=0$

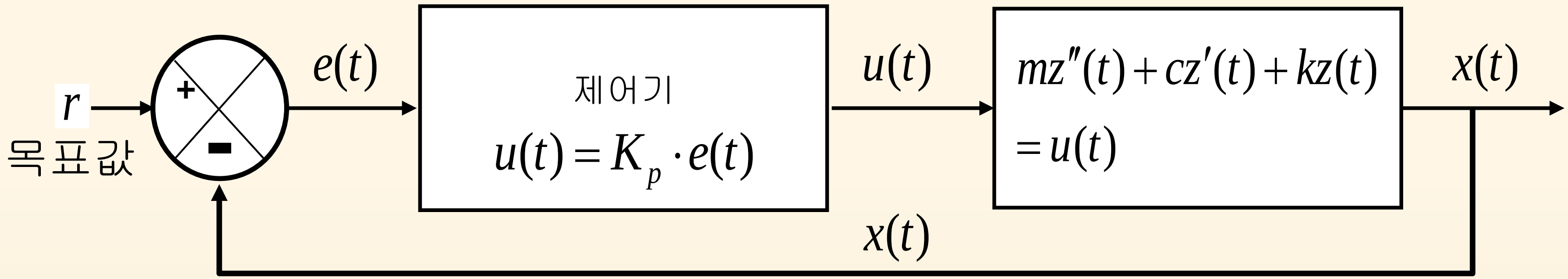
☑ 목표

- 수레의 거동 특성을 변경시키는 것
- 수레의 위치를 r로 유지시키는 것
- 수레의 위치를 변화시키기 위해서 수레에 가하는 제어력 u의 크기를 결정해야 함

Formulation of Linear Control System by Laplace Transform - Proportional Controller(비례 제어기)

$$mx''(t)+cx'(t)+kx(t)=u(t) \Leftarrow u(t)=K_p e(t)=K_p\left(r-x(t)\right)$$

$$e(t)=r-x(t)$$



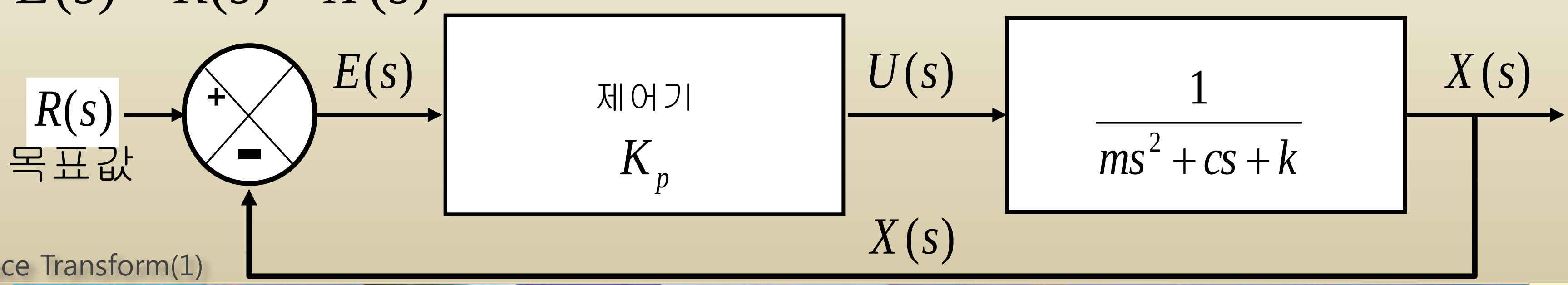
Laplace Transform

$$U(s)=K_p \cdot E(s)$$

$$K_p \cdot E(s)=U(s)$$

$$U(s)\cdot\frac{1}{ms^2+cs+k}=X(s)$$

$$E(s)=R(s)-X(s)$$

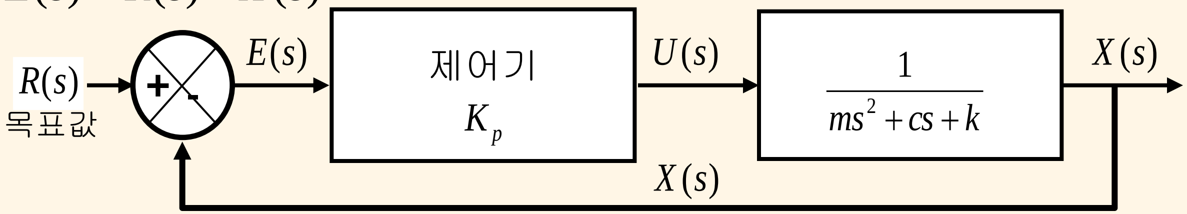


Katsuhiko Ogata, Modern Control System 4th edition, 2001, Prentice Hall, Chapter3 Mathematical Modeling of Dynamic System

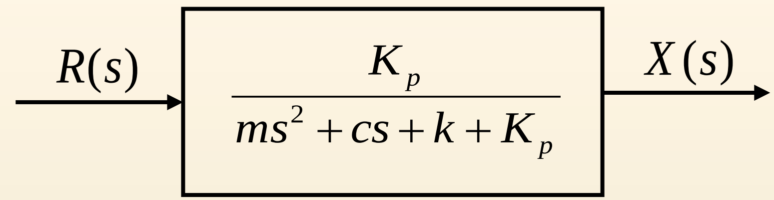
Formulation of Linear Control System by Laplace Transform - Proportional Controller(비례 제어기)

$$mx''(t) + cx'(t) + (k + K_p)x(t) = K_p r$$

$$E(s) = R(s) - X(s)$$



전체 거동을 살펴보기 위하여
페루프 전달함수를 이용하여
미분 방정식을 풀어보자



$$E(s) = R(s) - X(s)$$

$$\{R(s) - X(s)\}K_p = U(s)$$

$$\{R(s) - X(s)\}K_p \frac{1}{ms^2 + cs + k} = X(s)$$

$$X(s) = \frac{R(s)K_p}{ms^2 + cs + k + K_p} = \frac{rK_p}{s(ms^2 + cs + k + K_p)}$$

$$R(s) = \mathscr{L}(r(t)) = \frac{r}{s}, (\because r(t) = r)$$

$$x(t) = \mathscr{L}^{-1}(X(s)) = \mathscr{L}^{-1}\left(\frac{rK_p}{s(ms^2 + cs + k + K_p)}\right) = rK_p \times \mathscr{L}^{-1}\left(\frac{1}{s(ms^2 + cs + k + K_p)}\right)$$

Formulation of Linear Control System by Laplace Transform - Proportional Controller(비례 제어기)

$$x(t)=\mathscr{L}^{-1}\left(X(s)\right)=rK_p\times\mathscr{L}^{-1}\left(\frac{1}{s(ms^2+cs+k+K_p)}\right)$$

$$\begin{cases}\dfrac{1}{s(ms^2+cs+k+K_p)}=\dfrac{1}{s(ms^2+cs+K^*)}, \quad k+K_p=K^* \\[2ex] \qquad =\dfrac{1}{ms(s-\alpha)(s-\beta)}, \quad \alpha,\ \beta=\dfrac{-c\pm\sqrt{c^2-4mK*}}{2m} \\[2ex] \qquad =\dfrac{A}{s}+\dfrac{B}{s-\alpha}+\dfrac{C}{s-\beta} \\[2ex] \qquad A=\dfrac{1}{K^*},\ B=-\dfrac{c+\sqrt{c^2-4mK^*}}{2K^*\cdot\sqrt{c^2-4mK^*}},\ C=\dfrac{c-\sqrt{c^2-4mK^*}}{2K^*\cdot\sqrt{c^2-4mK^*}}\end{cases}$$

cf. 2nd ODE의 해

$$mx''(t)+cx'(t)+\left(k+K_p\right)x(t)=K_p r$$

$$x(t)=x_h(t)+x_p(t)$$

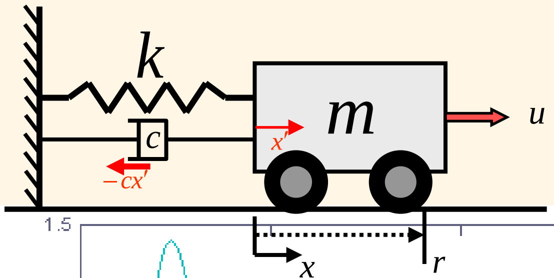
$$=Ae^{\lambda_1 t}+Be^{\lambda_2 t}+C$$

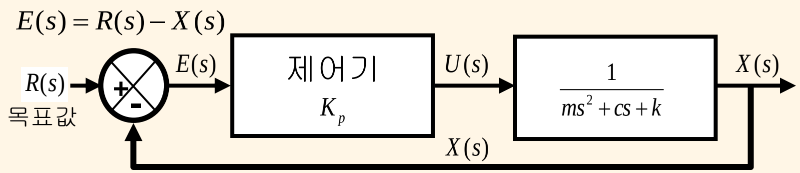
$$\mathscr{L}^{-1}\left(X(s)\right)=rK_p\times L^{-1}\left(\frac{A}{s}+\frac{B}{s-\alpha}+\frac{C}{s-\beta}\right)$$

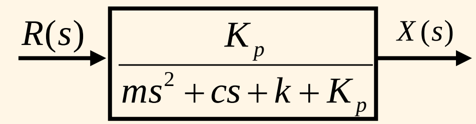
$$=rK_p\times(A+Be^{\alpha t}+Ce^{\beta t})$$

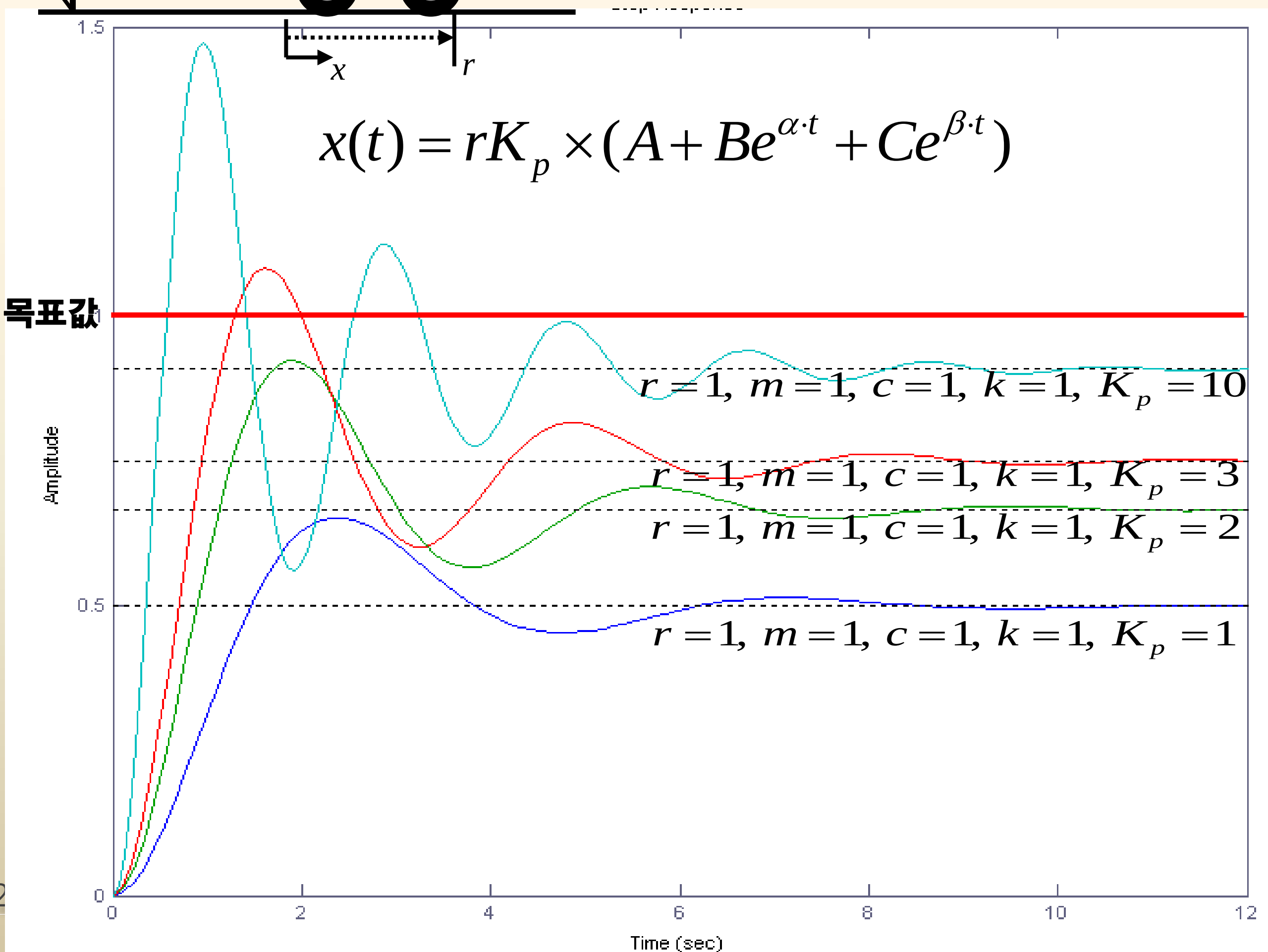
Formulation of Linear Control System by Laplace Transform - Proportional Controller(비례 제어기)

$$mx''(t)+cx'(t)+\left(k+K_p\right)x(t)=K_p r$$









$$x(t)=rK_p\times\left(A+Be^{\alpha\cdot t}+Ce^{\beta\cdot t}\right)$$

$$k+K_p=K^*$$

$$\alpha,\ \beta=\frac{-c\pm\sqrt{c^2-4mK^*}}{2m}$$

$$A=\frac{1}{K^*}$$

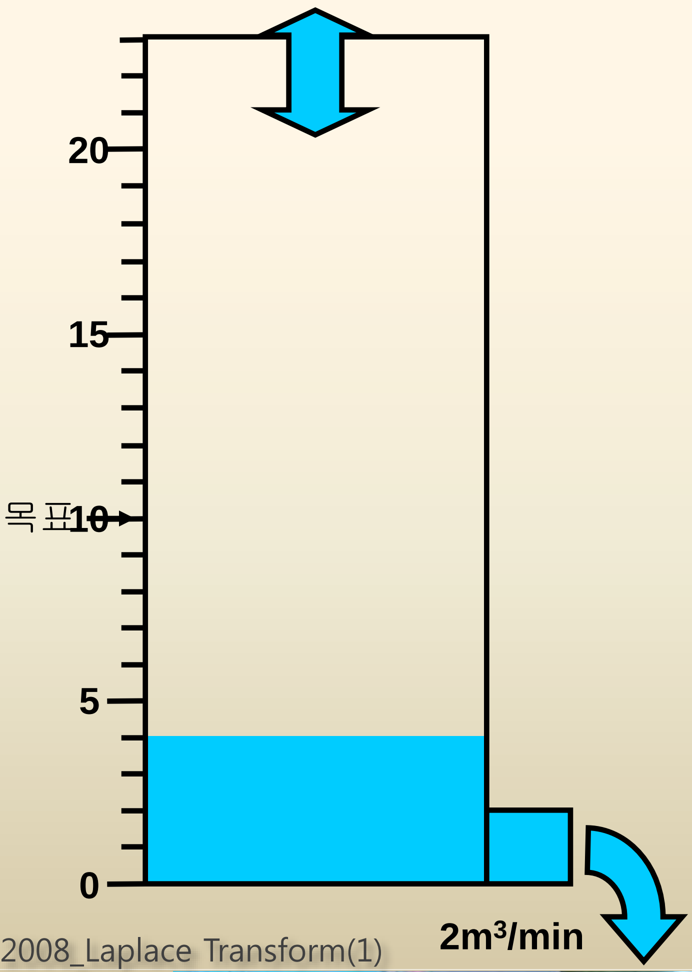
$$B=-\frac{c+\sqrt{c^2-4mK^*}}{2K^*\sqrt{c^2-4mK^*}}$$

$$C=\frac{c-\sqrt{c^2-4mK^*}}{2K^*\sqrt{c^2-4mK^*}}$$

Example of Linear Control System

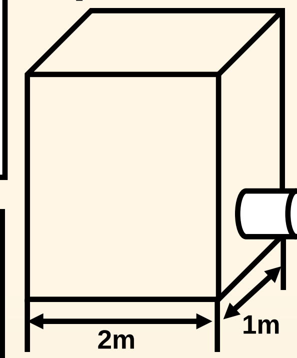
- Proportional Controller(비례 제어기)

$u(t) = K_p e(t) = 1 \cdot e(t)$ m³/min



문제 : 목표 수위를 유지하기 위하여 수조에
물 몇m³/min을 넣어야 하는가?
Given: - 수조의 밑 넓이 2m²
- 물이 수조에서 2m³/min씩 흘러나간다.

$K_p e(t) = u(t)$



다음과 같이 가정한다.

$e(t) =$ 목표 물의 높이 $r(t)$ – 현재 물의 높이 $c(t)$

1분당 넣는 물의 양 $u(t)$: $K_p e(t)$

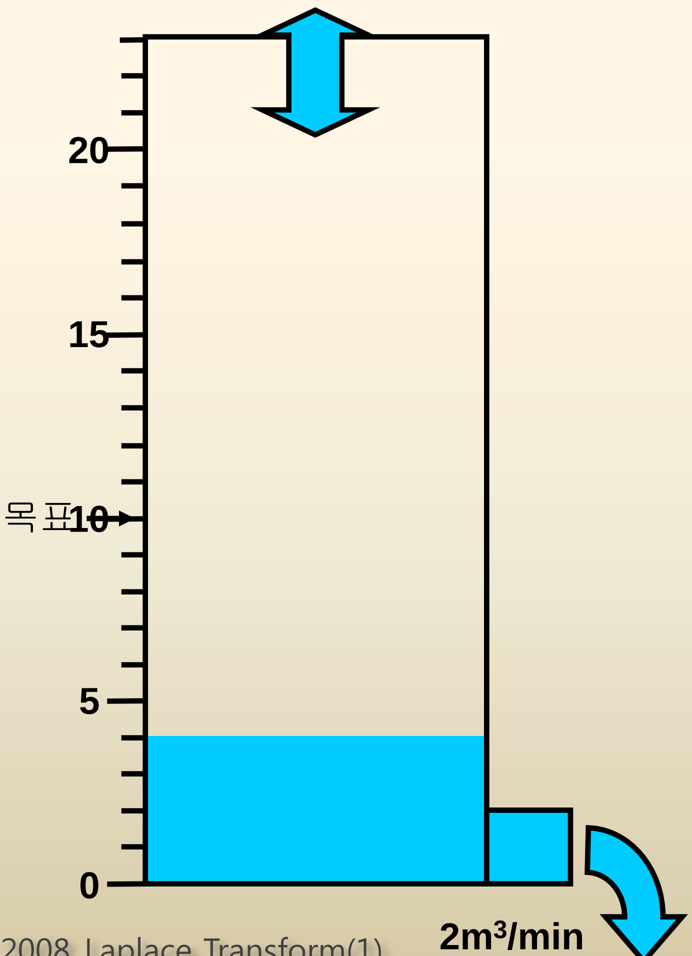
$K_p = 1,\ r = 10$

분	$c(t)$	r	$e(t)$	$u(t)=K_p e(t)$	물 증가량 $(u(t)-2)$	수위 증가 [물증가량/밑넓이]

Example of Linear Control System

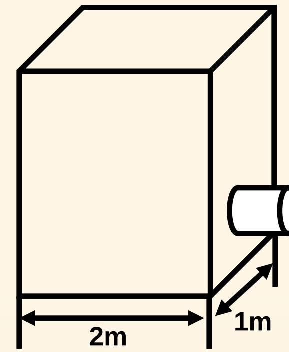
- Proportional Controller(비례 제어기)

$u(t) = K_p e(t) = 1 \cdot e(t)$ m³/min



문제 : 목표 수위를 유지하기 위하여 수조에 물 몇m³/min을 넣어야 하는가?
Given: - 수조의 밑 넓이 2m²
- 물이 수조에서 2m³/min씩 흘러나간다.

$K_p e(t) = u(t)$



다음과 같이 가정한다.

$e(t)$ = 목표 물의 높이 $r(t)$ – 현재 물의 높이 $c(t)$

1분당 넣는 물의 양 $u(t)$: $K_p e(t)$

$K_p = 1,\ r = 10$

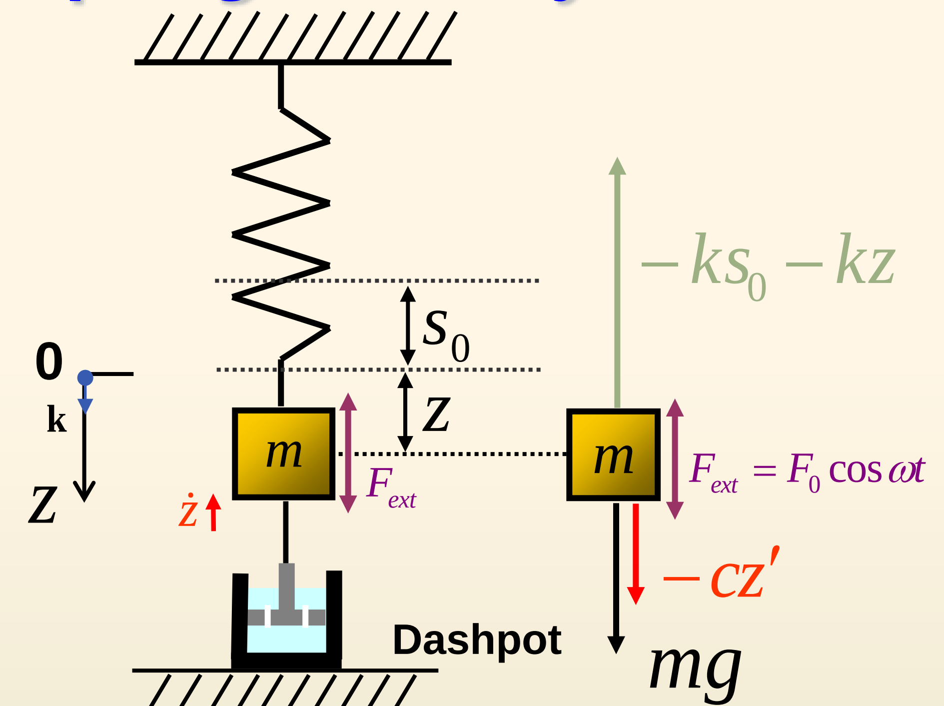
분	$c(t)$	r	$e(t)$	$u(t)=K_p e(t)$	물 증가량 $(u(t)-2))$	수위 증가 [물증가량/밑넓이]
0	4	10	6	6	4	2
1	6	10	4	4	2	1
2	7	10	3	3	1	0.5
3	7.5	10	2.5	2.5	0.5	0.25
4	7.75	10	2.25	2.25	0.25	0.125
…	…					
…	8	10	2	2	0	0
…	8	10	2	2	0	0

The Laplace Transform

1) Comparison
   1) Homogeneous & Nonhomogeneous Solutions.
   2) Transient & Steady-state Solutions
   3) Zero-Input & Zero_initial Solutions

Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



$$m\mathbf{z}'' = \mathbf{F}$$

$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0} \cos \omega t$$

$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0} \cos \omega t$$

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos \omega t$$

$$mz'' + cz' + kz = F_0 \cos \omega t$$ (z component)

- Solution of 2nd O.E.D.

