Naval Architecture & Ocean Engineering

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Engineering Mathematics 2

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Prof. Kyu-Yeul Lee

Department of Naval Architecture and Ocean Engineering,
Seoul National University of College of Engineering

Laplace Transforms (2)
: Properties

Definition of the Laplace Transform

The Inverse Transform

Transforms of Derivatives

Translation Theorems

Additional Operational Properties

The Dirac Delta Function

Systems of Linear Differential Equations

Definition of the Laplace Transform

☑ Introduction

- **Differentiation and integration are transform**
  - **Transform a function into another function**

$$\frac{d}{dx}x^2 = 2x,\ \int x^2 dx = \frac{1}{3}x^3 + c,\ \int_0^3 x^2 dx = 9$$

- **These two transform possess the <u>linearity property</u>**

$$\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha f'(x) + \beta g'(x)$$

$$\int [\alpha f(x) + \beta g(x)]dx = \alpha \int f'(x)dx + \int \beta g'(x)dx$$

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f'(x)dx + \int_a^b \beta g'(x)dx$$

Definition of the Laplace Transform

☑ Basic Definition

$$\int_0^\infty K(s,t)\,f(t)\,dt = \lim_{b\to\infty}\int_0^b K(s,t)\,f(t)\,dt$$

If the limit exists, the integral is exist (or convergent)
If the limit does not exists, the integral does not exist (or divergent)

If we choose $K(s,t)=e^{-st}$

Definition 4.1 — **Laplace Transform**

> **Let** f **be a function for** $t \ge 0$**, then the integral**
>
> $$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$
>
> **Is said to be the Laplace transform of** f**, provided the integral converges**

Result of Laplace transform is function of s**, so we can express it as**

$$\mathscr{L}\{f(t)\} = F(s) \qquad \mathscr{L}\{g(t)\} = G(s) \qquad \mathscr{L}\{H(t)\} = h(s)$$

Definition of the Laplace Transform

☑ **Some examples**

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \left.\frac{-e^{st}}{s}\right|_0^\infty = \frac{1}{s} \qquad s > 0$$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}tdt = \left.\frac{-te^{-st}}{s}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}(\frac{1}{s}) = \frac{1}{s^2} \qquad s > 0$$

$$\mathcal{L}\{e^{-3t}\} = \int_0^\infty e^{-st}e^{-3t}tdt = \int_0^\infty e^{-(s+3)t}dt = \left.\frac{-e^{-(s+3)}}{s+3}\right|_0^\infty = \frac{1}{s+3} \qquad s > -3$$

Definition of the Laplace Transform

☑ Some examples

$$\mathscr{L}\{\sin 2t\} = \int_0^\infty e^{-st} \sin 2t dt = \left. \frac{-e^{-st} \sin 2t}{s} \right|_0^\infty + \frac{2}{s} \int_0^\infty e^{-st} \cos 2t dt$$

$$= \frac{2}{s} \int_0^\infty e^{-st} \cos 2t dt \quad (s > 0)$$

$$= \frac{2}{s} \left[\left. \frac{-e^{-st} \cos 2t}{s} \right|_0^\infty - \frac{2}{s} \boxed{\int_0^\infty e^{-st} \sin 2t dt} \right] = \frac{2}{s^2} - \frac{4}{s^2} \mathscr{L}\{\sin 2t\}$$

→ $\mathscr{L}\{\sin 2t\}$

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2 + 4}, \quad s > 0$$

Definition of the Laplace Transform

☑ $\mathscr{L}$ is a Linear Transform

$$\int_0^\infty e^{-st}[\alpha f(t)+\beta g(t)]dt=\alpha\int_0^\infty e^{-st}f(t)dt+\beta\int_0^\infty e^{-st}g(t)dt$$

Whenever both integral converges for s>c

$$\mathscr{L}\{\alpha f(t)+\beta g(t)\}=\alpha\mathscr{L}\{f(t)\}+\beta\mathscr{L}\{g(t)\}=\alpha F(s)+\beta G(s)$$

→ Linearity

For example,

$$\mathscr{L}\{1+5t\}=\mathscr{L}\{1\}+5\mathscr{L}\{t\}=\frac{1}{s}+\frac{5}{s^2}$$

$$\mathscr{L}\{4e^{-3t}-10\sin 2t\}=4\mathscr{L}\{e^{-3t}\}-10\mathscr{L}\{\sin 2t\}=\frac{4}{s+3}-\frac{20}{s^2+4}$$

Definition of the Laplace Transform

☑ $\mathscr{L}$ is a Linear Transform

Theorem 4.1 **Transforms of Some Basic Functions**

$$(a)\ \mathscr{L}\{1\} = \frac{1}{s}$$

$$(b)\ \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}},\ n = 1,2,3,\ldots$$

$$(c)\ \mathscr{L}\{e^{at}\} = \frac{1}{s-a}$$

$$(d)\ \mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$(e)\ \mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$(f)\ \mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$$

$$(g)\ \mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

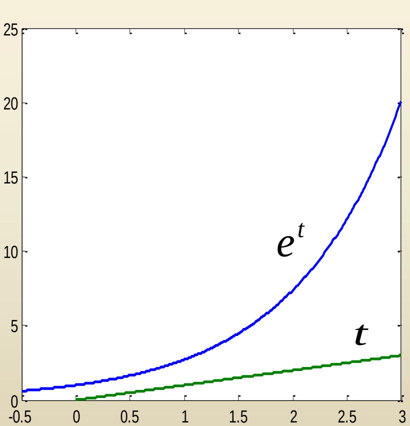
Definition of the Laplace Transform

☑Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

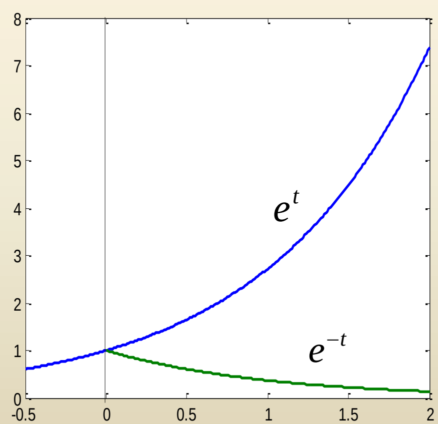
Definition 4.2 — **Exponential Order**

A function f is said to be of exponential order *c* if there exist constants *c, M*>0, and *T* >0 such that $|f(t)| \le Me^{ct}$ for all *t* >*T*

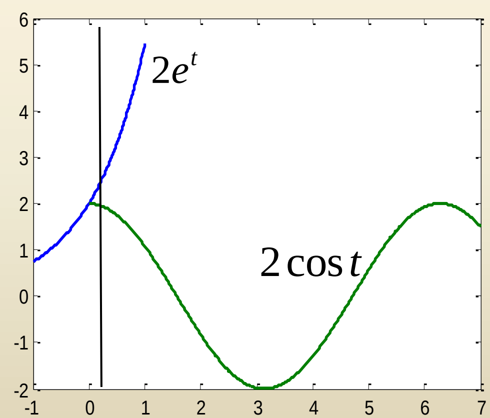
$f(t)=t, \quad f(t)=e^{-t}, \quad f(t)=2\cos t$ **Are all of exponential order *c* =1 for *t* >0**



$|t| \le e^t$



$|e^{-t}| \le e^t$



$|2\cos t| \le 2e^t$

Definition of the Laplace Transform

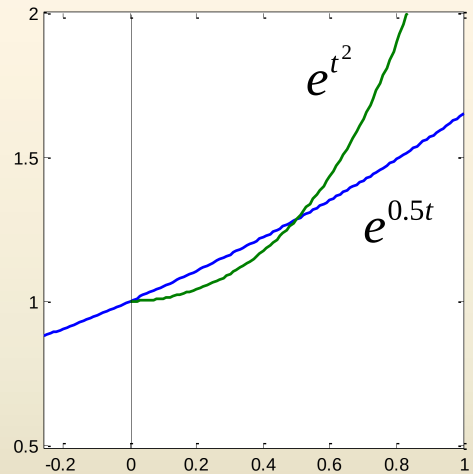
☑ Sufficient Conditions for Existence of ℒ{f(t)}

Definition 4.2 **Exponential Order**

A function f is said to be of exponential order *c* if there exist constants *c, M>0*, and *T* >0 such that $|f(t)| \le Me^{ct}$ for all *t* >*T*

$f(t) = e^{t^2}$ **Is not of exponential order**

Since it grows faster than any positive linear power of *e* for *t* > *c* >0



$$|e^{t^2}| > e^{0.5t}$$

Theorem 4.2 **Sufficient Conditions for Existence**

If $f(t)$ is piecewise continuous on the interval [0,∞) and of exponential order *c*, then ℒ{f(t)} exists for *s*>*c*.

Definition of the Laplace Transform

☑ **Example 5**

Transform of a Piecewise-Continuous Function

Evaluate

$$\mathscr{L}\{f(t)\} \text{ for } f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$$

solution)

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^3 e^{-st}(0)dt + \int_3^\infty e^{-st}(2)dt$$

$$= -\frac{2e^{-st}}{s}\Bigg|_3^\infty$$

$$= \frac{2e^{-3s}}{s}, \quad s > 0$$

Inverse Transforms

☑ The Inverse Problem

Theorem 4.3

Some Inverse T

$$(a)\ 1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$(b)\ t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1,2,3,\ldots \qquad (c)\ e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$(d)\ \sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \qquad (e)\ \cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$

$$(f)\ \sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} \qquad (g)\ \cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

Inverse Transforms

☑ $\mathscr{L}^{-1}$ is a Linear Transform

$$\mathscr{L}^{-1}\{\alpha F(s)+\beta G(s)\}=\alpha\,\mathscr{L}^{-1}\{F(s)\}+\beta\,\mathscr{L}^{-1}\{G(s)\}$$

☑ Partial Fractions

- Distinct linear factors

$$\frac{\alpha s^2+\beta s+\gamma}{(s+a)(s+b)(s+c)}=\frac{A}{(s+a)}+\frac{B}{(s+b)}+\frac{C}{(s+c)}$$

$$\mathscr{L}^{-1}\{\frac{\alpha s^2+\beta s+\gamma}{(s+a)(s+b)(s+c)}\}=A\mathscr{L}^{-1}\{\frac{1}{(s+a)}\}+B\mathscr{L}^{-1}\{\frac{1}{(s+b)}\}+C\,\mathscr{L}^{-1}\{\frac{1}{(s+c)}\}$$

Inverse Transforms

☑ Example 3
Partial Fractions and Linearity

Evaluate

$$\mathscr{L}^{-1}\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\}$$

solution)

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

$$=\frac{A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

$$s^2+6s+9$$
$$=A(s-2)(s+4)+B(s-1)(s+4)+C(s-1)(s-2)$$

set $s=1,\ s=2,\ s=-4$ respectively

$$16=A(-1)(5),\ 25=B(1)(6)\ 1=C(-5)(-6)$$

$$A=-\frac{16}{5},\ B=\frac{25}{6},\ C=\frac{1}{30}$$

Inverse Transforms

☑ Example 3
Partial Fractions and Linearity

Evaluate

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

solution)

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4}$$

$$A=-\frac{16}{5},\ B=\frac{25}{6},\ C=\frac{1}{30}$$

$$\therefore \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}$$

$$=-\frac{16}{5}\frac{1}{s-1}+\frac{25}{6}\frac{1}{s-2}+\frac{1}{30}\frac{1}{s+4}$$

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

$$=-\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

Inverse Transforms

☑ **Example 3**
Partial Fractions and Linearity

Evaluate

$$\mathscr{L}^{-1}\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\}$$

solution)

$$\mathscr{L}^{-1}\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\}$$

$$=-\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$=-\frac{16}{5}e^{t}+\frac{25}{6}e^{2t}+\frac{1}{30}e^{-4t}$$

Transforms of Derivatives

☑ Transform of a Derivative

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + s\mathscr{L}\{f(t)\}$$

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathscr{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)dt = e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)dt$$

$$= -f'(0) + s\mathscr{L}\{f'(t)\}$$

$$= s[sF(s) - f(0)] - f'(0)$$

$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}\{f'''(t)\} = s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$$

Transforms of Derivatives

☑ Transform of a Derivative

Theorem 4.4 **Transform of a Derivative**

If $f, f', \ldots, f^{(n-1)}$ **are continuous on [0,∞) and are of exponential order and if** $f^{(n)}(t)$ **is piecewise-continuous on [0, ∞), then**

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Where $F(s) = \mathscr{L}\{f(t)\}$

Transforms of Derivatives

☑ Solving linear ODEs

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t)$$

Linear differential equation with constant coefficient

$$y(0) = y_0,\ y'(0) = y_1, \ldots,\ y^{(n-1)}(0) = y_{n-1},$$

$$a_{0,} a_1, \ldots, a_n \text{ and } y_{0,} y_1, \ldots, y_{n-1} \text{ are constant}$$

⇩ **Laplace transform**

$$a_n \mathscr{L}\{\frac{d^n y}{dt^n}\} + a_{n-1} \mathscr{L}\{\frac{d^{n-1} y}{dt^{n-1}}\} + \cdots + a_0 \mathscr{L}\{y\} = \mathscr{L}\{g(t)\}$$

From theorem 4.4

$$a_n[s^n Y(s) - s^{n-1} y(0) - \ldots - y^{(n-1)}(0)]$$

$$+ a_{n-1}[s^{n-1} Y(s) - s^{n-2} y(0) - \ldots - y^{(n-2)}(0)] + \cdots + a_0 Y(s) = G(s)$$

Algebraic equation in Y(s)

Transforms of Derivatives

☑ Solving linear ODEs

$$a_n[s^nY(s)-s^{n-1}y(0)-\ldots-y^{(n-1)}(0)]$$
$$+a_{n-1}[s^{n-1}Y(s)-s^{n-2}y(0)-\ldots-y^{(n-2)}(0)]+\cdots+a_0Y(s)=G(s)$$

$$P(s)Y(s)=Q(s)+G(s),\qquad P(s)=a_ns^n+a_{n-1}s^{n-1}+\cdots+a_0$$

※ P(s) is the same as the nth degree auxiliary polynomial in Section 3.3(Homogeneous Linear equations with constant coefficient)

$$Y(s)=\frac{Q(s)}{P(s)}+\frac{G(s)}{P(s)}$$

$$y(t)=\mathscr{L}^{-1}\{Y(s)\}=\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}+\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$

Transforms of Derivatives

☑ Solving linear ODEs

Find unknown y(t) that satisfies a DE and initial condition

→ Apply Laplace transform $\mathcal{L}$ →

Transformed DE becomes an algebraic equation in Y(s)

↓

Solve transformed equation for Y(s)

← Apply inverse transform $\mathcal{L}^{-1}$ ←

Solution y(t) of original IVP

Transforms of Derivatives

☑ Example 4 Solving a First-Order IVP

Use the Laplace transform to solve the IVP

$$\frac{dy}{dt} + 3y = 13\sin 2t, \quad y(0) = 6$$

Solution)

take Laplace transform of each member of DE:

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + 3\mathscr{L}\{y\} = 13\mathscr{L}\{\sin 2t\}$$

$$sY(s) - y(0) + 3Y(s) = 13\frac{2}{s^2+4}, \qquad (s+3)Y(s) = 6 + \frac{26}{s^2+4}$$

$$Y(s) = \frac{6}{(s+3)} + \frac{26}{(s+3)(s^2+4)} = \frac{6s^2+50}{(s+3)(s^2+4)}$$

Transforms of Derivatives

☑ Example 4 **Solving a First-Order IVP**

Use the Laplace transform to solve the IVP

$$\frac{dy}{dt} + 3y = 13\sin 2t, \quad y(0) = 6$$

$$Y(s) = \frac{6s^2 + 50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs + C}{s^2 + 4}$$

$$6s^2 + 50 = A(s^2+4) + (Bs + C)(s+3)$$

$$= (A+B)s^2 + (3B + C)s + 4A + 3C$$

$$\begin{cases} A + B = 6 \\ 3B + C = 0 \\ 4A + 3C = 50 \end{cases} \qquad A = 8,\ B = -2,\ C = 6 \qquad \therefore Y(s) = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

Transforms of Derivatives

☑ Example 4　Solving a First-Order IVP

Use the Laplace transform to solve the IVP

$$\frac{dy}{dt} + 3y = 13\sin 2t, \quad y(0) = 6$$

$$Y(s) = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 8\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$\therefore\ y(t) = 8e^{3t} - 2\cos 2t + 3\sin 2t$$

Transforms of Derivatives

☑ Example 5 Solving a Second-Order IVP

solve

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1,\ y'(0) = 5$$

Solution)

take Laplace transform of each member of DE:

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

Transforms of Derivatives

☑ Example 5 Solving a Second-Order IVP

solve

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1,\ y'(0) = 5$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{(s^2 - 3s + 2)} + \frac{1}{(s^2 - 3s + 2)(s+4)} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}$$

$$= \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

Transforms of Derivatives

☑ Example 5 Solving a Second-Order IVP

solve

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1,\ y'(0) = 5$$

$$Y(s) = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$s^2 + 6s + 9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

Substituting s=1, s=2, s=-4 respectively,

$$16 = A(-1)(5), \qquad 4 + 12 + 9 = B(1)(6), \qquad 16 - 24 + 9 = C(-5)(-6)$$

$$A = -\frac{16}{5}, \qquad B = \frac{25}{6}, \qquad C = \frac{1}{30}$$

Transforms of Derivatives

☑ Example 5 Solving a Second-Order IVP

solve

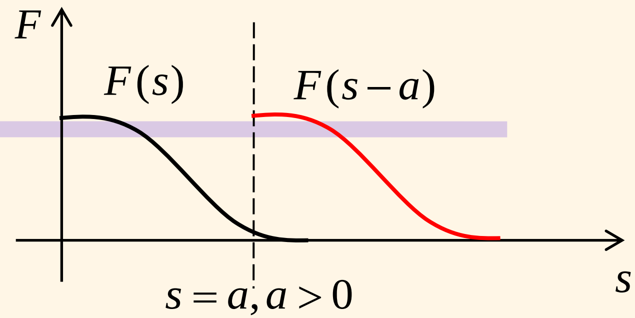
$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \; y'(0) = 5$$

$$Y(s) = -\frac{16}{15}\frac{1}{s-1} + \frac{25}{6}\frac{1}{s-2} + \frac{1}{30}\frac{1}{s+4}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = -\frac{16}{15}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\therefore y(t) = -\frac{16}{15}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$

Translation on the s-axis



☑ Translation on the s-axis

Theorem 4.6 **First Translation Theorem**

If $\mathscr{L}\{f(t)\} = F(s)$ **and a is any real number, then**

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a)$$

Proof)

$$\mathscr{L}\{e^{at} f(t)\} = \int_0^{\infty} e^{-st} e^{at} f(t) dt = \int_0^{\infty} e^{-(s-a)t} f(t) dt = F(s-a)$$

- **Inverse form of Theorem 4.6**

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)\big|_{s\to s-a}\} = e^{at} f(t)$$

Translation on the s-axis

☑ Example 1 Using the First Translation Theorem

Evaluate

$$(a)\ \mathscr{L}\{e^{5t}t^3\} \quad \text{and} \quad (b)\ \mathscr{L}\{e^{-2t}\cos 4t\}$$

Solution)

$$(a)\ \mathscr{L}\{e^{5t}t^3\} = \mathscr{L}\{t^3\}_{s\to s-5} = \left.\frac{3!}{s^4}\right|_{s\to s-5} = \frac{6}{(s-5)^4}$$

$$(b)\ \mathscr{L}\{e^{-2t}\cos 4t\} = \mathscr{L}\{\cos 4t\}_{s\to s-(-2)} = \left.\frac{s}{s^2+16}\right|_{s\to s+2} = \frac{s+2}{(s+2)^2+16}$$

Translation on the s-axis

☑ Example 2

Partial Fractions and Completing the Square

Evaluate

$$(a)\ \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} \quad \text{and} \quad (b)\ \mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}$$

Solution)

$$(a)\ \frac{2s+5}{(s-3)^2}=\frac{A}{s-3}+\frac{B}{(s-3)^2}$$

Multiply $(s-3)^2$ **both side**

$$2s+5=A(s-3)+B$$

Substitute s=3 gives B=11 and s=4 gives A=2

$$\therefore \mathcal{L}\left\{\frac{2s+5}{(s-3)^2}\right\}=2\mathcal{L}\left\{\frac{1}{s-3}\right\}+11\mathcal{L}\left\{\frac{1}{(s-3)^2}\right\}$$

Translation on the s-axis

☑ Example 2

Partial Fractions and Completing the Square

Evaluate

$$(a)\ \mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} \quad \text{and} \quad (b)\ \mathscr{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}$$

$$\mathscr{L}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathscr{L}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= 2e^{3t} + 11\mathscr{L}\left\{\left.\frac{1}{s^2}\right|_{s\to s-3}\right\} = 2e^{3t} + 11e^{3t}t$$

Translation on the s-axis

☑ Example 2

Partial Fractions and Completing the Square

Evaluate

$$(a)\ \mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} \quad \text{and} \quad (b)\ \mathscr{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}$$

$$(b)\ \frac{s/2+5/3}{s^2+4s+6}=\frac{s/2+5/3}{(s+2)^2+2}=\frac{(1/2)(s+2)+2/3}{(s+2)^2+2}$$

$$=\frac{1}{2}\frac{s+2}{(s+2)^2+2}+\frac{2}{3}\frac{1}{(s+2)^2+2}$$

Translation on the s-axis

☑ Example 2

Partial Fractions and Completing the Square

Evaluate

$$(a)\ \mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} \quad \text{and} \quad (b)\ \mathscr{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} + \frac{2}{3}\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$= \frac{1}{2}\mathscr{L}^{-1}\left\{\left.\frac{s}{s^2+2}\right|_{s\to s+2}\right\} + \frac{2}{3\sqrt{2}}\mathscr{L}^{-1}\left\{\left.\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right|_{s\to s+2}\right\}$$

$$= \frac{1}{2}e^{-2t}\cos\sqrt{2}t + \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

Translation on the s-axis

☑ Example 3 Initial-Value Problem

Solve

$$y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$$

---

solution) $\mathscr{L}\{y''\} - 6\mathscr{L}\{y'\} + 9\mathscr{L}\{y\} = \mathscr{L}\{t^2 e^{3t}\}$

$$s^2 Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y(s) = 2s + 17 - 12 + \frac{2}{(s-3)^3}$$

$$(s-3)^2 Y(s) = 2s + 5 + \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}$$

Translation on the s-axis

☑ Example 3 Initial-Value Problem

Solve

$$y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$$

$$Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5} = \frac{2(s-3)+11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$= \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 2\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \frac{2}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\}$$

$$= 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4 e^{3t}$$

Translation on the s-axis

☑ Example 4 Initial-Value Problem

Solve

$$y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$$

---

solution) $\mathscr{L}\{y''\} + 4\mathscr{L}\{y'\} + 6\mathscr{L}\{y\} = \mathscr{L}\{1\} + \mathscr{L}\{e^{-t}\}$

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 6Y(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$(s^2 + 4s + 6)Y(s) = \frac{2s+1}{s(s+1)}$$

$$Y(s) = \frac{2s+1}{s(s+1)(s^2+4s+6)} = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{s/2 + 5/3}{s^2+4s+6}$$

Translation on the s-axis

☑ Example 4 **Initial-Value Problem**

Solve

$$y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 0, \quad y'(0) = 0$$

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{s/2 + 5/3}{s^2 + 4s + 6} = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{\frac{1}{2}(s+2) + \frac{2}{3}}{(s+2)^2 + 2}$$

$$y(t) = \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+2}\right\} - \frac{2}{3\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$

$$= \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t$$

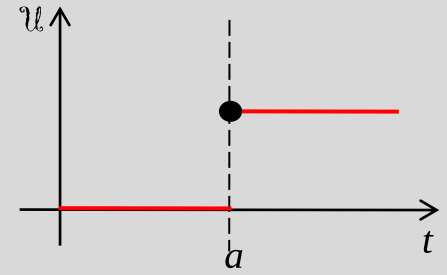
Translation on the t-axis

☑ Unit step Function

Definition 4.3 — **Unit Step Function**

The unit step function $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a)=\begin{cases}0, & 0\le t<a\\ 1, & t\ge a\end{cases}$$



$$f(t)=\begin{cases}g(t), & 0\le t<a\\ h(t), & t\ge a\end{cases} \quad = \quad f(t)=g(t)-g(t)\mathcal{U}(t-a)+h(t)\mathcal{U}(t-a)$$

$$f(t)=\begin{cases}0 & 0\le t<a\\ g(t), & a\le t<b\\ 0, & t\ge b\end{cases} \quad = \quad f(t)=g(t)\{\mathcal{U}(t-a)-\mathcal{U}(t-b)\}$$

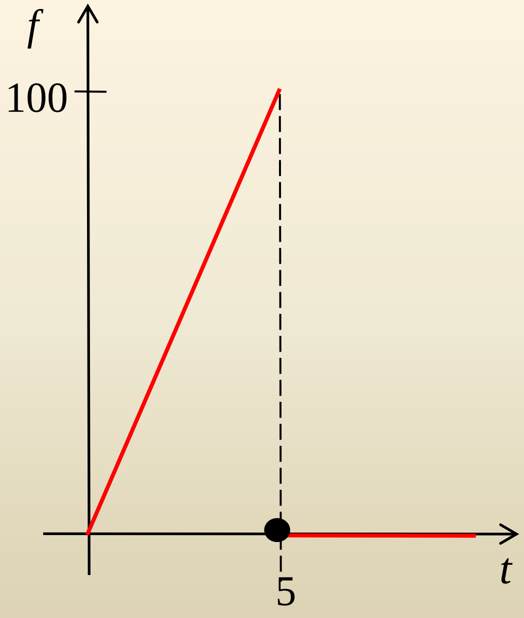
Translation on the t-axis

☑ Example 5 A Piecewise-Defined Function

Express $f(t)=\begin{cases}20t, & 0\le t\le 5\\ 0, & t\ge 5\end{cases}$ **In terms of unit step functions. graph**

Solution)

$$f(t)=20t-20t\,\mathcal{U}(t-5)$$



Translation on the t-axis

☑ Translation on the t-axis

Theorem 4.7 **Second Translation Theorem**

If $\mathscr{L}\{f(t)\}=F(s)$ **and a >0, then**

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\}=e^{-as}F(s)$$

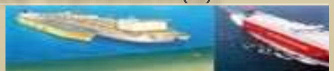
Proof)

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\}=\int_0^a e^{-st}f(t-a)\underbrace{\mathscr{U}(t-a)}_{0\text{ for }0\le t<a}dt+\int_a^\infty e^{-st}f(t-a)\underbrace{\mathscr{U}(t-a)}_{1\text{ for }t\ge a}dt$$

$$=\int_a^\infty e^{-st}f(t-a)dt$$

$$=\int_a^\infty e^{-s(v+a)}f(v)dv$$

$$=e^{-as}\int_0^\infty e^{-sv}f(v)dv=e^{-as}\mathscr{L}\{f(t)\}=e^{-as}F(s)$$



Translation on the t-axis

☑Translation on the t-axis

Theorem 4.7 **Second Translation Theorem**

If $\mathscr{L}\{f(t)\} = F(s)$ **and a >0, then**

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$$

If $f(t-a) = 1$, then $F(s) = 1/s$. So

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}$$

☑Inverse form of theorem 4.7

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Translation on the t-axis

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a) \cdots (15)$$

☑ Example 6 **Using Formula (15)**

Evaluate $(a)\ \mathscr{L}^{-1}\left\{\dfrac{1}{s-4}e^{-2s}\right\}$ *and* $(b)\ \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+9}e^{-\pi s/2}\right\}$

Solution) **(a)** $a = 2, F(s) = \dfrac{1}{(s-4)}, \mathscr{L}^{-1}\{F(s)\} = f(t) = e^{4t}$

$$\mathscr{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}\mathcal{U}(t-2)$$

(b) $a = \dfrac{\pi}{2}, F(s) = \dfrac{s}{s^2+9}, \mathscr{L}^{-1}\{F(s)\} = f(t) = \cos 3t$

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} = \cos 3(t-\frac{\pi}{2})\mathcal{U}(t-\frac{\pi}{2})$$

Translation on the t-axis

☑ Alternate Form of Theorem 4.7

Theorem 4.7 **Second Translation Theorem**

If $\mathscr{L}\{f(t)\} = F(s)$ **and a >0, then**

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s)$$

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = \int_a^\infty e^{-st}g(t)dt = \int_a^\infty e^{-s(u+a)}g(u+a)du$$

$$\therefore \mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

"Alternate Form of Theorem 4.7"

Translation on the t-axis

☑ Example 7

Second Translation Theorem-Alternative form

Evaluate $\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi)\}.$

Solution)

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}\cdots(16)$$

$$g(t) = \cos t,\ a = \pi, \text{then, } g(t+\pi) = \cos(t+\pi) = -\cos t$$

$$\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi)\} = -e^{\pi s}\mathscr{L}\{\cos t\} = -\frac{s}{s^2+1}e^{-\pi s}$$

Translation on the t-axis

☑ Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t), \quad y(0) = 5, \ \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

Solution)

We can rewrite $f(t)$ **as** $f(t) = 3\cos t\,\mathcal{U}(t-\pi)$

$$\mathscr{L}\{y'\} + \mathscr{L}\{y\} = 3\mathscr{L}\{\cos t\,\mathcal{U}(t-\pi)\}$$

$$sY(s) - y(0) + Y(s) = -3\frac{s}{s^2+1}e^{-\pi s}$$

$$(s+1)Y(s) = 5 - 3\frac{s}{s^2+1}e^{-\pi s}$$

$$Y(s) = \frac{5}{(s+1)} - 3\frac{s}{(s+1)(s^2+1)}e^{-\pi s}$$

Translation on the t-axis

☑ Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t), \quad y(0) = 5, \ \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

$$Y(s) = \frac{5}{(s+1)} - 3\frac{s}{(s+1)(s^2+1)}e^{-\pi s}$$

$$\frac{s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} = \frac{(A+B)s^2 + (B+C)s + A + C}{(s+1)(s^2+1)}$$

$$A+B=0, \ B+C=1, \ A+C=0 \qquad A=-\frac{1}{2}, \ B=\frac{1}{2}, \ C=\frac{1}{2}$$

$$\frac{s}{(s+1)(s^2+1)} = -\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{s+1}{s^2+1}$$

Translation on the t-axis

☑ Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t),\quad y(0) = 5,\ \text{where}\ f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

$$Y(s) = \frac{5}{(s+1)} - \frac{3}{2}\left[-\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s}\right]$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s+1}e^{-\pi s}\right\} = e^{-(t-\pi)}\mathcal{U}(t-\pi),\quad \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\} = \sin(t-\pi)\mathcal{U}(t-\pi)$$

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\} = \cos(t-\pi)\mathcal{U}(t-\pi)$$

Translation on the t-axis

☑ Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t), \quad y(0) = 5, \text{ where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

$$y(t) = 5\mathscr{L}^{-1}\left\{\frac{1}{(s+1)}\right\} - \frac{3}{2}[-\mathscr{L}^{-1}\left\{\frac{1}{s+1}e^{-\pi s}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\} + \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\}]$$

$$= 5e^{-t} + \frac{3}{2}e^{-(t-\pi)}\mathcal{U}(t-\pi) - \frac{3}{2}\sin(t-\pi)\mathcal{U}(t-\pi) - \frac{3}{2}\cos(t-\pi)\mathcal{U}(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin(\pi - t) - \cos(\pi - t)\right]\mathcal{U}(t-\pi)$$

$$= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin t + \cos t\right]\mathcal{U}(t-\pi)$$

Translation on the t-axis

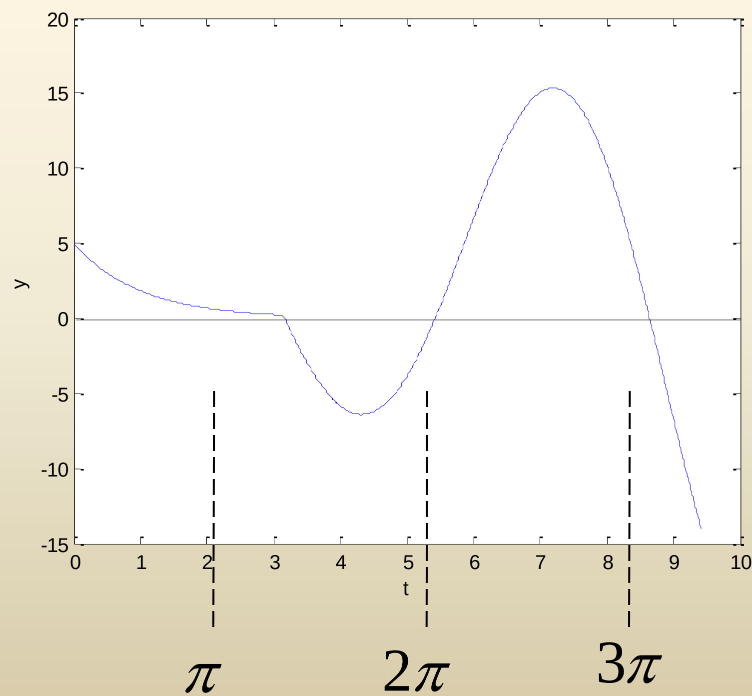
☑ Example 8 An Initial-Value Problem

Solve

$$y' + y = f(t), \quad y(0) = 5, \ \text{where } f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases}$$

$$y(t) = 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin t + \cos t\right]\mathcal{U}(t-\pi)$$

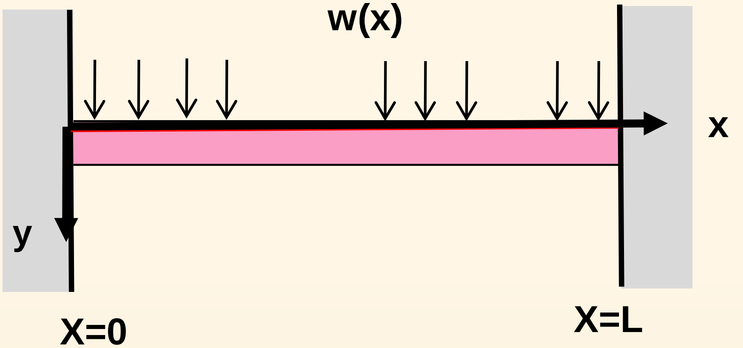
$$y(t) = \begin{cases} 5e^{-t} & , 0 \le t < \pi \\ 5e^{-t} + \frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\sin t + \frac{3}{2}\cos t & , t \ge \pi \end{cases}$$



Translation on the t-axis

☑ **Beams**

$$EI\frac{d^4y}{dx^4} = w(x)$$



When w(x) is piecewise defined,
it is useful to use Laplace transform.

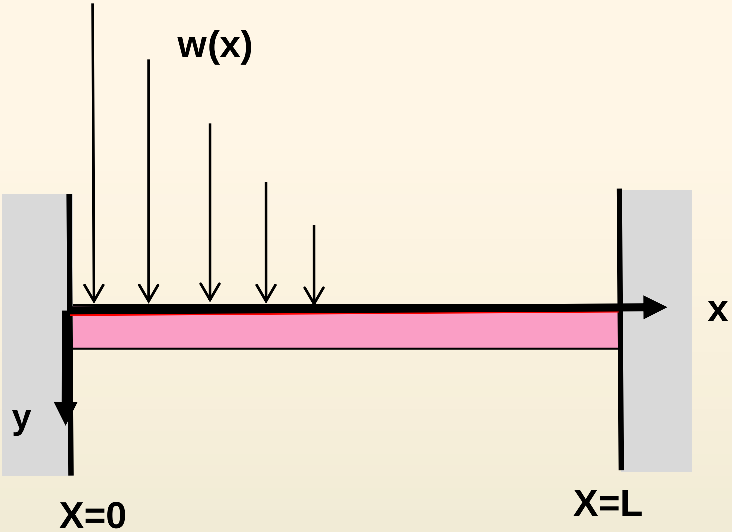
But, in order to use Laplace transform,
we have to assume that y(x) and w(x) are defined on (0,∞)

And this is a <u>boundary-value problem</u>.

Translation on the t-axis

☑ Example 9 An Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0(1-\dfrac{2}{L}x), & 0<x<\dfrac{L}{2} \\ 0, & \dfrac{L}{2}<x<L, \end{cases}$$

Where w_0 is constant

Solution) Boundary conditions are

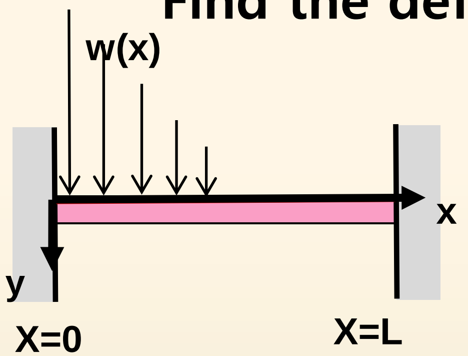
$$y(0)=0, y'(0)=0, y(L)=0, y'(L)=0$$

$$w(x)=w_0(1-\frac{2}{L}x)-w_0(1-\frac{2}{L}x)\mathcal{U}(x-\frac{L}{2})=\frac{2w_0}{L}\left[\frac{L}{2}-x+(x-\frac{L}{2})\mathcal{U}(x-\frac{L}{2})\right]$$

Translation on the t-axis

☑ Example 9 An Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0(1-\frac{2}{L}x), & 0<x<\frac{L}{2} \\ 0, & \frac{L}{2}<x<L, \end{cases}$$

w_0 is constant

B.C.: $y(0)=0, y'(0)=0, y(L)=0, y'(L)=0$

$$w(x)=\frac{2w_0}{L}\left[\frac{L}{2}-x+(x-\frac{L}{2})\mathcal{U}(x-\frac{L}{2})\right]$$

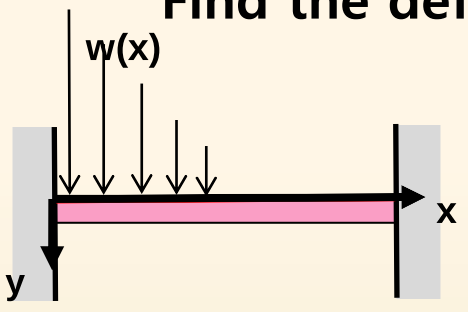
$$EI(s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0))=\frac{2w_0}{L}\left[\frac{L/2}{s}-\frac{1}{s^2}+\frac{1}{s^2}e^{-Ls/2}\right]$$

$$s^4Y(s)-sy''(0)-y'''(0)=\frac{2w_0}{EIL}\left[\frac{L/2}{s}-\frac{1}{s^2}+\frac{1}{s^2}e^{-Ls/2}\right]$$

Translation on the t-axis

☑ Example 9 An Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x) = \begin{cases} w_0(1 - \frac{2}{L}x), & 0 < x < \frac{L}{2} \\ 0, & \frac{L}{2} < x < L, \end{cases}$$

w_0 is constant

B.C.: $y(0) = 0, y'(0) = 0, y(L) = 0, y'(L) = 0$

$$s^4 Y(s) - sy''(0) - y'''(0) = \frac{2w_0}{EIL}\left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right]$$

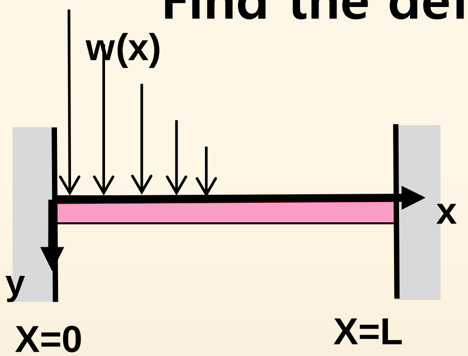
Let $c_1 = y''(0), c_2 = y'''(0)$ **then,**

$$Y(s) = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{2w_0}{EIL}\left[\frac{L/2}{s^5} - \frac{1}{s^6} + \frac{1}{s^6}e^{-Ls/2}\right]$$

Translation on the t-axis

☑ Example 9 An Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0\left(1-\frac{2}{L}x\right), & 0<x<\frac{L}{2} \\ 0, & \frac{L}{2}<x<L, \end{cases}$$

w_0 is constant

B.C.: $y(0)=0,\ y'(0)=0,\ y(L)=0,\ y'(L)=0$

$$Y(s)=\frac{c_1}{s^3}+\frac{c_2}{s^4}+\frac{2w_0}{EIL}\left[\frac{L/2}{s^5}-\frac{1}{s^6}+\frac{1}{s^6}e^{-Ls/2}\right]$$

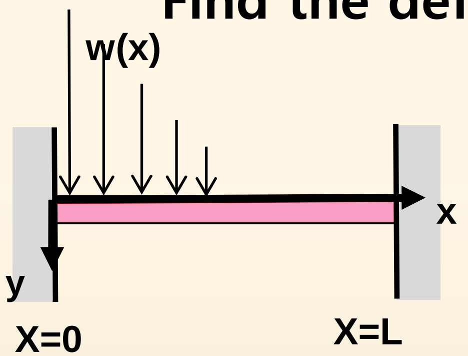
$$y(x)=\frac{c_1}{2!}\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\}+\frac{c_2}{3!}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$+\frac{2w_0}{EIL}\left[\frac{L}{2\cdot 4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\}-\frac{1}{5!}\mathscr{L}^{-1}\left\{\frac{5!}{s^6}\right\}+\frac{1}{5!}\mathscr{L}^{-1}\left\{\frac{5!}{s^6}e^{-Ls/2}\right\}\right]$$

Translation on the t-axis

☑ Example 9 An Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0(1-\frac{2}{L}x), & 0<x<\frac{L}{2} \\ 0, & \frac{L}{2}<x<L, \end{cases}$$

w_0 is constant

B.C.: $y(0)=0, y'(0)=0, y(L)=0, y'(L)=0$

$$y(x)=\frac{c_1}{2}x^2+\frac{c_2}{6}x^3+\frac{w_0}{60EIL}\left[\frac{5L}{2}x^4-x^5+\left(x-\frac{L}{2}\right)^5\mathcal{U}\left(x-\frac{L}{2}\right)\right]$$

According to Boundary condition $y(L)=0, y'(L)=0$

$$c_1\frac{L^2}{2}+c_2\frac{L^3}{6}+\frac{49w_0L^4}{1920EI}=0, \quad c_1L+c_2\frac{L^2}{2}+\frac{85w_0L^4}{960EI}=0$$

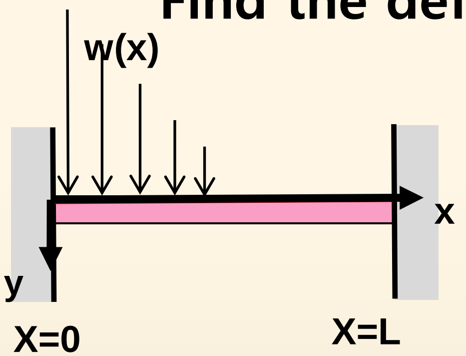
$$c_1=\frac{23w_0L^2}{960EI}$$

$$c_2=\frac{-9w_0L^2}{40EI}$$

Translation on the t-axis

☑ Example 9 An Boundary-Value Problem

Find the deflection of the beam when the load is given by



$$w(x)=\begin{cases} w_0\left(1-\dfrac{2}{L}x\right), & 0<x<\dfrac{L}{2} \\ 0, & \dfrac{L}{2}<x<L, \end{cases}$$

w_0 is constant

B.C.: $y(0)=0,\ y'(0)=0,\ y(L)=0,\ y'(L)=0$

$$y(x)=\frac{c_1}{2}x^2+\frac{c_2}{6}x^3+\frac{w_0}{60EIL}\left[\frac{5L}{2}x^4-x^5+\left(x-\frac{L}{2}\right)^5\mathcal{U}\left(x-\frac{L}{2}\right)\right]$$

$$c_1=\frac{23w_0L^2}{960EI},\quad c_2=\frac{-9w_0L^2}{40EI}$$

$$\therefore y(x)=\frac{23w_0L^2}{1920EI}x^2-\frac{3w_0L}{80EI}x^3+\frac{w_0}{60EIL}\left[\frac{5L}{2}x^4-x^5+\left(x-\frac{L}{2}\right)^5\mathcal{U}\left(x-\frac{L}{2}\right)\right]$$

Derivatives of Transform

☑ Multiplying a Function by t^n

$$\frac{d}{ds}F(s)=\frac{d}{ds}\int_0^{\infty}e^{-st}f(t)dt=\int_0^{\infty}\frac{\partial}{\partial s}[e^{-st}f(t)]dt=-\int_0^{\infty}e^{-st}tf(t)dt=-\mathscr{L}\{tf(t)\}$$

$$\mathscr{L}\{tf(t)\}=-\frac{d}{ds}F(s)$$

Similarly, $\mathscr{L}\{t^2 f(t)\}=\mathscr{L}\{t\cdot tf(t)\}=-\mathscr{L}\{tf(t)\}$

$$=-\frac{d}{ds}\left(-\frac{d}{ds}\mathscr{L}\{f(t)\}\right)=\frac{d^2}{ds^2}\mathscr{L}\{f(t)\}$$

The proceeding two cases suggest the general result

Theorem 4.8 **Derivatives of Transform**

If $\mathscr{L}\{f(t)\}=F(s)$ **and n=1,2,3,…, then**

$$\mathscr{L}\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}F(s)$$

Derivatives of Transform

☑ Example 1 Using Theorem 4.8

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Evaluate $\mathscr{L}\{t \sin kt\}$

Solution)

$$f(t) = \sin kt,\ F(s) = \frac{k}{s^2 + k^2},\ n = 1$$

$$\mathscr{L}\{t \sin kt\} = -\frac{d}{ds}\mathscr{L}\{\sin kt\} = -\frac{d}{ds}\left(\frac{k}{s^2 + k^2}\right) = \frac{2ks}{(s^2 + k^2)^2}$$

Derivatives of Transform

☑ Example 2 An Initial-Value Problem

Solve

$$x'' + 16x = \cos 4t, \quad x(0) = 0, \; x'(0) = 1$$

Solution)

$$\mathscr{L}\{x''\} + 16\mathscr{L}\{x\} = \mathscr{L}\{\cos 4t\}$$

$$s^2 X(s) - sx(0) - x'(0) + 16sX(s) - 16x(0) = \frac{s}{s^2+16}$$

$$(s^2+16s)X(s) = 1 + \frac{s}{s^2+16}$$

$$X(s) = \frac{1}{s^2+16s} + \frac{s}{(s^2+16)^2}$$

Derivatives of Transform

☑ Example 2 An Initial-Value Problem

Solve

$$x'' + 16x = \cos 4t, \quad x(0) = 0, \ x'(0) = 1$$

$$X(s) = \frac{1}{s^2 + 16s} + \frac{s}{(s^2+16)^2}$$

In Example 1,

$$\mathscr{L}\{t \sin kt\} = \frac{2ks}{(s^2+k^2)^2} \ \Rightarrow \ \mathscr{L}^{-1}\left\{\frac{2ks}{(s^2+k^2)^2}\right\} = t \sin kt$$

$$x(t) = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{4}{s^2+16}\right\} + \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{8s}{(s^2+16)^2}\right\} = \frac{1}{4}\sin 4t + \frac{1}{8} t \sin 4t$$

Transform of Integrals

☑ Convolution

$$f * g = \int_0^t f(\tau) g(t-\tau) d\tau$$

Ref : Properties of convolution

Theorem 4.9 **Convolution Theorem**

If $f(t)$ **and** $g(t)$ **are piecewise continuous on [0,∞) and of exponential order, then**

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$$

Proof)

Additional Operational Properties

☑ Example 3 Transform of a Convolution

Evaluate

$$\mathscr{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right\}$$

Solution)

$$f(t)=e^{t},\ \ g(t)=\sin t$$

$$\mathscr{L}\left\{\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right\}=F(s)G(s)=\mathscr{L}\{e^{t}\}\mathscr{L}\{\sin t\}$$

$$=\frac{1}{s-1}\cdot\frac{1}{s^2+1}=\frac{1}{(s-1)(s^2+1)}$$

Transform of Integrals

Theorem 4.9 **Convolution Theorem**

If $f(t)$ and $g(t)$ are piecewise continuous on [0,∞) and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$$

☑ Inverse Form of Theorem 4.9

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g$$

Transform of Integrals

☑ Example 4 Inverse Transform as a Convolution

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$

Solution) Let $F(s)=G(s)=\dfrac{1}{(s^2+k^2)^2}$

$$f(t)=g(t)=\frac{1}{k}L^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\frac{1}{k}\sin kt$$

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}=\frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)d\tau$$

$$=\frac{1}{k^2}\int_0^t \frac{1}{2}[\cos(k\tau-kt+k\tau)-\cos(k\tau+kt-k\tau)]d\tau$$

$$=\frac{1}{2k^2}\int_0^t[\cos(2k\tau-kt)-\cos(kt)]d\tau$$

Transform of Integrals

☑ Example 4 Inverse Transform as a Convolution

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+k^2)^2}\right\}$

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^2}\int_0^t [\cos(2k\tau - kt) - \cos(kt)]d\tau$$

$$= \frac{1}{2k^2}\left[\frac{1}{2k}\sin k(2\tau - t) - \tau\cos kt\right]_0^t$$

$$= \frac{\sin kt - kt\cos kt}{2k^3}$$

Transform of Integrals

☑ Transform of an integral

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$ **Convolution Theorem**

When $g(t)=1,\ G(s)=1/s$ **So,**

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \cdots (7)$$

Inverse of (7)

$$\int_0^\infty f(\tau)d\tau = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} \cdots (8)$$

By (8)

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau\, d\tau = 1-\cos t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos\tau)\, d\tau = t-\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (\tau-\sin\tau)\, d\tau = \frac{t^2}{2}-1+\cos t$$ **And so on.**

Transform of Integrals

☑ Volterra integral Equation

Unknown

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)\,d\tau$$

: Volterra integral equation for f(t)

Additional Operational Properties

☑ Example 5 An Integral Equation

Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau$ for $f(t)$

Solution) $h(t-\tau) = e^{t-\tau}$, $h(t) = e^t$

$$\mathscr{L}\{f(t)\} = 3\mathscr{L}\{t^2\} - \mathscr{L}\{e^{-t}\} - \mathscr{L}\left\{\int_0^t f(\tau)e^{t-\tau}d\tau\right\}$$

$$F(s) = 3\frac{2!}{s^3} - \frac{1}{s+1} - F(s)\cdot\frac{1}{s-1}$$

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

Additional Operational Properties

☑ Example 5 An Integral Equation

Solve $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau) e^{t-\tau} d\tau$ for $f(t)$

Solution)

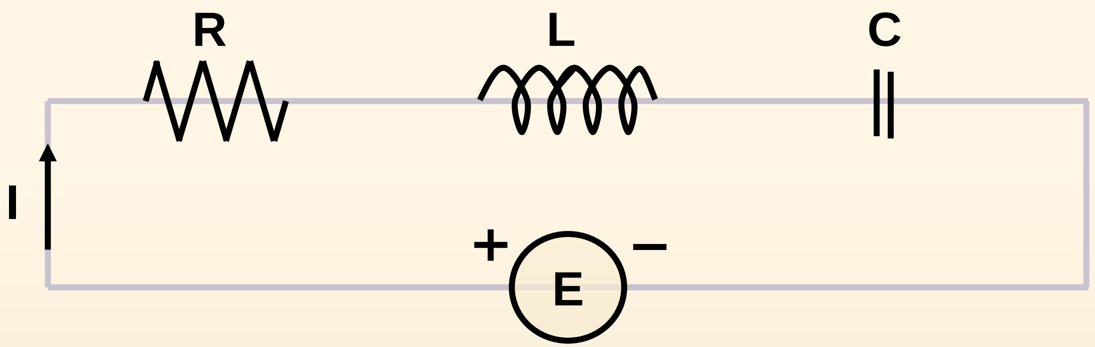
$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$f(t) = 3\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 3t^2 - t^3 + 1 - 2e^{-t}$$

Transform of Integrals

☑ **Series Circuits**

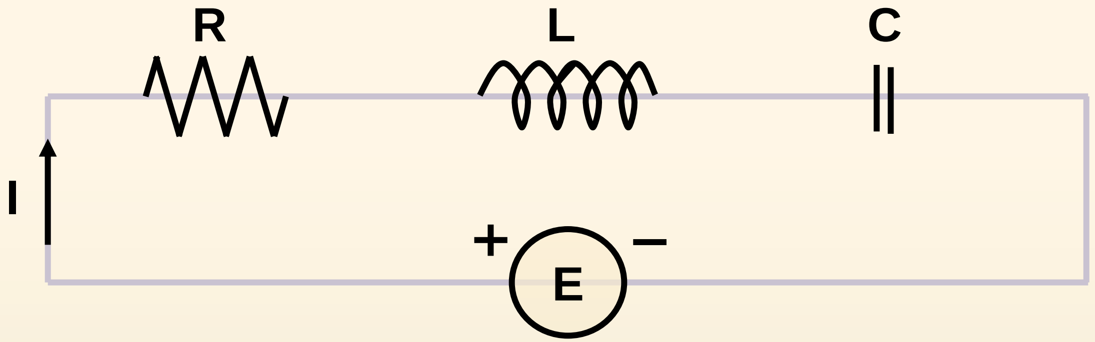


Integrodifferential equation

In chapter 3.8 we differentiate equation to eliminate integral, but by Using Laplace transform, we don't have to differentiate equation

Transform of Integrals

☑ **Series Circuits**



$$L\frac{di}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$

Integrodifferential equation

In chapter 3.8 we differentiate equation to eliminate integral, but by Using Laplace transform, we don't have to differentiate equation

Transform of Integrals

☑ Example 6 An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit
When L=0.1[h], R=2[Ω], C=0.1[f], $i(0)=0$
and the impressed voltage is

$$E(t) = 120t - 120t\,\mathcal{U}(t-1)$$

Solution)

$$0.1\frac{di}{dt} + 2i + 10\int_0^t i(\tau)\,d\tau = 120t - 120t\,\mathcal{U}(t-1)$$

$$0.1\mathscr{L}\left\{\frac{di}{dt}\right\} + 2\mathscr{L}\{i\} + 10\mathscr{L}\left\{\int_0^t i(\tau)\,d\tau\right\} = 120\mathscr{L}\{t\} - 120\mathscr{L}\{t\,\mathcal{U}(t-1)\}$$

$$0.1sI(s) + 2I(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right]$$

Transform of Integrals

☑ Example 6 An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit When L=0.1[h], R=2[Ω], C=0.1[f], $i(0)=0$ and the impressed voltage is

$$E(t)=120t-120t\,\mathcal{U}(t-1)$$

$$0.1sI(s)+2I(s)+10\frac{I(s)}{s}=120\left[\frac{1}{s^2}-\frac{1}{s^2}e^{-s}-\frac{1}{s}e^{-s}\right]$$

$$(s^2+20s+100)I(s)=1200\left[\frac{1}{s}-\frac{1}{s}e^{-s}-e^{-s}\right]$$

$$I(s)=1200\left[\frac{1}{s(s+10)^2}-\frac{1}{s(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\right]$$

Transform of Integrals

☑ Example 6 An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit When L=0.1[h], R=2[Ω], C=0.1[f], $i(0)=0$ and the impressed voltage is

$$E(t)=120t-120t\,\mathcal{U}(t-1)$$

$$I(s)=1200\left[\frac{1}{s(s+10)^2}-\frac{1}{s(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\right]$$

By partial fraction,

$$I(s)=1200\left[\frac{1/100}{s}-\frac{1/100}{s+10}-\frac{1/10}{(s+10)^2}\right.$$

$$\left.-\frac{1/100}{s}e^{-s}+\frac{1/100}{s+10}e^{-s}+\frac{1/10}{(s+10)^2}e^{-s}-\frac{1}{(s+10)^2}e^{-s}\right]$$

Transform of Integrals

☑ Example 6 An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop LRC-circuit When L=0.1[h], R=2[Ω], C=0.1[f], $i(0)=0$ and the impressed voltage is

$$E(t) = 120t - 120t\,\mathcal{U}(t-1)$$

$$I(s) = 1200\left[\frac{1/100}{s} - \frac{1/100}{s+10} - \frac{1/10}{(s+10)^2} - \frac{1/100}{s}e^{-s} + \frac{1/100}{s+10}e^{-s} + \frac{1/10}{(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right]$$

$$i(t) = 12[1 - \mathcal{U}(t-1)] - 12[e^{-10t} - e^{-10(t-1)}\mathcal{U}(t-1)] - 120te^{-10t} - 1080(t-1)e^{-10(t-1)}\mathcal{U}(t-1)$$

Transform of a Periodic Function

☑ Periodic function

$$f(t+T) = f(t) \text{ (period: } T,\ T>0)$$

Theorem 4.10 **Transform of a Periodic Function**

If $f(t)$ is piecewise continuous on [0,∞), of exponential order, and periodic with period *T*, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

Proof)

$$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

Let *t=u+T*

$$\int_T^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-s(u+T)} f(u+T)\,du = e^{-sT}\int_0^\infty e^{-su} f(u)\,du = e^{-sT}\mathscr{L}\{f(t)\}$$

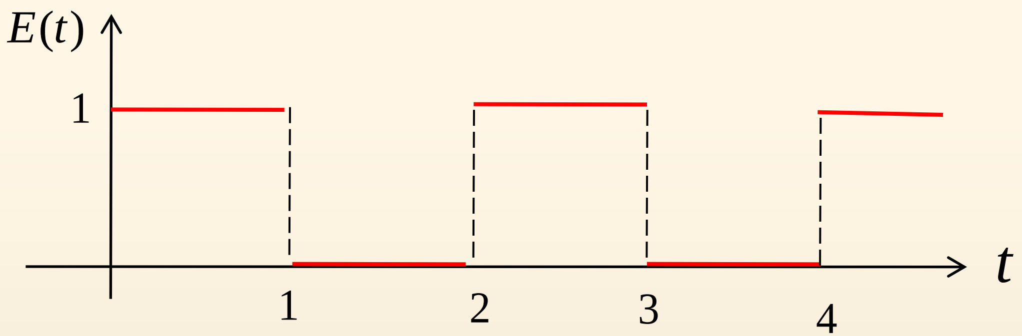
Therefore $$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + e^{-st}\mathscr{L}\{f(t)\}$$

Solving for $\mathscr{L}\{f(t)\}$ Proves the theorem

Transform of a Periodic Function

☑ Example 7 Transform of a Periodic Function

Find the Laplace transform of the periodic function shown in Figure



('Square wave')

Solution) $T = 2, \quad E(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases} \quad \text{on } 0 \le t < 2$

$$\mathscr{L}\{E(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} E(t)\,dt = \frac{1}{1-e^{-2s}} \int_0^1 e^{-st}\,dt$$

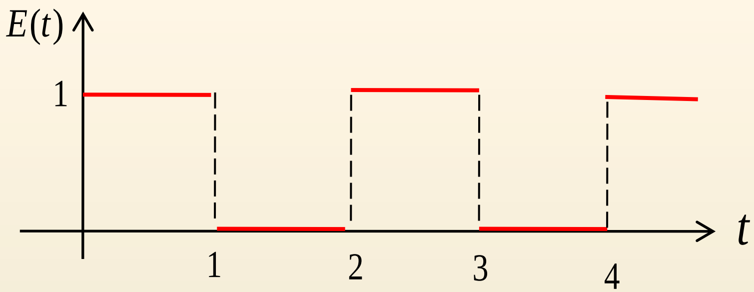
$$= \frac{1}{1-e^{-2s}} \frac{1-e^{-s}}{s} = \frac{1}{s(1+e^{-s})}$$

Transform of a Periodic Function

☑ Example 8 A Periodic Impressed Voltage

$$L\frac{di}{dt}+Ri=E(t)$$

Determine the $i(t)$ **When** $i(0)=0$ and $E(t)$ is shown in figure



('Square wave')

Solution)

$$L\mathscr{L}\left\{\frac{di}{dt}\right\}+R\mathscr{L}\{i\}=\mathscr{L}\{E(t)\}$$

In example 7

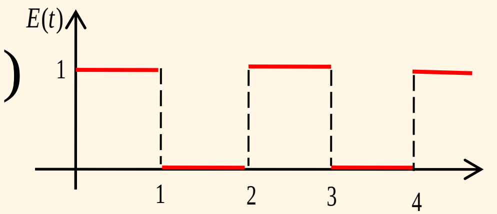
$$\mathscr{L}\{E(t)\}=\frac{1}{s(1+e^{-s})}$$

$$LsI(s)+RI(s)=\frac{1}{s(1+e^{-s})} \quad or \quad I(s)=\frac{1/L}{s(s+R/L)}\cdot\frac{1}{1+e^{-s}}$$

Transform of a Periodic Function

☑ Example 8 A Periodic Impressed Voltage

$$L\frac{di}{dt}+Ri=E(t) \qquad E(t)=\begin{cases}1, & 0\le t<1\\ 0, & 1\le t<2\end{cases} \qquad E(t+2)=E(t)$$