Linear Model (Linear Equation)

$$mz'' + cz' + kz = F_0 \cos \omega t$$

General Solution -Nonghomogeneous

$$z = z_h + z_p$$

-Superposition

z_h: **General Solution - Homogeneous** ($mz'' + cz' + kz = 0$)

z_p: **Particular Solution** ← **Another Superposition**

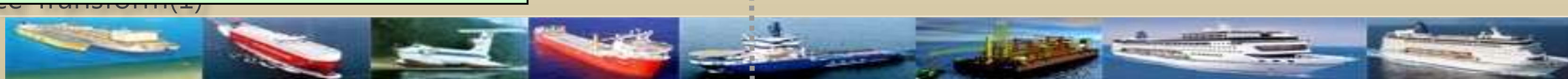
$$z_h = c_1 z_1 + c_2 z_2$$

-Linear Combination
-Superposition

-Fundamental set of solutions (Basis): z_1, z_2

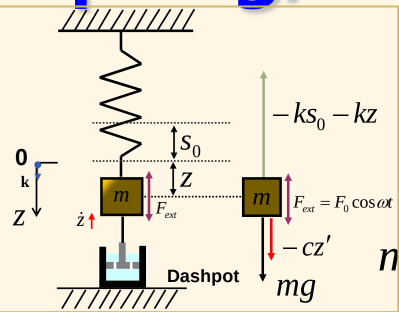
Linearly Independent

Try: $z = e^{\lambda t}$



Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



Linear Model (Linear Equation)

$$mz'' + cz' + kz = F_0 \cos\omega t$$

General Solution -Nonghomogeneous

$$z = z_h + z_p$$

-Superposition

$$mz'' + cz' + kz = 0$$

z_h → **General Solution - Homogeneous**

z_p → **Particular Solution**

Another Superposition

$$z_h = c_1 z_1 + c_2 z_2$$

-Linear Combination
-Superposition

-Fundamental set of solutions (Basis)

z_1 , z_2

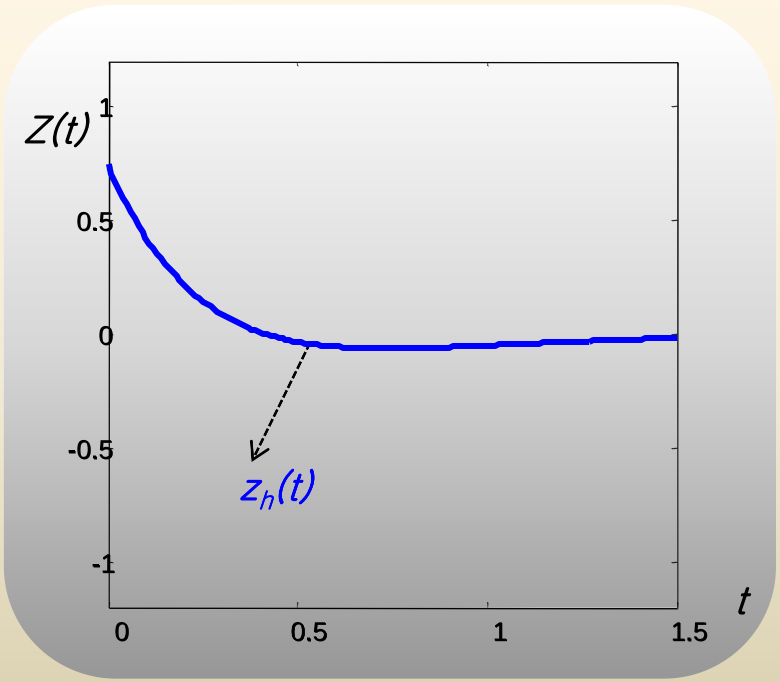
Linearly Independent

Try:
$z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos\omega t$$

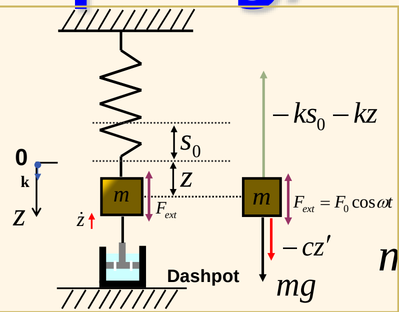
$$0.2z'' + 1.2z' + 2z = 5\cos 4t$$

$$z(0) = \frac{1}{2},\ z'(0) = 0$$



Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



Linear Model (Linear Equation)

$$mz'' + cz' + kz = F_0 \cos \omega t$$

General Solution -Nonghomogeneous

$$z = z_h + z_p$$

-Superposition

$$mz'' + cz' + kz = 0$$

z_h — **General Solution - Homogeneous**

z_p — **Particular Solution**

Another Superposition

$$z_h = c_1 z_1 + c_2 z_2$$

-Linear Combination
-Superposition

-Fundamental set of solutions (Basis): z_1, z_2

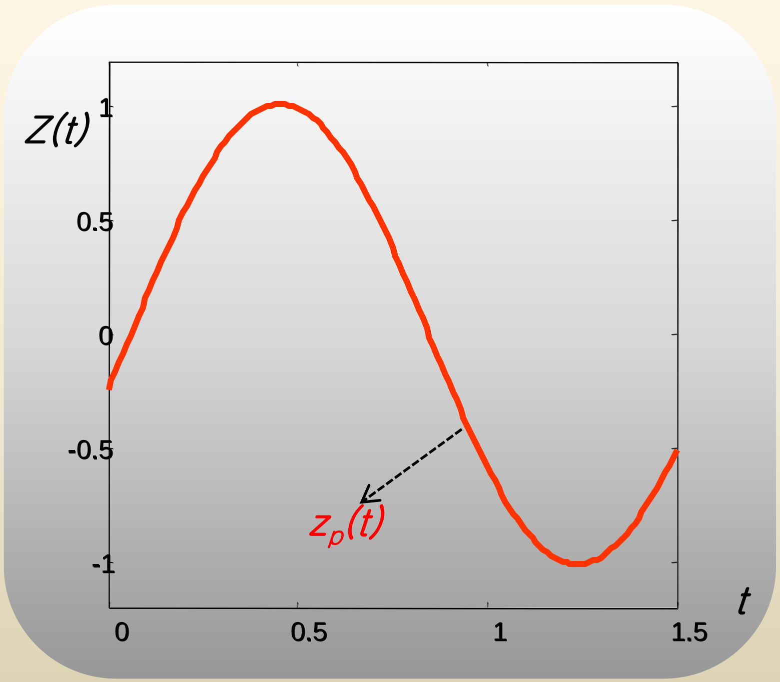
Linearly Independent

Try: $z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

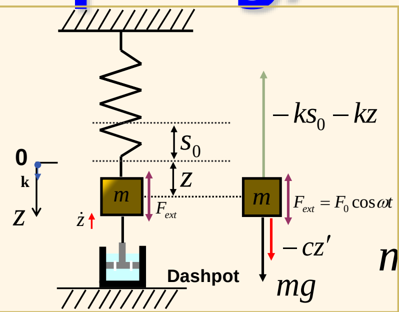
$$0.2z'' + 1.2z' + 2z = 5\cos 4t$$

$$z(0) = \frac{1}{2},\ z'(0) = 0$$



Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



Linear Model (Linear Equation)

$$mz'' + cz' + kz = F_0 \cos\omega t$$

General Solution -Nonghomogeneous

$$z = z_h + z_p$$

-Superposition

$$mz'' + cz' + kz = 0$$

z_h → **General Solution - Homogeneous**

z_p → **Particular Solution**

Another Superposition

$$z_h = c_1 z_1 + c_2 z_2$$

-Linear Combination
-Superposition

-Fundamental set of solutions (Basis)

z_1 z_2

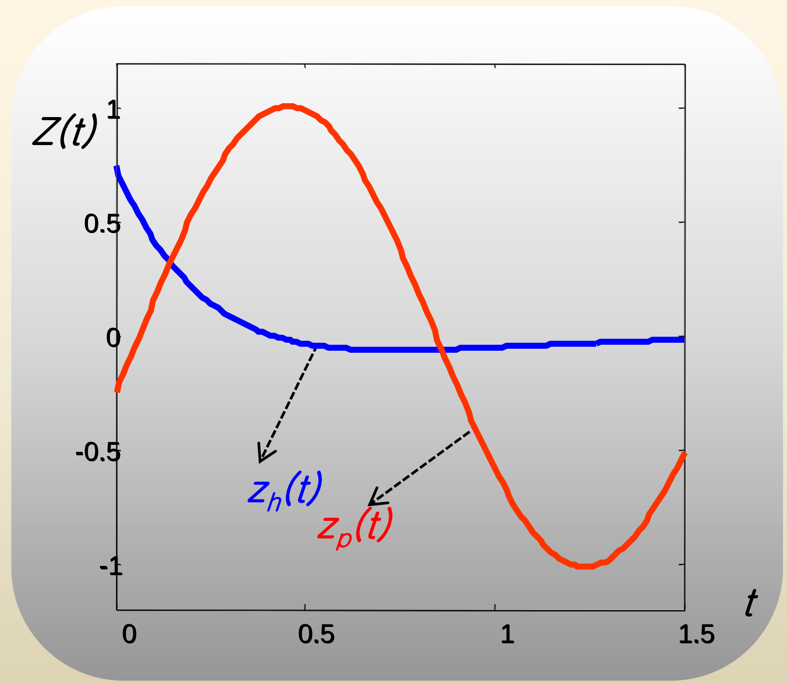
Linearly Independent

Try:
$z = e^{\lambda t}$

$$mz'' + cz' + kz = F_0 \cos\omega t$$

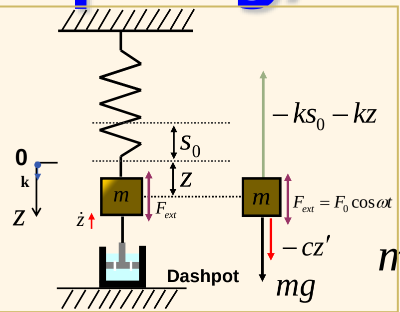
$$0.2z'' + 1.2z' + 2z = 5\cos 4t$$

$$z(0) = \frac{1}{2},\ z'(0) = 0$$



Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



Linear Model (Linear Equation)

$$mz'' + cz' + kz = F_0 \cos\omega t$$

General Solution -Nonghomogeneous

$$z = z_h + z_p$$

-Superposition

$$mz'' + cz' + kz = 0$$

z_h | z_p

General Solution - Homogeneous | **Particular Solution**

$$z_h = c_1 z_1 + c_2 z_2$$

-Linear Combination
-Superposition

Another Superposition

-Fundamental set of solutions (Basis)

z_1 | z_2

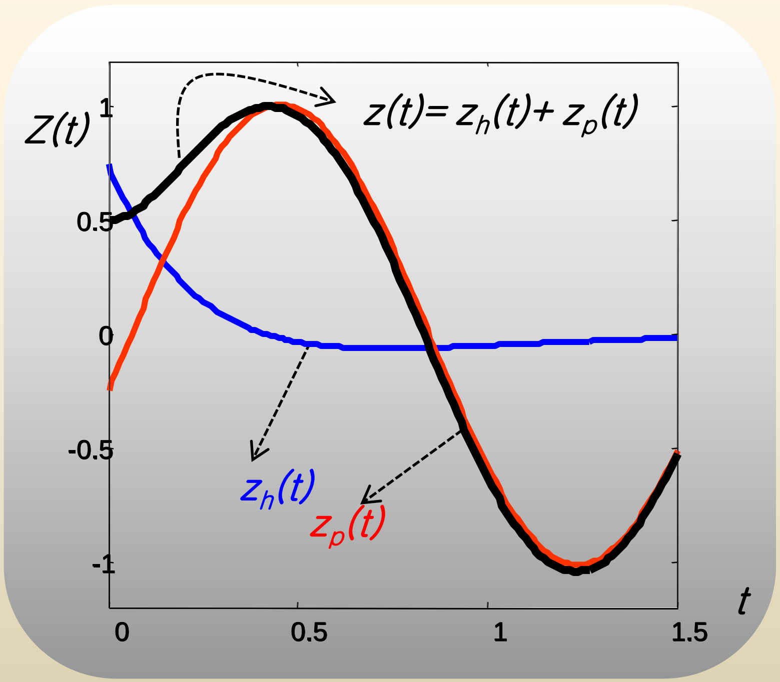
Try:
$z = e^{\lambda t}$

Linearly Independent

$$mz'' + cz' + kz = F_0 \cos\omega t$$

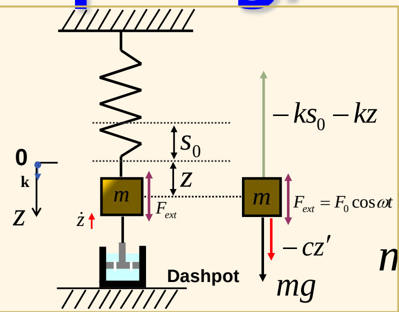
$$0.2z'' + 1.2z' + 2z = 5\cos 4t$$

$$z(0) = \frac{1}{2},\ z'(0) = 0$$



Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$



Linear Model (Linear Equation)

$$mz'' + cz' + kz = F_0 \cos\omega t$$

General Solution -Nonghomogeneous

$$z = z_h + z_p$$

-Superposition

$$mz'' + cz' + kz = 0$$

z_h → **General Solution - Homogeneous**

z_p → **Particular Solution**

Another Superposition

$$z_h = c_1 z_1 + c_2 z_2$$

-Linear Combination
-Superposition

-Fundamental set of solutions (Basis)

z_1 z_2

Try: $z = e^{\lambda t}$

Linearly Independent

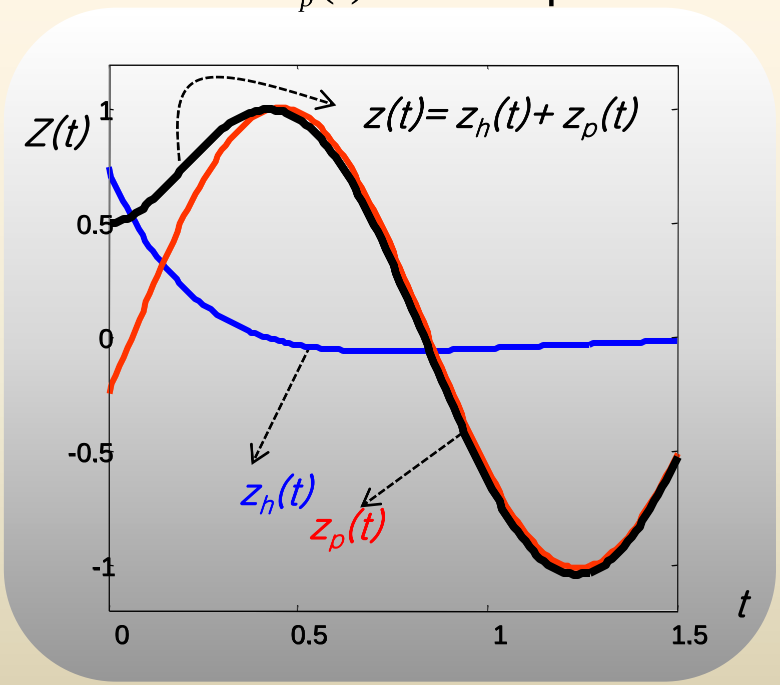
$$mz'' + cz' + kz = F_0 \cos\omega t$$

Transient and Steady-State Terms

When *F(t)* is periodic function, general solution have

$z_h(t)$ **nonperiodic function**

$z_p(t)$ **and periodic function**



$z_h(t)$ **: transient solution**

$z_p(t)$ **: steady-state solution**

Comparison

Method to find the solution of the 2nd-order O.D.E

$$mz'' + cz' + kz = u(t)$$

$$z(t) = e^{\lambda t} \xrightarrow{\text{Solution}} z_{\text{homogeneous}},\ z_{\text{particular}}$$

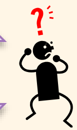
Laplace Transform $\mathscr{L}\{ z(t) \}$ $\xrightarrow{\text{Solution}}$ $z_{\text{zero-input}},\ z_{\text{zero-state}}$

Comparison

Method to find the solution of the 2nd-order O.D.E

$$mz'' + cz' + kz = u(t)$$

$$\rightarrow z(t) = e^{\lambda t} \xrightarrow{\text{Solution}} z_{\text{transient}},\ z_{\text{steady}}$$



$$\rightarrow \text{Laplace Transform}\ \mathscr{L}\{z(t)\} \xrightarrow{\text{Solution}} z_{\text{zero-input}},\ z_{\text{zero-state}}$$

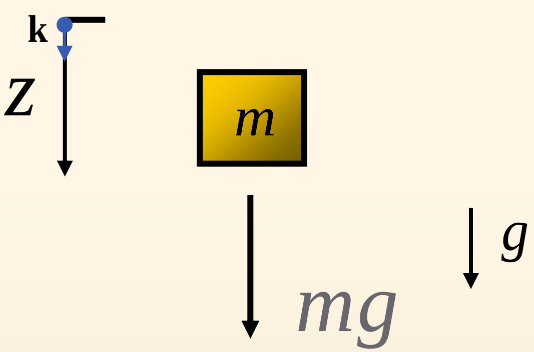
$$z_{\text{general}} = z_{\text{transient}} + z_{\text{steady}}$$

Example

$$\begin{array}{ccccc} z_{\text{general}} & = & z_{\text{transient}} & + & z_{\text{steady}} \\ \parallel & & \parallel & & \parallel \\ z_{\text{zero-input}} & = & z_{\text{transient of zero-input}} & + & z_{\text{steady of zero-input}} \\ + & & + & & + \\ z_{\text{zero-state}} & = & z_{\text{transient of zero-state}} & + & z_{\text{steady of zero-state}} \end{array}$$

Action/Reaction, Equilibrium

①

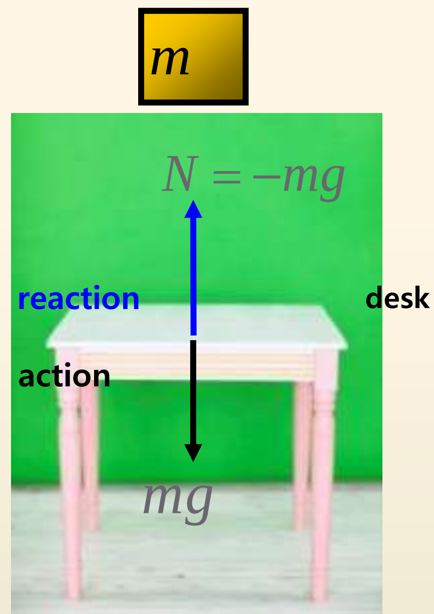


:Gravity force

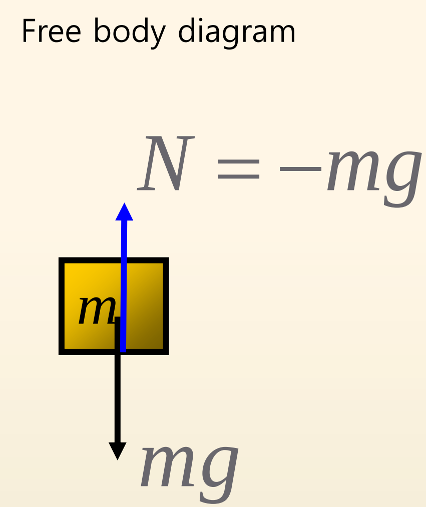
By Newton's second law,

$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k}$$

②



③



By Newton's second law,

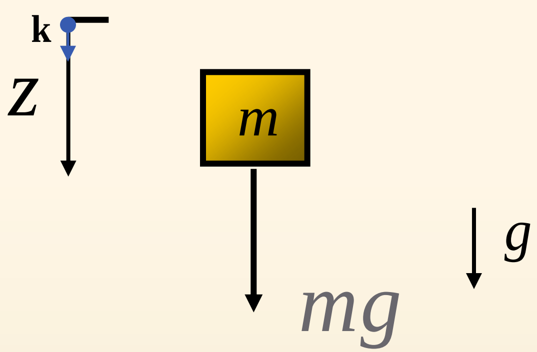
$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - mg\mathbf{k}$$
$$= 0$$