Determine the $i(t)$ **When** $i(0)=0$

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots \xrightarrow{x=e^{-s}} \frac{1}{1+e^{-s}}=1-e^{-s}+e^{-2s}-e^{-3s}+\cdots$$

$$\frac{1}{s(s+R/L)}=\frac{L/R}{s}-\frac{L/R}{s+R/L}$$

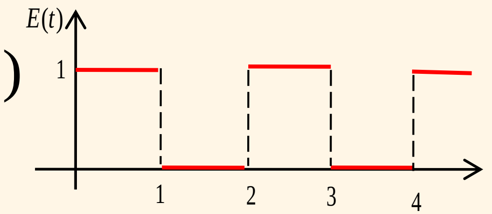
$$I(s)=\frac{1/L}{s(s+R/L)}\cdot\frac{1}{1+e^{-s}}=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+R/L}\right)(1-e^{-s}+e^{-2s}-e^{-3s}+\cdots)$$

$$=\frac{1}{R}\left(\frac{1}{s}-\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-\frac{e^{-3s}}{s}+\cdots\right)-\frac{1}{R}\left(\frac{1}{s+R/L}-\frac{e^{-s}}{s+R/L}+\frac{e^{-2s}}{s+R/L}-\cdots\right)$$

Transform of a Periodic Function

☑ Example 8 A Periodic Impressed Voltage

$$L\frac{di}{dt}+Ri=E(t) \qquad E(t)=\begin{cases}1, & 0\le t<1\\ 0, & 1\le t<2\end{cases} \qquad E(t+2)=E(t)$$

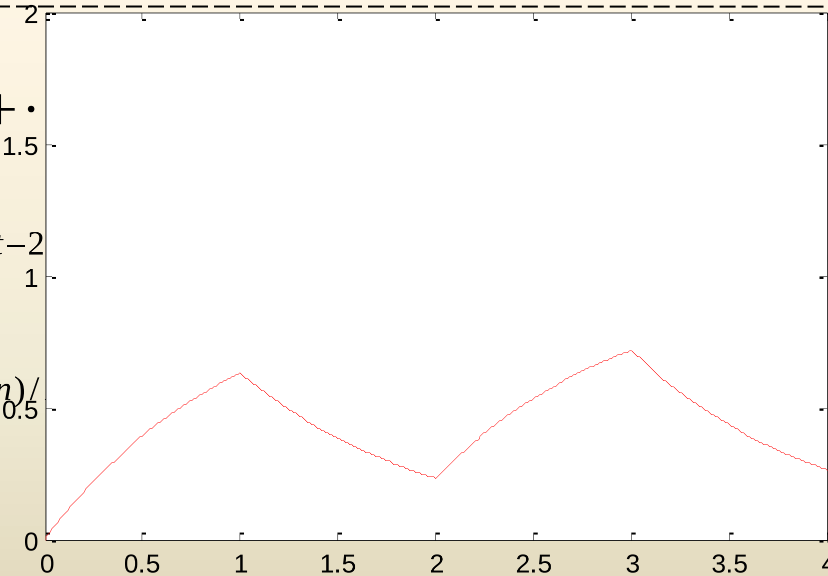


Determine the $i(t)$ **When** $i(0)=0$

$$i(t)=\frac{1}{R}\left(1-\mathcal{U}(t-1)+\mathcal{U}(t-2)-\mathcal{U}(t-3)+\cdot\right.$$

$$-\frac{1}{R}\left(e^{-Rt/L}-e^{-R(t-1)/L}\mathcal{U}(t-1)+e^{-R(t-2}\right.$$

$$i(t)=\frac{1}{R}\left(1-e^{-Rt/L}\right)+\frac{1}{R}\sum_{n=1}^{\infty}(-1)^n\left(1-e^{-R(t-n)/}\right.$$



when $R=1, L=1$ *and* $0\le t<4$

$$i(t)=1-e^{-t}-\left(1-e^{-(t-1)}\right)\mathcal{U}(t-1)+\left(1-e^{-(t-2)}\right)\mathcal{U}(t-2)-\left(1-e^{-(t-3)}\right)\mathcal{U}(t-3)$$

The Dirac Delta Function

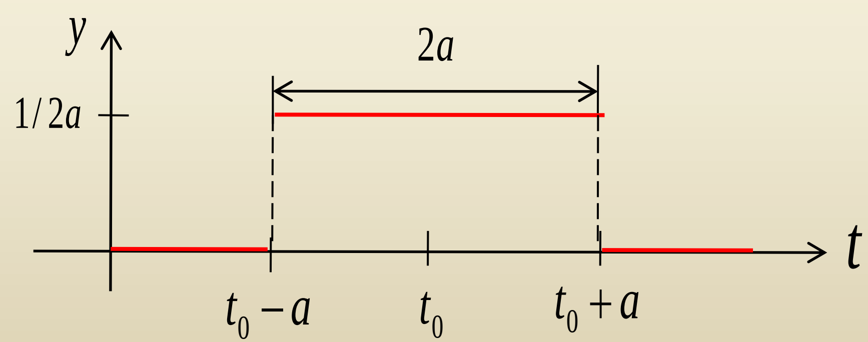
☑ Unit Impulse

- External force of large magnitude that acts only for a very short period of time

$$\delta_a(t-t_0)=\begin{cases} 0, & 0\le t<t_0-a \\ \dfrac{1}{2a}, & t_0-a\le t<t_0+a \\ 0, & t\ge t_0+a \end{cases}$$

$$\int_0^{\infty}\delta_a(t-t_0)\,dt=1$$

->'Unit' impulse



For small *a*, $\delta_a(t-t_0)$ have large magnitude.

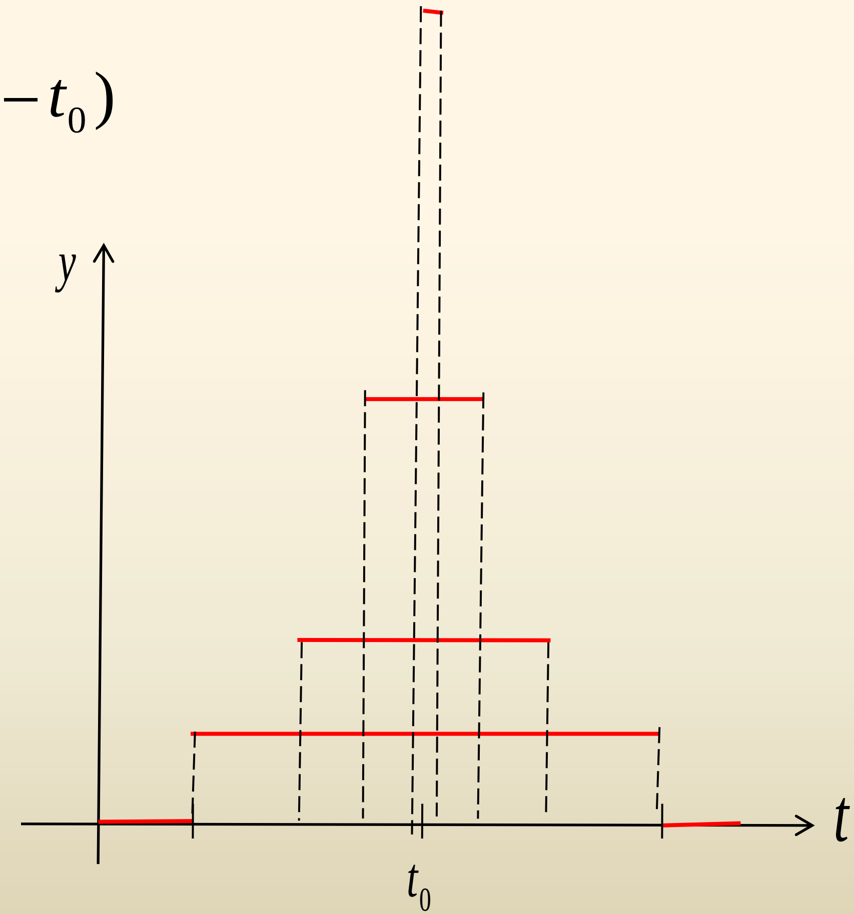
The Dirac Delta Function

☑ The Dirac Delta Function

$$\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$$

$$(i)\ \delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases}$$

$$(ii)\int_0^x \delta(t-t_0)\,dt=1$$



The Dirac Delta Function

☑ The Dirac Delta Function

$$\delta(t-t_0)=\lim_{a\to 0}\delta_a(t-t_0)$$

Theorem 4.11 — **Transform of the Dirac Delta Function**

For $t_0 > 0$ $$\mathscr{L}\{\delta(t-t_0)\}=e^{-st_0}$$

Proof) $$\delta_a(t-t_0)=\frac{1}{2a}[\mathscr{U}(t-(t_0-a))-\mathscr{U}(t-(t_0+a))]$$

$$\mathscr{L}\{\delta_a(t-t_0)\}=\frac{1}{2a}\left[\frac{e^{-s(t_0-a)}}{s}-\frac{e^{-s(t_0+a)}}{s}\right]=e^{-st_0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right)$$

$$\mathscr{L}\{\delta(t-t_0)\}=\lim_{a\to 0}\mathscr{L}\{\delta_a(t-t_0)\}=\lim_{a\to 0}e^{-st_0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right)=e^{-st_0}$$

When $t_0 = 0$, $$\mathscr{L}\{\delta(t)\}=1$$

The Dirac Delta Function

☑ Example 1 Two Initial-Value Problems

Solve $y'' + y = 4\delta(t - 2\pi)$ **subject to**

$(a)\ y(0) = 1,\ \ y'(0) = 0 \qquad\qquad (b)\ y(0) = 0,\ \ y'(0) = 0$

$$\mathscr{L}\{y''\} + \mathscr{L}\{y\} = 4\mathscr{L}\{\delta(t - 2\pi)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + sY(s) - y(0) = 4e^{-2\pi s}$$

$$(a)\quad s^2 Y(s) - s + Y(s) = 4e^{-2\pi s} \ \ or\ \ Y(s) = \frac{s}{s^2 + 1} + \frac{4e^{-2\pi s}}{s^2 + 1}$$

$$y(t) = \cos t + 4\sin(t - 2\pi)\mathscr{U}(t - 2\pi)$$

$$y(t) = \begin{cases} \cos t & 0 \le t < 2\pi \\ \cos t + 4\sin t & t \ge 2\pi \end{cases}$$

The Dirac Delta Function

☑ Example 1 Two Initial-Value Problems

Solve $y'' + y = 4\delta(t - 2\pi)$ **subject to**

$(a)\ y(0) = 1,\ \ y'(0) = 0 \qquad\qquad (b)\ y(0) = 0,\ \ y'(0) = 0$

$$\mathscr{L}\{y''\} + \mathscr{L}\{y\} = 4\mathscr{L}\{\delta(t - 2\pi)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + sY(s) - y(0) = 4e^{-2\pi s}$$

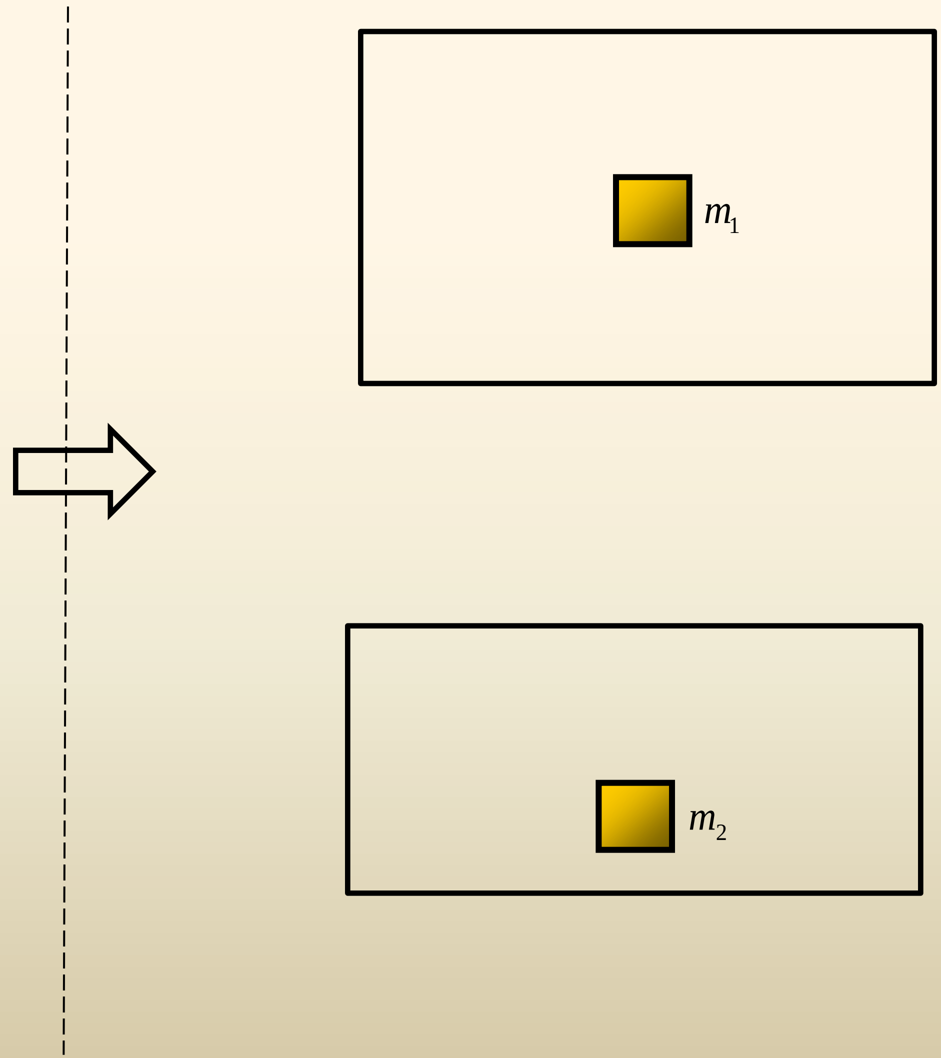
$$(b) \quad s^2 Y(s) + Y(s) = 4e^{-2\pi s} \ \ or\ \ Y(s) = \frac{4e^{-2\pi s}}{s^2 + 1}$$

$$y(t) = 4\sin(t - 2\pi)\mathscr{U}(t - 2\pi)$$

$$y(t) = \begin{cases} 0 & 0 \le t < 2\pi \\ 4\sin t & t \ge 2\pi \end{cases}$$

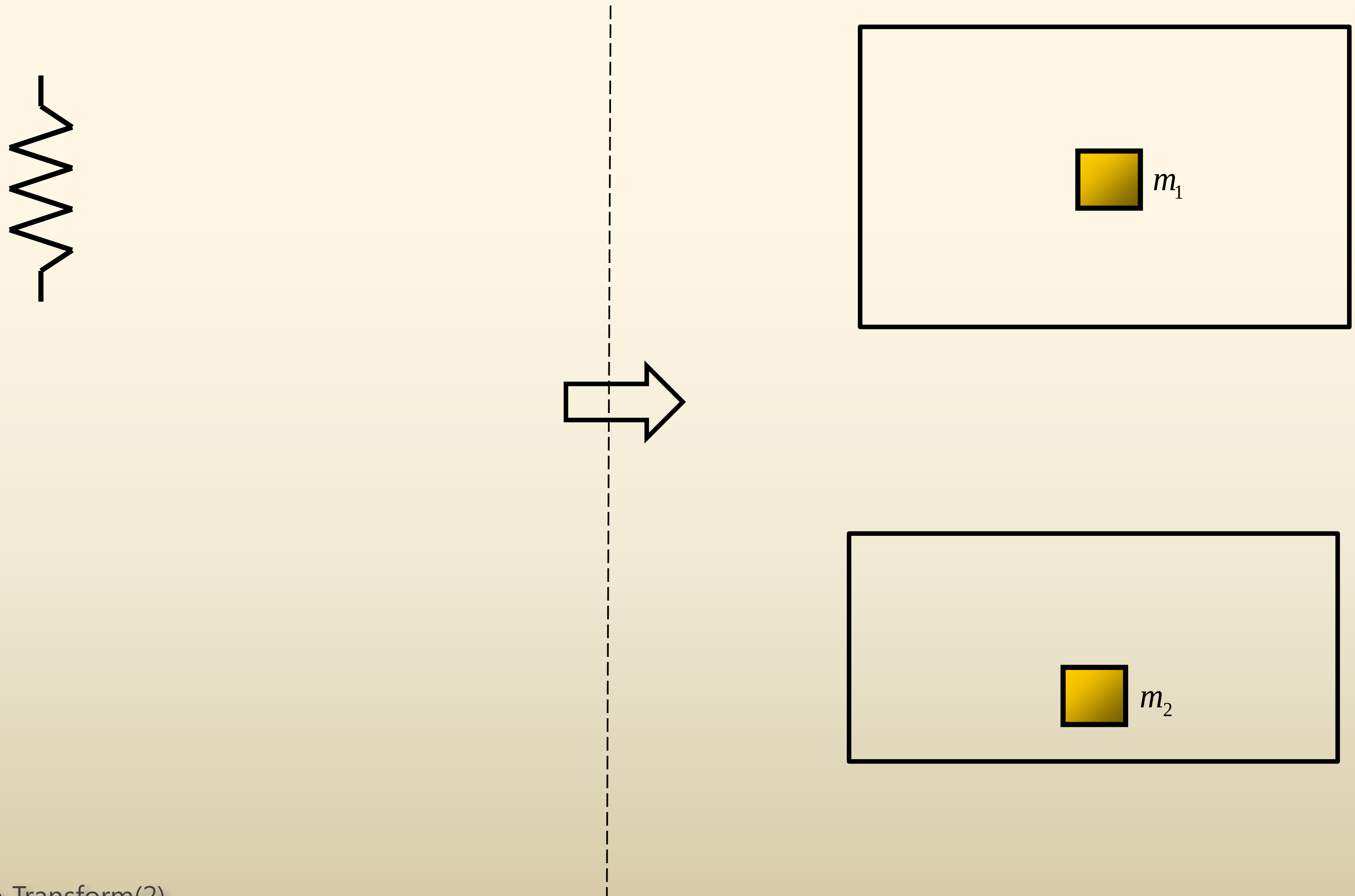
Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



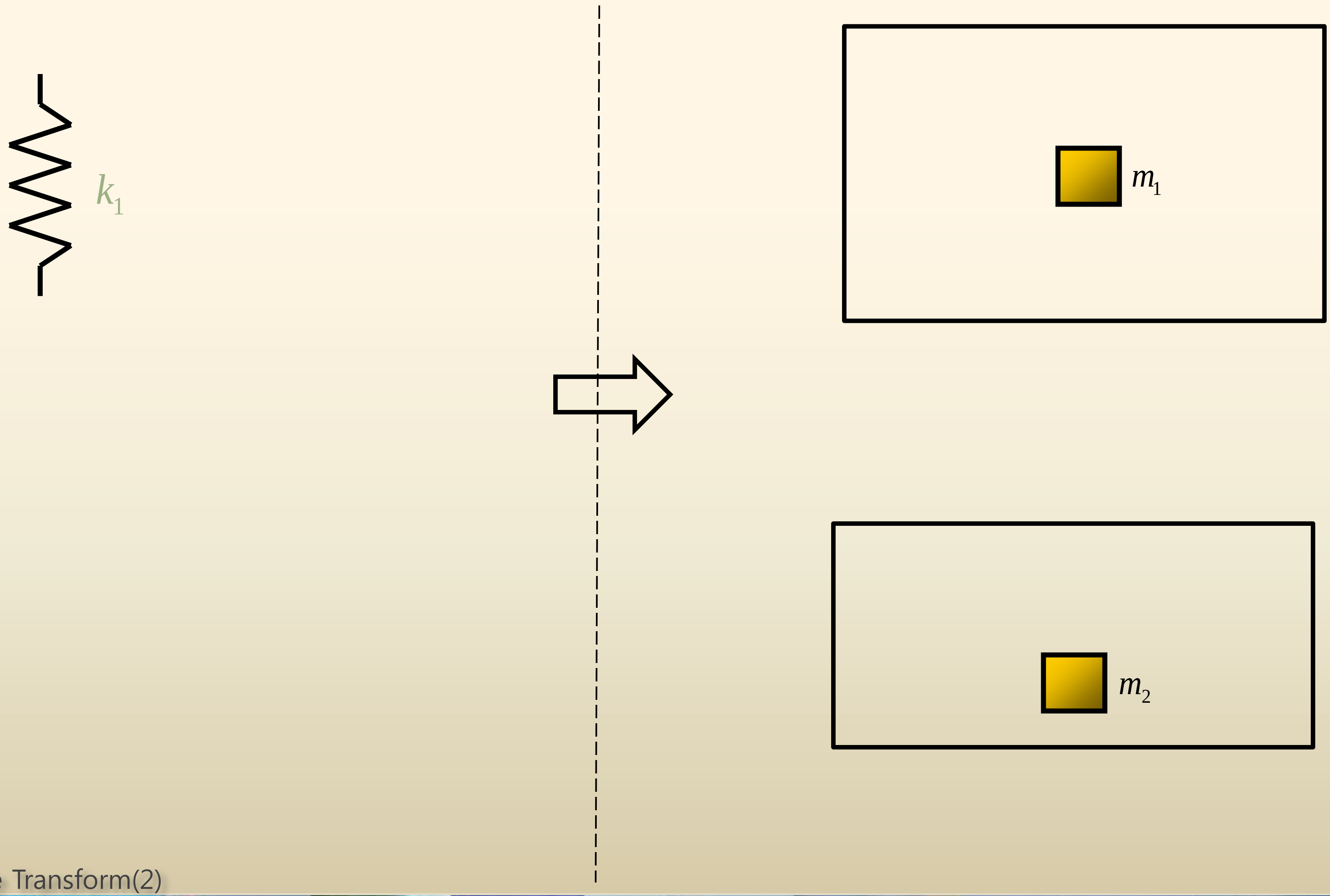
Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



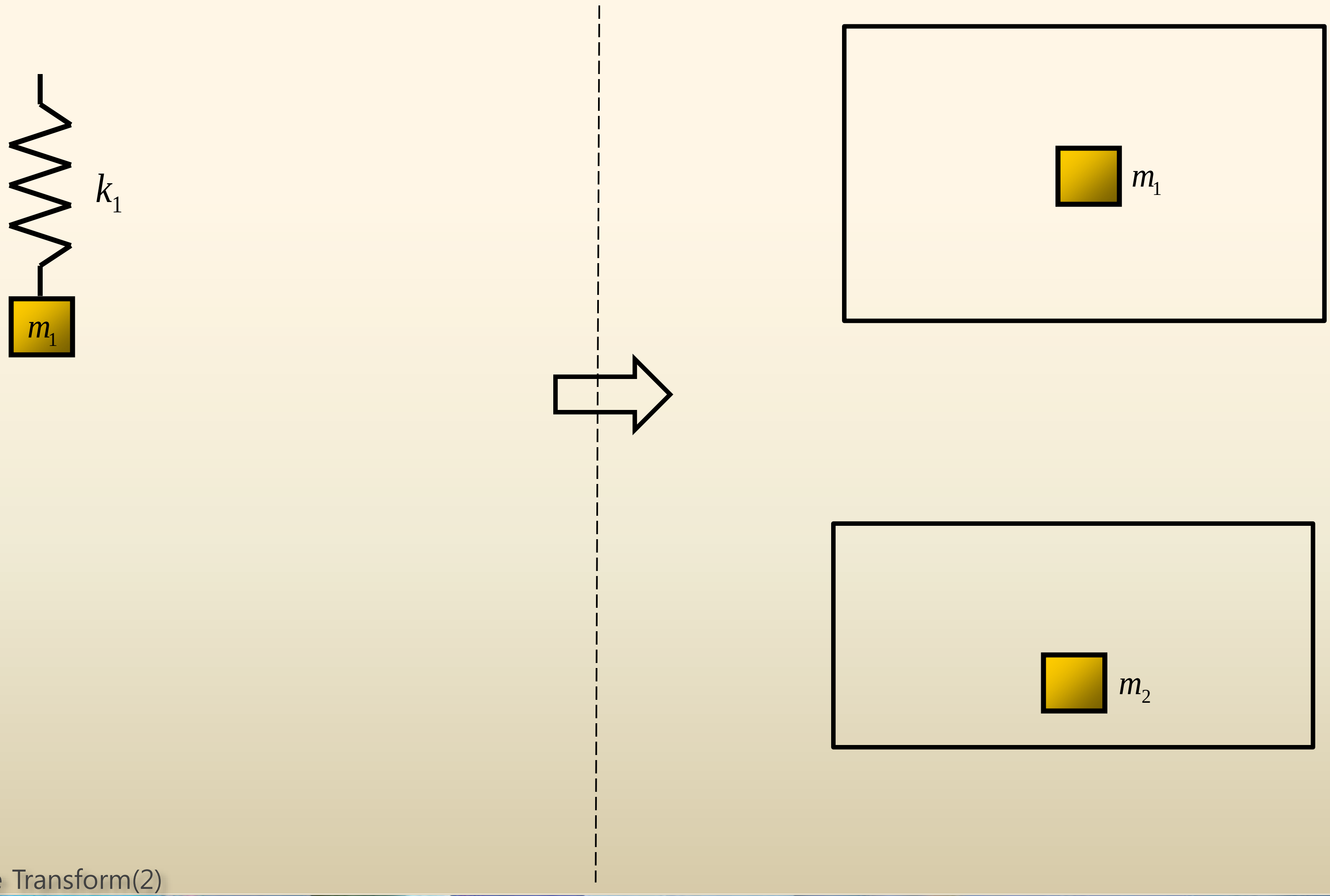
Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

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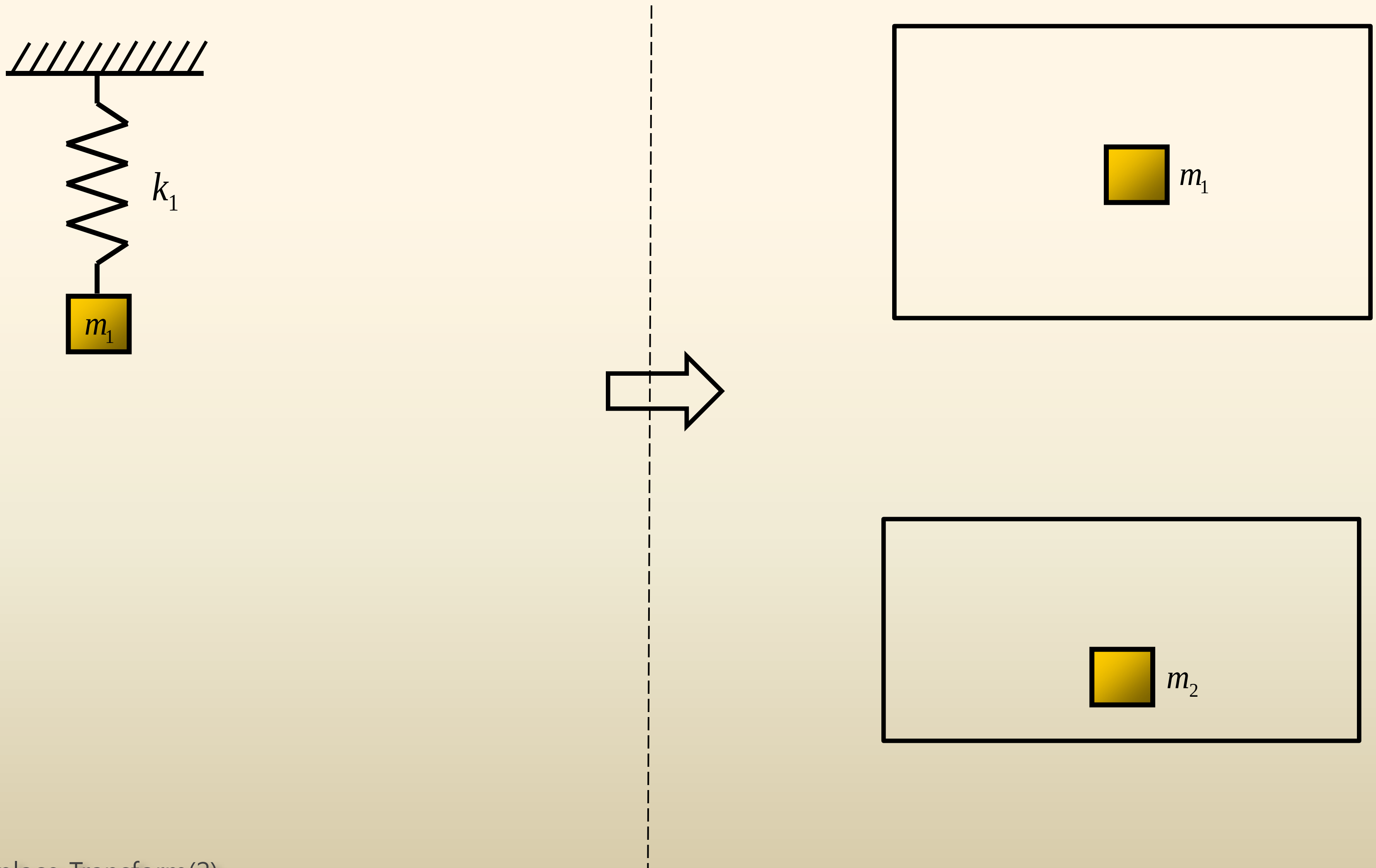
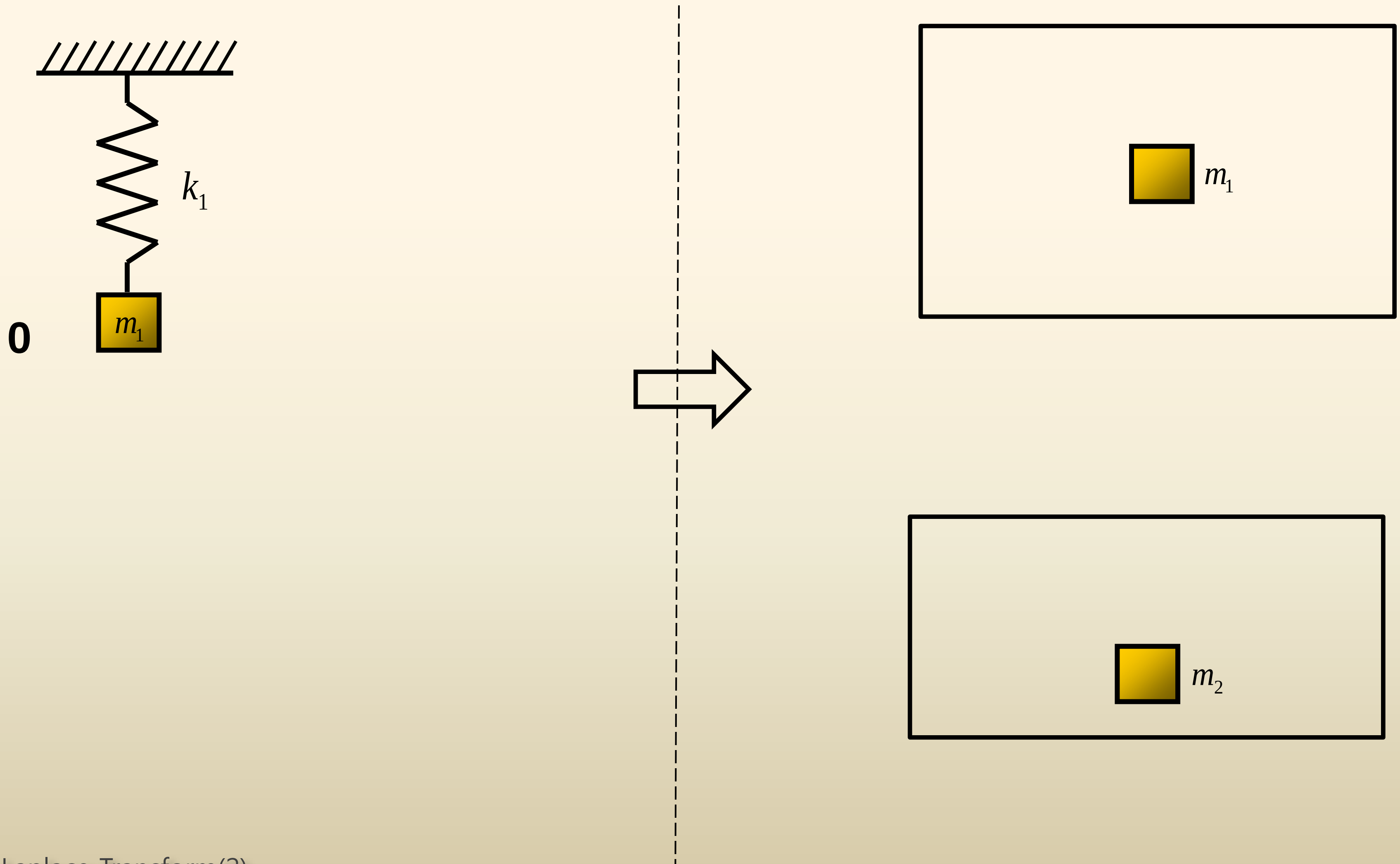


Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

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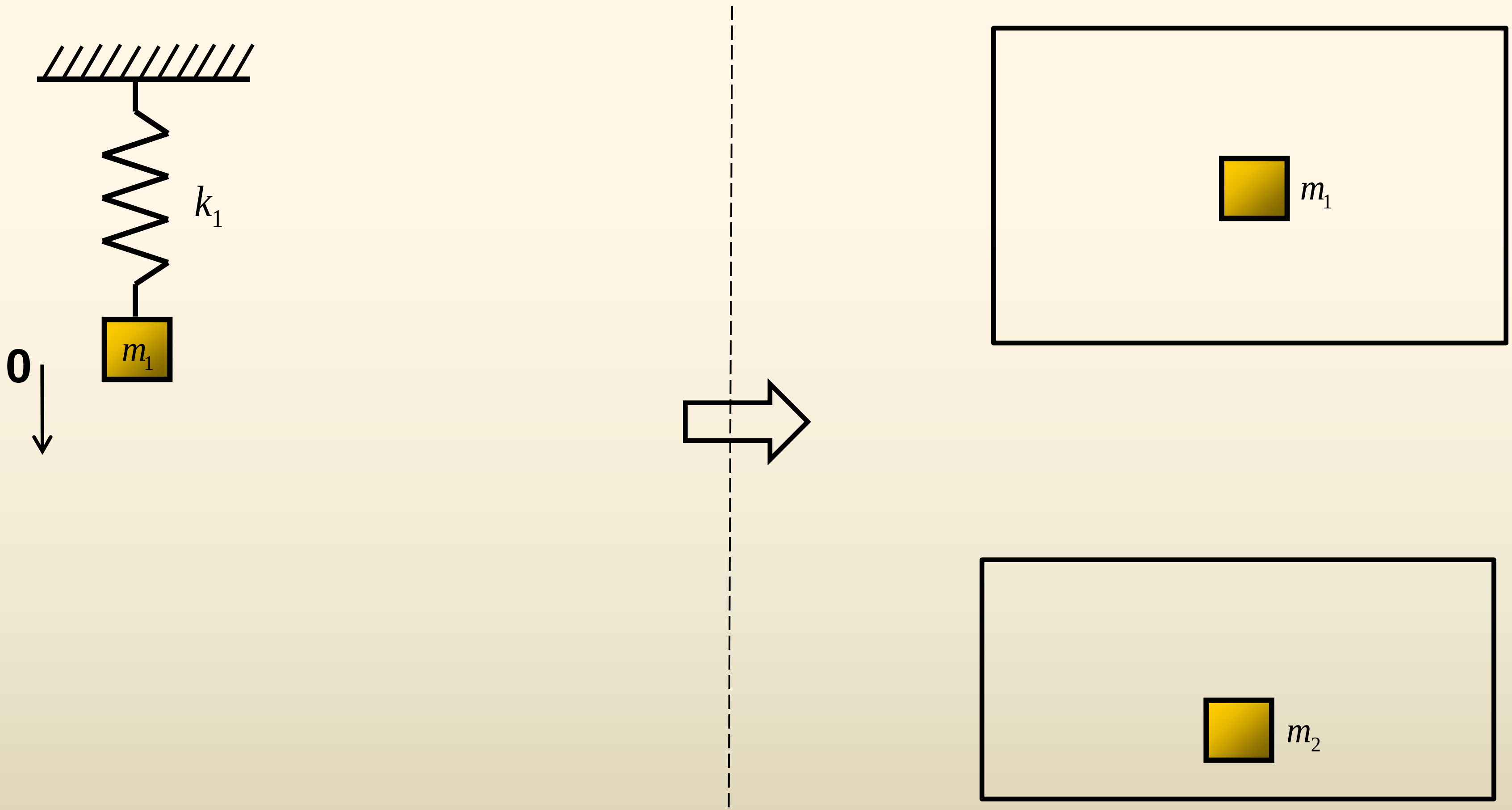
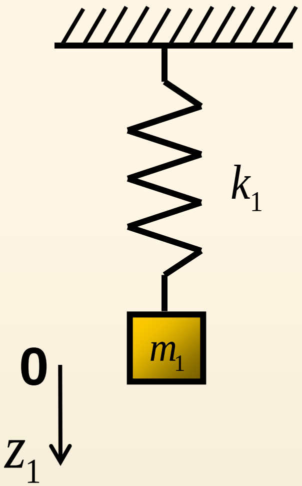
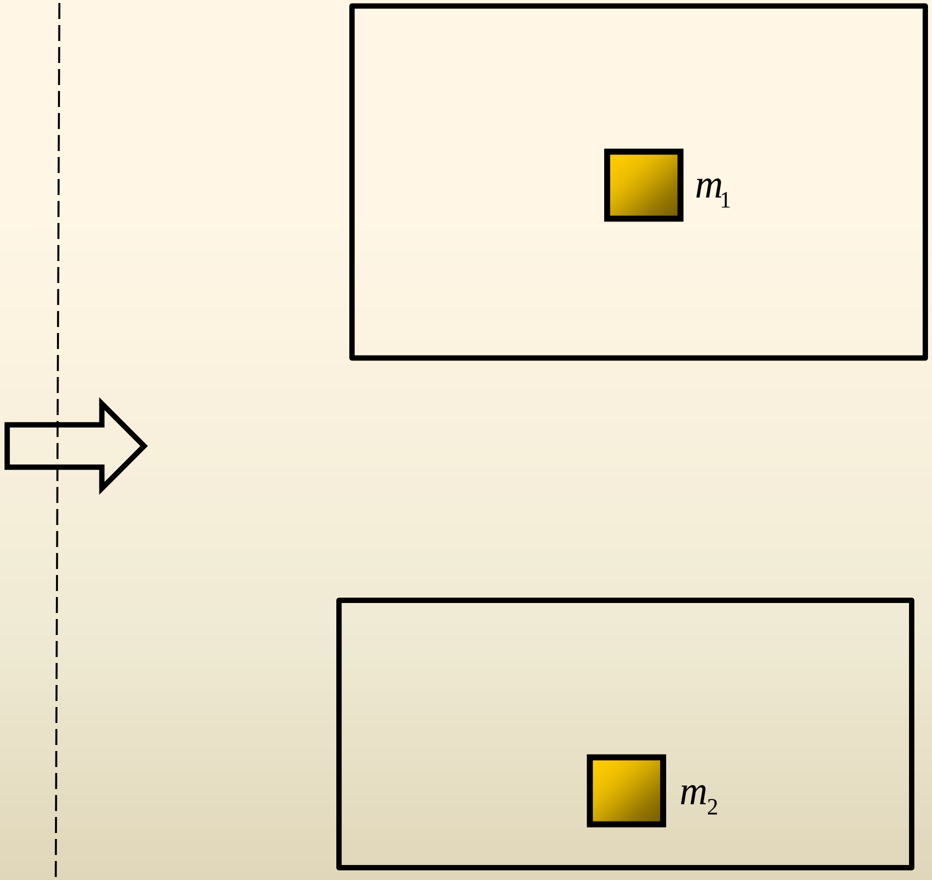


Systems of Linear Differential Equations

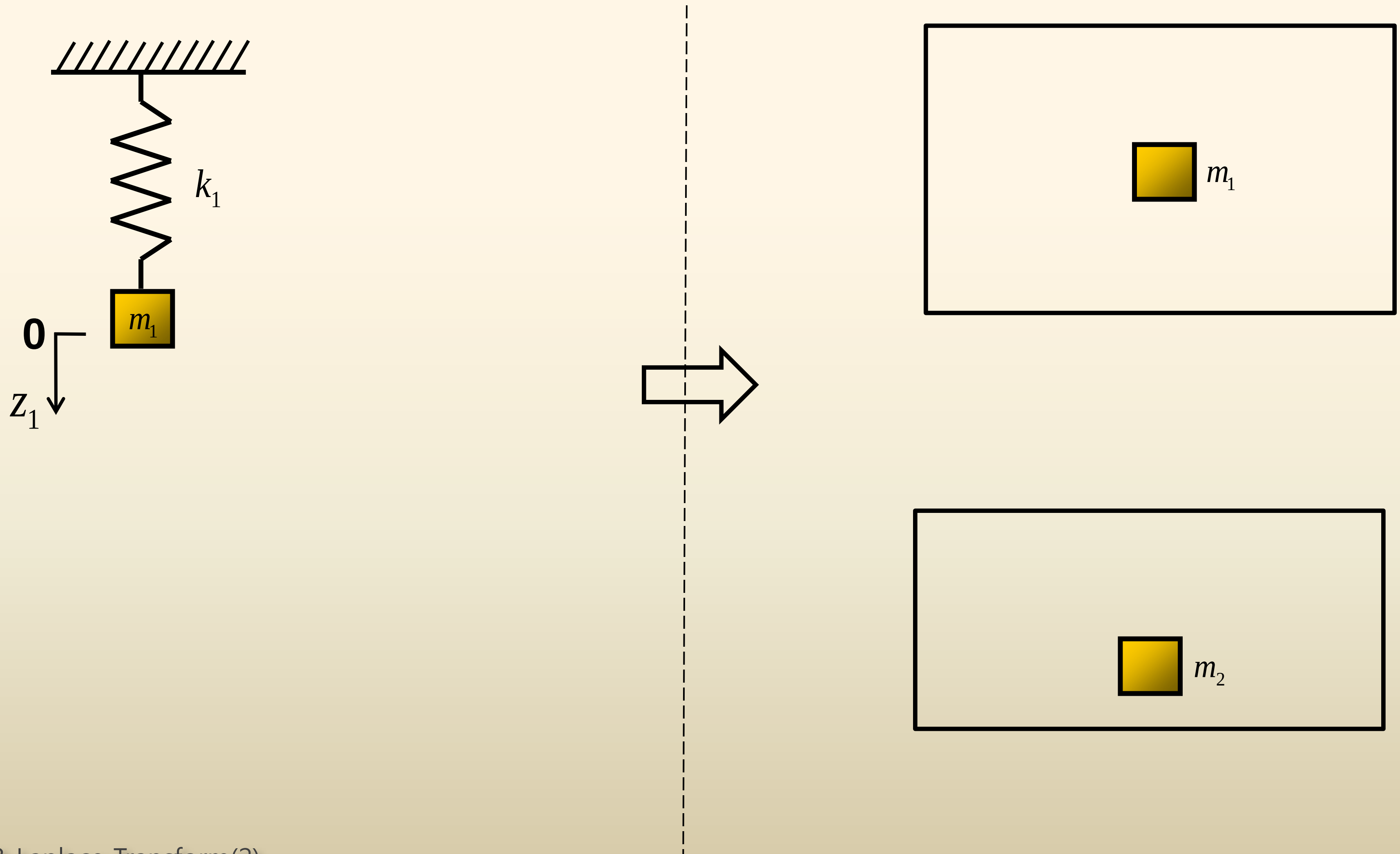
☑ Coupled Spring/Mass System





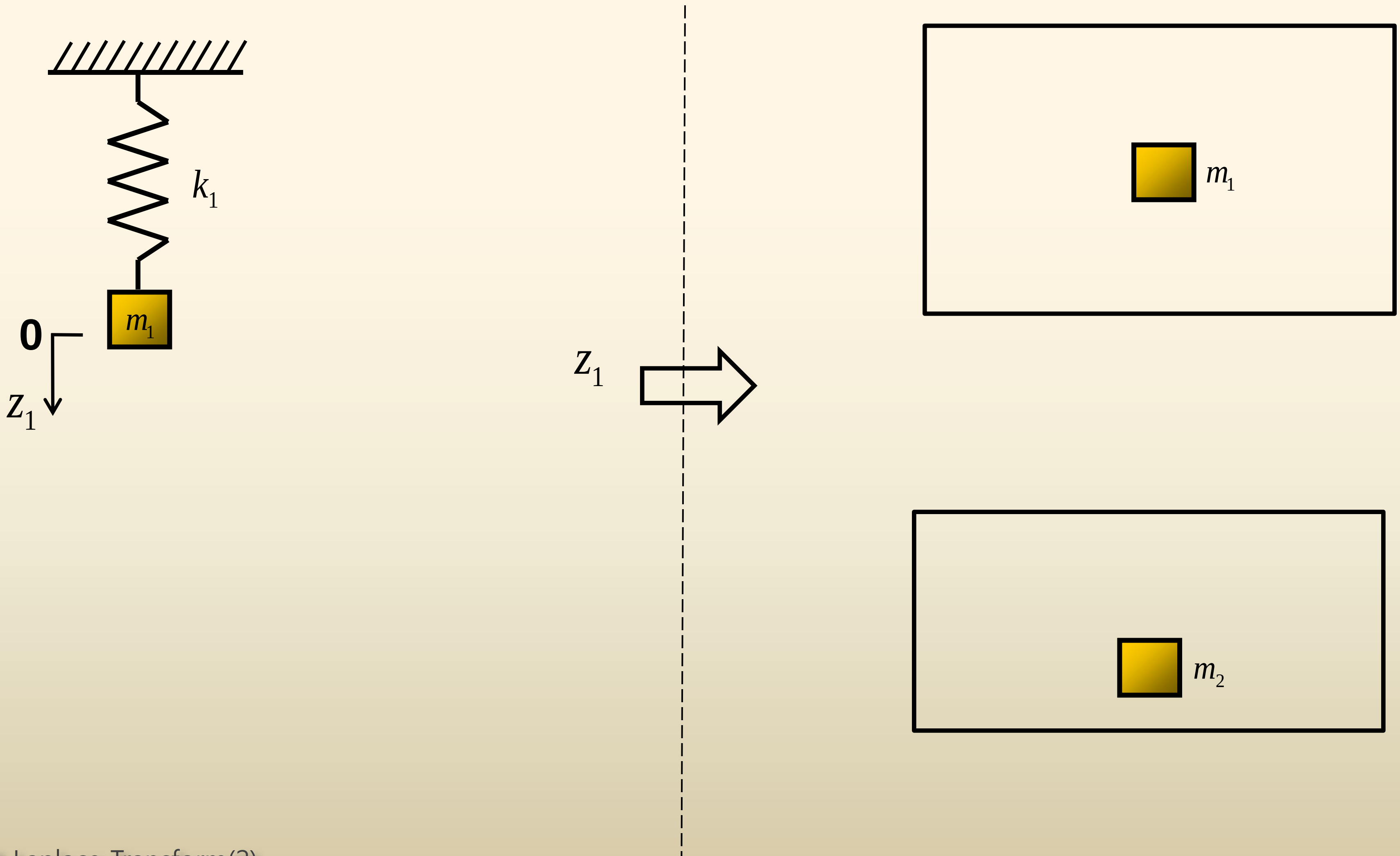
Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

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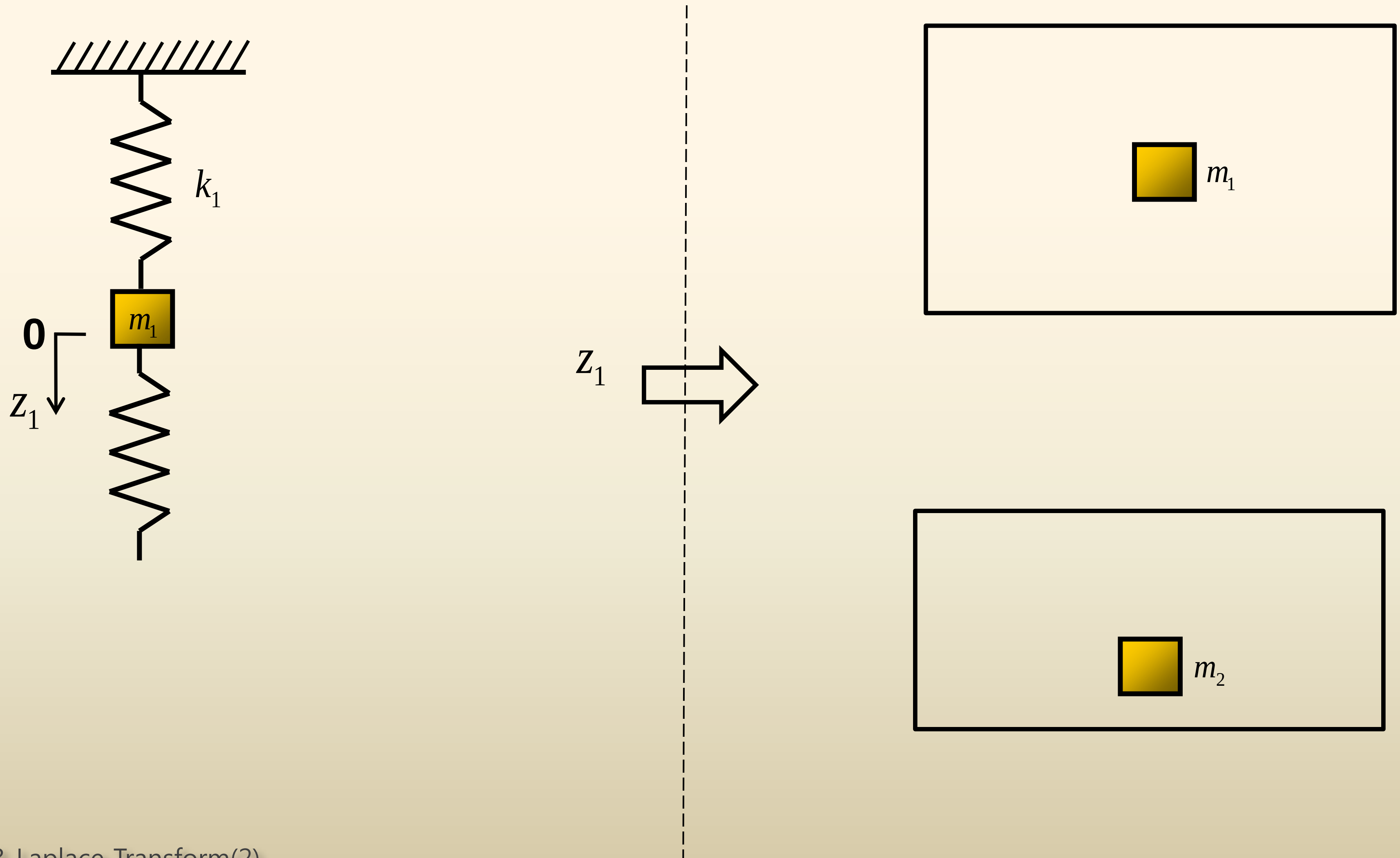

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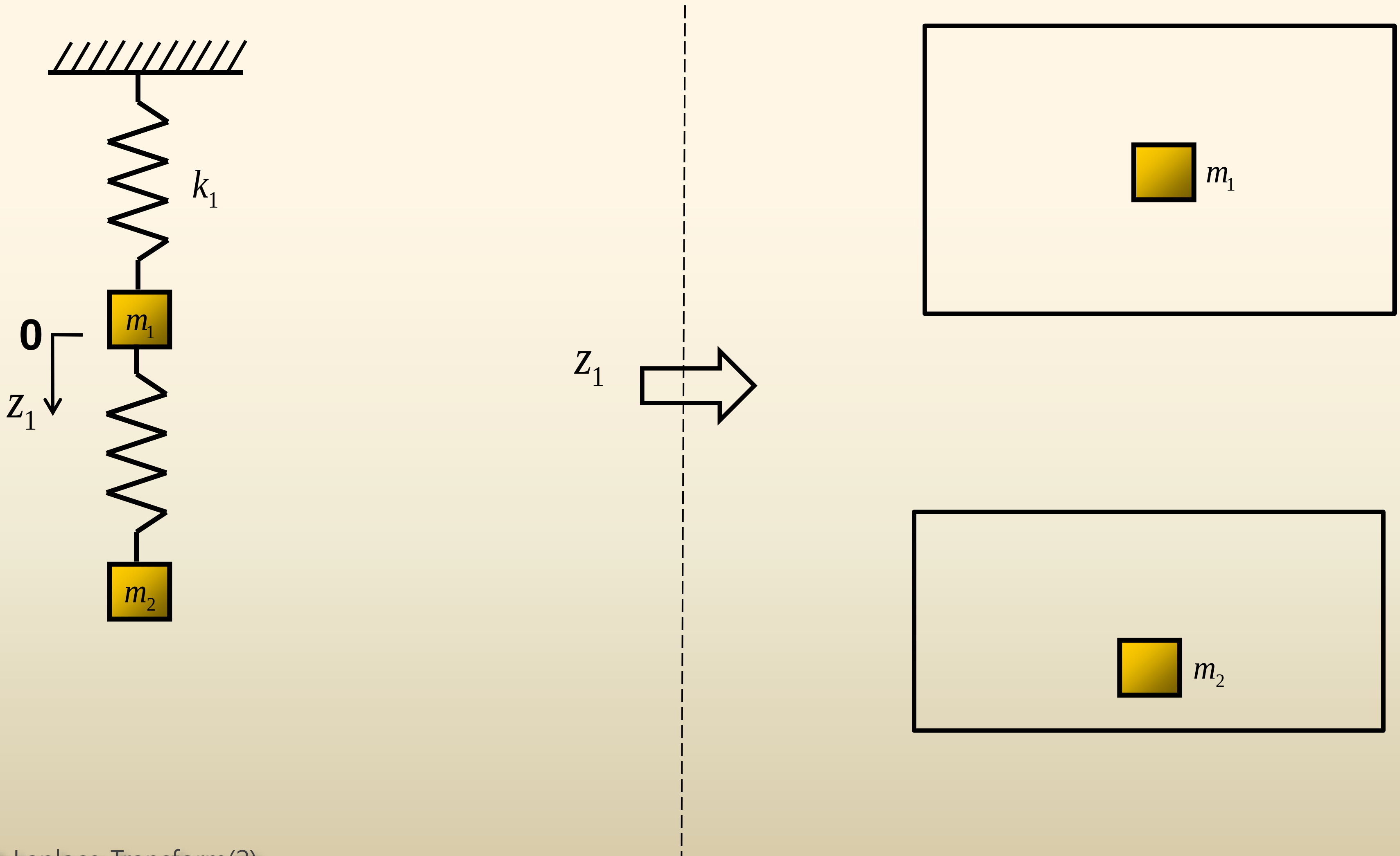

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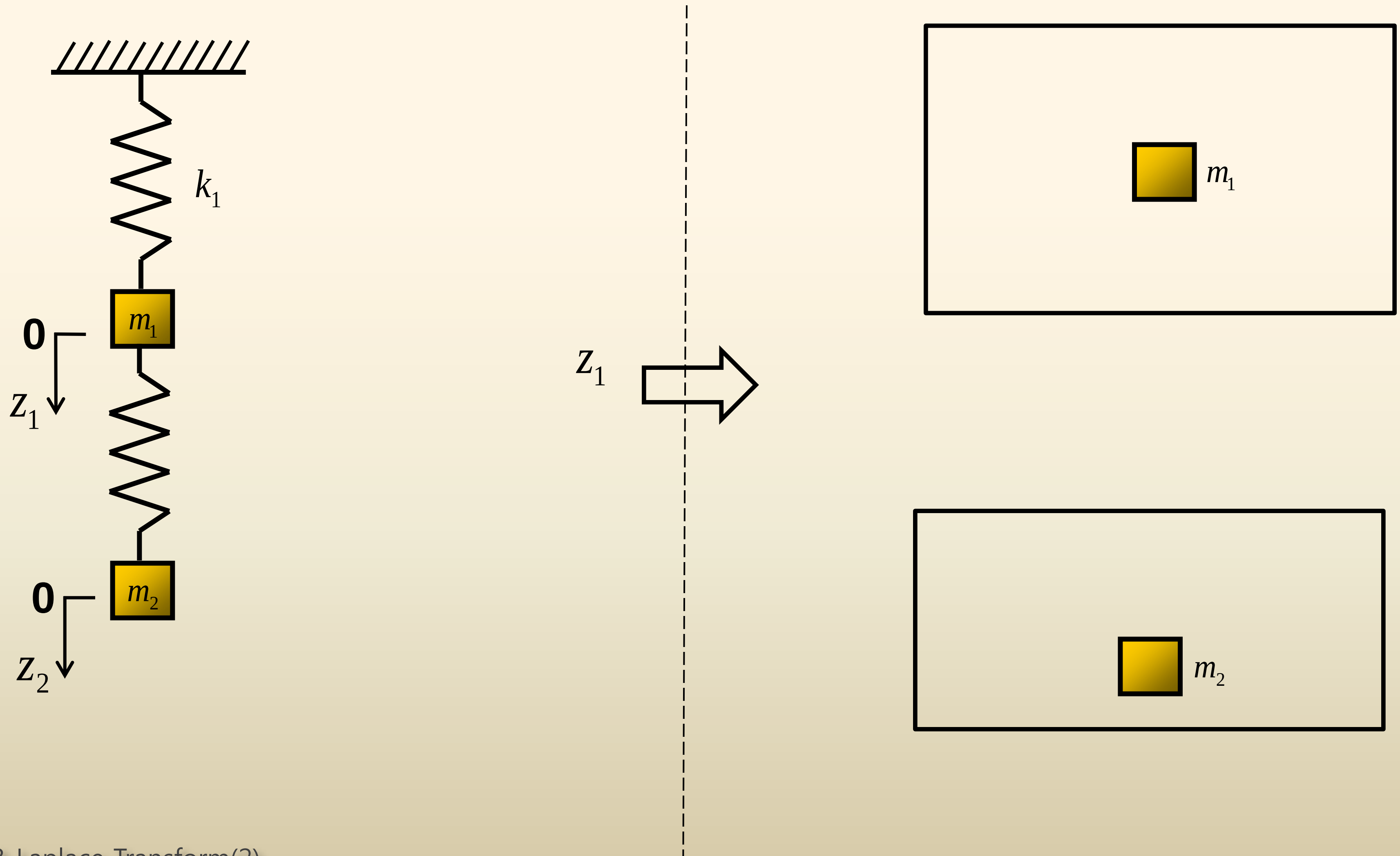

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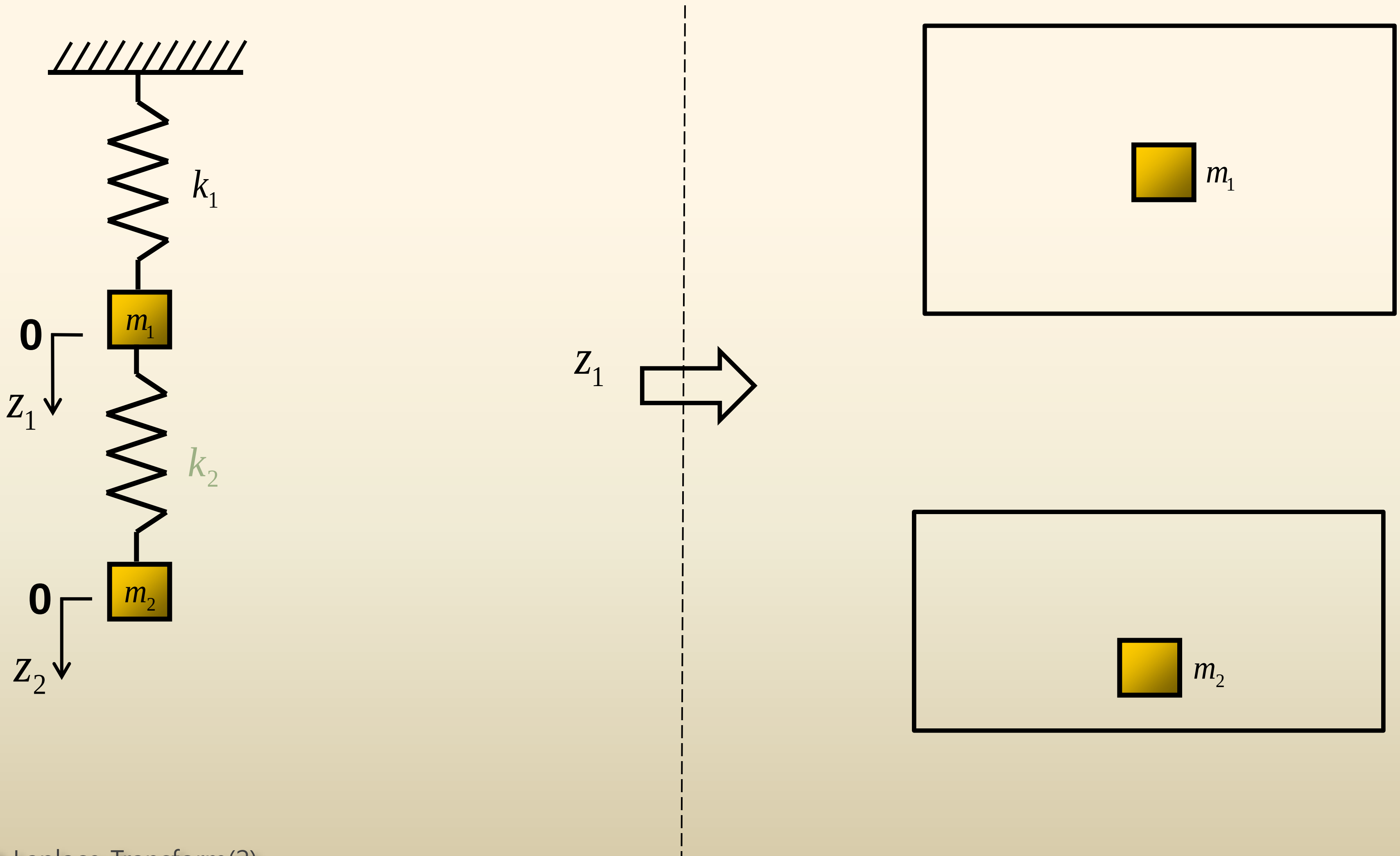

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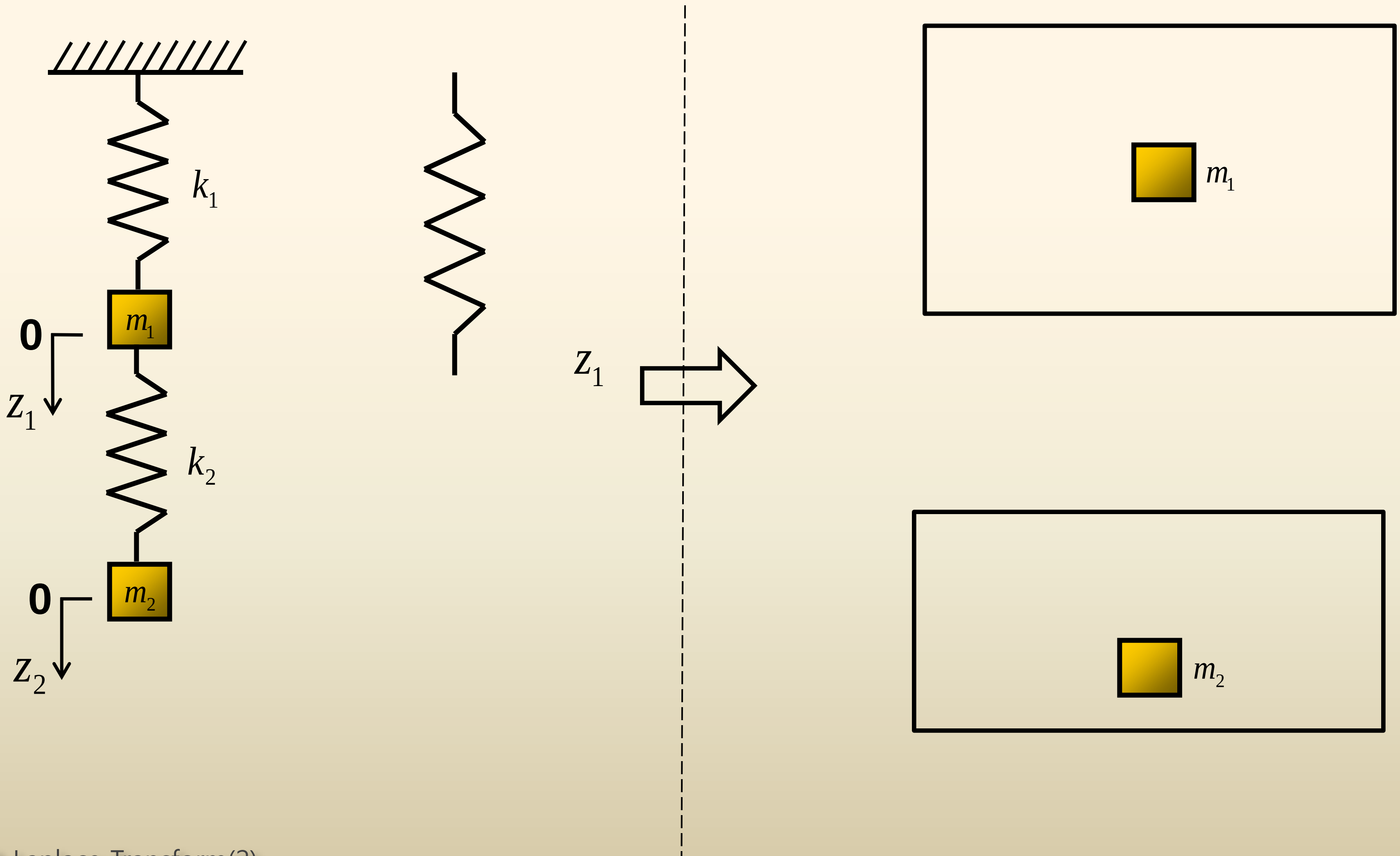

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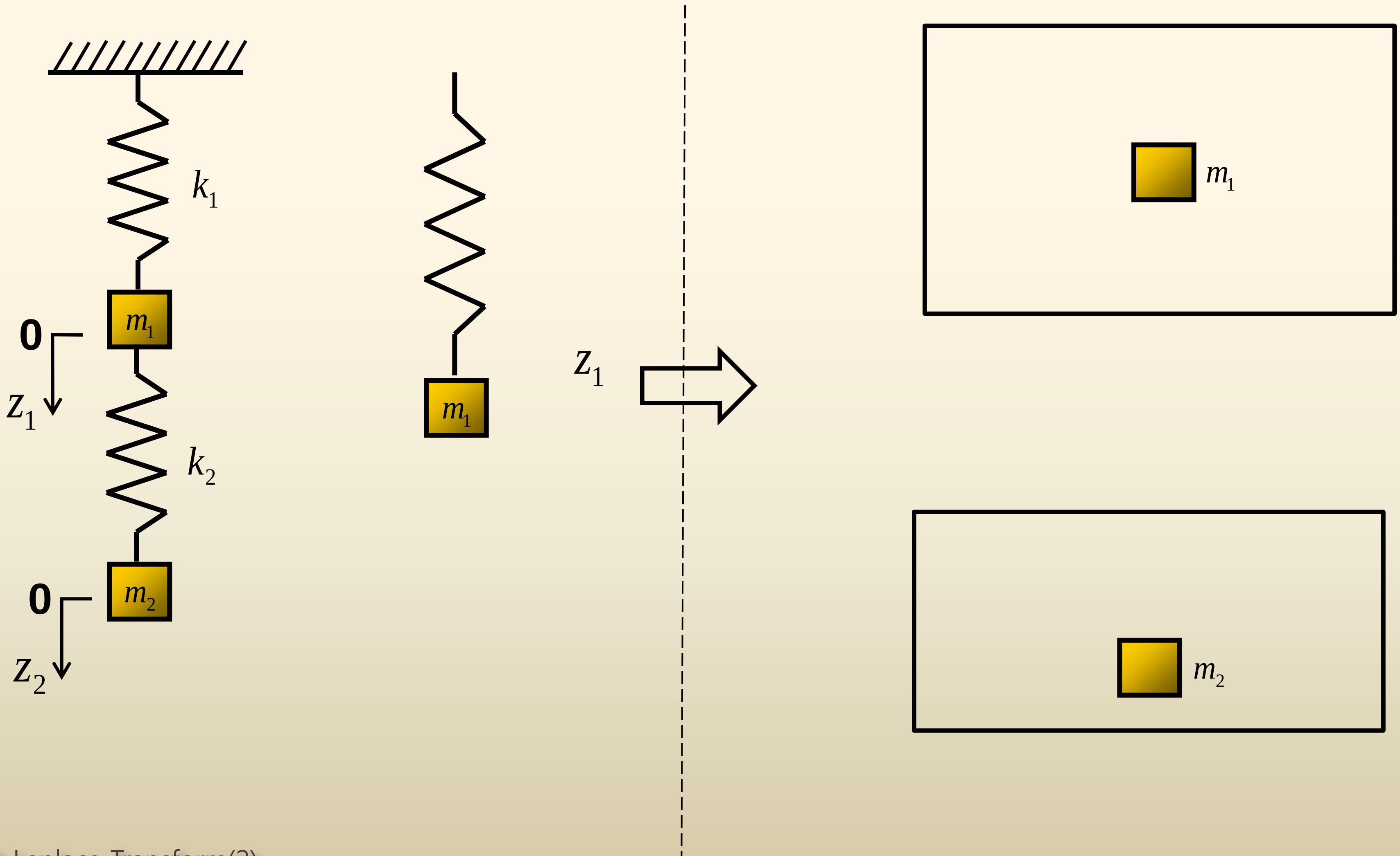

Systems of Linear Differential Equations

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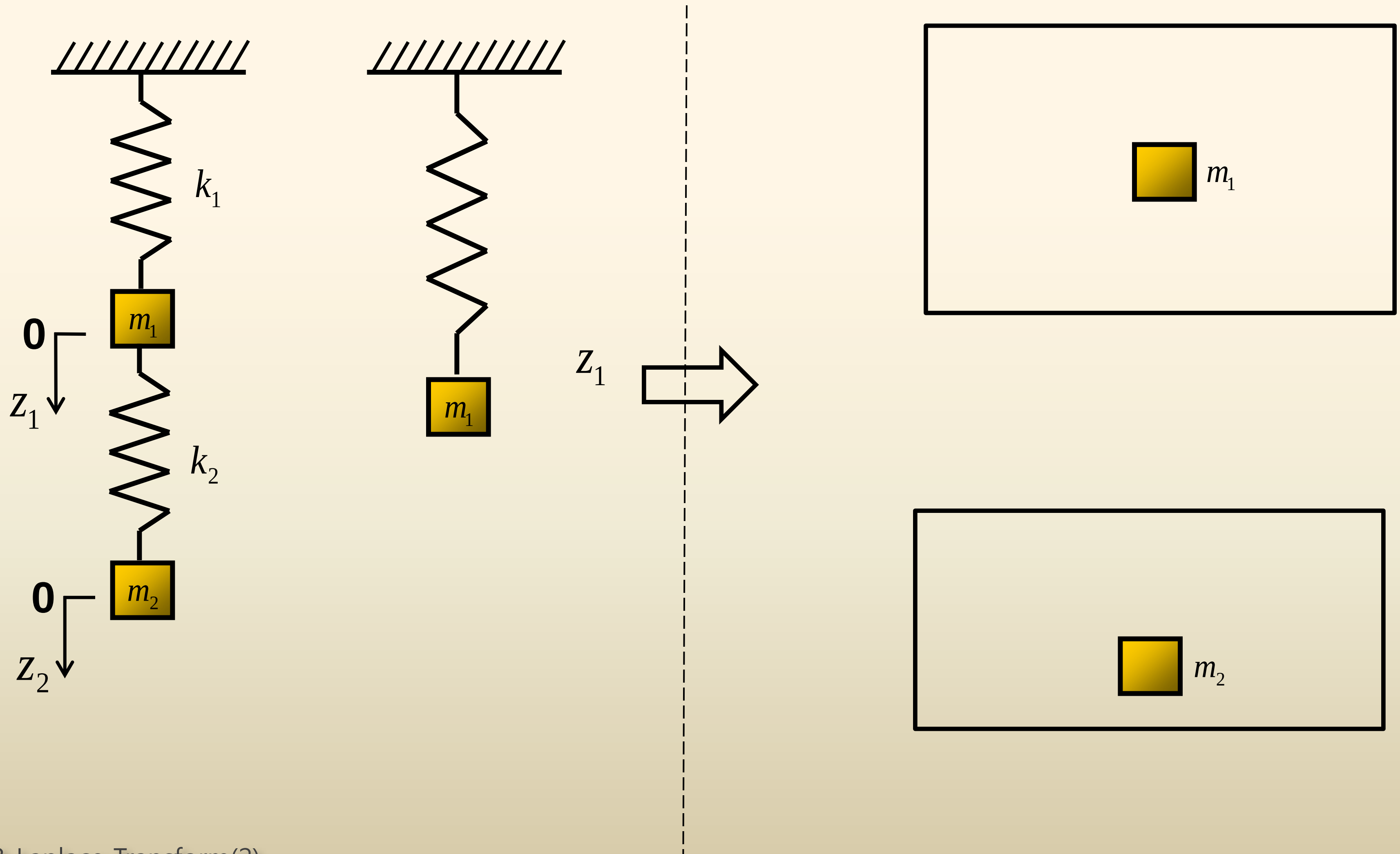

Systems of Linear Differential Equations

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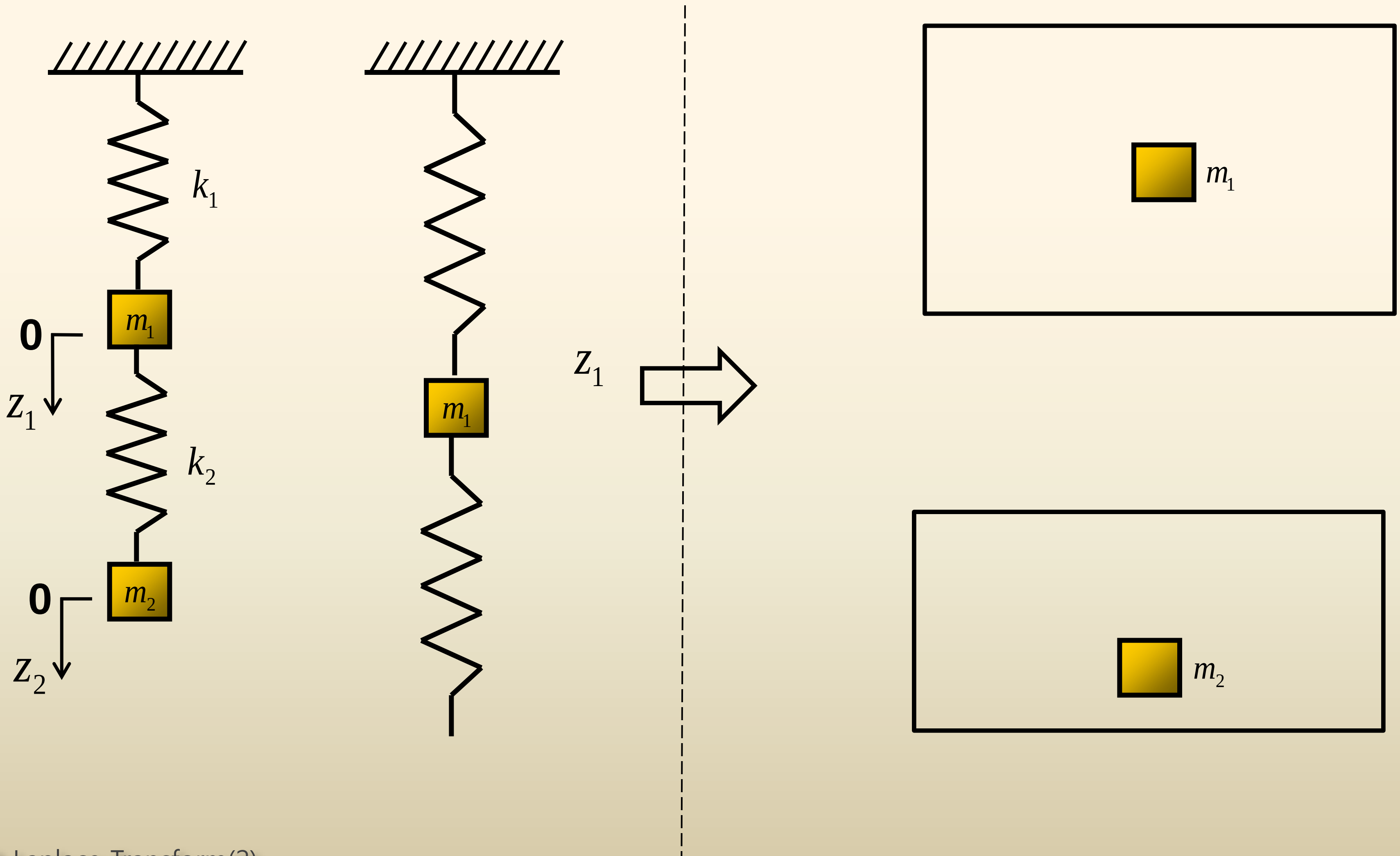

Systems of Linear Differential Equations

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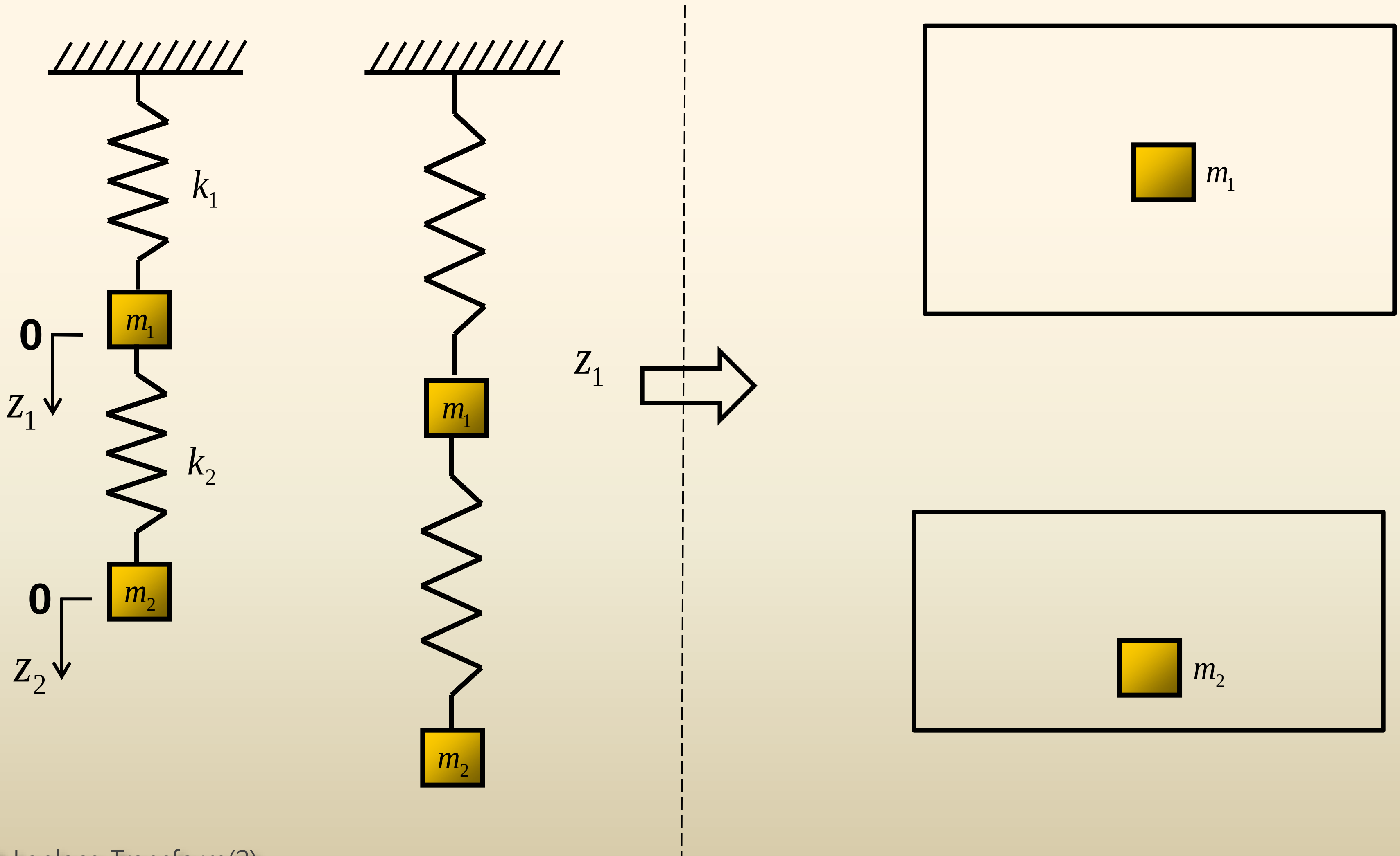
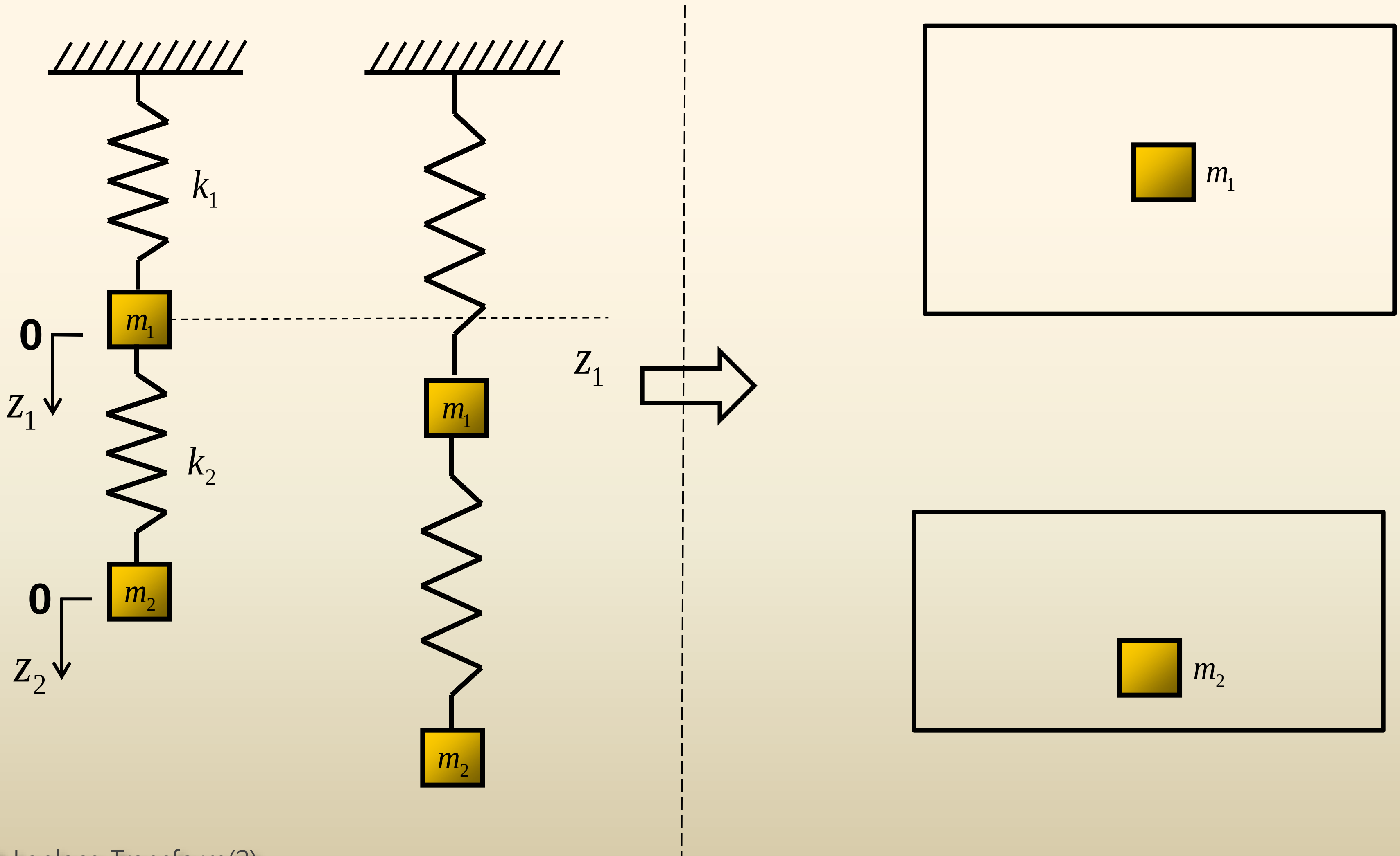


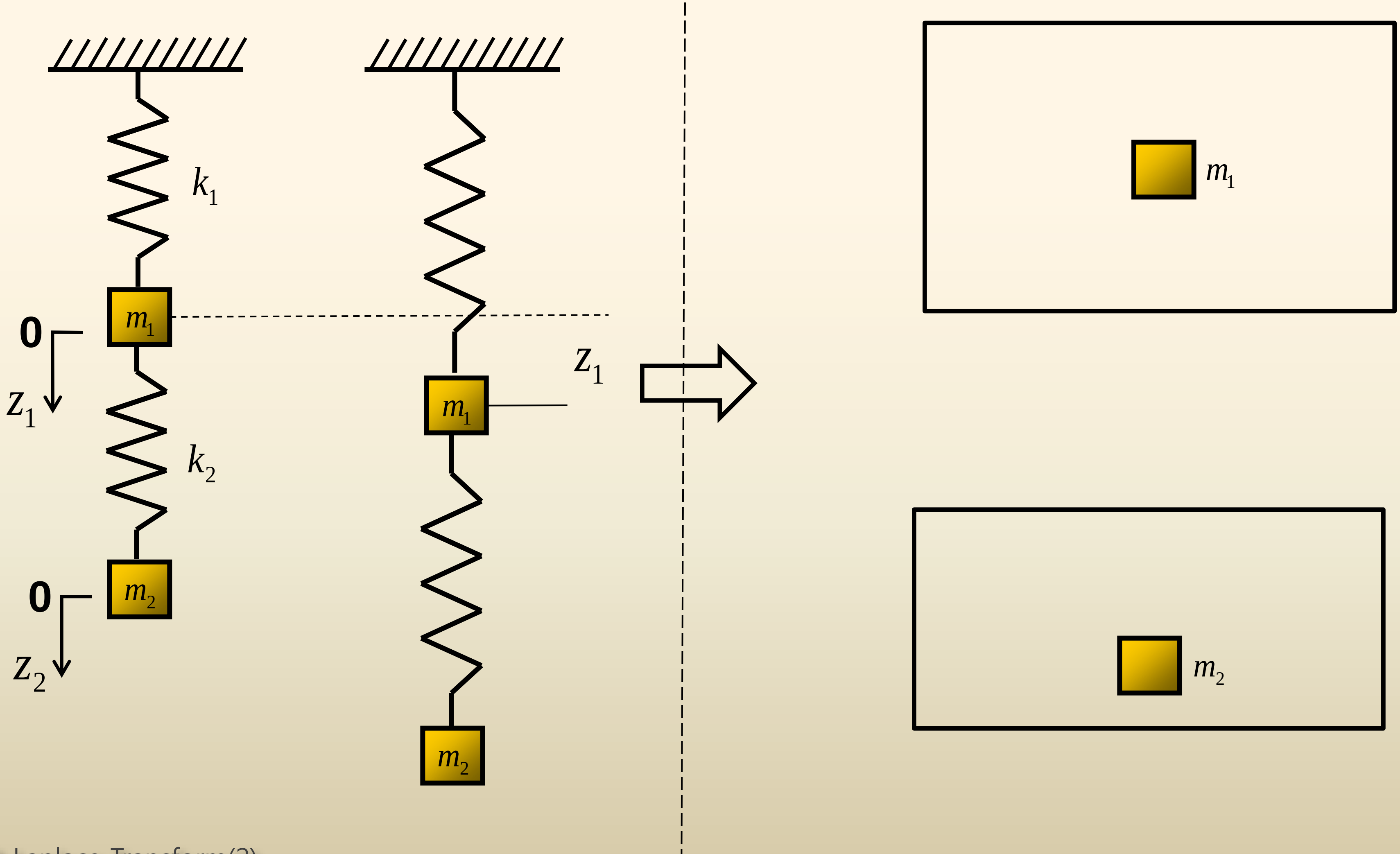
Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



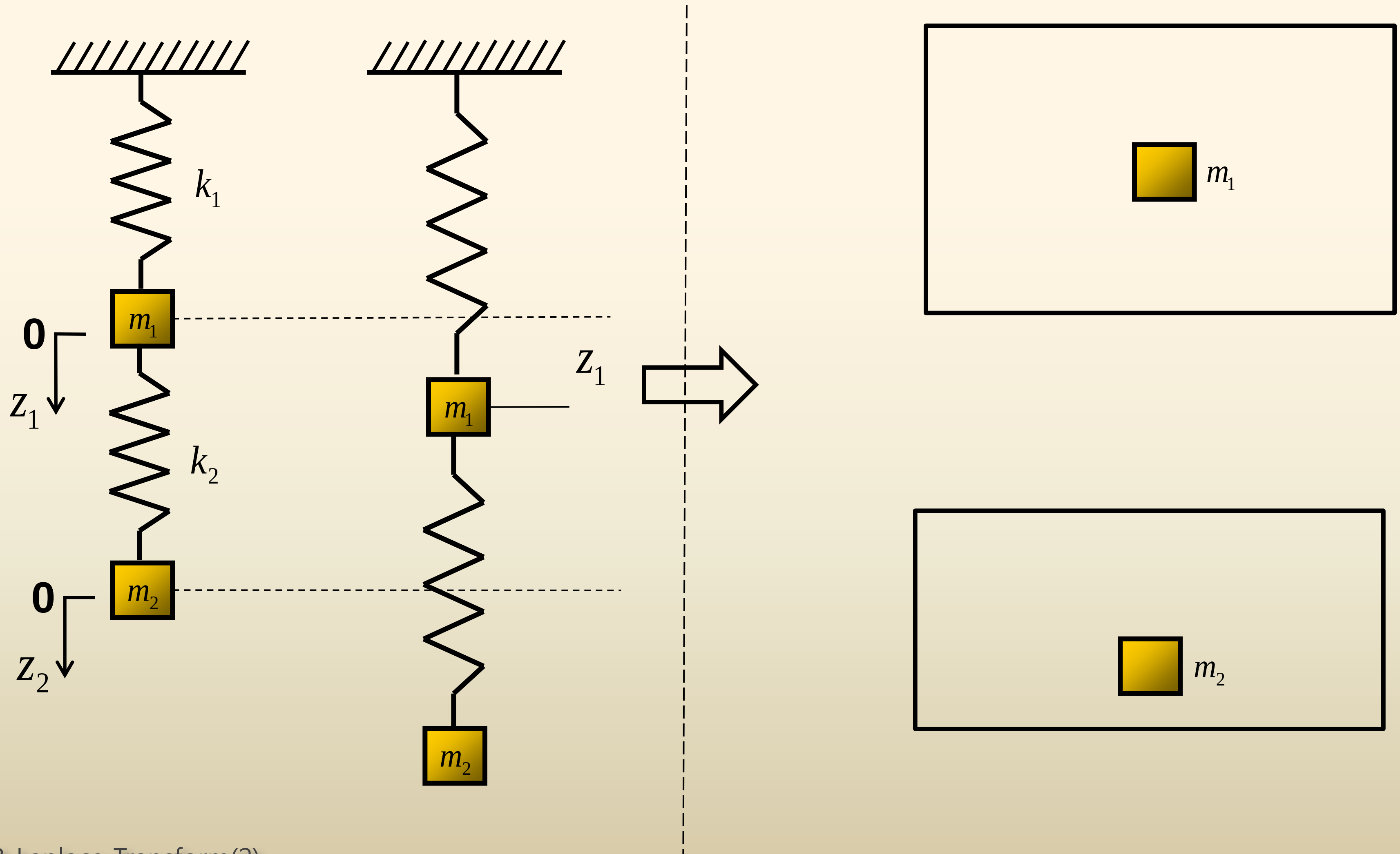
Systems of Linear Differential Equations