: Static equilibrium

Spring/Mass Systems: Driven Motion

$z = z(t)\ , \quad z'' = \dfrac{d^2 z}{dt^2}$

①

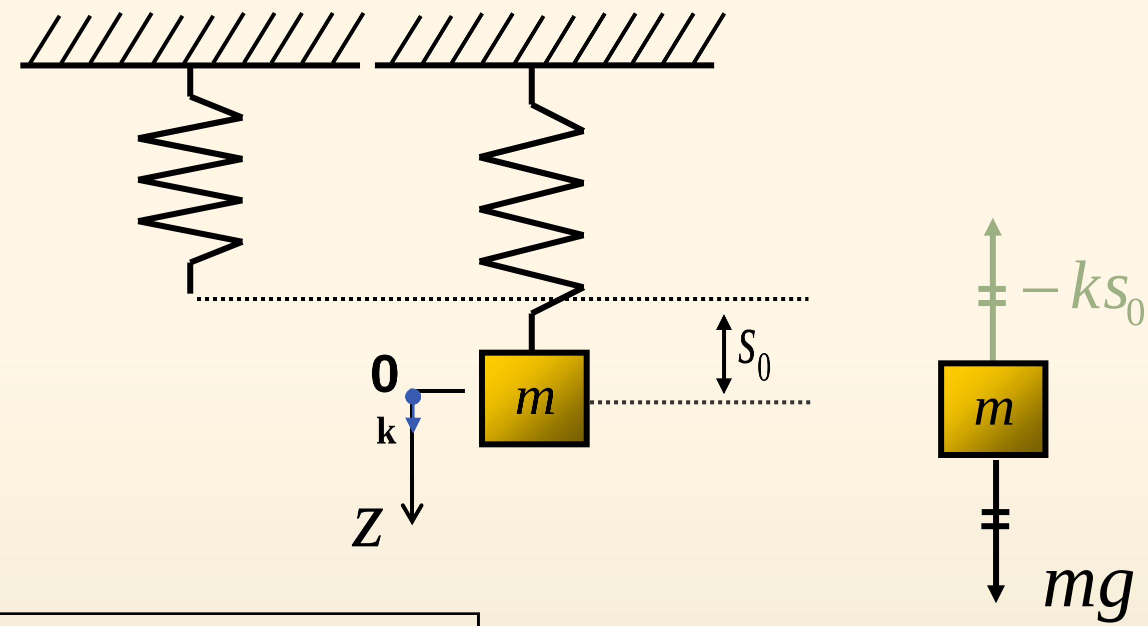


:Gravity force

By Newton's 2nd law,

$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k}$$

②



Nonlinearity of spring

$$\mathbf{F}(z) = -k\mathbf{z} - k_1\mathbf{z}^3$$

linearize

Hooke's law

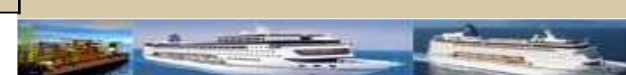
$$F \propto z$$
$$\mathbf{F}_{spring} = -k\mathbf{z}$$

k : spring constant

$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k}$$
$$= 0 \quad (\because z'' = 0)$$

: static equilibrium

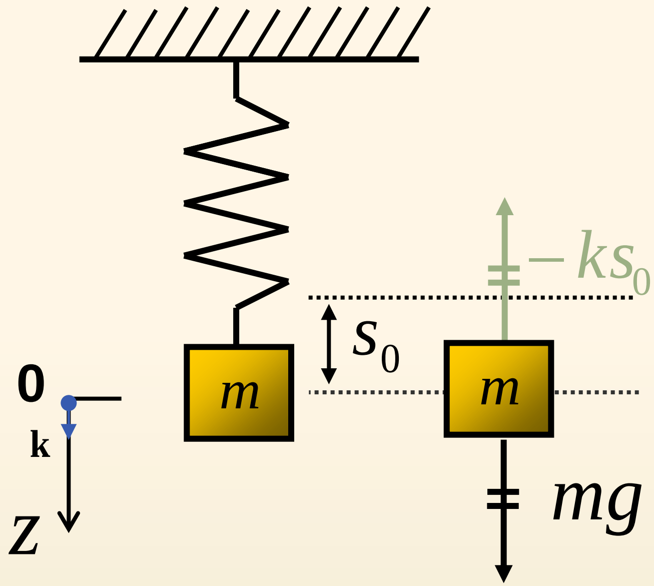
○ opposite to the direction of displacement



Spring/Mass Systems: Driven Motion

$z = z(t)\ , \quad z'' = \dfrac{d^2 z}{dt^2}$

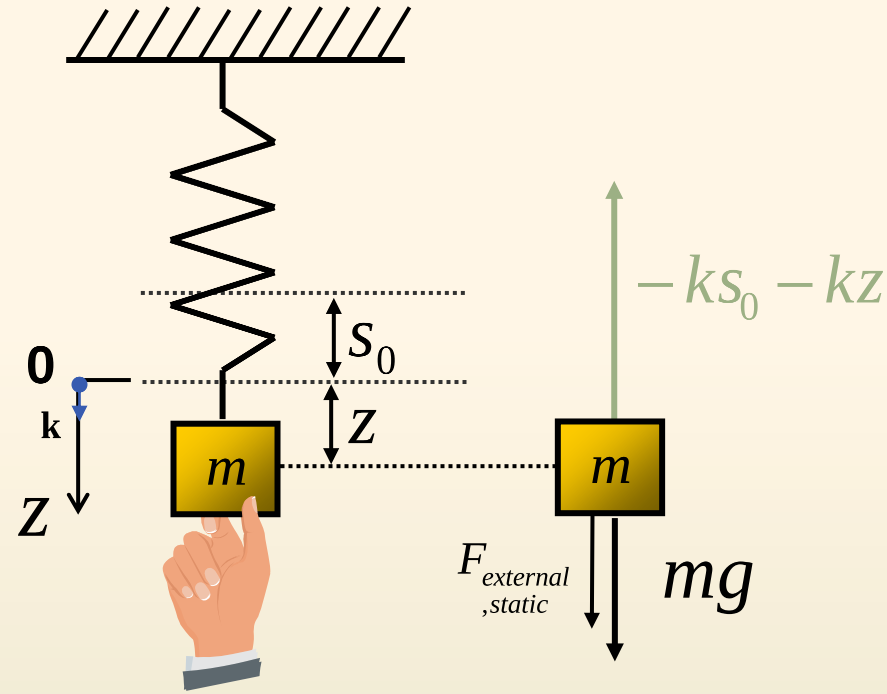
②



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k}$$
$$= 0 \quad (\because z'' = 0)$$

: static equilibrium

③

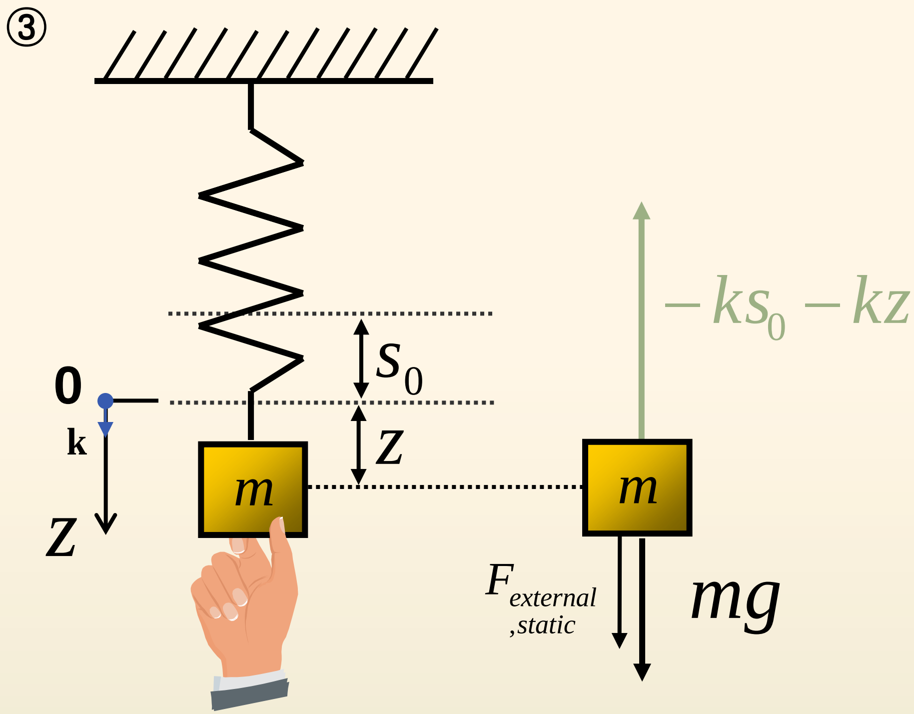


$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= 0 \quad (\because z'' = 0)$$

Spring/Mass Systems: Driven Motion

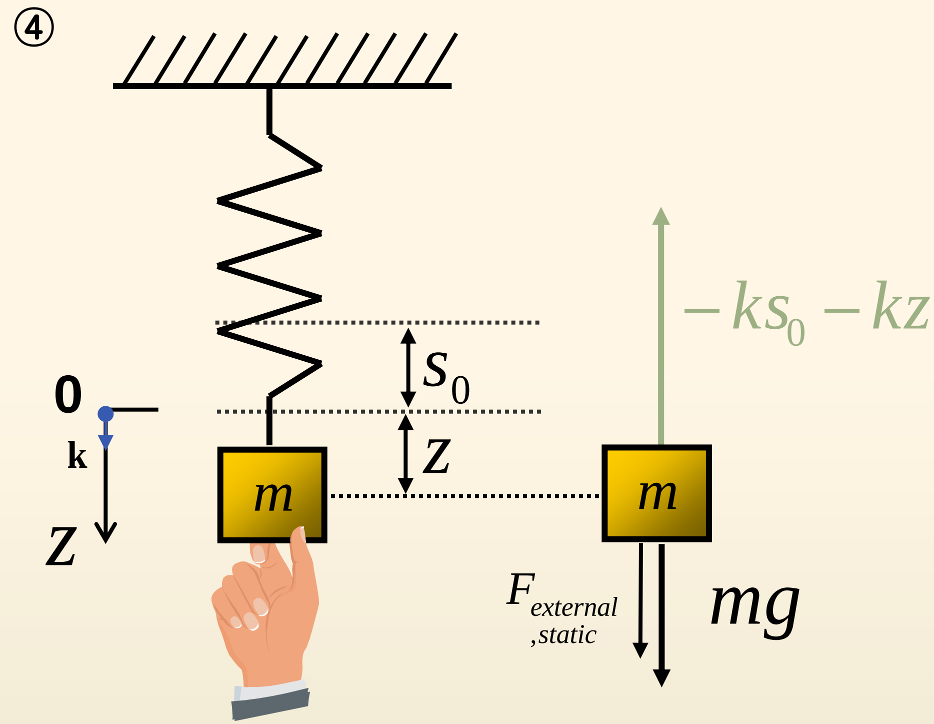
$z = z(t)\ , \quad z'' = \dfrac{d^2 z}{dt^2}$

③



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= 0 \quad (\because z'' = 0)$$

④



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$

restoring force

Physical Phenomenon

Mathematical Equation

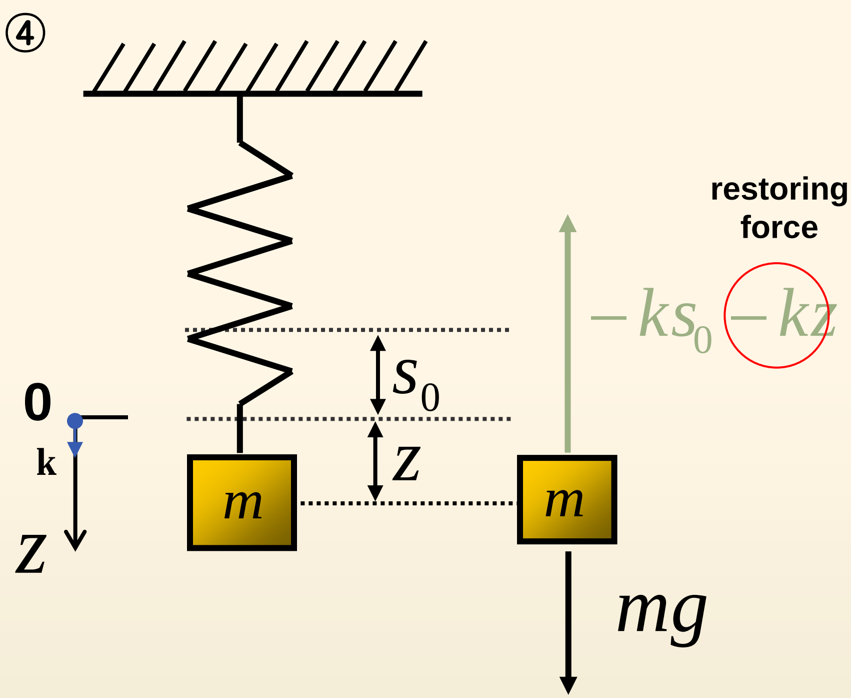
$$mz'' + kz = 0$$

Oscillation by the restoring force

Spring/Mass Systems: Driven Motion

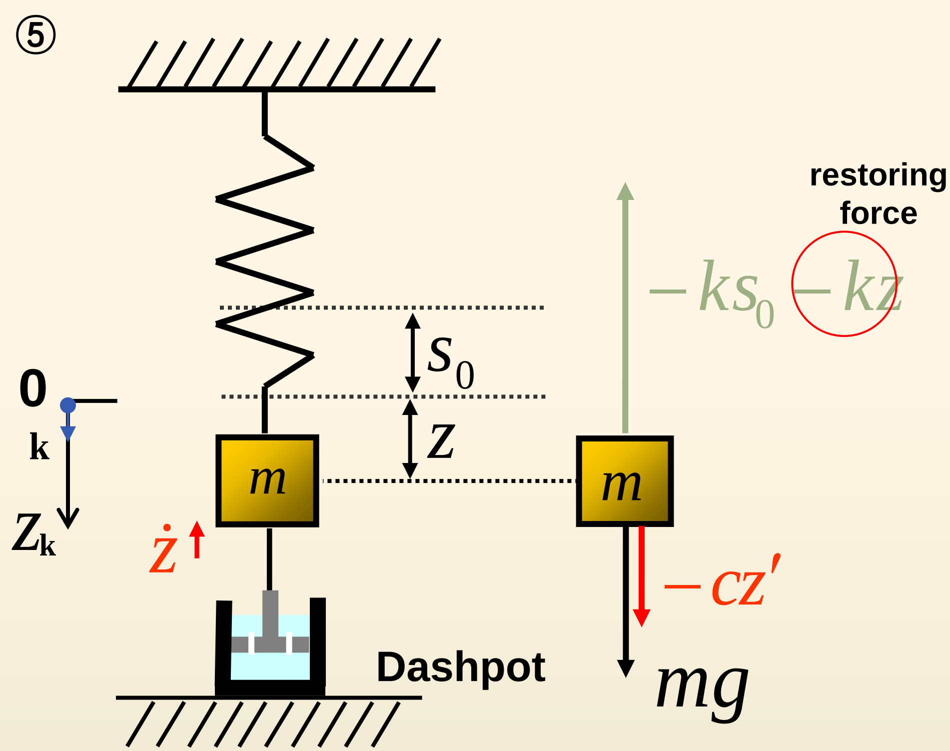
$$z = z(t)\ , \quad z'' = \frac{d^2 z}{dt^2}$$

④



$$
\begin{aligned}
m\mathbf{z}'' &= \mathbf{F} \\
&= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} \\
&= -kz\mathbf{k}
\end{aligned}
$$

⑤



$$
\begin{aligned}
m\mathbf{z}'' &= \mathbf{F} \\
&= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} \\
&= -kz\mathbf{k} - cz'\mathbf{k}
\end{aligned}
$$

Physical Phenomenon

Mathematical Equation

$$m\mathbf{z}'' + k\mathbf{z} = 0$$

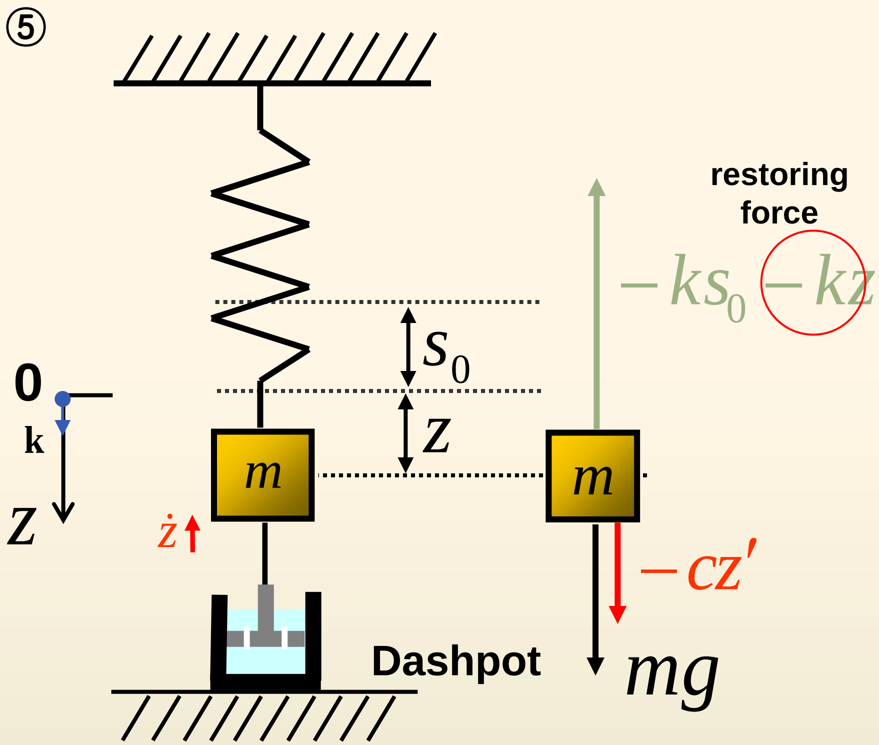
oscillation by the restoring force

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = 0$$

Spring/Mass Systems: Driven Motion

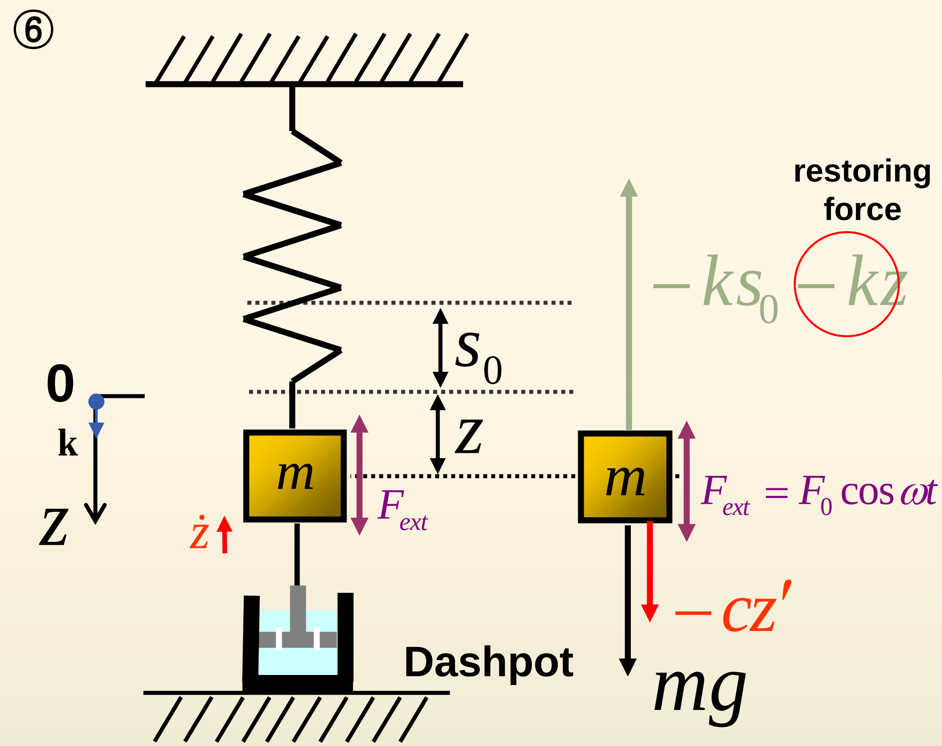
$$z = z(t)\,, \quad z'' = \frac{d^2 z}{dt^2}$$

⑤



$$\mathbf{m}\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} \quad -cz'\mathbf{k}$$
$$= -kz\mathbf{k} \quad -cz'\mathbf{k}$$

⑥



$$\mathbf{m}\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$

Physical Phenomenon

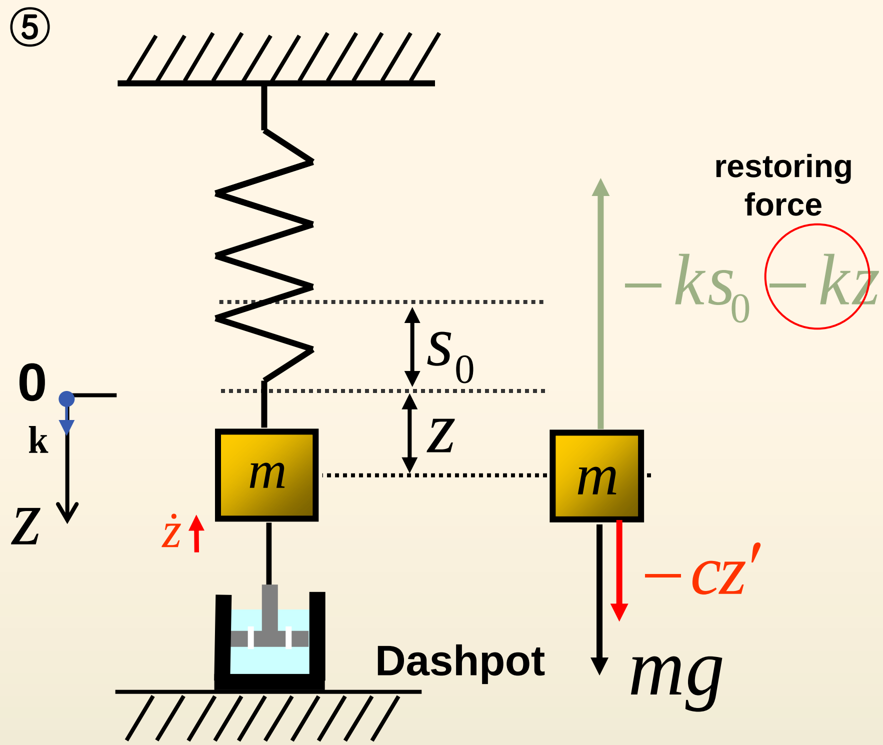
Mathematical Equation

$$\mathbf{m}\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = 0$$

$$\mathbf{m}\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos \omega t$$

Spring/Mass Systems: Driven Motion

$z = z(t)\,, \quad z'' = \dfrac{d^2 z}{dt^2}$

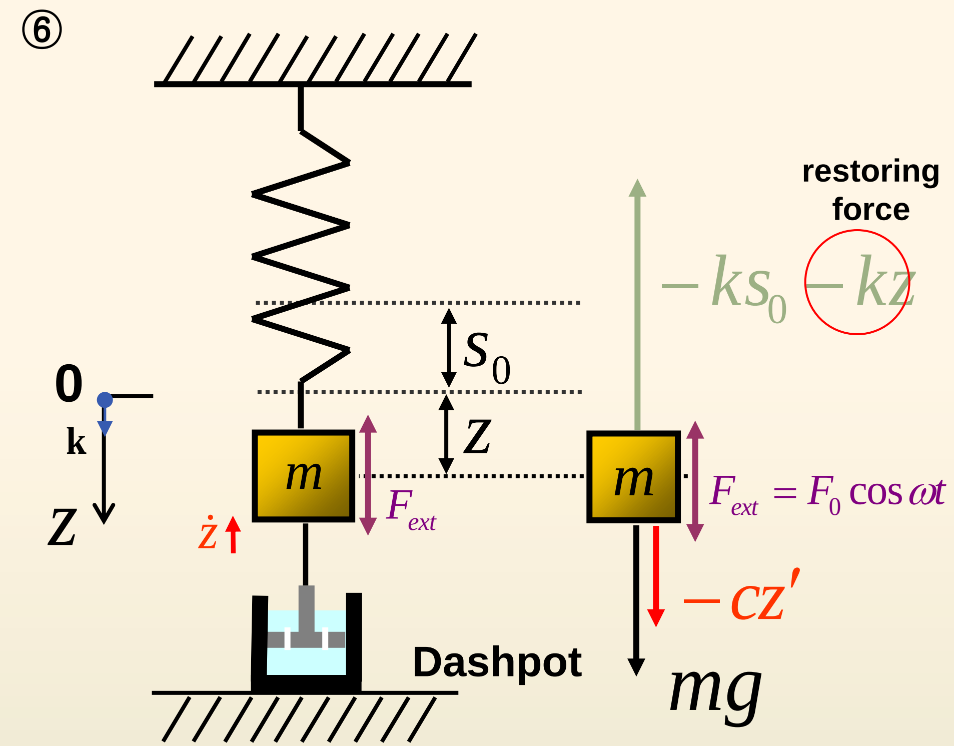


$$\begin{aligned} m\mathbf{z}'' &= \mathbf{F} \\ &= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} \\ &= -kz\mathbf{k} - cz'\mathbf{k} \end{aligned}$$