☑ **Coupled Spring/Mass System**



Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

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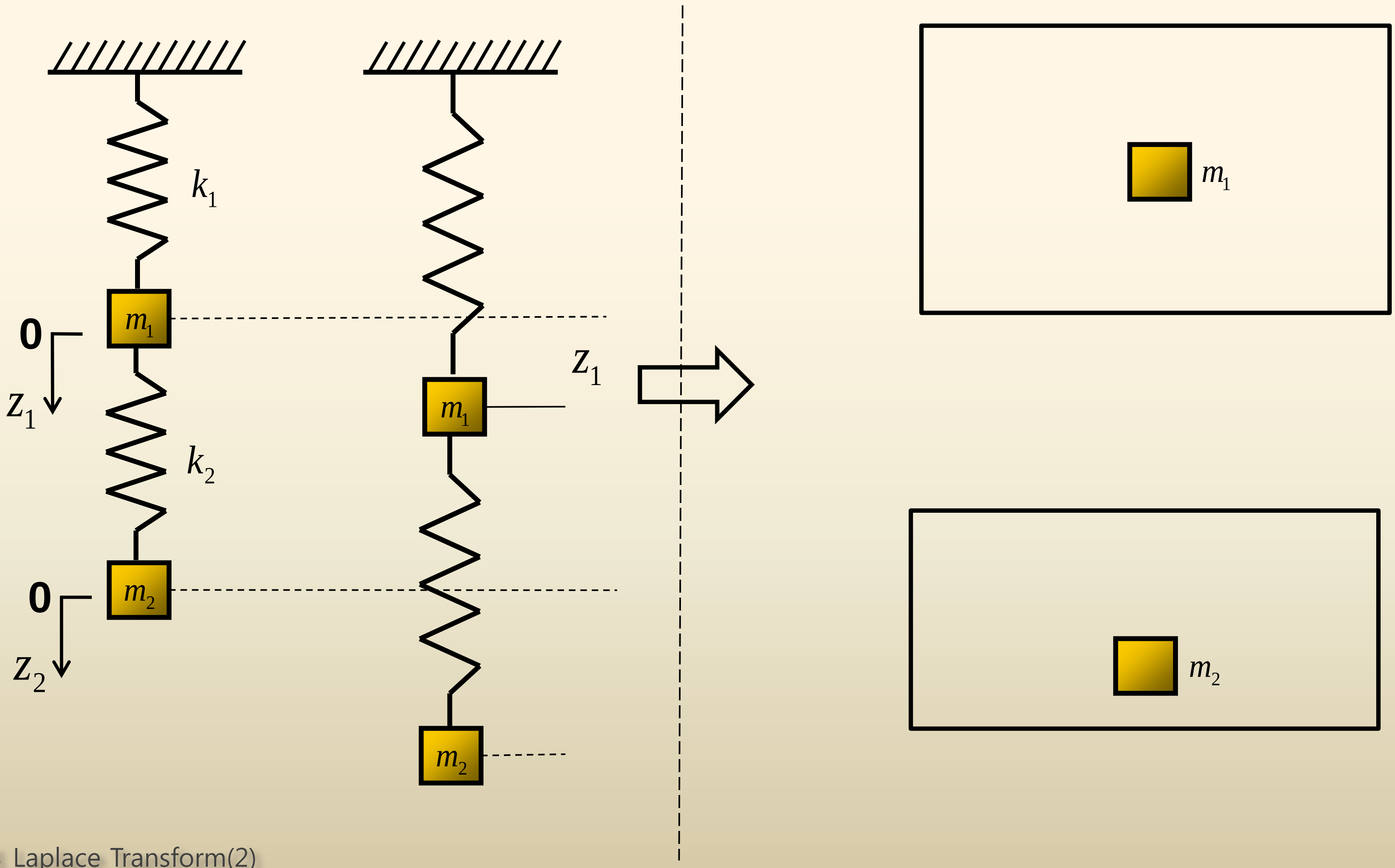

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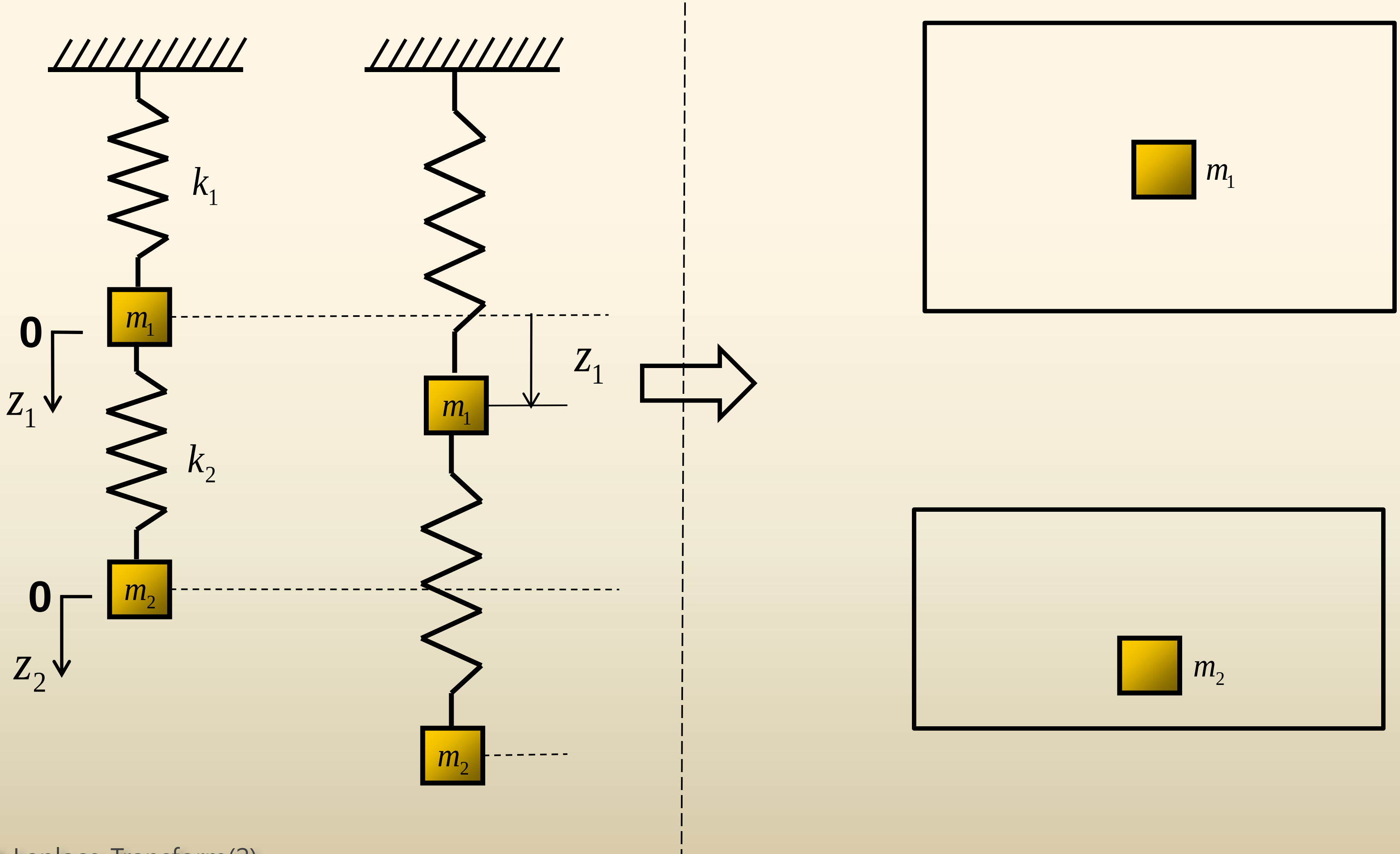

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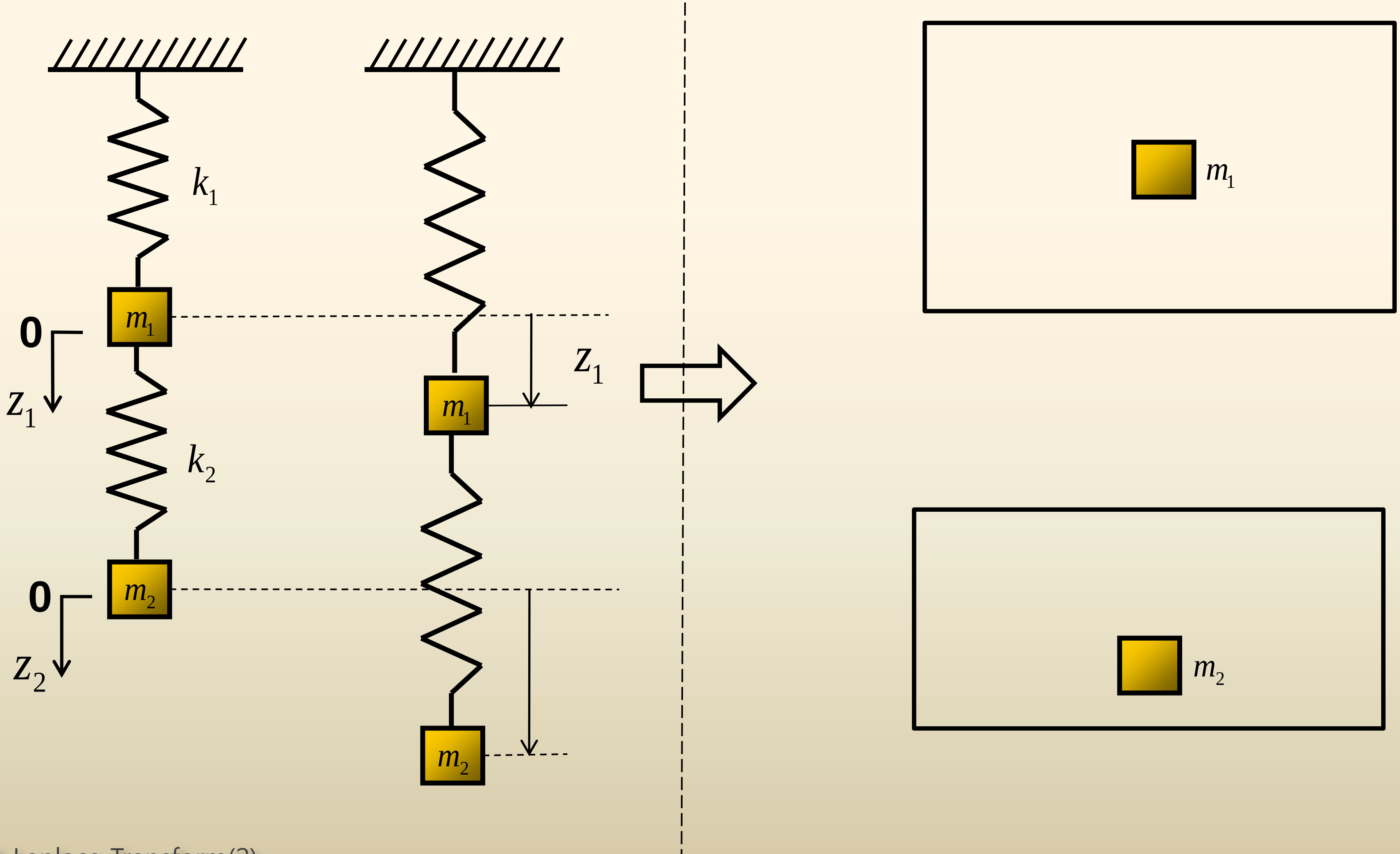

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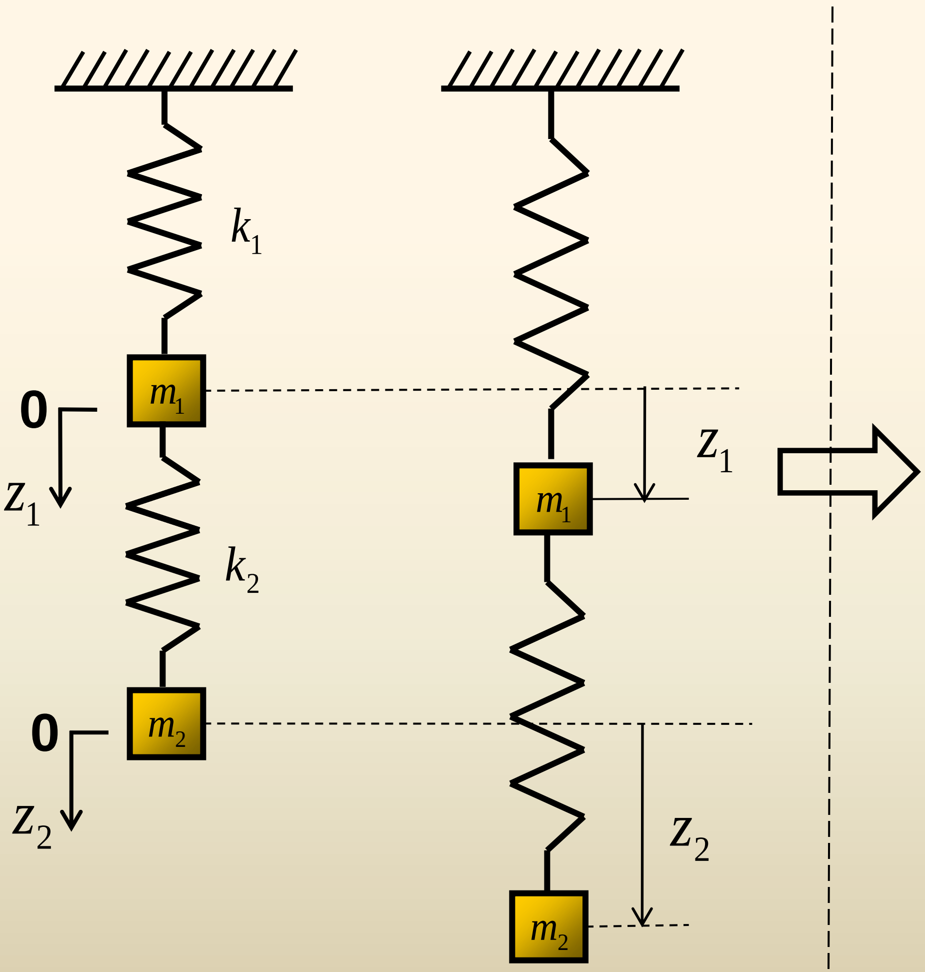

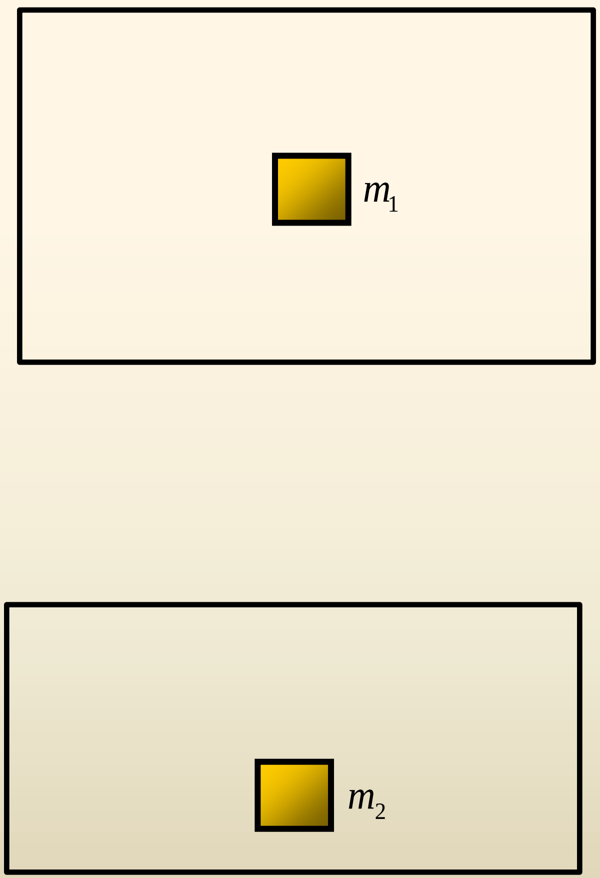

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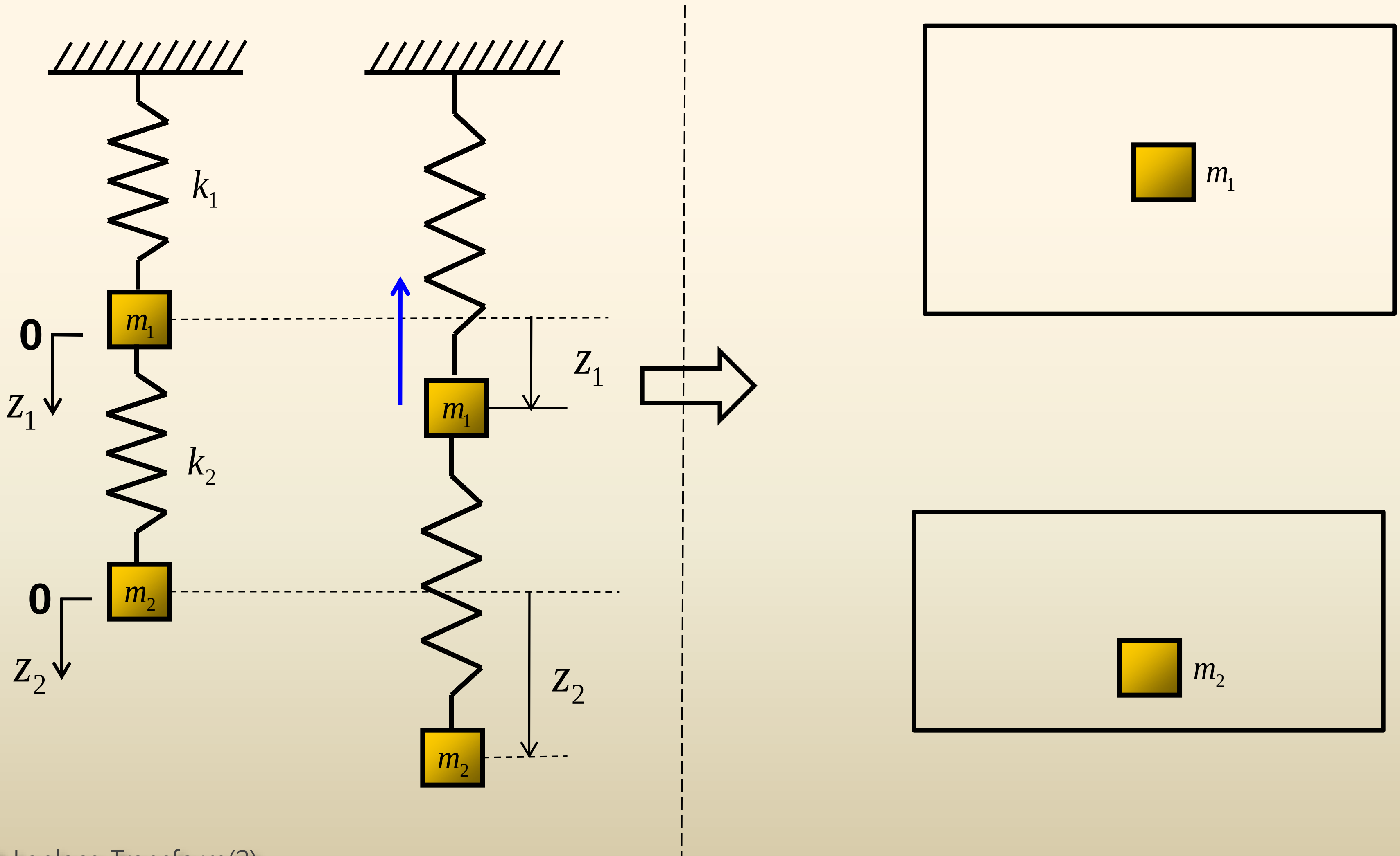

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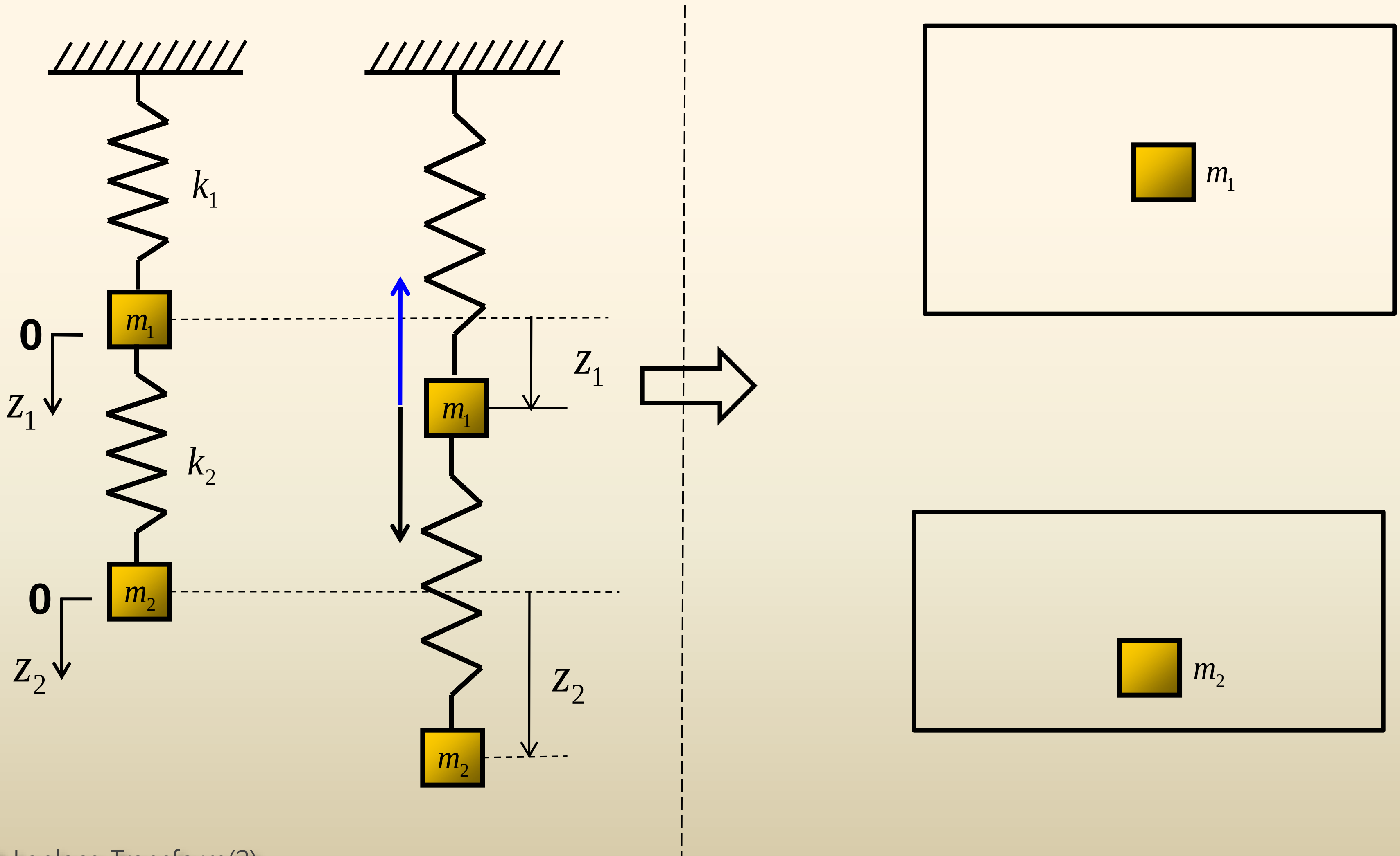

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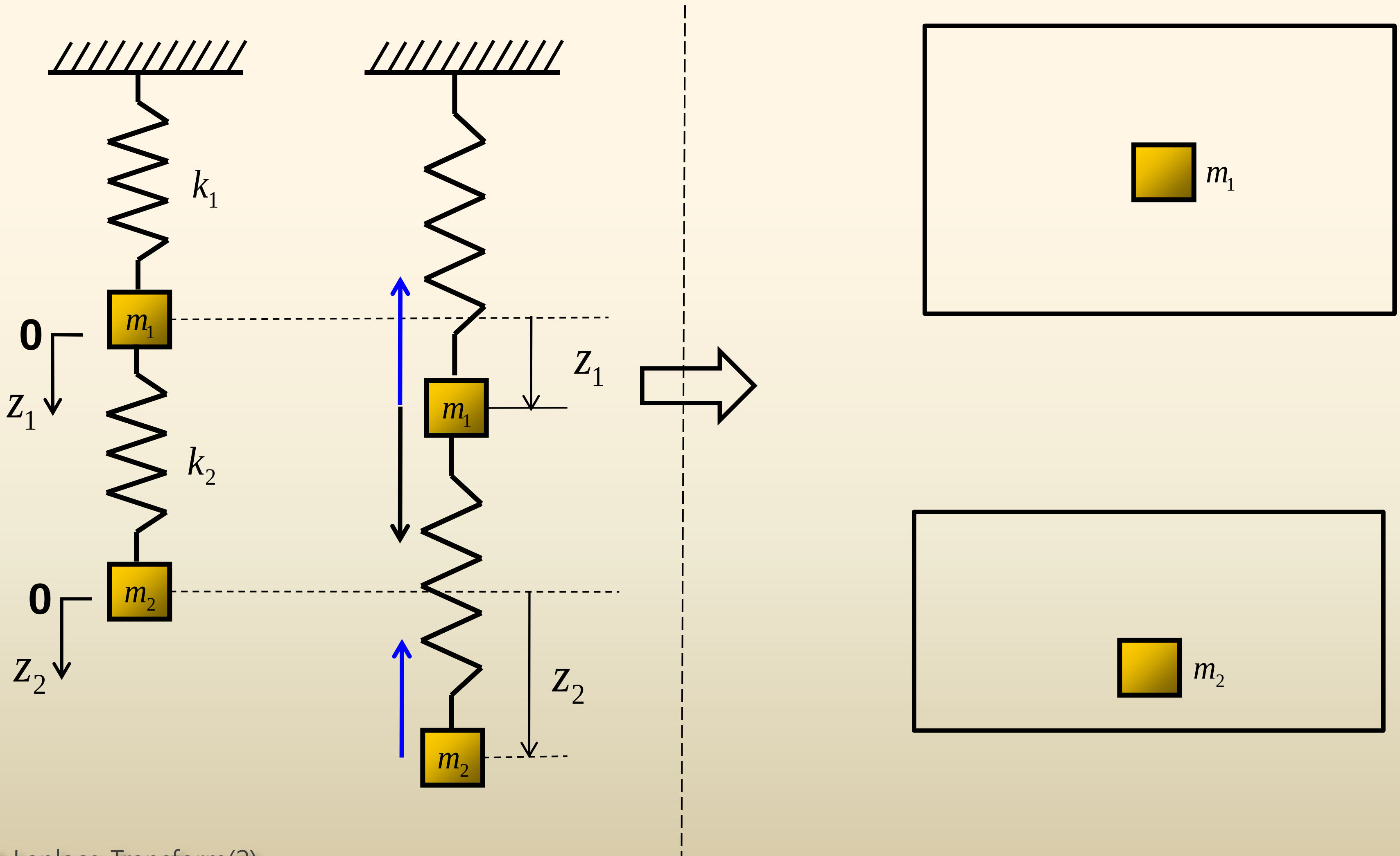

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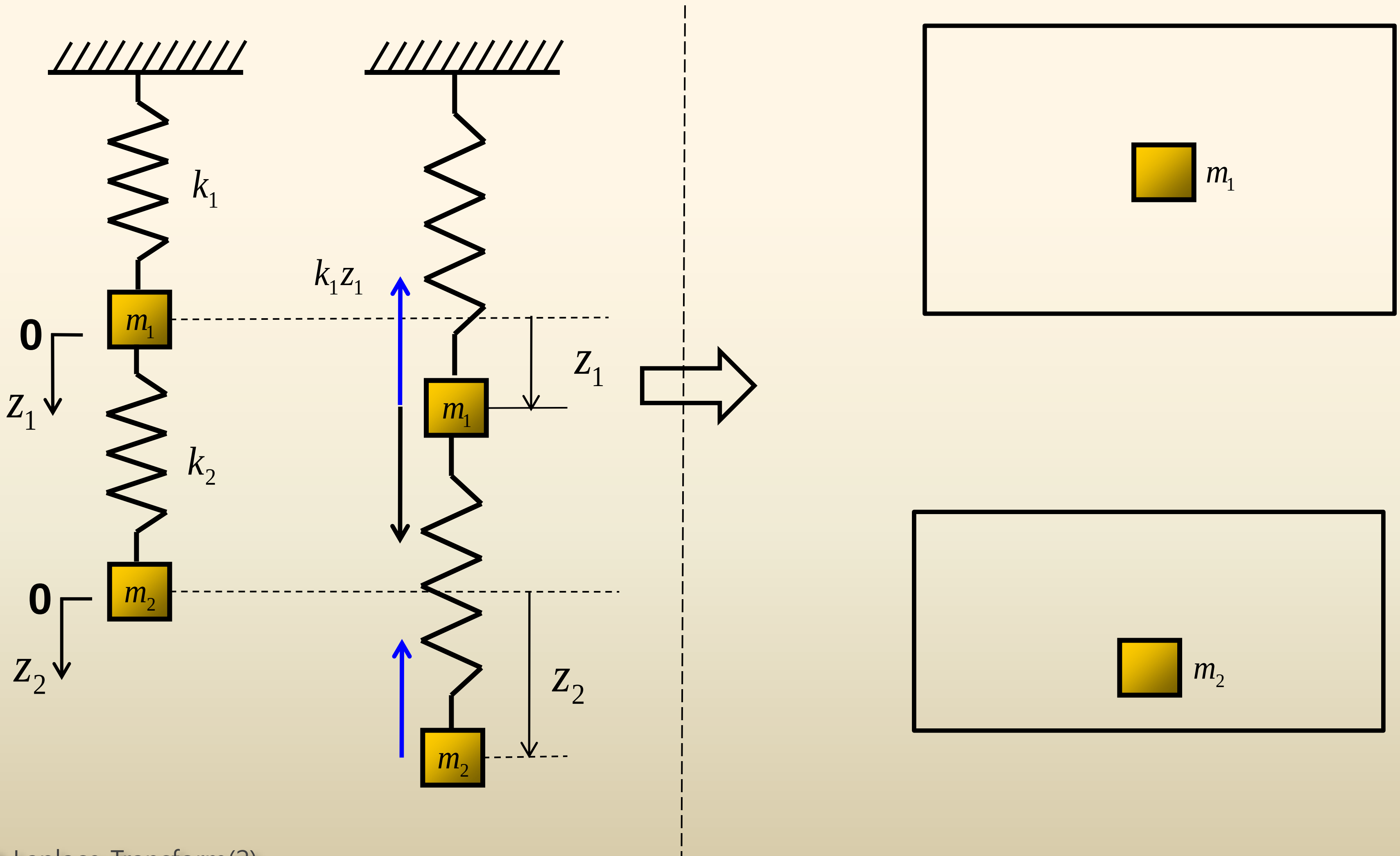