Physical Phenomenon

Mathematical Equation

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = 0$$



$$\begin{aligned} m\mathbf{z}'' &= \mathbf{F} \\ &= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos\omega t \\ &= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos\omega t \end{aligned}$$

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos\omega t$$

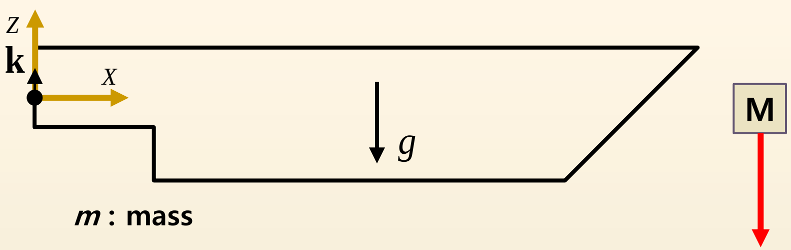
Nonlinearity

Nonlinearity of the nature → Nonlinear Mathematical Model → Numerical Method

↓ **Linearization**

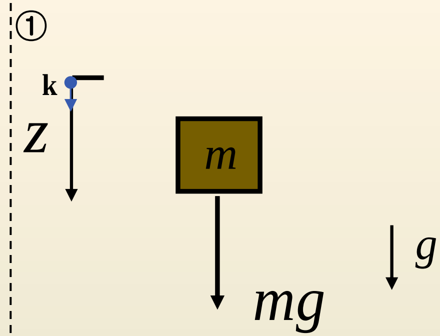
Linear Mathematical Model → Analytic Solution

Ex) Heave Motion of a Ship – step 1



$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity}$$
$$= -mg\mathbf{k}$$

✓ Mass-Spring-Damper system



By Newton's 2nd law,

$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k}$$

Nonlinearity

Nonlinearity of the nature ⇨ Nonlinear Mathematical Model ⇨ Numerical Method

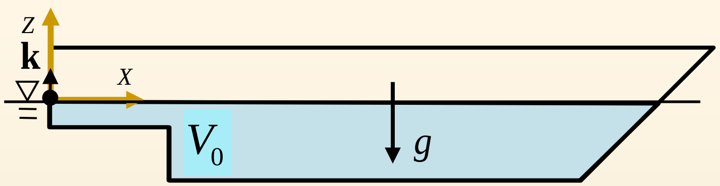
⇩ Linearization

Linear Mathematical Model ⇨ Analytic Solution

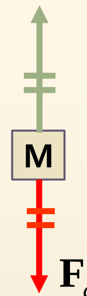
Ex) Heave Motion of a Ship – step 2

$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k}$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$



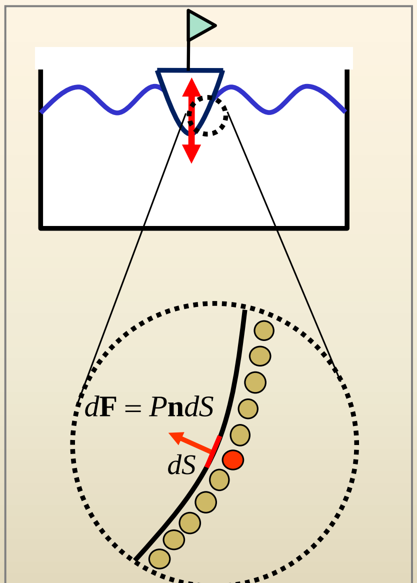
m : mass ρ : density of sea water
V_0: submerged volume
S_B : submerged surface area



$\mathbf{F}_{gravity}$
$= -mg\mathbf{k}$

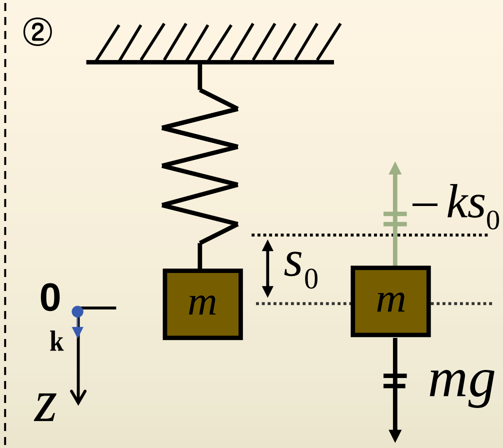
$$
\begin{aligned}
m\ddot{\mathbf{z}} &= \mathbf{F} \\
&= \mathbf{F}_{gravity} + \mathbf{F}_{static} \\
&= -mg\mathbf{k} + \rho g V_0 \mathbf{k} \\
&= 0 \ (\because \ddot{z} = 0)
\end{aligned}
$$

: static equilibrium



dS : infinitesimal submerged surface area
$d\mathbf{F}$: force exerted by the infinitesimal fluid element on dS
$\mathbf{n}$: normal vector of dS

✓ Mass-Spring-Damper system



$$
\begin{aligned}
m\mathbf{z}'' &= \mathbf{F} \\
&= mg\mathbf{k} - ks_0\mathbf{k} \\
&= 0 \ (\because z'' = 0)
\end{aligned}
$$

: static equilibrium

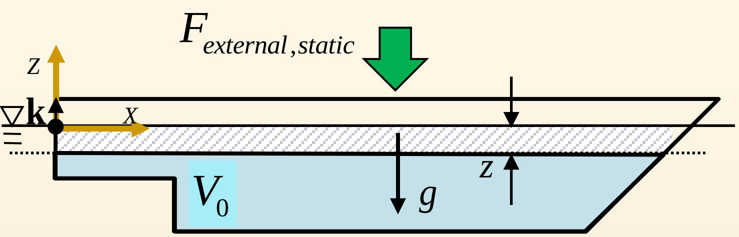
Nonlinearity

Nonlinearity of the nature ⇒ **Nonlinear Mathematical Model** ⇒ **Numerical Method**

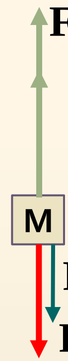
⇓ **Linearization**

Linear Mathematical Model ⇒ **Analytic Solution**

Ex) Heave Motion of a Ship – step 3



m : mass
ρ : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0 \mathbf{k} + \mathbf{F}_{addtional\,bouyancy}$$

additional buoyancy caused by additional displacement z

$$\mathbf{F}_{external,static}$$
$$\mathbf{F}_{gravity}$$
$$= -mg\mathbf{k}$$

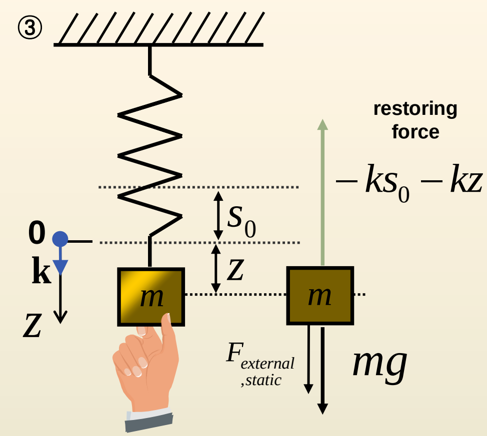
if, z is small

$$\mathbf{F}_{addtional\,bouyancy}$$
$$= -\rho g A_{WP}\mathbf{z}$$
$$= -k\mathbf{z}$$
$$, k = \rho g A_{WP}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{external,static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -\rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -k\mathbf{z} + \mathbf{F}_{external,static}$$
$$= 0 \ (\because \ddot{z} = 0)$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= 0 \quad (\because z'' = 0)$$

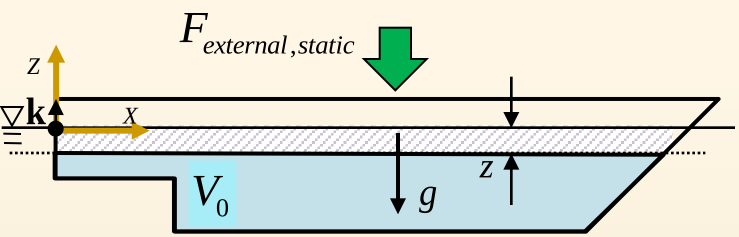
Nonlinearity

Nonlinearity of the nature ⇒ **Nonlinear Mathematical Model** ⇒ Numerical Method

⇓ **Linearization**

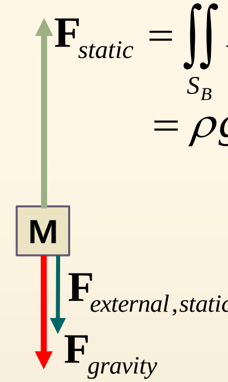
Linear Mathematical Model ⇒ **Analytic Solution**

Ex) Heave Motion of a Ship – step 4



m : mass　　*ρ* : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{external,static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -\rho g A_{wp}\mathbf{z} + \mathbf{F}_{external,static}$$
$$= -k\mathbf{z} + \mathbf{F}_{external,static} \;, k = \rho g A_{WP}$$
$$= 0 \;(\because \ddot{z} = 0)$$



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0 \mathbf{k} + \mathbf{F}_{addtional\ bouyancy}$$

additional buoyancy caused by additional displacement z

$$= -mg\mathbf{k}$$

if, z is small

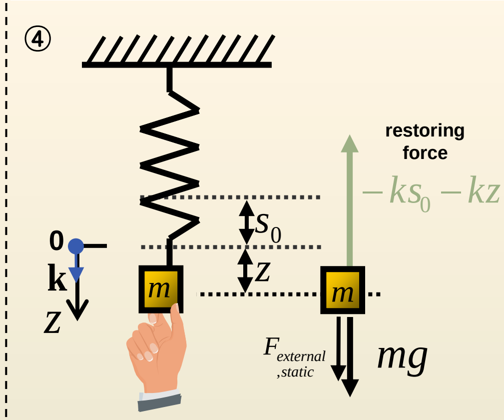
$$\mathbf{F}_{addtional\ bouyancy}$$
$$= -\rho g A_{WP}\mathbf{z}$$
$$= -k\mathbf{z}$$
$$, k = \rho g A_{WP}$$

Linearized Restoring Force

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ **Mass-Spring-Damper system**

④



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} + \mathbf{F}_{external,static}$$
$$= -kz\mathbf{k} \quad + \mathbf{F}_{external,static}$$

$$\boxed{m\mathbf{z}'' + k\mathbf{z} = 0}$$ Oscillation by the restoring force

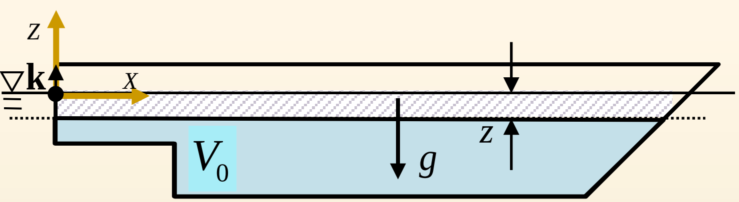
Nonlinearity

Nonlinearity of the nature ⇒ **Nonlinear Mathematical Model** ⇒ Numerical Method

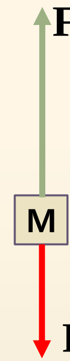
⇓ **Linearization**

Linear Mathematical Model ⇒ **Analytic Solution**

Ex) Heave Motion of a Ship – step 4



m : mass ρ : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z}$$
$$= \rho g V_0 \mathbf{k} - k\mathbf{z}$$

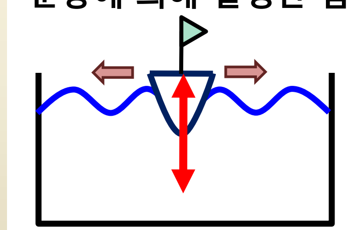
$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z}$$
$$= -\rho g A_{wp}\mathbf{z}$$
$$= -k\mathbf{z}$$

Ship will oscillate forever?

Energy is dissipated by radiation wave ⇒

정수 중 선박의 강제
운동에 의해 발생한 힘



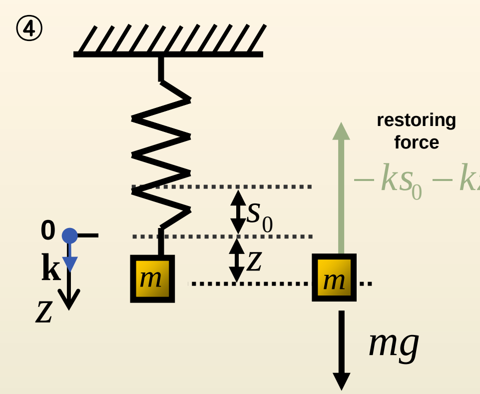
Radiation Force

$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

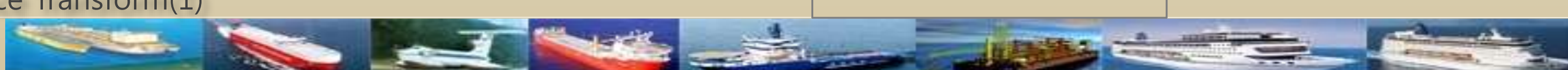
✓ Mass-Spring-Damper system

④



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k}$$
$$= -kz\mathbf{k}$$

$$m\mathbf{z}'' + k\mathbf{z} = 0$$ Oscillation by the restoring force



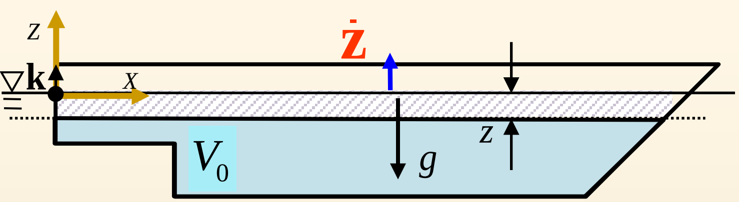
Nonlinearity

Nonlinearity of the nature ⇒ Nonlinear Mathematical Model ⇒ Numerical Method

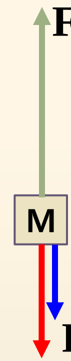
⇓ Linearization

Linear Mathematical Model ⇒ Analytic Solution

Ex) Heave Motion of a Ship – step 5



m : mass ρ : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS = \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z} = \rho g V_0 \mathbf{k} - k\mathbf{z}$$

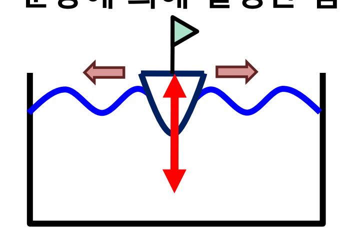
$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F} = \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} = -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} = -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} = -k\mathbf{z} - c\dot{\mathbf{z}}$$

opposite to velocity

정수 중 선박의 강제
운동에 의해 발생한 힘



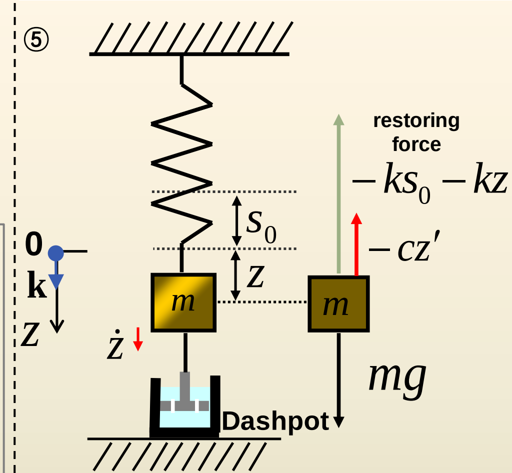
Radiation Force

$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS = -c\dot{\mathbf{z}}$$

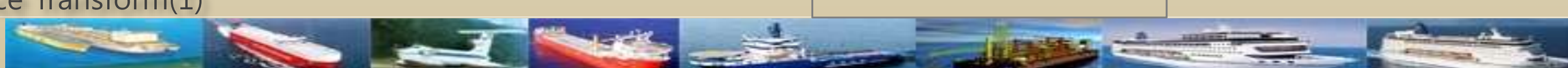
c : damping coefficient

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system



$$m\mathbf{z}'' = \mathbf{F} = mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} = -kz\mathbf{k} - cz'\mathbf{k}$$



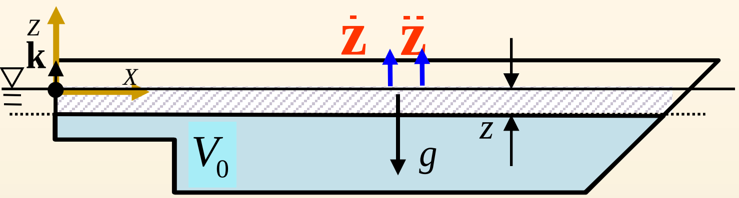
Nonlinearity

Nonlinearity of the nature → **Nonlinear Mathematical Model** → **Numerical Method**

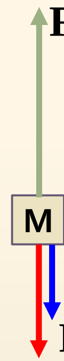
↓ **Linearization**

Linear Mathematical Model → **Analytic Solution**

Ex) Heave Motion of a Ship – step 5



m : mass
ρ : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z}$$
$$= \rho g V_0 \mathbf{k} - k\mathbf{z}$$

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

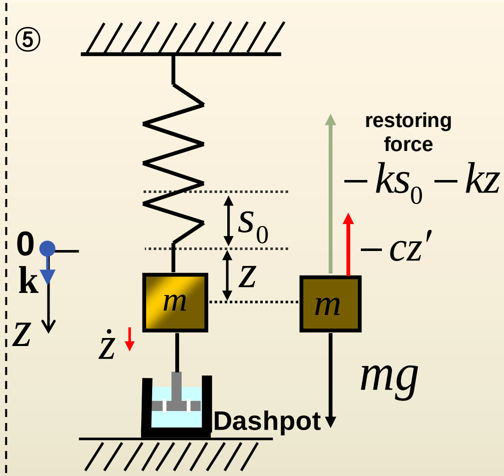
$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} \ - \rho g A_{wp}\mathbf{z} \ - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$
$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

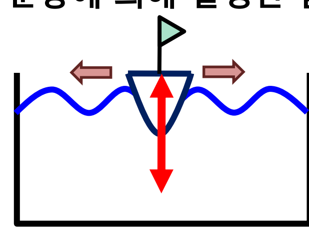
✓ Mass-Spring-Damper system

⑤



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} \ - cz'\mathbf{k}$$
$$= -kz\mathbf{k} - cz'\mathbf{k}$$

정수 중 선박의 강제 운동에 의해 발생한 힘

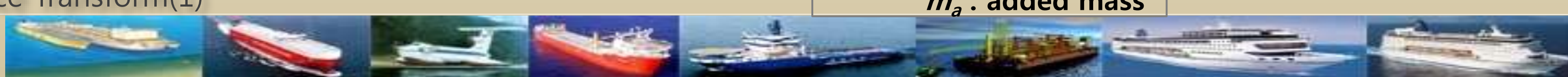


Radiation Force

$$\mathbf{F}_{radiation} = \iint_{S_B} P_{radiation}\mathbf{n}dS$$
$$= -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

opposite to velocity
opposite to acceleration

c : damping coefficient
m_a : added mass



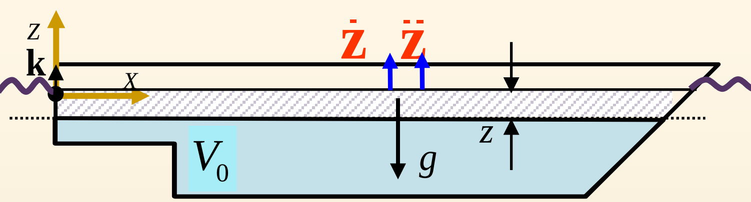
Nonlinearity

Nonlinearity of the nature → **Nonlinear Mathematical Model** → **Numerical Method**

↓ **Linearization**

Linear Mathematical Model → **Analytic Solution**

Ex) Heave Motion of a Ship – step 6

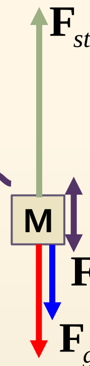


m : mass ρ : density of sea water

V_0: submerged volume

S_B : submerged surface area

Awp : waterplane area



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z}$$
$$= \rho g V_0 \mathbf{k} - k\mathbf{z}$$

$$\mathbf{F}_{exciting}$$

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

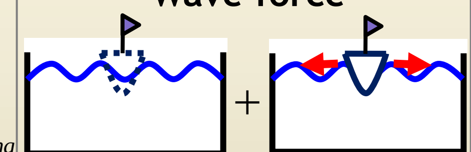
$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} + \mathbf{F}_{exciting}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

c : damping coefficient

m_a : added mass

Wave force

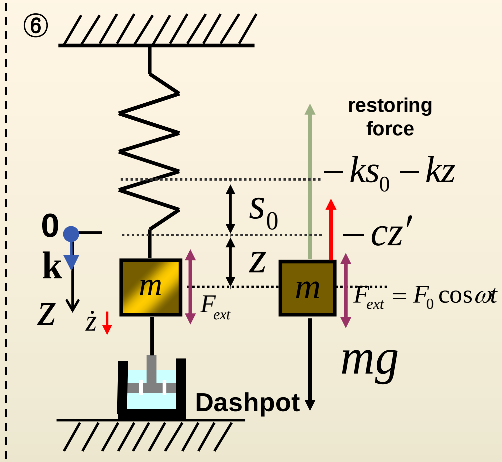


Froude-Kriloff Force + Diffraction Force

$$\mathbf{F}_{wave\ exciting}$$
$$= \iint_{S_B} P_{wave\ exciting}\mathbf{n}dS$$
$$\left(= \mathbf{F}_{exciting}\right)$$