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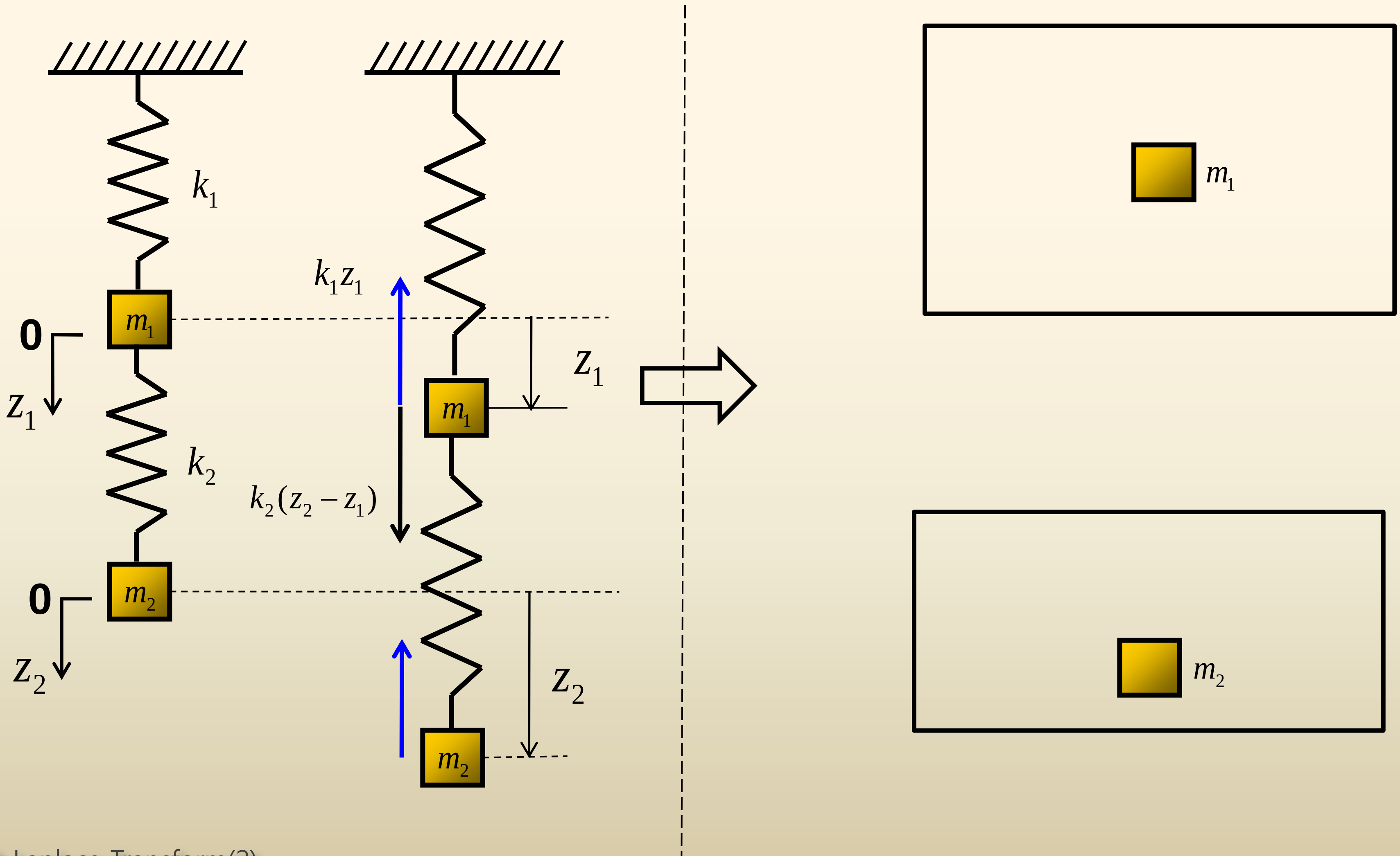

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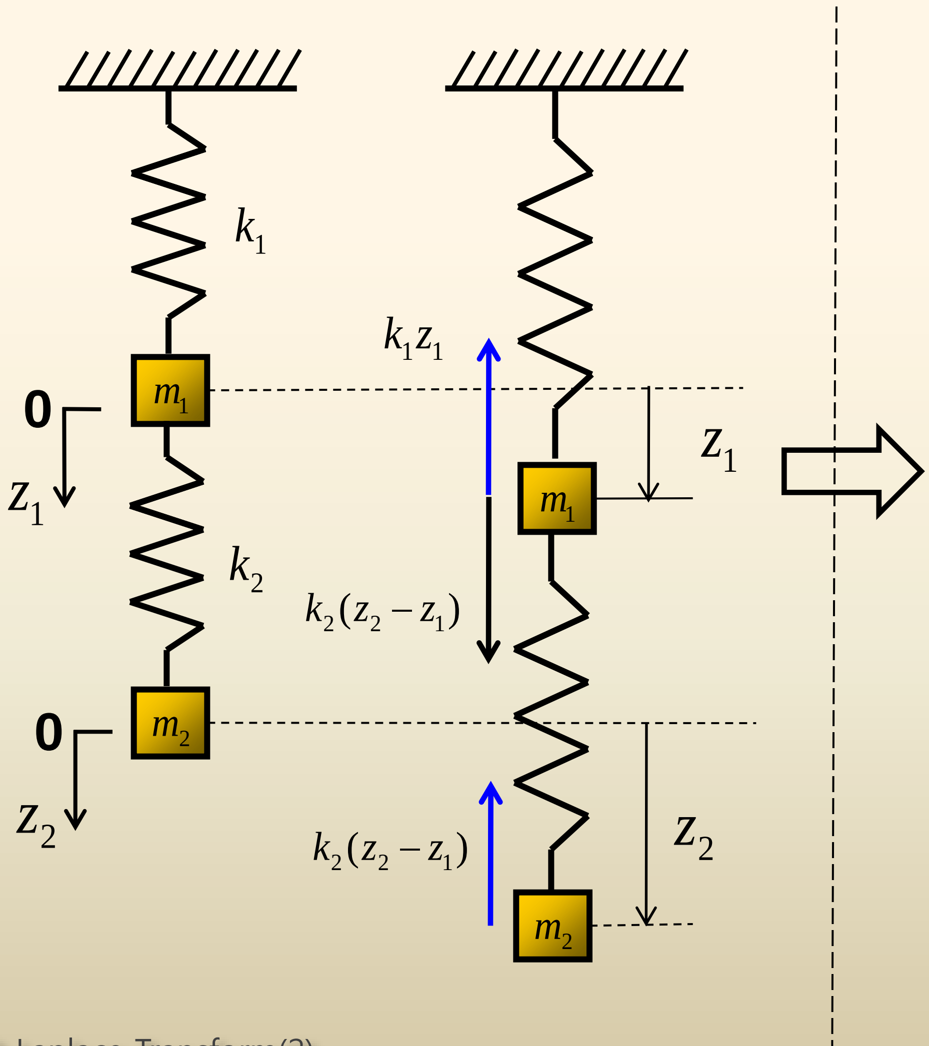

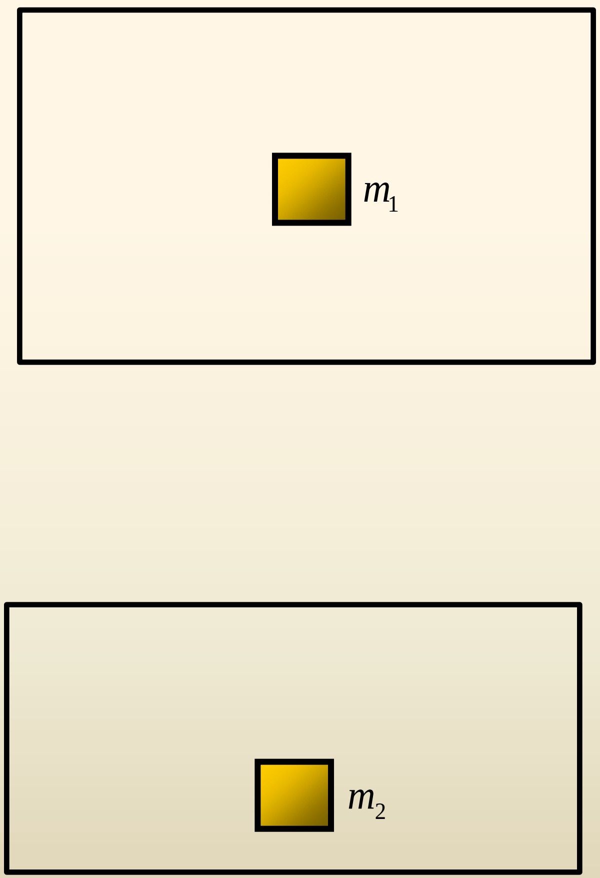

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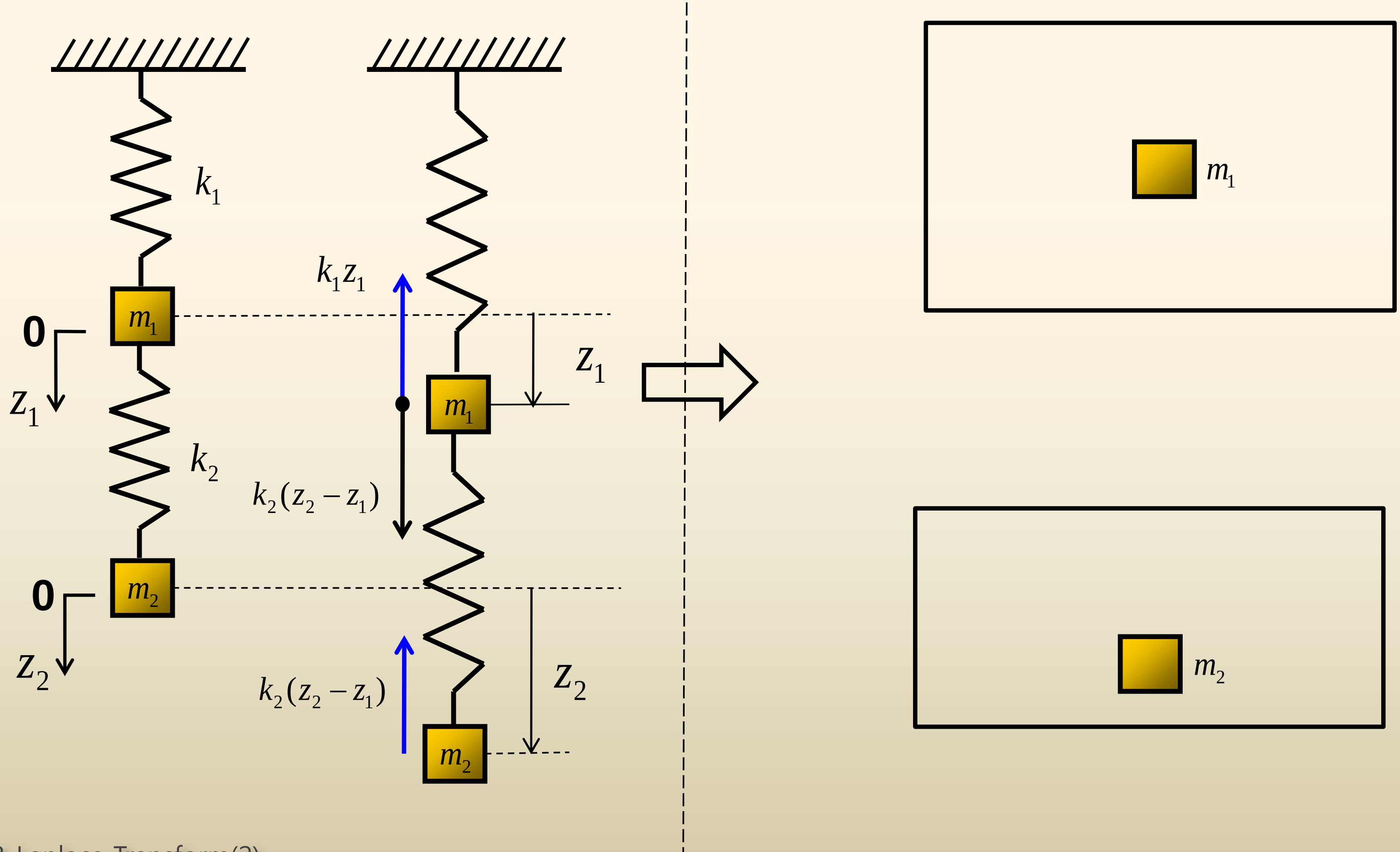

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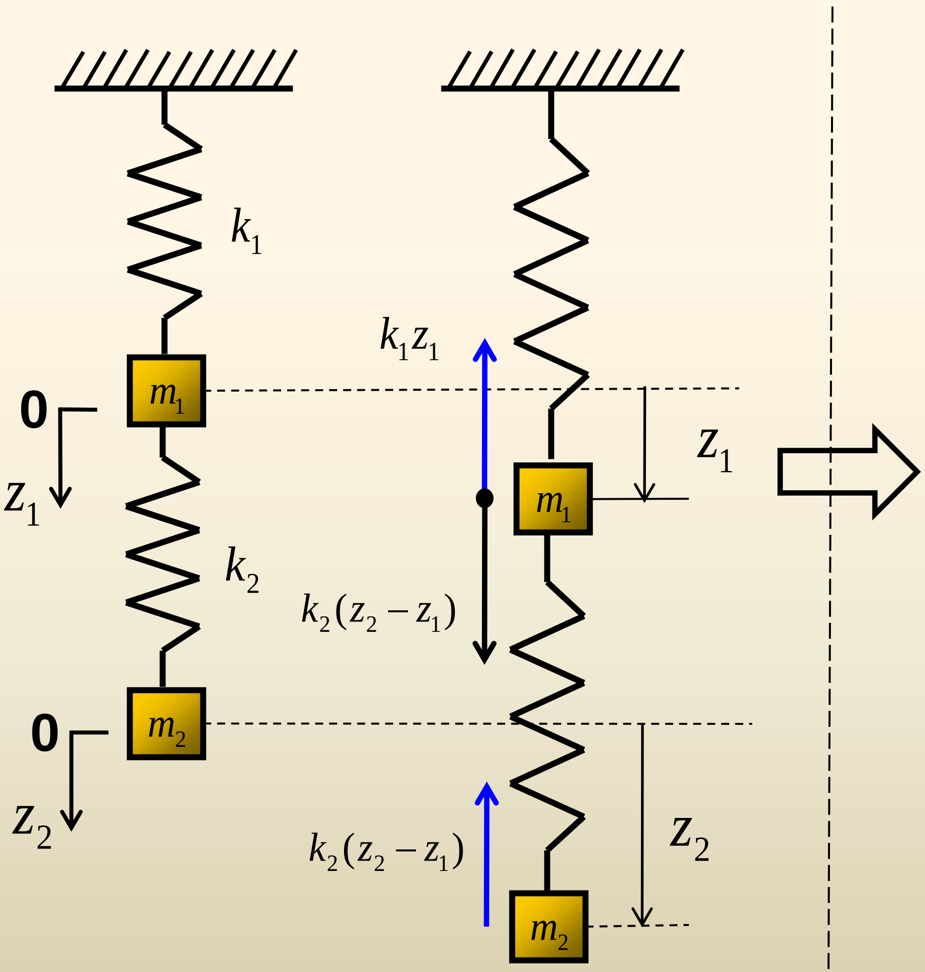
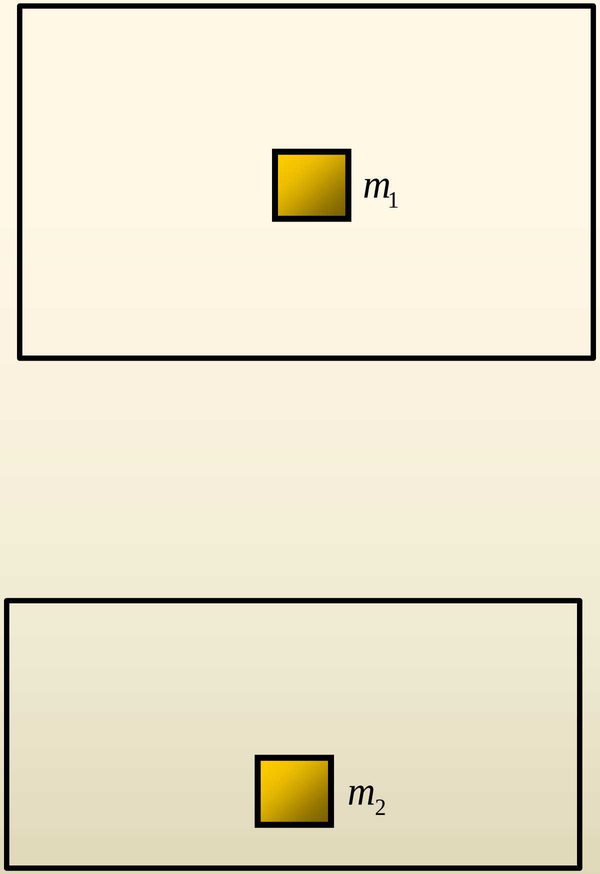


(a) equilibrium



Systems of Linear Differential Equations

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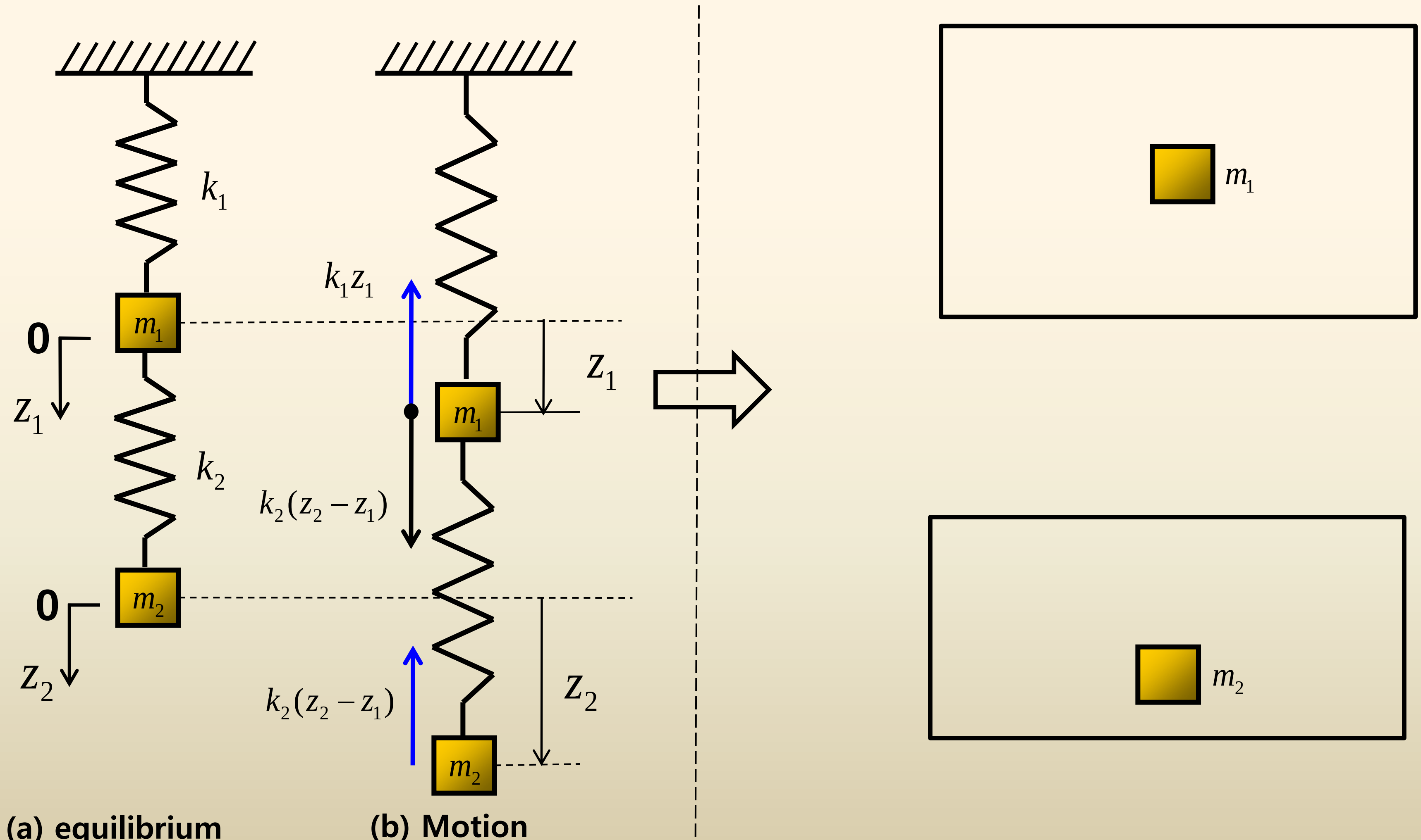

Systems of Linear Differential Equations

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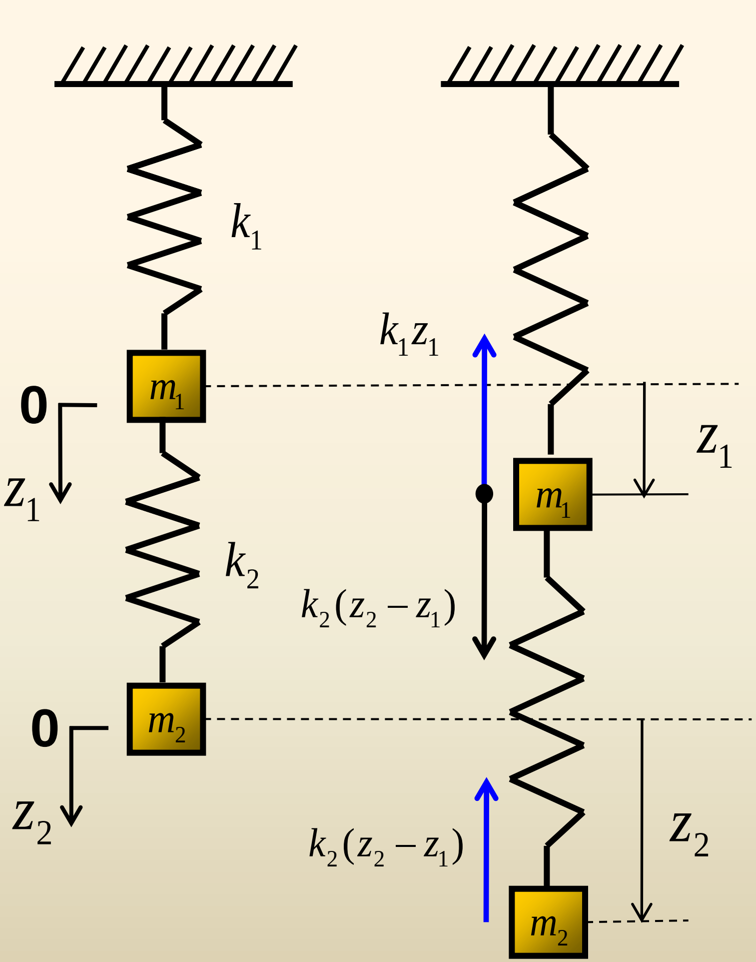
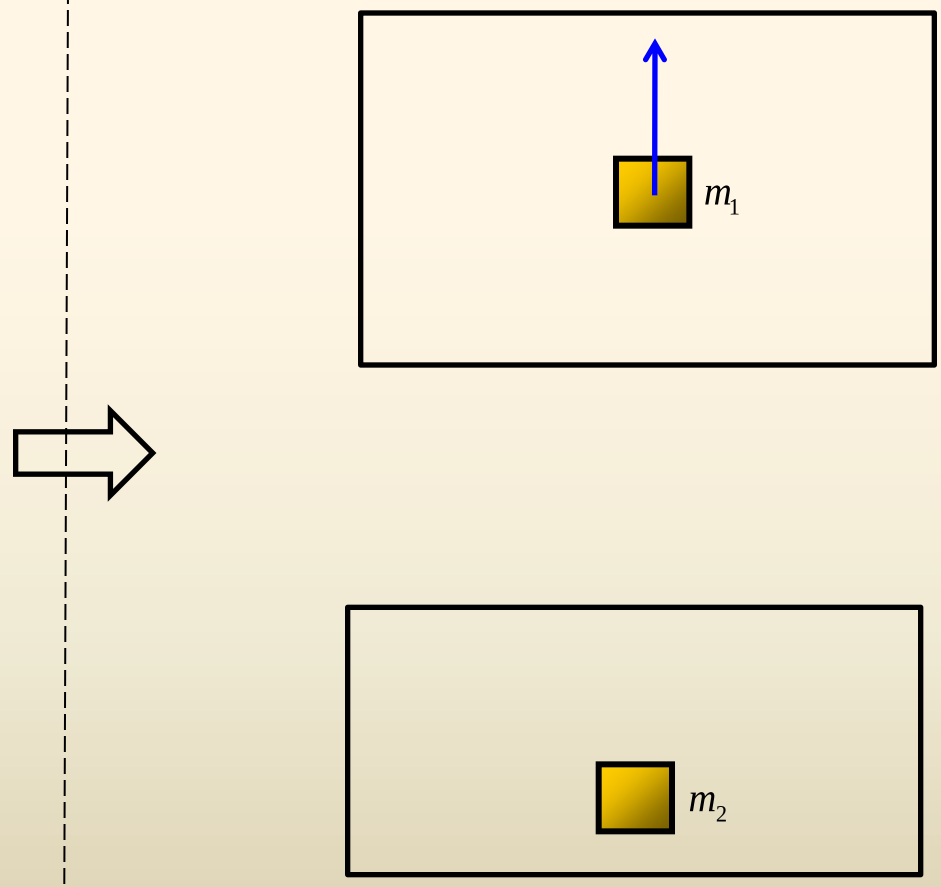


(a) equilibrium (b) Motion



Systems of Linear Differential Equations

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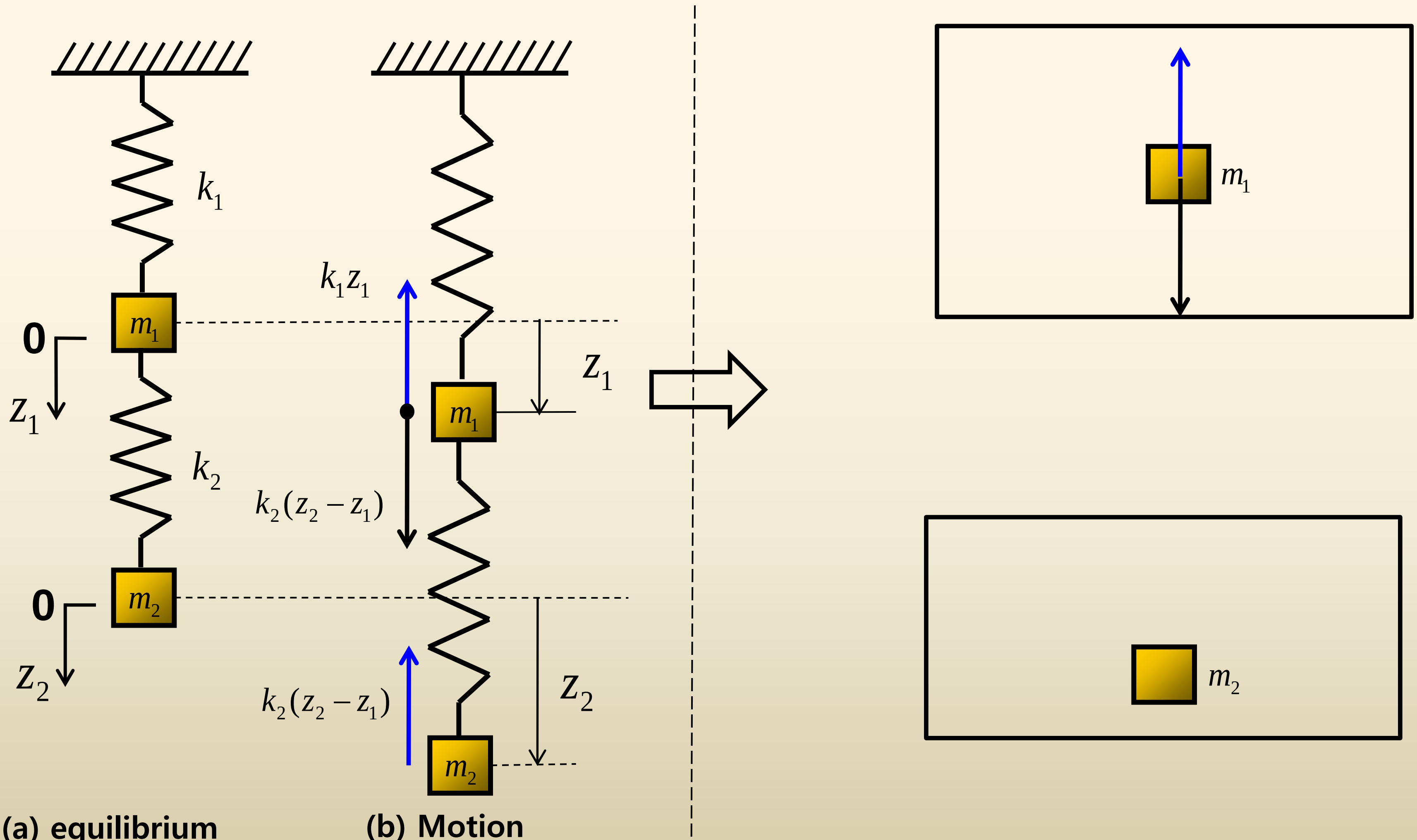