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system



$$mz'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos\omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos\omega t$$

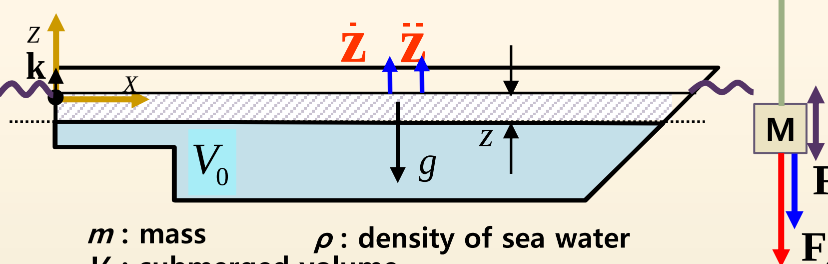
Nonlinearity

Nonlinearity of the nature ⇒ Nonlinear Mathematical Model ⇒ Numerical Method

⇓ Linearization

Linear Mathematical Model ⇒ Analytic Solution

Ex) Heave Motion of a Ship – step 6



$$\mathbf{F}_{static} = \iint_{S_B} P_{static}\mathbf{n}dS$$
$$= \rho g V_0 \mathbf{k} - \rho g A_{WP}\mathbf{z}$$
$$= \rho g V_0 \mathbf{k} - k\mathbf{z}$$

$\mathbf{F}_{exciting}$

$$\mathbf{F}_{radiation} = -c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}}$$

$$\mathbf{F}_{gravity} = -mg\mathbf{k}$$

m : mass
ρ : density of sea water
V_0: submerged volume
S_B : submerged surface area
Awp : waterplane area

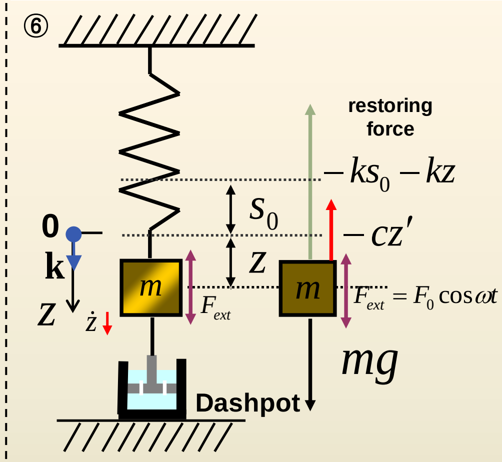
$$m\ddot{\mathbf{z}} = \mathbf{F}$$
$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{radiation} + \mathbf{F}_{exciting}$$
$$= -mg\mathbf{k} + \rho g V_0 \mathbf{k} - \rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -\rho g A_{wp}\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$
$$= -k\mathbf{z} - c\dot{\mathbf{z}} - m_a\ddot{\mathbf{z}} + \mathbf{F}_{exciting}$$

$$\boxed{(m + m_a)\ddot{\mathbf{z}} + c\dot{\mathbf{z}} + k\mathbf{z} = \mathbf{F}_{exciting}}$$

c : damping coefficient
m_a : added mass

✓ Archimedes' Principle $\mathbf{F}_{static} = \rho g V_0 \mathbf{k}$

✓ Mass-Spring-Damper system



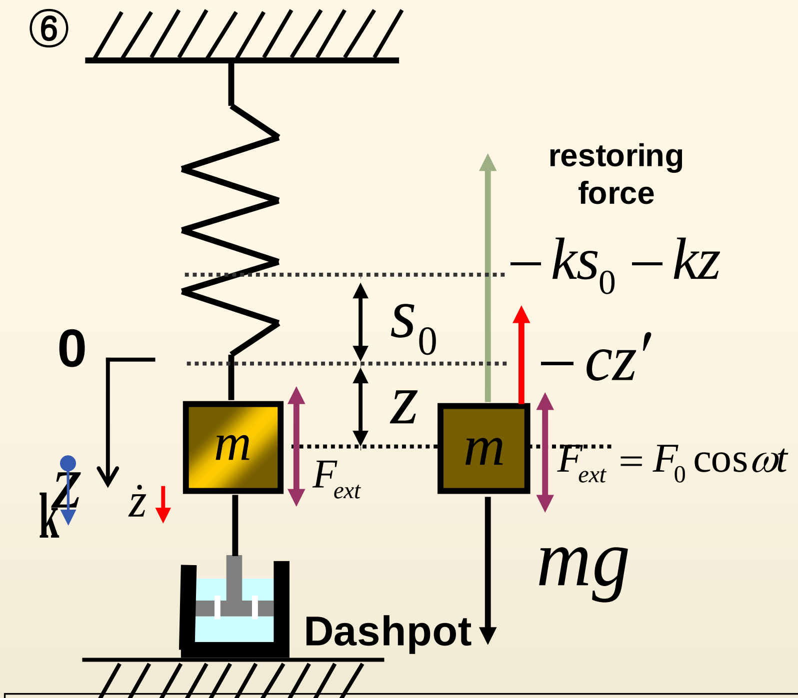
$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0}\cos\omega t$$
$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0}\cos\omega t$$

$$\boxed{m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos\omega t}$$

Linear Model
Spring/Mass Systems: Driven Motion

$$z = z(t)\ ,\ z'' = \frac{d^2 z}{dt^2}$$

✓ Mass-Spring-Damper system

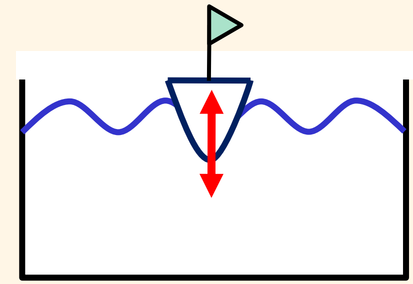


$$m\mathbf{z}'' = \mathbf{F}$$

$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0}\cos\omega t$$

$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0}\cos\omega t$$

$$\boxed{m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos\omega t}$$



✓선박의 6자유도 운동 방정식

① Coordinate system 정의

(Global & Body-fixed coordinate)

② Newton's 2nd Law

$$\mathbf{M}\ddot{\mathbf{x}} = \sum \mathbf{F} = \mathbf{F}_{body} + \mathbf{F}_{surface}$$

$$= \mathbf{F}_{gravity} + \mathbf{F}_{static} + \mathbf{F}_{F.K} + \mathbf{F}_{D} + \mathbf{F}_{R}$$

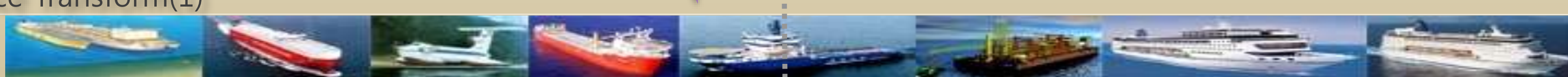
$$= \mathbf{F}_{restoring} + \mathbf{F}_{exciting} - \mathbf{A}\ddot{\mathbf{x}} - \mathbf{B}\dot{\mathbf{x}}$$

$$\boxed{(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} = \mathbf{F}_{restoring} + \mathbf{F}_{exciting}}$$

cf) 선형화 된 복원력 ($\mathbf{F}_{restoring} = -\mathbf{C}\mathbf{x}$)

$$(\mathbf{M}+\mathbf{A})\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{F}_{exciting}$$

Identical



Spring/Mass Systems: Driven Motion

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try}: z = e^{\lambda t}$$

$$z = c_1 z_1 + c_2 z_2 (= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}) \quad where,\ \lambda_{1,2} = \frac{-c \pm \sqrt{\boxed{c^2 - 4mk}}}{2m}$$

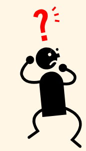
are you sure Z_1, Z_2 are linearly independent?

There could be three cases depending on the condition

Spring/Mass Systems: Driven Motion

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**



are you sure z_1, z_2 are linearly independent?

$$mz'' + cz' + kz = 0 \quad \text{try}: z = e^{\lambda t}$$

$$z = c_1 z_1 + c_2 z_2 (= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}) \qquad where,\ \lambda_{1,2} = \frac{-c \pm \sqrt{\boxed{c^2 - 4mk}}}{2m}$$

There could be three cases depending on the condition

cases	$c^2 - 4mk$	Root	
Case I	$c^2 - 4mk > 0$	Distinct Real Roots	λ_1, λ_2
Case II	$c^2 - 4mk = 0$	Repeated Real Roots	λ_1
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

Spring/Mass Systems: Driven Motion

homogeneous
$$z = e^{\lambda t}$$

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try}: z = e^{\lambda t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{\boxed{c^2 - 4mk}}}{2m}$$

cases	$c^2 - 4mk$	Root	
Case I	$c^2 - 4mk > 0$	Distinct Real Roots	$\lambda_1 \neq \lambda_2$

Solution
$z = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \Rightarrow z = \boxed{e^{\lambda_1 t}}$$

$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \Rightarrow z = \boxed{e^{\lambda_2 t}}$$

Linearly independent

Linear combination

Recall the example

$$c_1 e^{t} + c_2 e^{2t} = 0$$

Satisfied only when $c_1 = c_2 = 0$

Linearly Independent

Spring/Mass Systems: Driven Motion

homogeneous

$$z = e^{\lambda t}$$

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try}: z = e^{\lambda t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{\boxed{c^2 - 4mk}}}{2m}$$

cases	$c^2 - 4mk$	**Root**	
Case II	$c^2 - 4mk = 0$	Repeated Real Roots	λ_1

Solution

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \quad \Rightarrow \quad z_1 = e^{\lambda_1 t}$$

To create another linearly independent solution , multiply t **to** z_1 $\Rightarrow$ $z_2 = te^{\lambda_1 t}$

Reduction of order

Spring/Mass Systems: Driven Motion

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try}: z = e^{\lambda t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{\boxed{c^2 - 4mk}}}{2m}$$

cases	$c^2 - 4mk$	Root	
Case II	$c^2 - 4mk = 0$	Repeated Real Roots	λ_1

Solution
$z = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a}$$

$z_1 = \boxed{e^{\lambda_1 t}}$

$z_2 = \boxed{te^{\lambda_2 t}}$

Linearly independent

Linear combination

Spring/Mass Systems: Driven Motion

homogeneous

$$z = e^{\lambda t}$$

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try}: z = e^{\lambda t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{\boxed{c^2 - 4mk}}}{2m}$$

cases	$c^2 - 4mk$	Root	
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

Solution

$$\lambda_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a} = \alpha + i\beta$$

$$\lambda_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = \underbrace{\boxed{\frac{-b}{2a}}}_{\alpha} - i\underbrace{\boxed{\frac{\sqrt{4ac - b^2}}{2a}}}_{\beta} = \alpha - i\beta$$

$$z_1 = \boxed{e^{(\alpha + i\beta)t}}$$

$$z_2 = \boxed{e^{(\alpha - i\beta)t}}$$

Linearly independent → Linear combination

$$z = C_1 z_1 + C_2 z_2$$

homogeneous
$z = e^{\lambda t}$

Spring/Mass Systems: Driven Motion

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try}: z = e^{\lambda t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{\boxed{c^2 - 4mk}}}{2m}$$

cases	$c^2 - 4mk$	Root	
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

Solution

Linear combination

$$z = C_1 e^{(\alpha + i\beta)t} + C_2 e^{(\alpha - i\beta)t}$$

Two solutions by the choices

$$C_1 = 1, C_2 = 1$$

$$z_1 = e^{(\alpha + i\beta)t} + e^{(\alpha - i\beta)t}$$

$$C_1 = 1, C_2 = -1$$

$$z_2 = e^{(\alpha + i\beta)t} - e^{(\alpha - i\beta)t}$$

Spring/Mass Systems: Driven Motion

homogeneous

$$z = e^{\lambda t}$$

Euler's formular

$$e^{it} = \cos t + i \sin t$$

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try} : z = e^{\lambda t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{\boxed{c^2 - 4mk}}}{2m}$$

cases	$c^2 - 4mk$	Root	
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

Solution

$$z_1 = e^{(\alpha + i\beta)t} + e^{(\alpha - i\beta)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t) + e^{\alpha t}(\cos \beta t - i \sin \beta t) = 2\boxed{e^{\alpha t} \cos \beta t}$$

$$z_2 = e^{(\alpha + i\beta)t} - e^{(\alpha - i\beta)t} = e^{\alpha t}(\cos \beta t + i \sin \beta t) - e^{\alpha t}(\cos \beta t - i \sin \beta t) = 2\boxed{e^{\alpha t} \sin \beta t}$$

New fundamental set of solution

Spring/Mass Systems: Driven Motion

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \quad \text{try}: z = e^{\lambda t}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{\boxed{c^2 - 4mk}}}{2m}$$

cases	$c^2 - 4mk$	Root	
Case III	$c^2 - 4mk < 0$	Conjugated Complex Roots	$\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

Solution
$y = e^{\alpha t}(c_1 \cos \beta t + c_2 \cos \beta t)$

$$z_1 = 2e^{\alpha t} \cos \beta t$$

$$z_2 = 2e^{\alpha t} \sin \beta t$$

$$\boxed{e^{\alpha t} \cos \beta t}$$

$$\boxed{e^{\alpha t} \sin \beta t}$$

Linearly independent → **Linear combination**

Spring/Mass Systems: Driven Motion

homogeneous
$z = e^{\lambda t}$

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step1. homogeneous solution**

$$mz'' + cz' + kz = 0 \qquad \textbf{try:} \quad z = e^{\lambda t} \longrightarrow a\lambda^2 + b\lambda + c = 0$$

Case 1 $c^2 - 4mk > 0$

$$z_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \left(\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \right), (\lambda_1, \lambda_2 < 0)$$

Case 2 $c^2 - 4mk = 0$

$$z_h = (c_1 + c_2 \cdot t) \cdot e^{\frac{-c}{2m} \cdot t}, \left(\frac{-c}{2m} < 0 \right)$$

Case 3 $c^2 - 4mk < 0$

$$z_h = e^{\frac{-c}{2m} t} \cdot \left(A \cdot \cos(\omega_0 t) + B \cdot \sin(\omega_0 t) \right), \left(\omega_0 = \sqrt{\left(\frac{k}{m}\right) - \frac{1}{4} \cdot \left(\frac{c}{m}\right)^2} = \frac{1}{2m}\sqrt{4mk - c^2} \right)$$

Spring/Mass Systems: Driven Motion

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step2. particular solution**

By the method of Undetermined Coefficient, we choose that

$$z_p(t) = a\cos\omega t + b\sin\omega t$$

$$z'_p(t) = -a\omega\sin\omega t + b\omega\cos\omega t$$

$$z''_p(t) = -a\omega^2\cos\omega t - b\omega^2\sin\omega t$$

$y'' + ay' + by = r(x)$

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

$$\boxed{m\left(-\omega^2 a\cos\omega t - \omega^2 b\sin\omega t\right) + c\left(-\omega a\sin\omega t + \omega b\cos\omega t\right) + k\left(a\cos\omega t + b\sin\omega t\right) = F_0\cos\omega t}$$

Spring/Mass Systems: Driven Motion

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

$$\boxed{z_p(t) = a\cos\omega t + b\sin\omega t}$$

$$m\left(-\omega^2 a\cos\omega t - \omega^2 b\sin\omega t\right) + c\left(-\omega a\sin\omega t + \omega b\cos\omega t\right) + k\left(a\cos\omega t + b\sin\omega t\right) = F_0\cos\omega t$$

$$\boxed{[(k - m\omega^2)a + \omega cb]}\cos\omega t + \boxed{[-\omega ca + (k - m\omega^2)b]}\sin\omega t = \boxed{F_0}\cos\omega t$$

$$\begin{cases} (k - m\omega^2)a + \omega cb = F_0 \\ -\omega ca + (k - m\omega^2)b = 0 \end{cases}$$

$$\therefore a = F_0 \frac{m({\omega^2}_0 - \omega^2)}{m^2({\omega^2}_0 - \omega^2)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2({\omega^2}_0 - \omega^2)^2 + \omega^2 c^2}$$

$$\boxed{z_p(t) = a\cos\omega t + b\sin\omega t}$$

Spring/Mass Systems: Driven Motion

$$\boxed{mz'' + cz' + kz = F_0 \cos \omega t}$$

- **Step3. general solution**

$$z(t) = z_h(t) + z_p(t)$$

$$z_h(t) = \begin{cases} c_1 e^{-\left(\frac{c}{2m} - \frac{1}{2m}\sqrt{c^2 - 4mk}\right)t} + c_2 e^{-\left(\frac{c}{2m} + \frac{1}{2m}\sqrt{c^2 - 4mk}\right)t} & ,(c^2 - 4mk > 0) \\ (c_1 + c_2 t) e^{-\frac{c}{2m}t} & ,(c^2 - 4mk = 0) \\ e^{-\frac{c}{2m}t}(A\cos\omega_0 t + B\sin\omega_0 t) = Ce^{-\frac{c}{2m}t}\cos(\omega_0 t - \delta) & ,(c^2 - 4mk < 0) \end{cases}$$

$$z_p(t) = a\cos\omega t + b\sin\omega t \quad , \left(a = F_0 \frac{m({\omega^2}_0 - \omega^2)}{m^2({\omega^2}_0 - \omega^2)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2({\omega^2}_0 - \omega^2)^2 + \omega^2 c^2} \right)$$

$$\text{as } t \to \infty,\ z_h(t) \to 0, \text{ so } z(t) \to z_p(t)$$

Spring/Mass Systems: Driven Motion

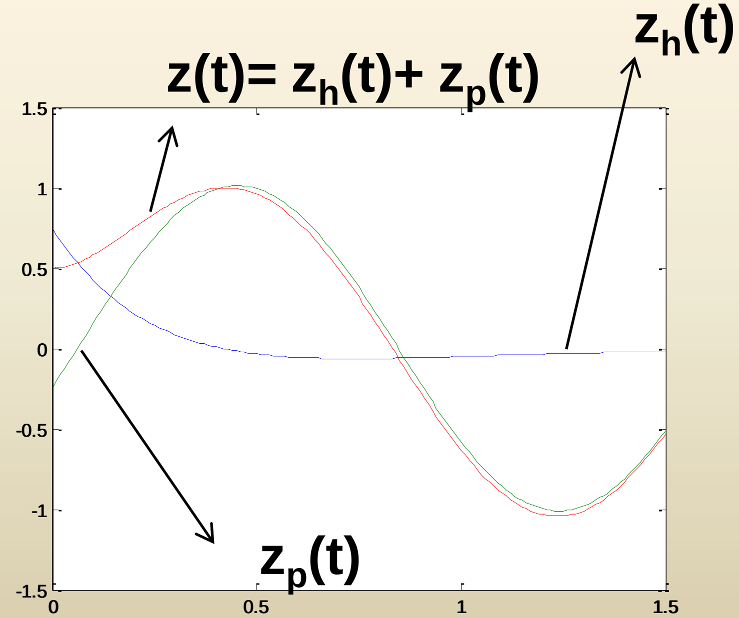
☑ Transient and Steady-State Terms

$$mz'' + cz' + kz = F(t) = F_0 \cos \omega t$$

- When F is periodic function, general solution have nonperiodic function $z_h(t)$ and periodic function $z_p(t)$

$z_h(t)$: transient solution

$z_p(t)$: steady-state solution



Comparison : example

$$y(t) = y_h(t) + y_p(t)$$

example*

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t \quad y(0) = 1, \dot{y}(0) = 5$$

solution

1) Homogeneous Solution

Homogeneous

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0$$

Try : $y_h(t) = e^{mt}$

$$(\ddot{m} + 3\dot{m} + 2)e^{mt} = 0$$

$$(m+1)(m+2) = 0$$

$$\therefore m = -1, m = -2$$

$$y_h(t) = e^{mt}$$

e^{-t} and e^{-2t} : linearly independent

$$\therefore y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$$

! Notation changed in the example

$$z(t) \rightarrow y(t)$$

$$z'(t), z''(t) \rightarrow y'(t), y''(t)$$

$$z_{\text{transient}}, z_{\text{steady}} \rightarrow y_h(t), y_p(t)$$

$$z_{\text{zero-input}}, z_{\text{zero-state}} \rightarrow y_0(t), y_1(t)$$

*Zill & Cullen, Advanced Engineering Mathematics,3rd Edition p204, example 5 modified

Comparison : example

$$y(t) = y_h(t) + y_p(t)$$

example

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t \quad y(0) = 1, \dot{y}(0) = 5$$

Solution

2) Particular Solution

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t$$

Try : $y_p(t) = A\sin 2t + B\cos 2t$

$$\ddot{y}_p = -4A\sin 2t - 4B\cos 2t$$

$$\dot{y}_p = 2A\cos 2t - 2B\sin 2t$$

L.H.S.: $-4\left(A\sin 2t + B\cos 2t\right) + 6\left(A\cos 2t - B\sin 2t\right) + 2\left(A\sin 2t + B\cos 2t\right) = (-2A - 6B)\sin 2t + (6A - 2B)\cos 2t$

R.H.S.: $\sin 2t$

L.H.S. = R.H.S.

$$\begin{aligned} -2A - 6B &= 1 \\ 6A - 2B &= 0 \end{aligned} \Rightarrow A = -\frac{1}{20}, B = -\frac{3}{20}$$

$$\therefore y_p = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : example

$$y(t) = y_h(t) + y_p(t)$$

example

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \sin 2t \quad y(0) = 1, \dot{y}(0) = 5$$

solution

3) General Solution

$$y_h(t) = c_1 e^{-t} + c_2 e^{-2t}$$

$$y_p = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Initial condition : $y(0) = 1, \dot{y}(0) = 5$

$$y(0) : c_1 + c_2 - \frac{3}{20} = 1$$

$$\dot{y}(0) : -c_1 - 2c_2 - \frac{2}{20} = 5$$

$$c_1 = \frac{37}{5}$$

$$c_2 = -\frac{25}{4}$$

$$y_h(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t}, \quad y_p(t) = -\frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$

$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$

$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

Zero Input:

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$$

$$y_{0\_h}(t)=e^{mt}$$

$$(m+1)(m+2)e^{mt}=0$$

$$y_{0\_h}(t)=c_1e^{-t}+c_2e^{-2t}$$

Zero State:

$$\ddot{y}(t)-3\dot{y}(t)+2y(t)=0$$

$$y_{1\_h}(t)=e^{mt}$$

$$(m+1)(m+2)e^{mt}=0$$

$$y_{1\_h}(t)=c_1e^{-t}+c_2e^{-2t}$$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$
$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$
$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0 \qquad \ddot{y}(t)-3\dot{y}(t)+2y(t)=0$$

$$y_{0\_h}(t)=c_1e^{-t}+c_2e^{-2t} \qquad y_{1\_h}(t)=c_1e^{-t}+c_2e^{-2t}$$

2) Particular Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0 \qquad \ddot{y}(t)-3\dot{y}(t)+2y(t)=\sin 2t$$

$$y_{1\_p}=A\sin 2t+B\cos 2t$$

L.H.S.: $(-2A-6B)\sin 2t+(6A-2B)\cos 2t$

R.H.S.: $\sin 2t$

L.H.S. = R.H.S. $A=-\frac{1}{20}, B=-\frac{3}{20}$

$$y_{0\_p}(t)=0 \qquad y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Comparison : example

예제

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$

$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$

$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0 \qquad \ddot{y}(t)-3\dot{y}(t)+2y(t)=0$$

$$y_{0\_h}(t)=c_1e^{-t}+c_2e^{-2t} \qquad y_{1\_h}(t)=c_1e^{-t}+c_2e^{-2t}$$

2) Particular Solution

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0 \qquad \ddot{y}(t)-3\dot{y}(t)+2y(t)=\sin 2t$$

$$y_{0\_p}(t)=0 \qquad y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

3) General Solution

$$y_{0\_g}(t)=c_1e^{-t}+c_2e^{-2t} \qquad y_{1\_g}(t)=c_1e^{-t}+c_2e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Initial condition

$$y(0)=1, \dot{y}(0)=5 \qquad y(0)=0 , \dot{y}(0)=0$$

$$c_1+c_2=1 \qquad c_1=7$$

$$-c_1-2c_2=5 \qquad c_2=-6$$

$$y(0): c_1+c_2-\frac{3}{20}=1 \qquad c_1=\frac{2}{5}$$

$$\dot{y}(0): -c_1-2c_2-\frac{2}{20}=5 \qquad c_2=-\frac{1}{4}$$

Comparison : example

example
$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

	Zero Input	Zero State
	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$ $y(0)=1, \dot{y}(0)=5$	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$ $y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$
1) Homogeneous Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=0$
	$y_{0\_h}(t)=c_1e^{-t}+c_2e^{-2t}$	$y_{1\_h}(t)=c_1e^{-t}+c_2e^{-2t}$
2) Particular Solution	$\ddot{y}(t)+3\dot{y}(t)+2y(t)=0$	$\ddot{y}(t)-3\dot{y}(t)+2y(t)=\sin 2t$
	$y_{0\_p}(t)=0$	$y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
3) General Solution	$y_{0\_g}(t)=c_1e^{-t}+c_2e^{-2t}$	$y_{1\_g}(t)=c_1e^{-t}+c_2e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$
Initial condition	$y(0)=1, \dot{y}(0)=5$	$y(0)=0\ , \dot{y}(0)=0$
General Solution	$y_{0\_g}(t)=7e^{-t}-6e^{-2t}$	$y_{1\_g}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$

$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$

$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

$$y_{0\_h}(t)=7e^{-t}-6e^{-2t}$$

$$y_{1\_h}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}$$

2) Particular Solution

$$y_{0\_p}(t)=0$$

$$y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

3)General Solution

$$y_{0\_g}(t)=y_0(t)=7e^{-t}-6e^{-2t}$$

$$y_{1\_g}(t)=y_1(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\left(\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}\right)-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t)$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$

$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$

$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

$$y_{0\_h}(t)=7e^{-t}-6e^{-2t}$$

$$y_{1\_h}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}$$

2) Particular Solution

$$y_{0\_p}(t)=(0)$$

$$y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

3)General Solution

$$y_{0\_g}(t)=y_0(t)=(7e^{-t}-6e^{-2t})$$

$$y_{1\_g}(t)=y_1(t)=\left(\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}\right)-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t)$$

$$y(t) \quad = \qquad\qquad + $$

$$\| \qquad\qquad \| \qquad\qquad\qquad \|$$

$$y_0(t) \quad = \qquad\qquad + $$

$$+ \qquad\qquad + \qquad\qquad\qquad +$$

$$y_1(t) \quad = \qquad\qquad + $$

Comparison : example

example

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t \quad y(0)=1, \dot{y}(0)=5$$

$$y(t)=\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

Zero Input

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\boxed{0}$$

$$y(0)=1, \dot{y}(0)=5$$

Zero State

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=\sin 2t$$

$$y(0)=\boxed{0}, \dot{y}(0)=\boxed{0}$$

1) Homogeneous Solution

$$y_{0\_h}(t)=7e^{-t}-6e^{-2t} \qquad y_{1\_h}(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}$$

2) Particular Solution

$$y_{0\_p}(t)=0 \qquad y_{1\_p}=-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

3)General Solution

$$y_{0\_g}(t)=y_0(t)=7e^{-t}-6e^{-2t} \qquad y_{1\_g}(t)=y_1(t)=\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t$$

$$\begin{array}{ccccc}
y(t) & = & (\frac{37}{5}e^{-t}-\frac{25}{4}e^{-2t}) & + & (-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t) \\
|| & & || & & || \\
y_0(t) & = & (7e^{-t}-6e^{-2t}) & + & (0) \\
+ & & + & & + \\
y_1(t) & = & (\frac{2}{5}e^{-t}-\frac{1}{4}e^{-2t}) & + & (-\frac{1}{20}\sin 2t-\frac{3}{20}\cos 2t)
\end{array}$$

Comparison : example-proof

$y(t) = y_0(t) + y_1(t)$

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t) \quad , u(t) \neq 0 \quad \cdots(1)$$

$$y(0) = a, \dot{y}(0) = b \qquad , a \neq 0, b \neq 0 \quad \cdots(2)$$

Zero Input solution : $y_0(t)$

$\boxed{m\ddot{y}_0(t) + c\dot{y}_0(t) + ky_0(t) = 0}$

$\boxed{y_0(0) = a}, \boxed{\dot{y}_0(0) = b}$

Zero state solution : $y_1(t)$

$\boxed{m\ddot{y}_1(t) + c\dot{y}_1(t) + ky_1(t) = u(t)}$

$\boxed{y_1(0) = 0}, \boxed{\dot{y}_1(0) = 0}$

assum. : $y(t) = y_0(t) + y_1(t)$

$\therefore y(t) = y_0(t) + y_1(t)$

y_t→ (1) :

L.H.S.:

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t)$$

$$= m(\ddot{y}_0(t) + \ddot{y}_1(t)) + c(\dot{y}_0(t) + \dot{y}_1(t)) + k(y_0(t) + y_1(t))$$

$$= \boxed{\left[m\ddot{y}_0(t) + c\dot{y}_0(t) + ky_0(t)\right]} + \boxed{\left[m\ddot{y}_1(t) + c\dot{y}_1(t) + ky_1(t)\right]}$$

$$= \boxed{0} + \boxed{u(t)}$$

R.H.S.: $u(t)$

$\therefore$ L.H.S.=R.H.S "(1) satisfied"

y_t→ (2) :

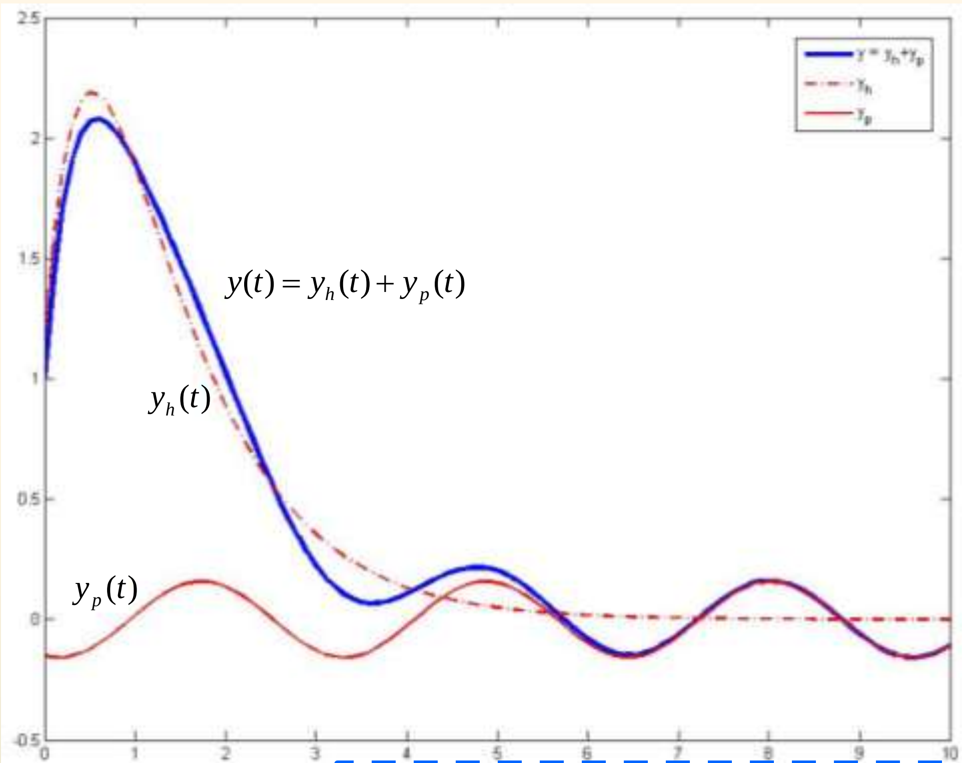
$$y(0) = \boxed{y_0(0)} + \boxed{y_1(0)} = \boxed{a} + \boxed{0}$$

$$\dot{y}(0) = \boxed{\dot{y}_0(0)} + \boxed{\dot{y}_1(0)} = \boxed{b} + \boxed{0}$$

$$\therefore y(0) = a \qquad \text{"(2) satisfied"}$$

$$\dot{y}(0) = b$$

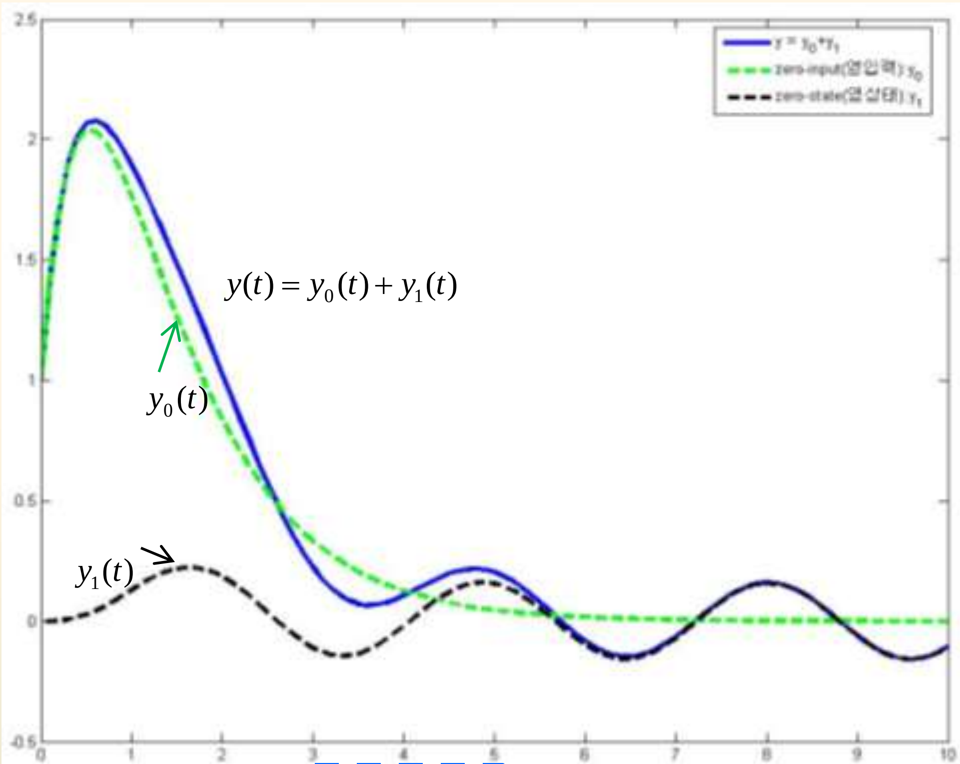
Comparison : graph



$$\boxed{y(t) = y_h(t) + y_p(t)}$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\|$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

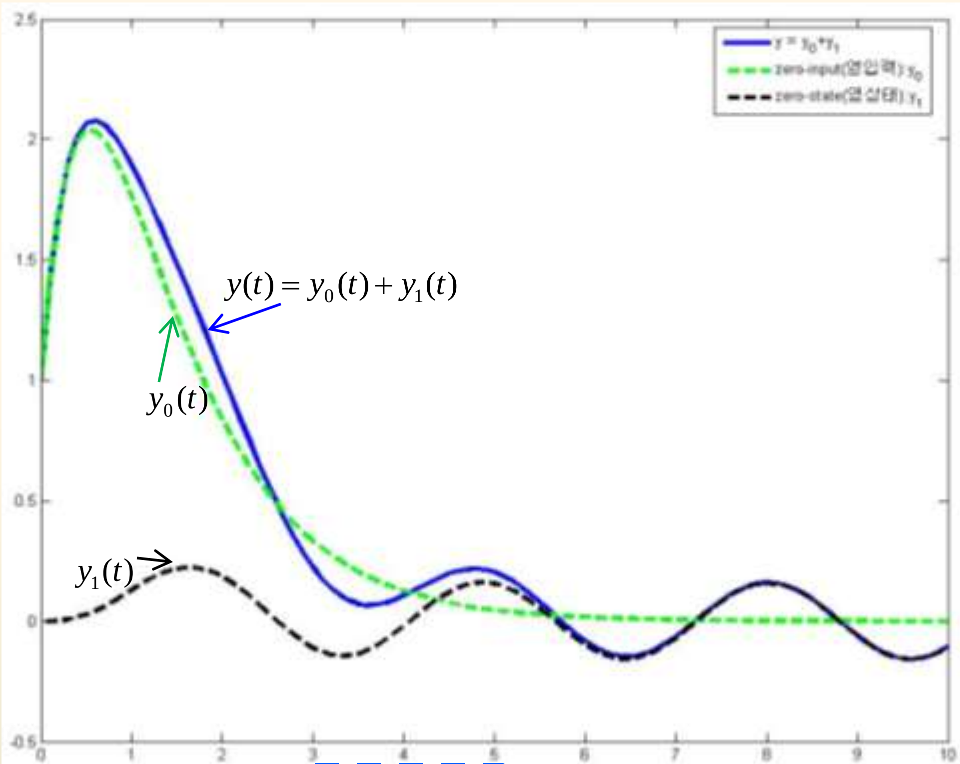
$$\|$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

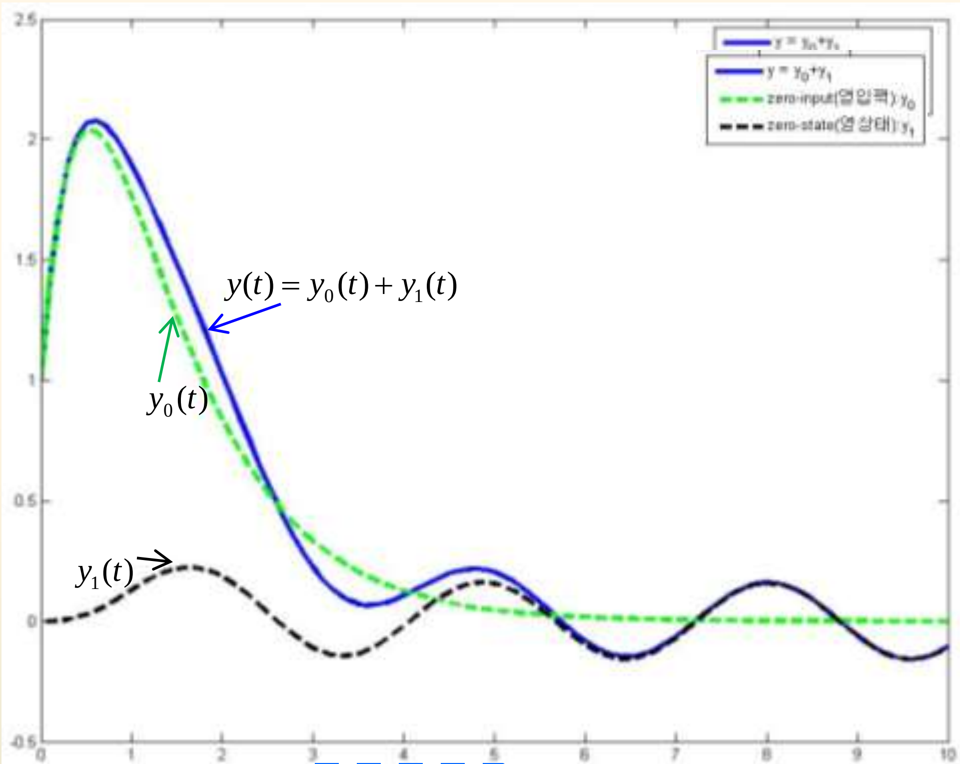
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)$=0

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

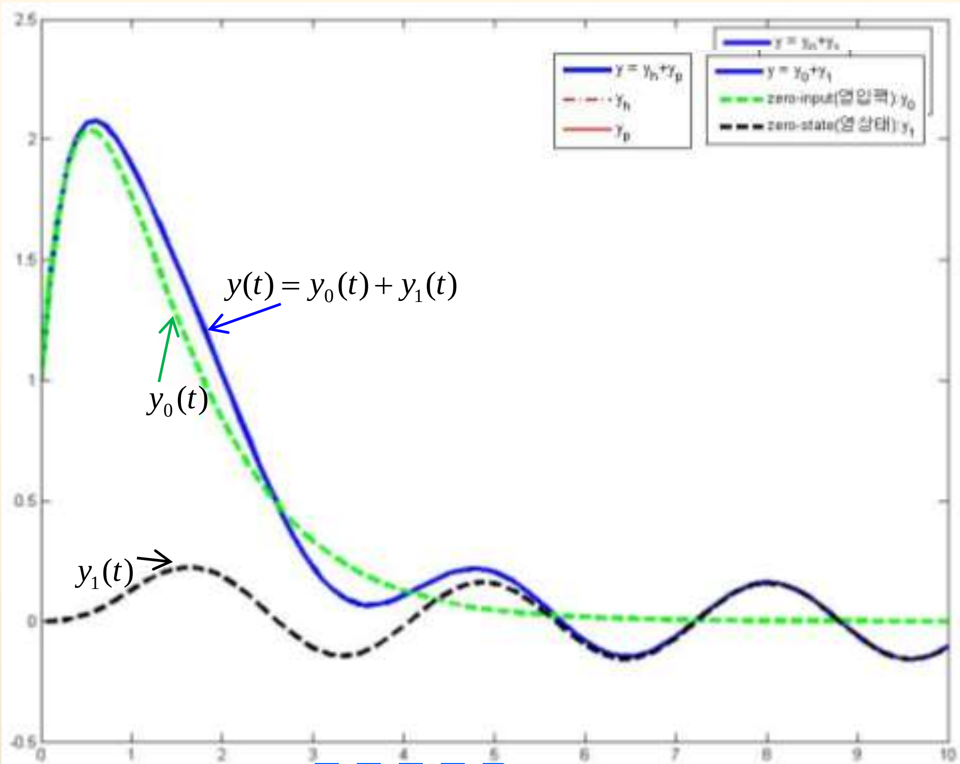
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

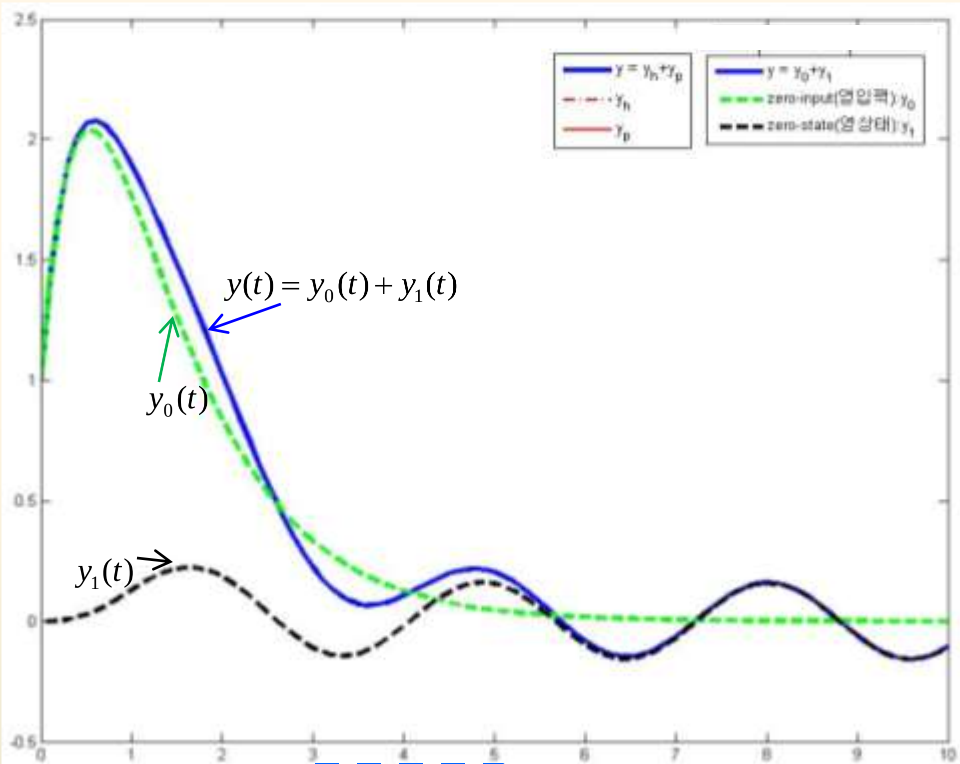
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$
$$||$$

Zero input solution
$u(t)=0$

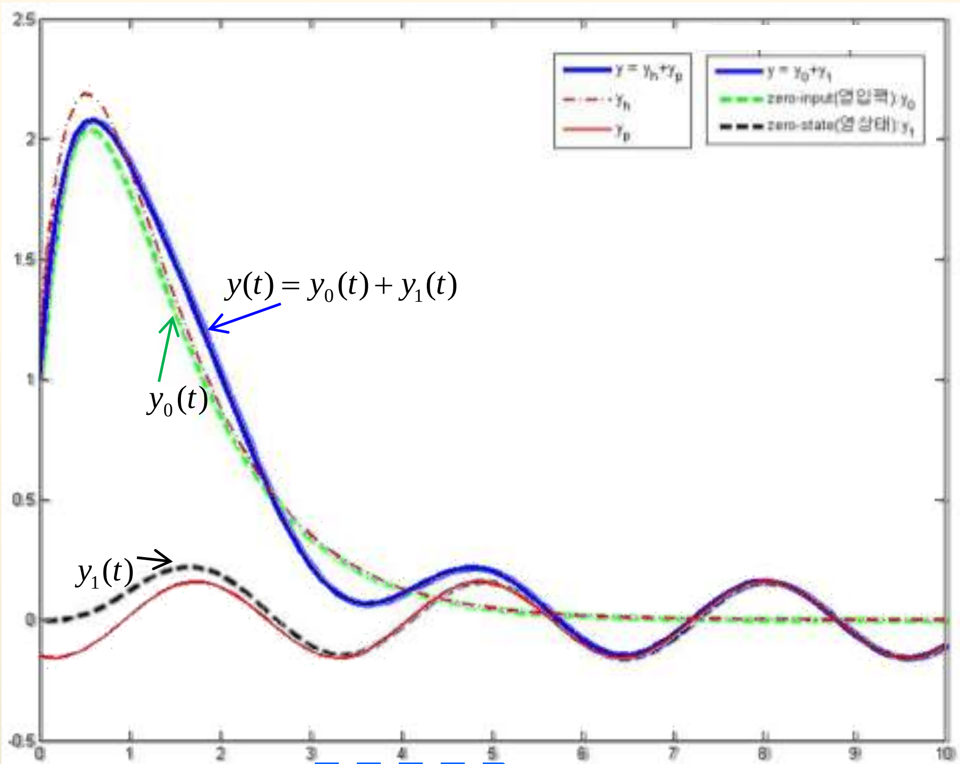
$$y_0(t)$$
$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$
$$||$$
$$y_0(t) = 7e^{-t} - 6e^{-2t}$$
$$+$$
$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

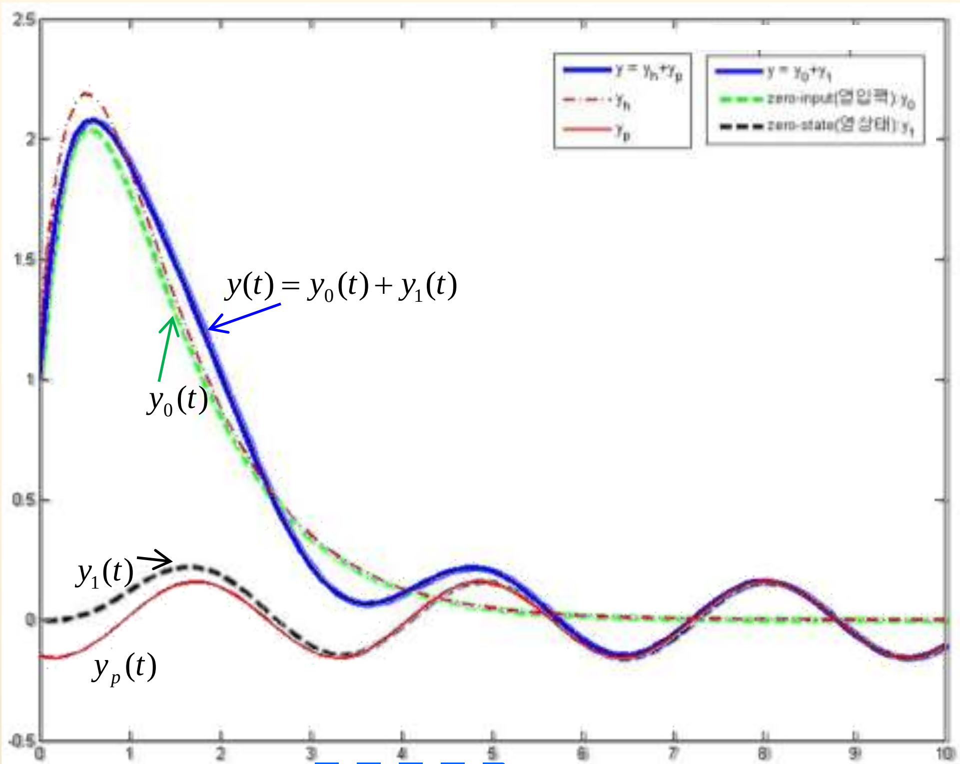
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

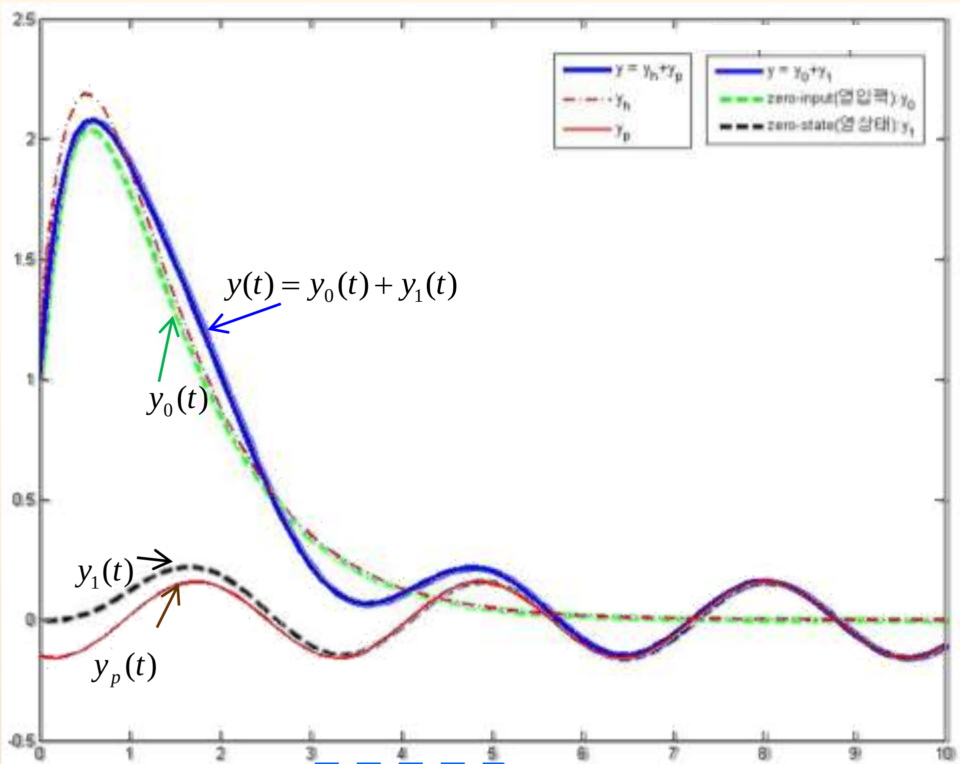
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

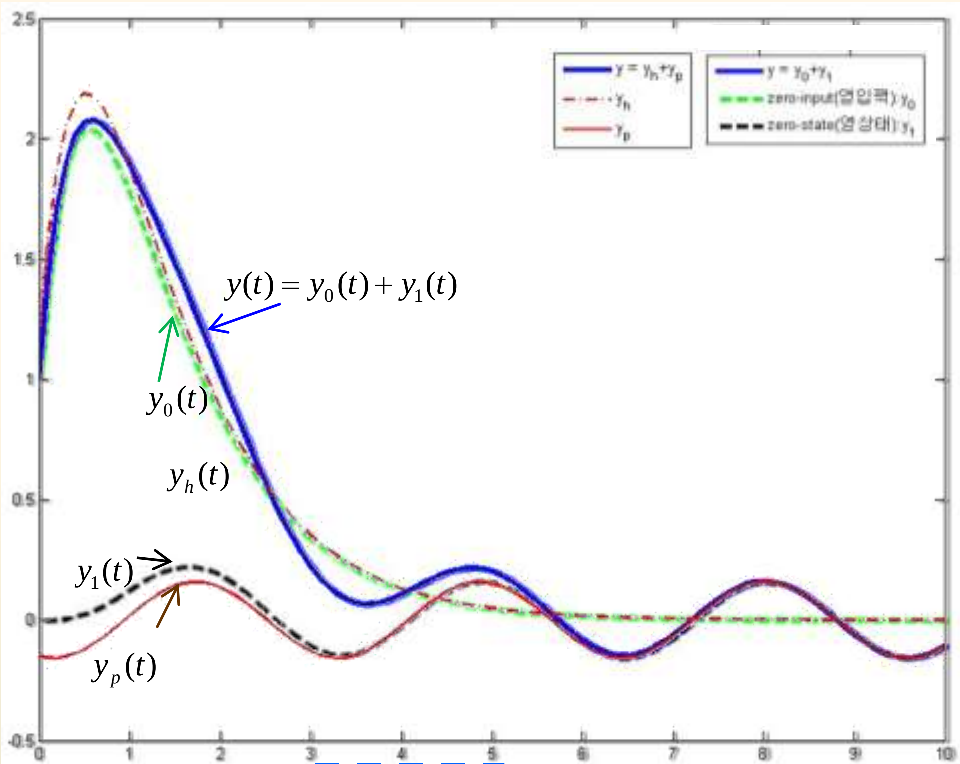
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

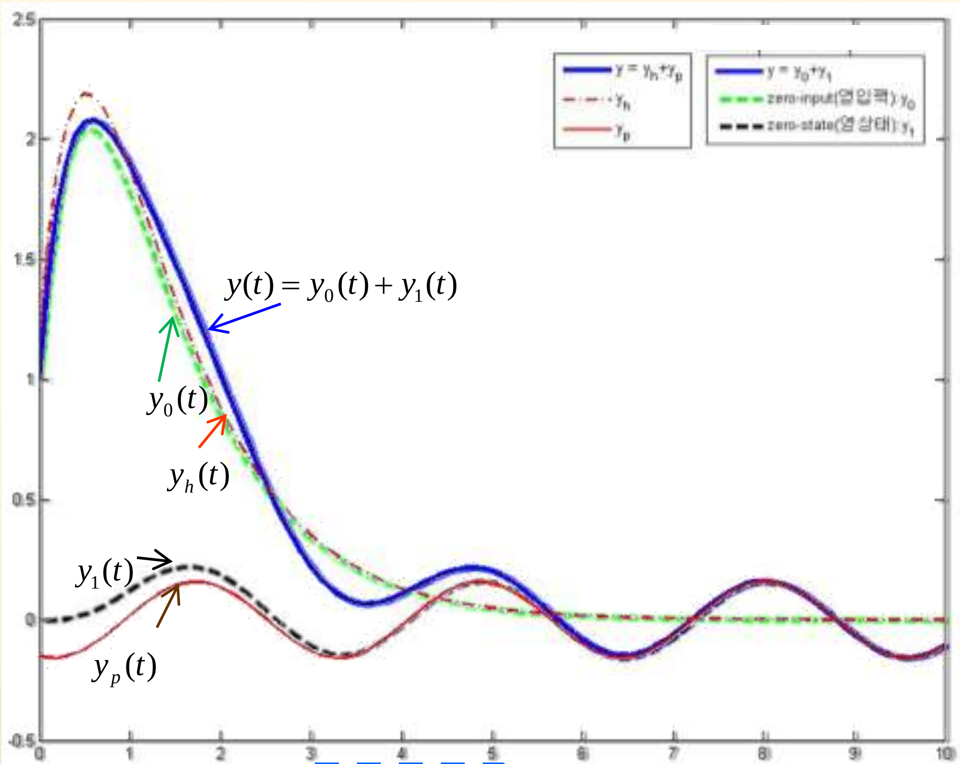
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

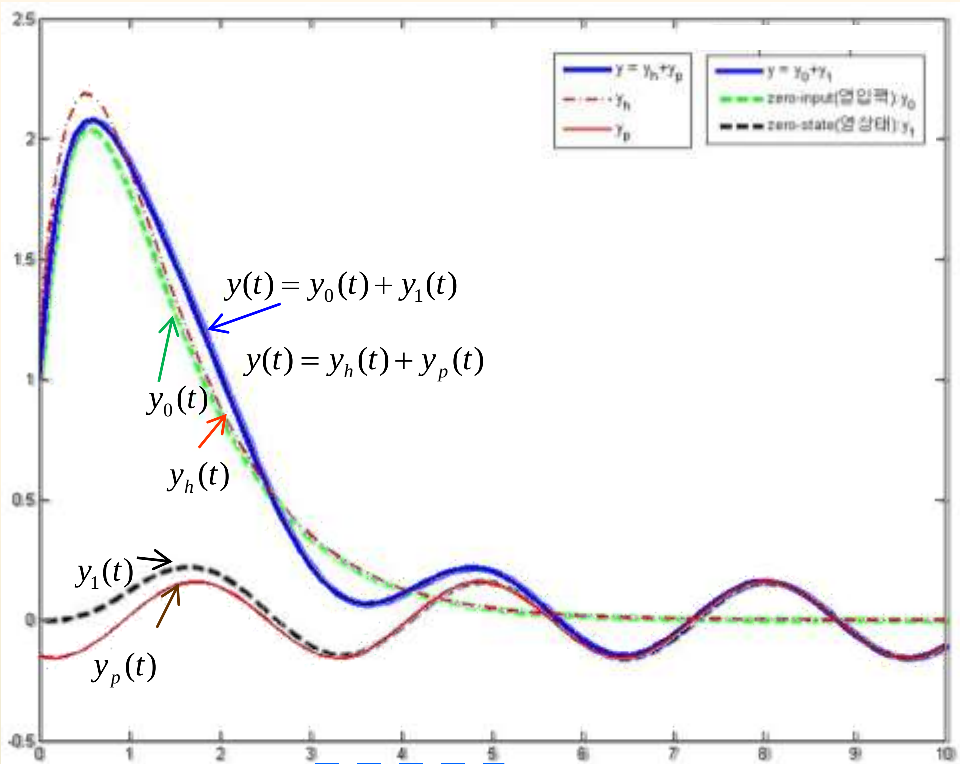
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

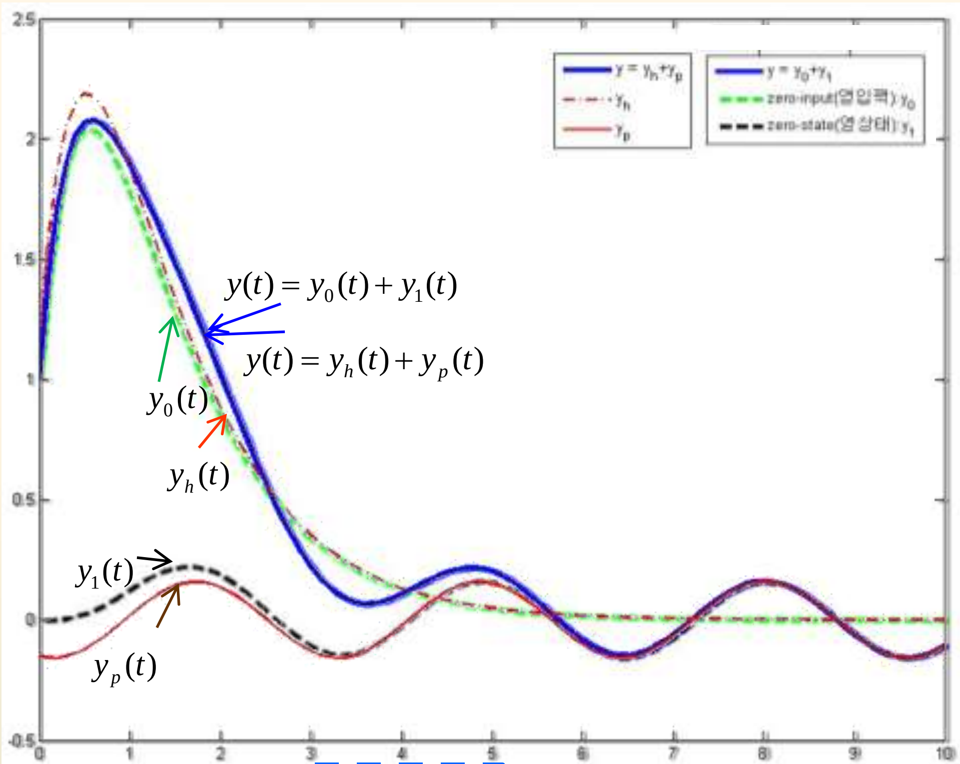
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$||$$

Zero input solution
$u(t)=0$

$$y_0(t)$$

$$+$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

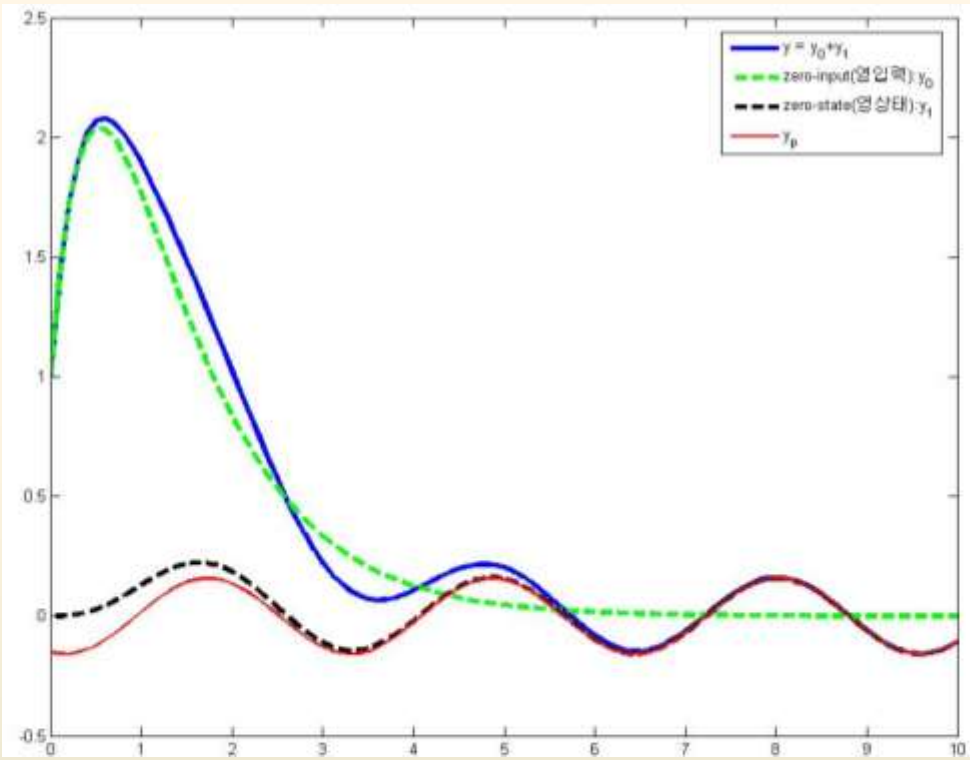
$$||$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$+$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\| \qquad \| \qquad \|$$

Zero Input solution
$u(t)=0$

$$y_0(t) = y_{0\_h}(t) + y_{0\_p}(t)$$

$$+ \qquad + \qquad +$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

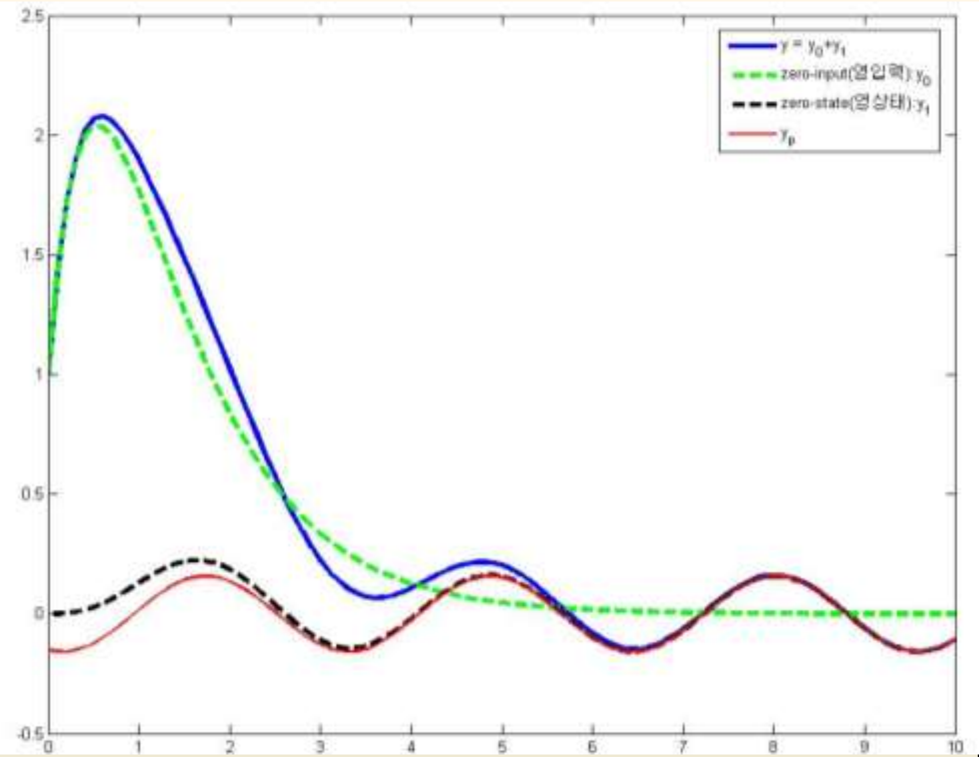
$$y_1(t) = y_{1\_h}(t) + y_{1\_p}(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\| \qquad \| \qquad \|$$

Zero Input solution
$u(t)=0$

$$y_0(t) = y_{0\_h}(t) + y_{0\_p}(t)$$

$$+ \qquad + \qquad +$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

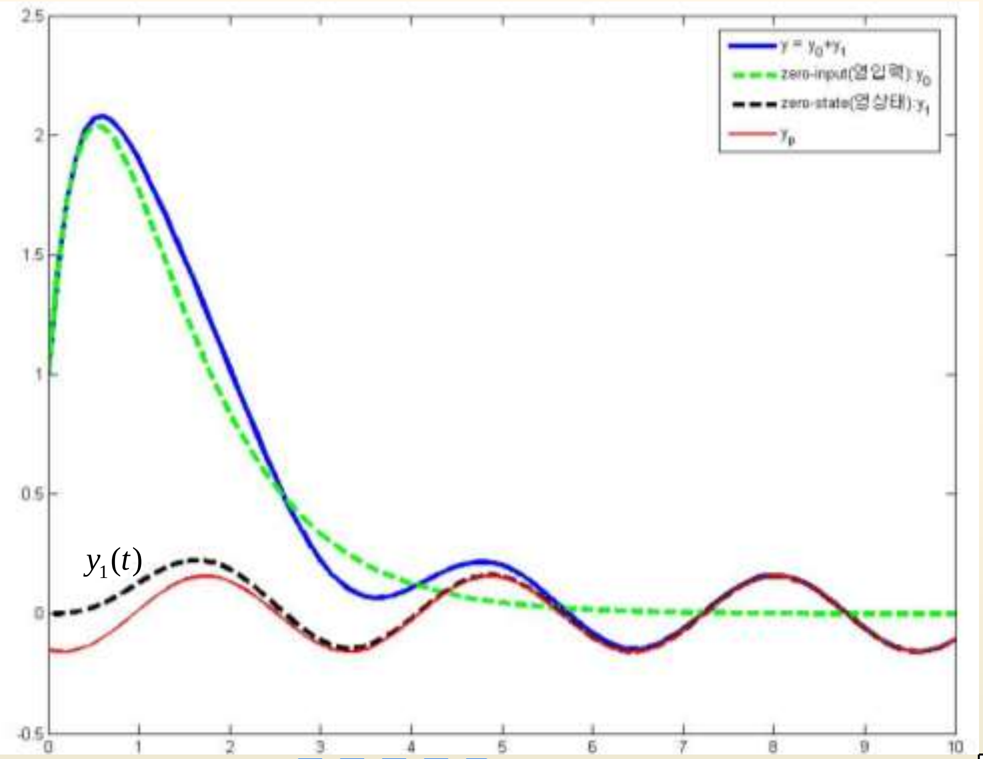
$$y_1(t) = y_{1\_h}(t) + y_{1\_p}(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\| \qquad \| \qquad \|$$

Zero Input solution
$u(t)=0$

$$y_0(t) = y_{0\_h}(t) + y_{0\_p}(t)$$

$$+ \qquad + \qquad +$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

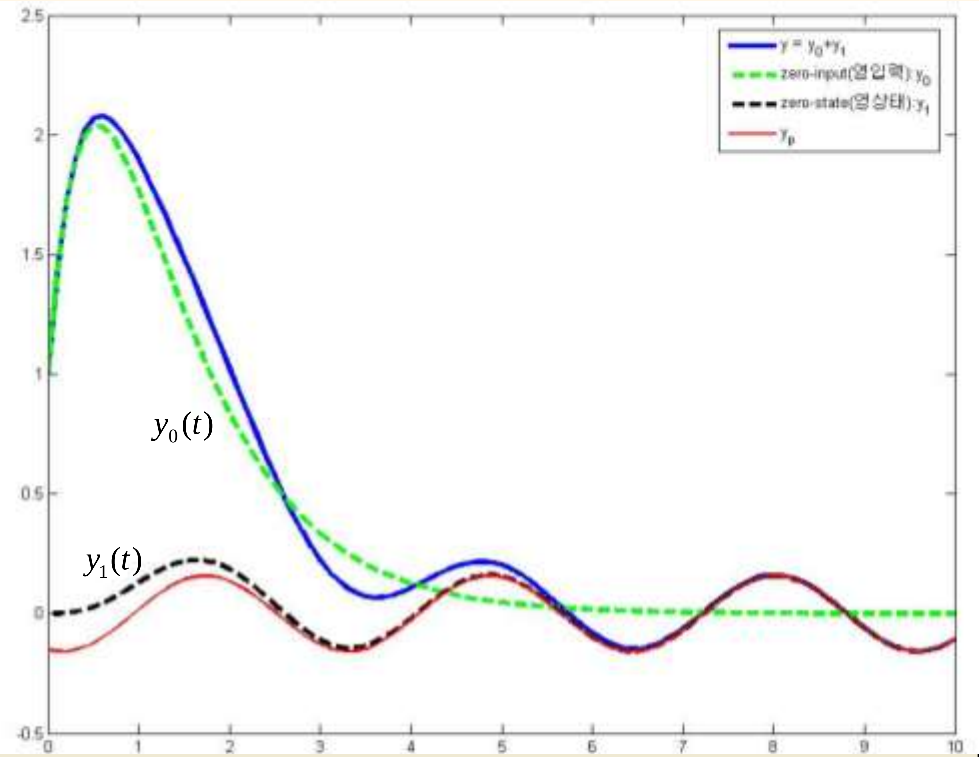
$$y_1(t) = y_{1\_h}(t) + y_{1\_p}(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$
\begin{array}{ccccc}
y(t) & = & y_h(t) & + & y_p(t) \\
\| & & \| & & \| \\
y_0(t) & = & y_{0\_h}(t) & + & y_{0\_p}(t) \\
+ & & + & & + \\
y_1(t) & = & y_{1\_h}(t) & + & y_{1\_p}(t)
\end{array}
$$

Zero Input solution
$u(t)=0$

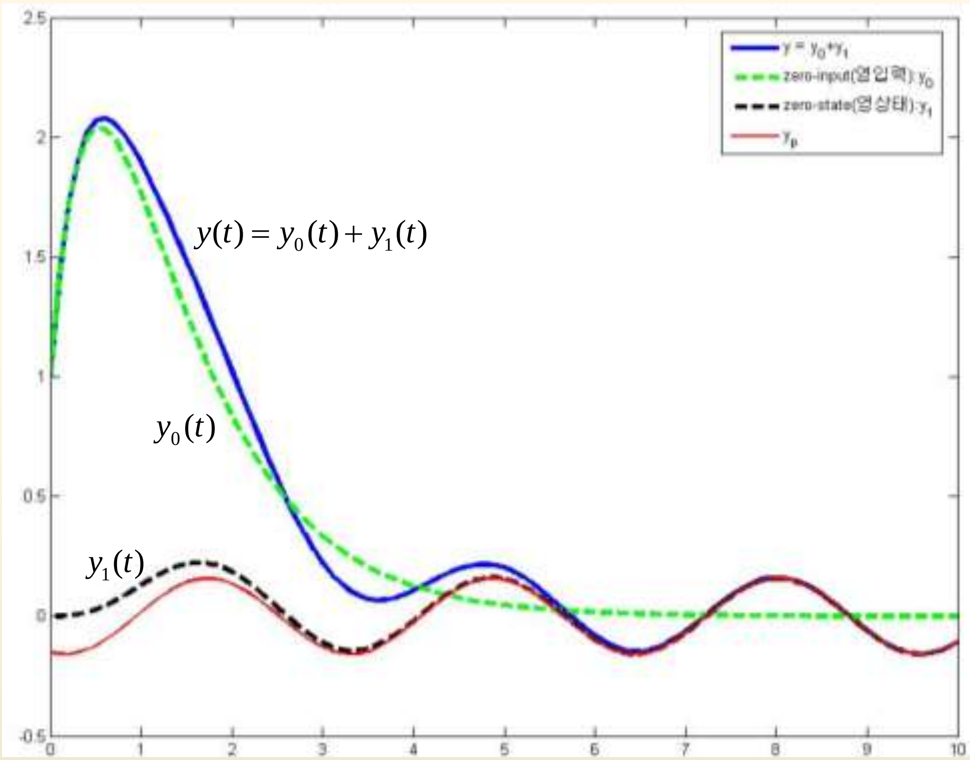
Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\| \qquad \| \qquad \|$$

Zero Input solution
$u(t)=0$

$$y_0(t) = y_{0\_h}(t) + y_{0\_p}(t)$$

$$+ \qquad + \qquad +$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

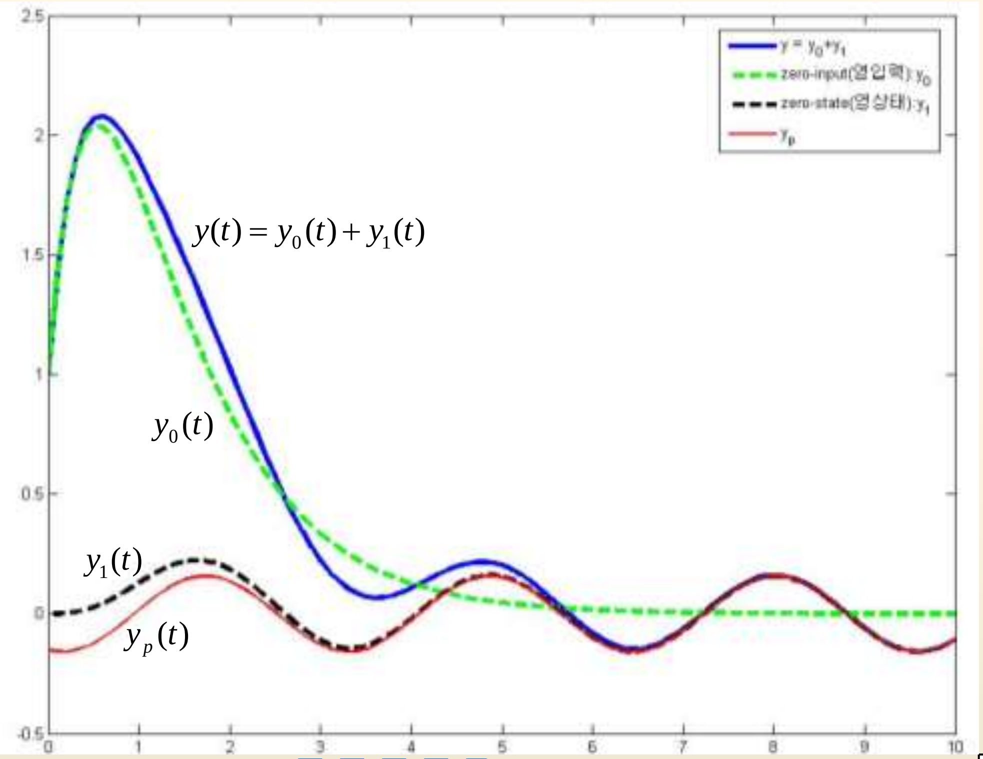
$$y_1(t) = y_{1\_h}(t) + y_{1\_p}(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$\boxed{y(t)} = y_h(t) + y_p(t)$$

$$\Vert \qquad \Vert \qquad \Vert$$

Zero Input solution
$u(t)$=0

$$\boxed{y_0(t)} = y_{0\_h}(t) + y_{0\_p}(t)$$

$$+ \qquad + \qquad +$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

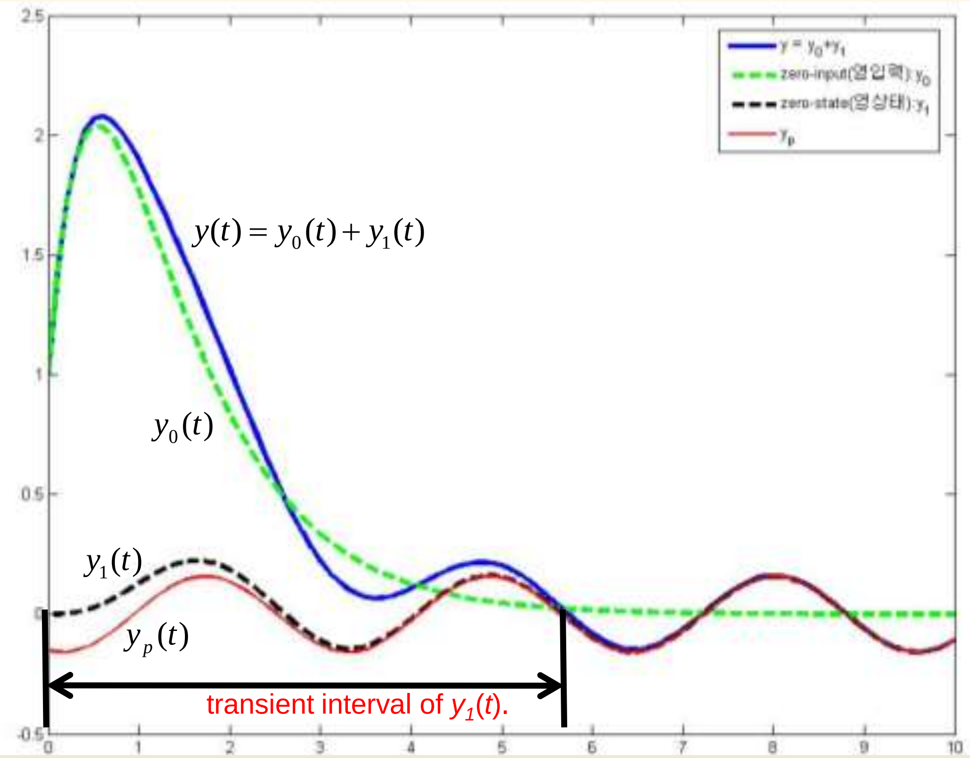
$$\boxed{y_1(t)} = y_{1\_h}(t) + y_{1\_p}(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$
\begin{array}{ccccc}
y(t) & = & y_h(t) & + & y_p(t) \\
\| & & \| & & \| \\
y_0(t) & = & y_{0\_h}(t) & + & y_{0\_p}(t) \\
+ & & + & & + \\
y_1(t) & = & y_{1\_h}(t) & + & y_{1\_p}(t)
\end{array}
$$

Zero Input solution
$u(t)=0$

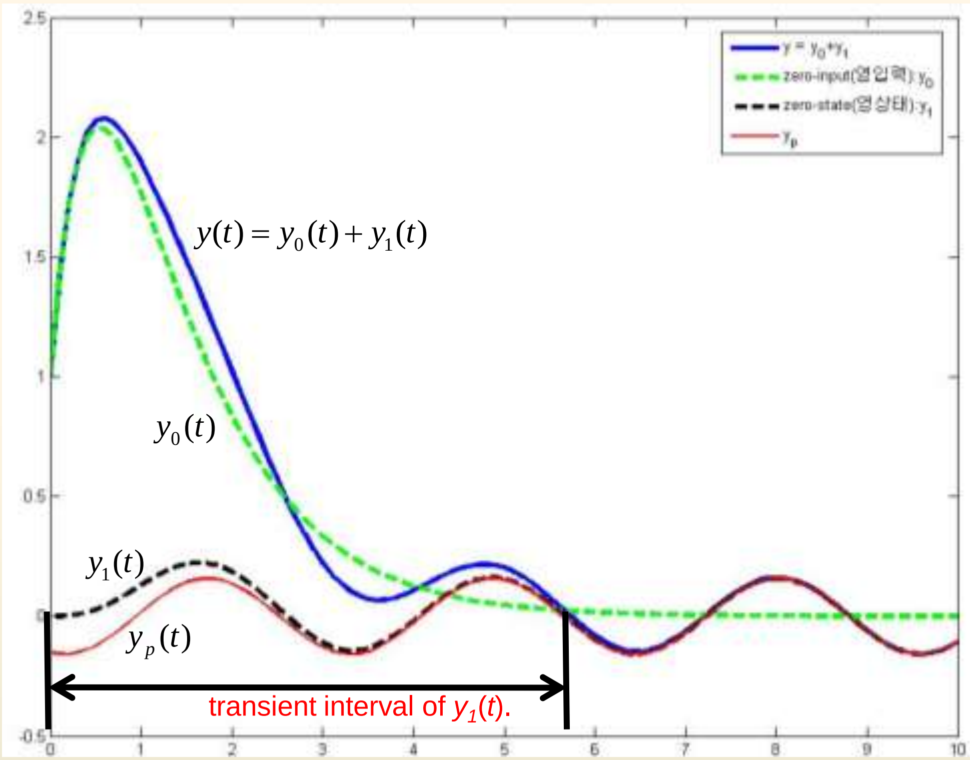
Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$
\begin{array}{ccccc}
y(t) & = & y_h(t) & + & y_p(t) \\
\| & & \| & & \| \\
y_0(t) & = & y_{0\_h}(t) & + & y_{0\_p}(t) \\
+ & & + & & + \\
y_1(t) & = & y_{1\_h}(t) & + & y_{1\_p}(t)
\end{array}
$$

Zero Input solution
$u(t)=0$

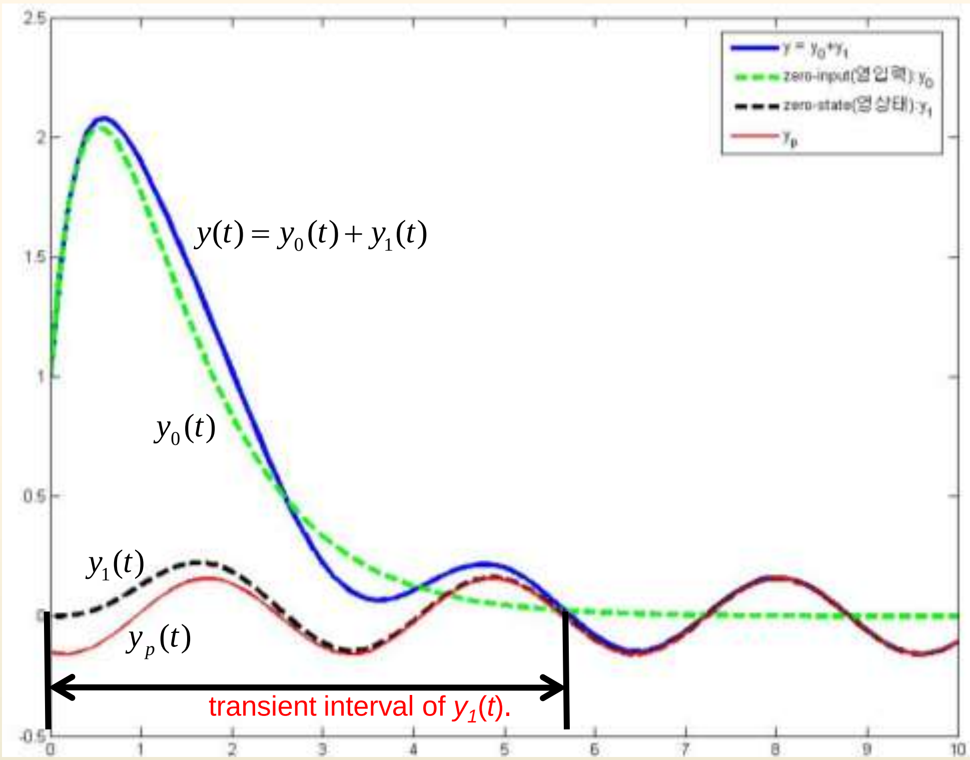
Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \boxed{\frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t}} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Comparison : graph



$$y(t) = y_h(t) + y_p(t)$$

$$\| \qquad \| \qquad \|$$

Zero Input solution
$u(t)=0$

$$y_0(t) = y_{0\_h}(t) + y_{0\_p}(t)$$

$$+ \qquad + \qquad +$$

Zero state solution
$y(0)=0, \dot{y}(0)=0$

$$y_1(t) = \boxed{y_{1\_h}(t)} + y_{1\_p}(t)$$

$$y(t) = \frac{37}{5}e^{-t} - \frac{25}{4}e^{-2t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

$$y_0(t) = 7e^{-t} - 6e^{-2t}$$

$$y_1(t) = \boxed{\frac{2}{5}e^{-t} - \frac{1}{4}e^{-2t}} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Reference slides

Fourier Transform

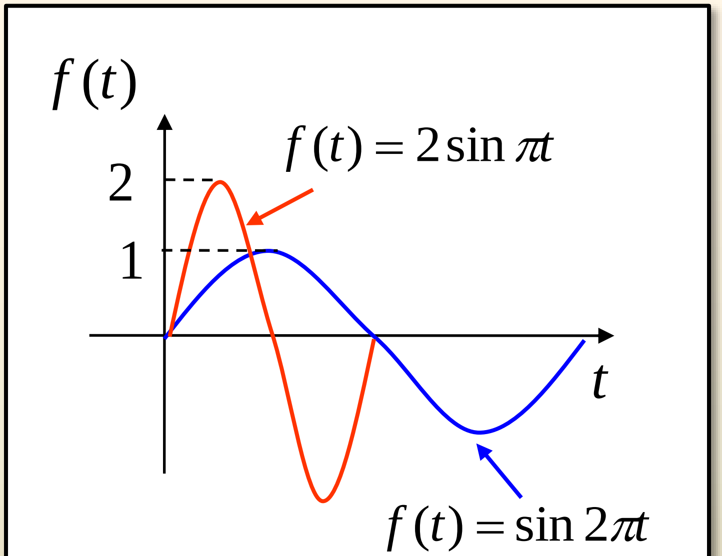
Fourier Transform

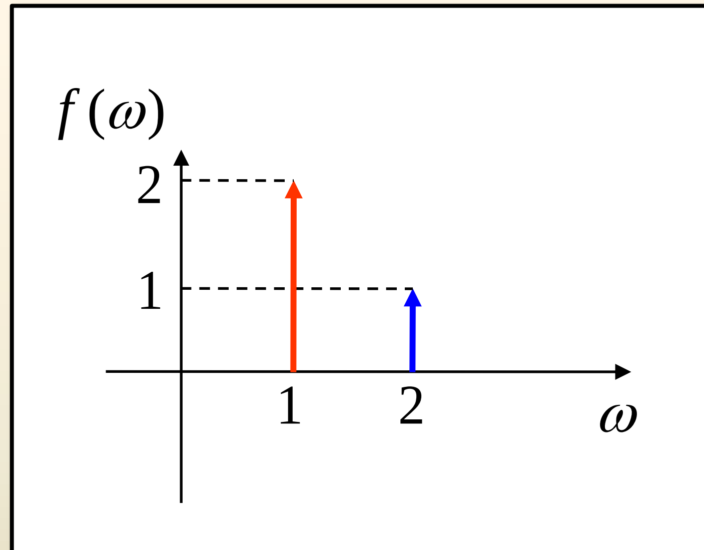
Transform between time domain and frequency domain.

ex) Interpretation of the Fourier transform

Time Domain $\xleftarrow{\text{Inverse Fourier Transform}}$ Frequency Domain

Time Domain $\xrightarrow[\text{Fourier Transform}]{}$ Frequency Domain





Frequency와 Amplitude로 표현됨

=> 시계열의 운동 복원 가능