Systems of Linear Differential Equations

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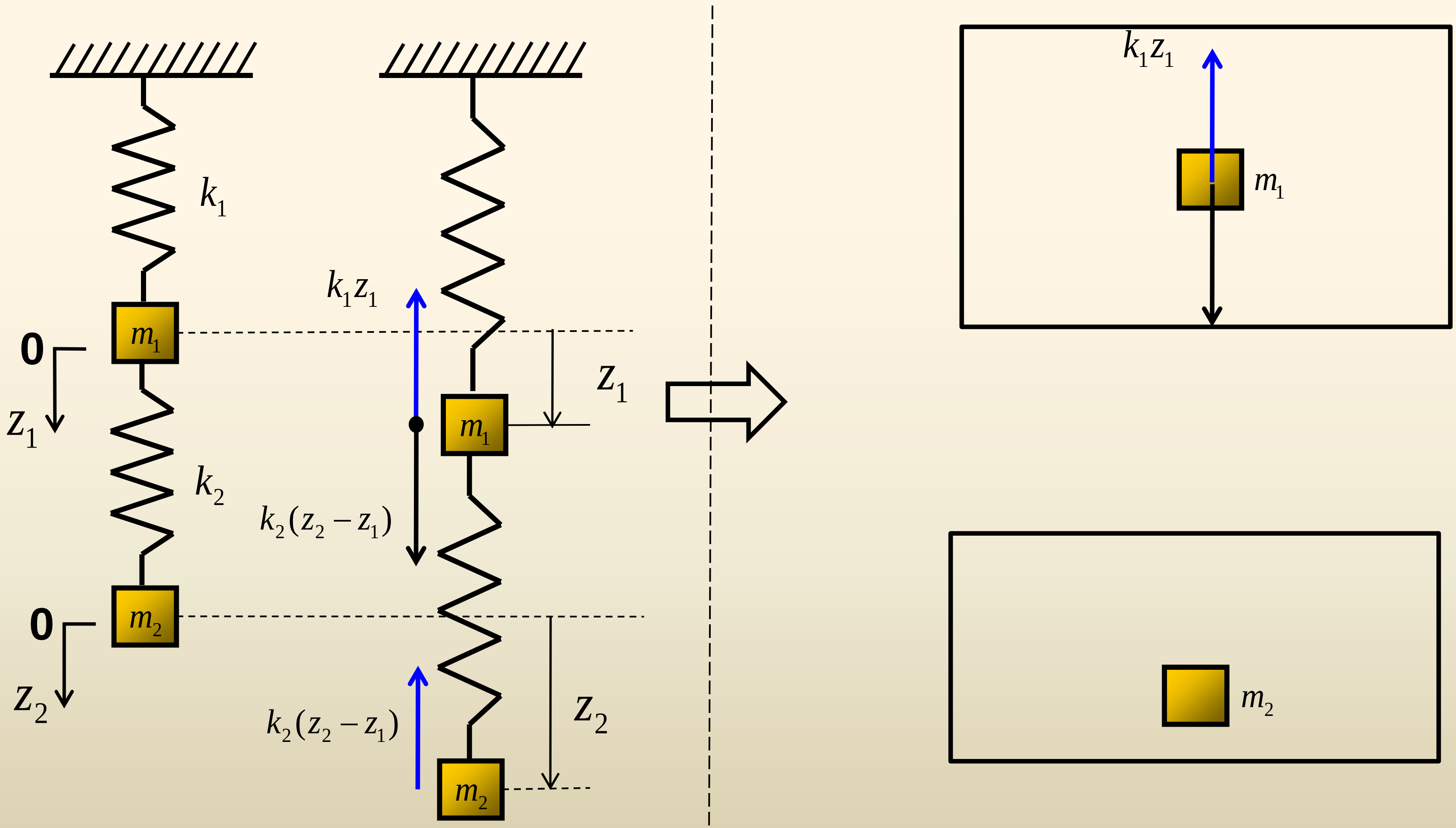
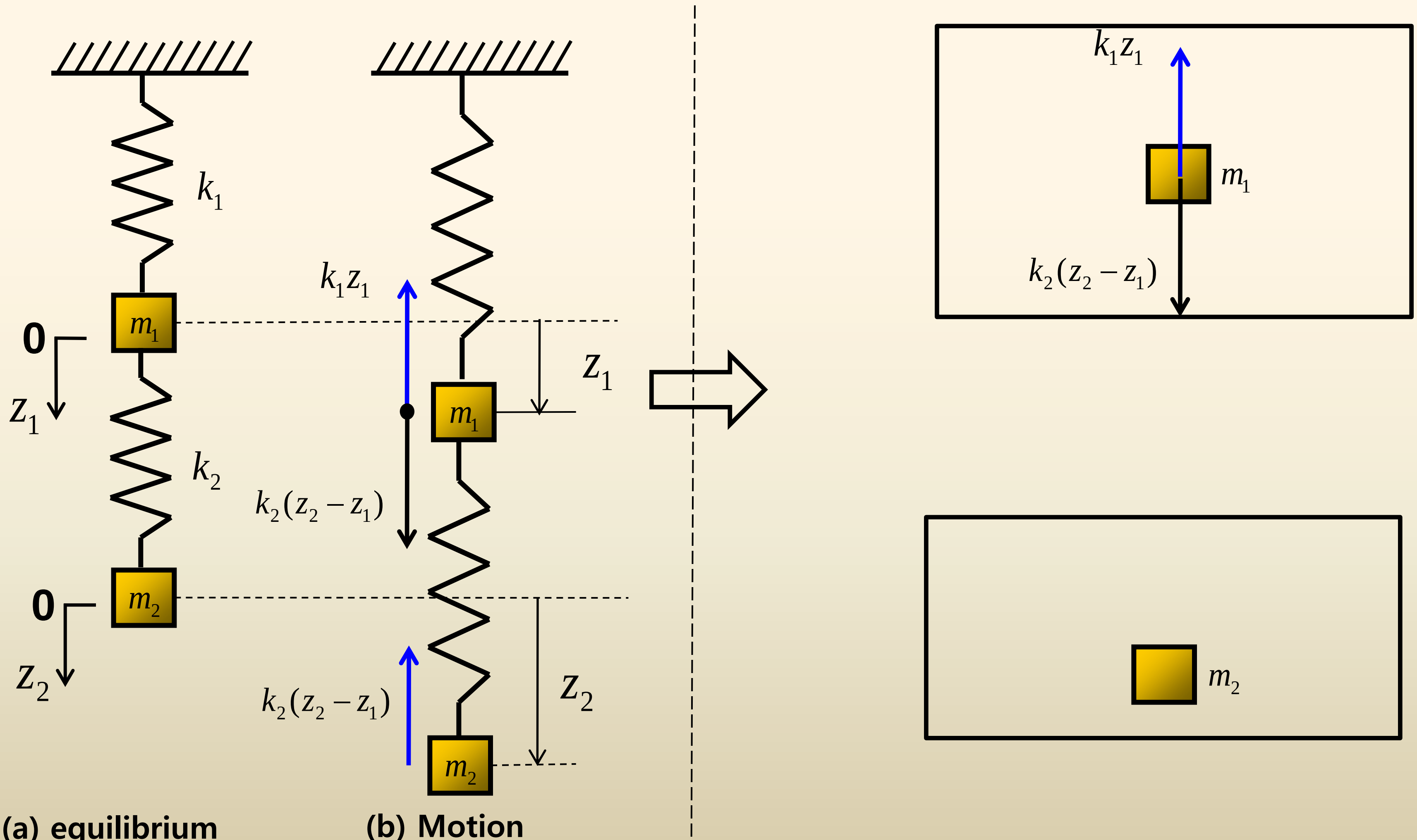


(a) equilibrium

(b) Motion

Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

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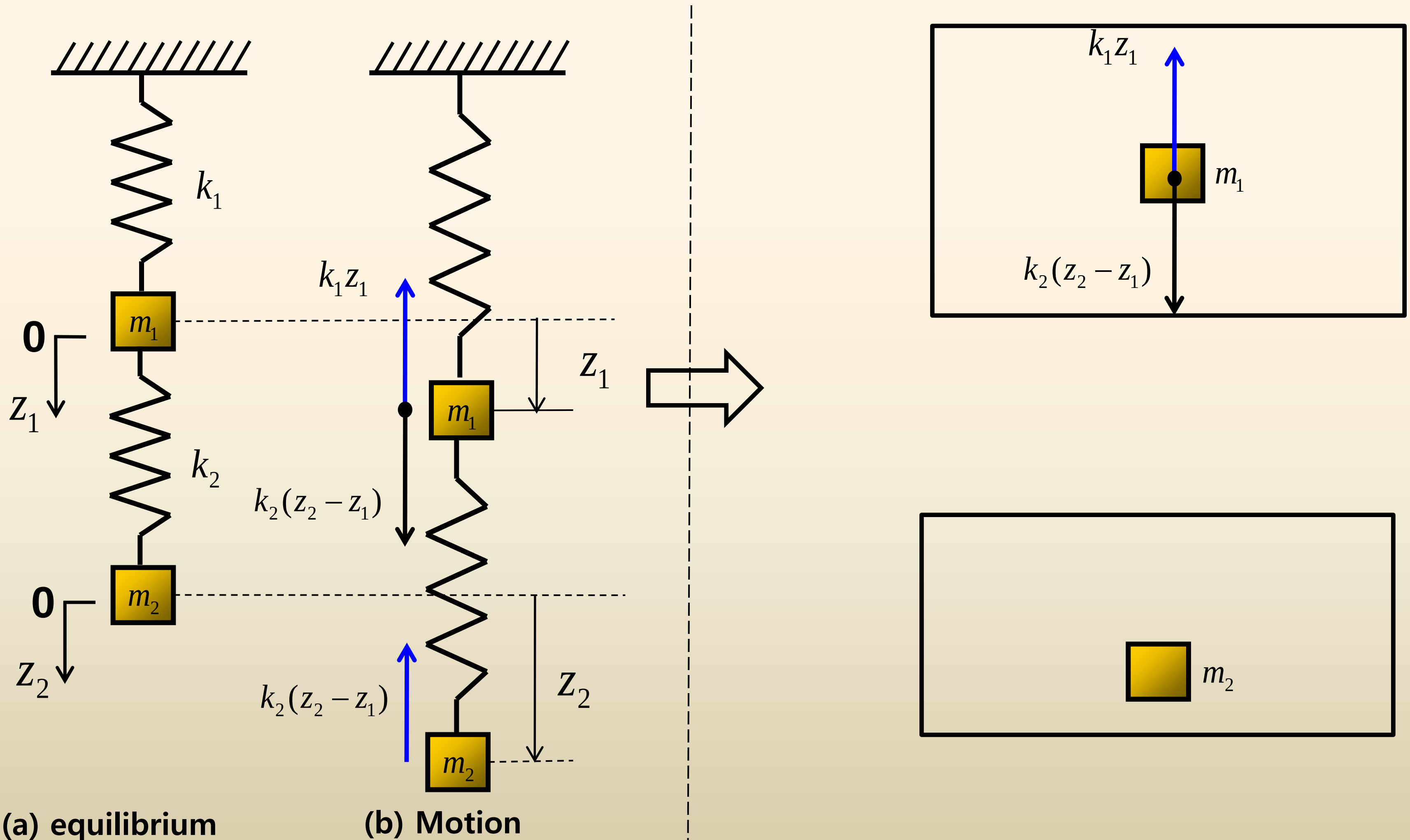

Systems of Linear Differential Equations

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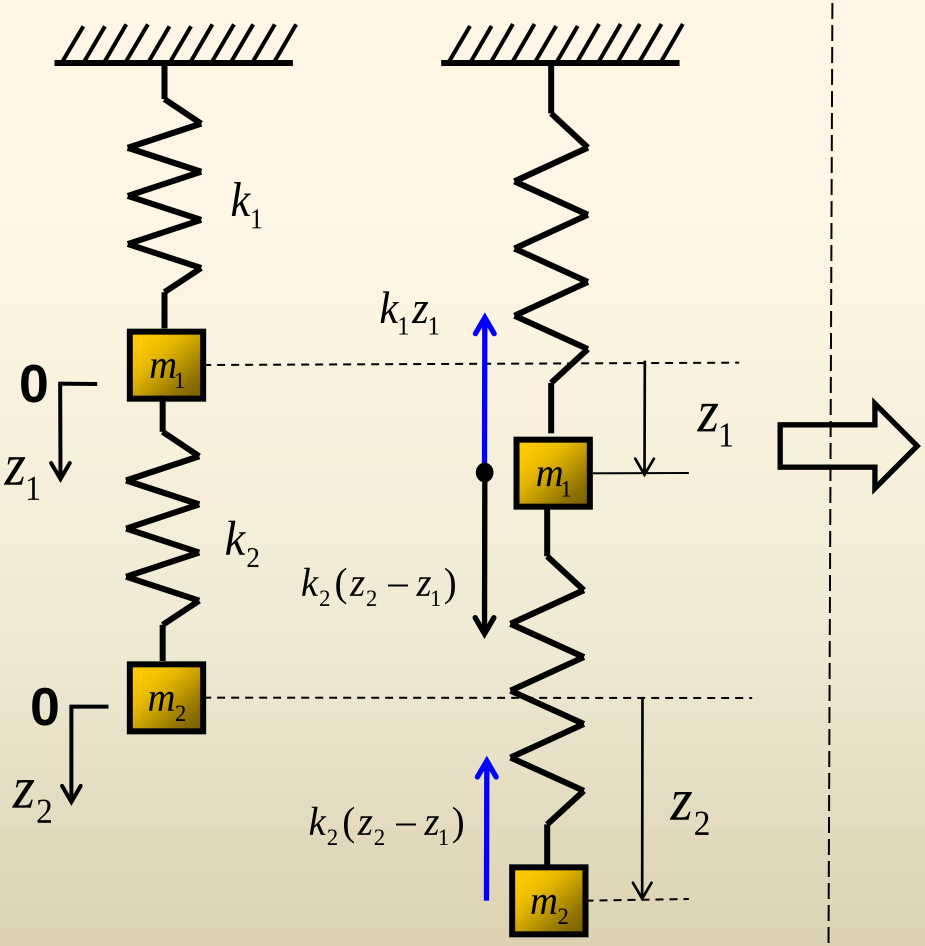
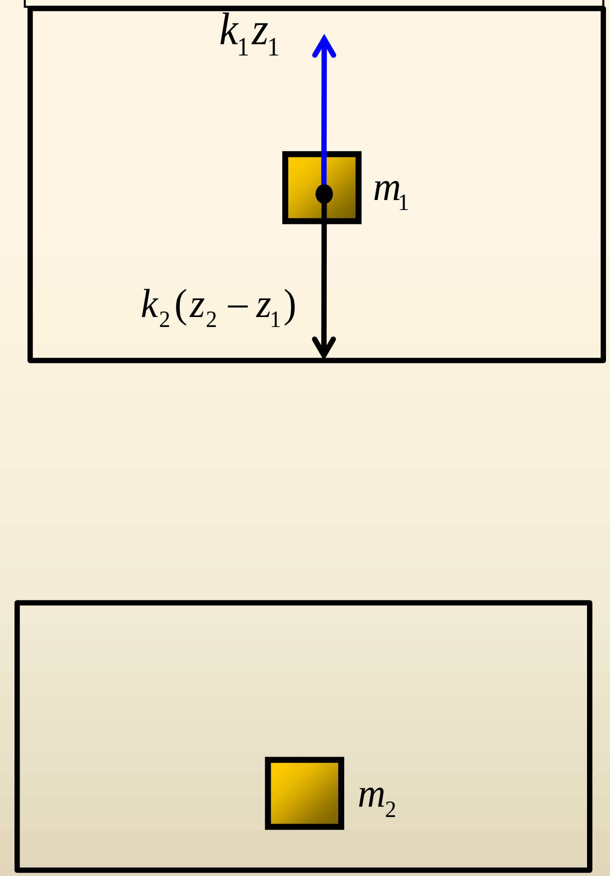


(a) equilibrium (b) Motion

Free body diagram for m_1



Systems of Linear Differential Equations

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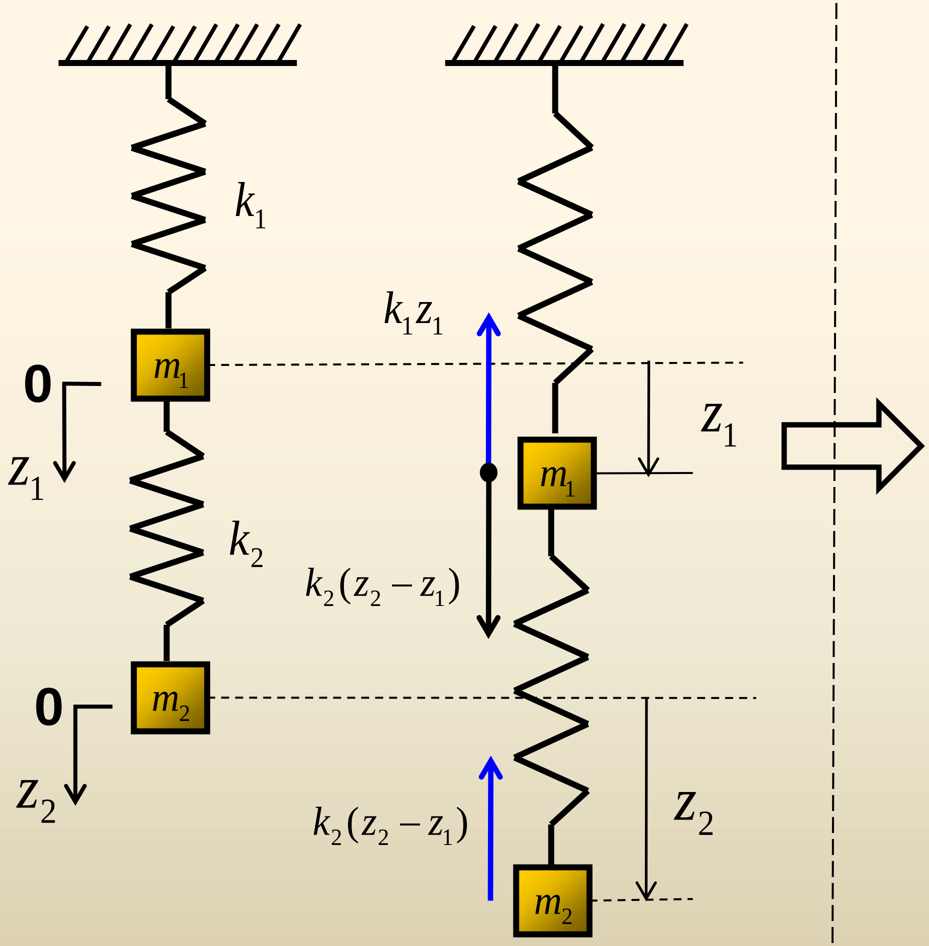

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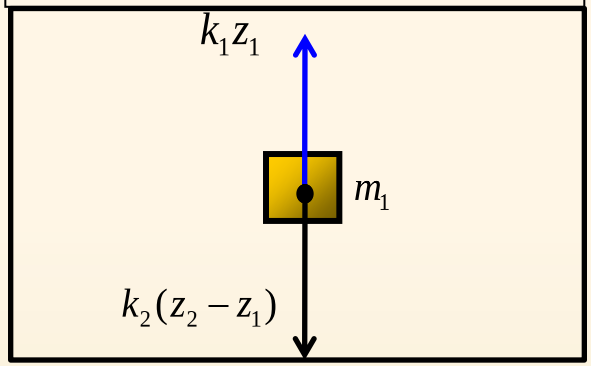
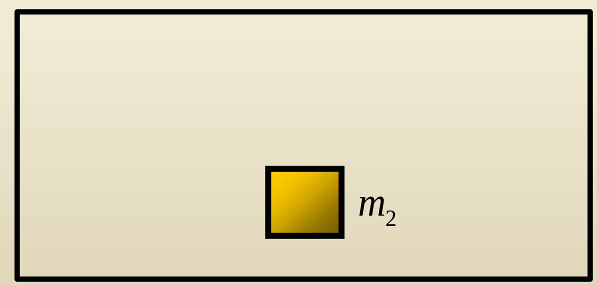


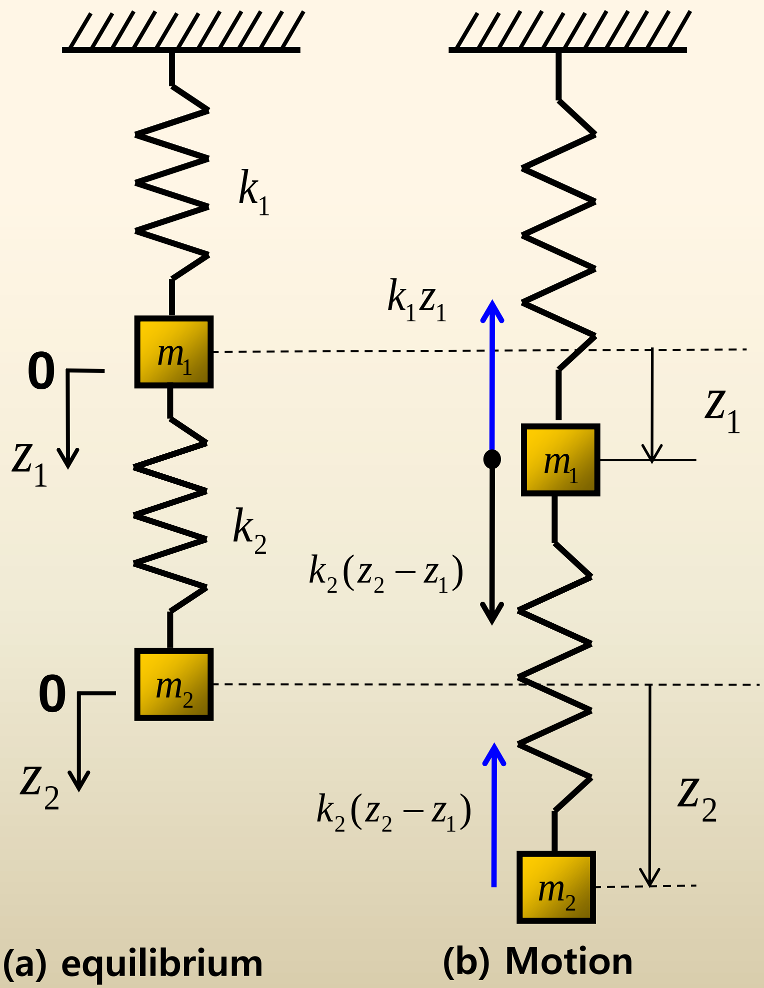


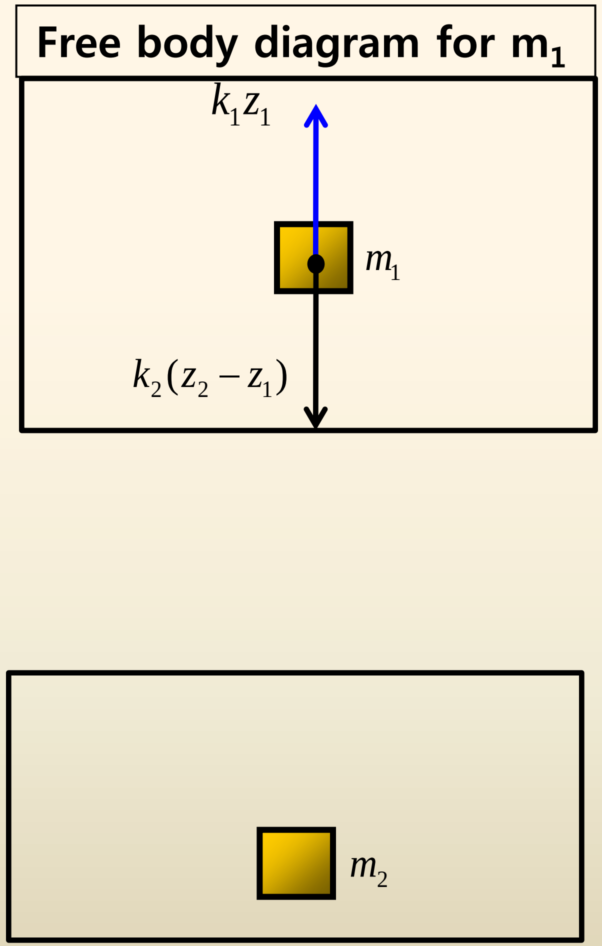
$$m_2 \frac{d^2 z_2}{dt^2} = -k_2(z_2 - z_1)$$

Systems of Linear Differential Equations

☑ Coupled Spring/Mass System

From Newton's 2nd raw,





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Systems of Linear Differential Equations

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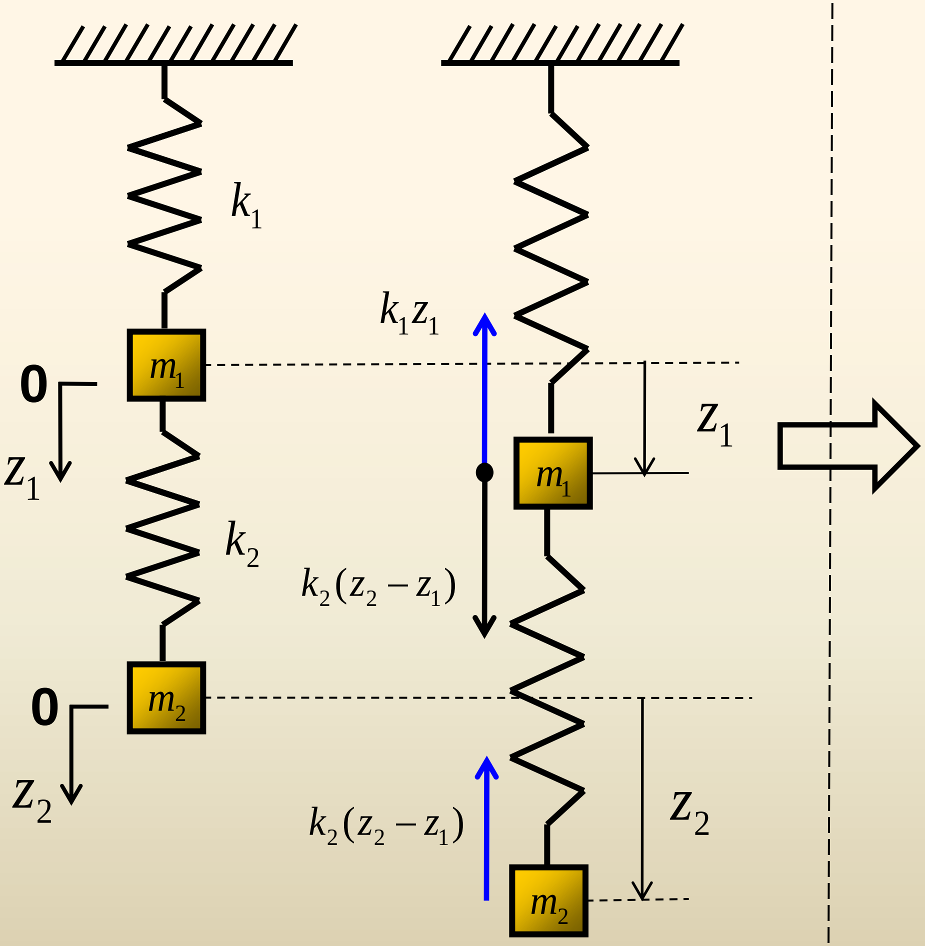
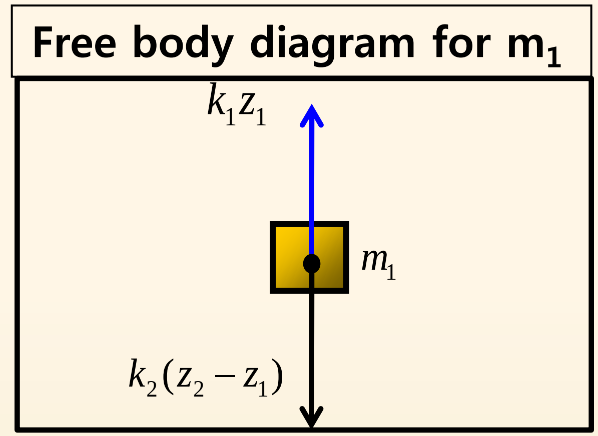
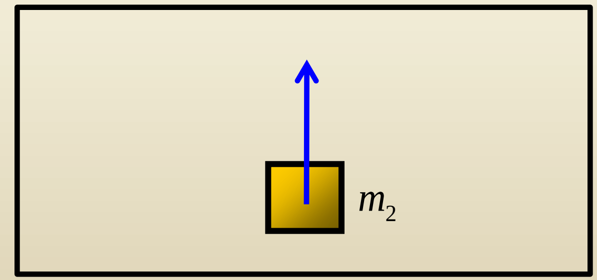


(a) equilibrium **(b) Motion**

From Newton's 2nd raw,

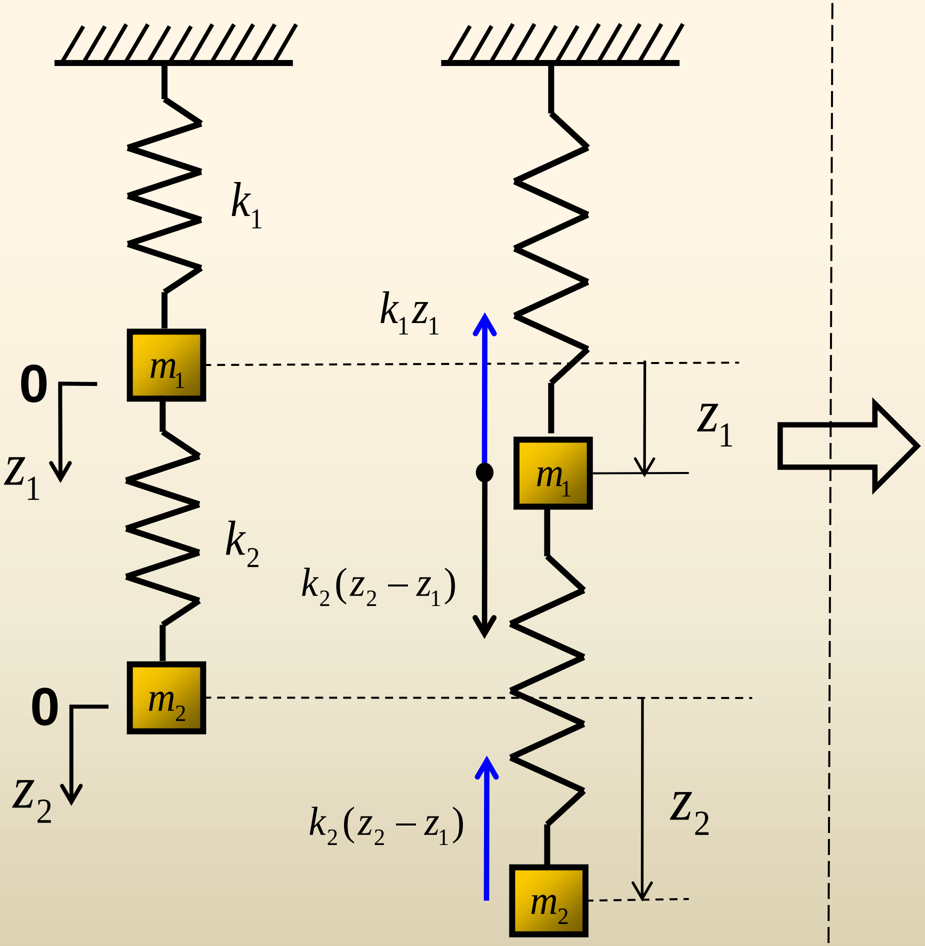




$$m_2 \frac{d^2 z_2}{dt^2} = -k_2(z_2 - z_1)$$

Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



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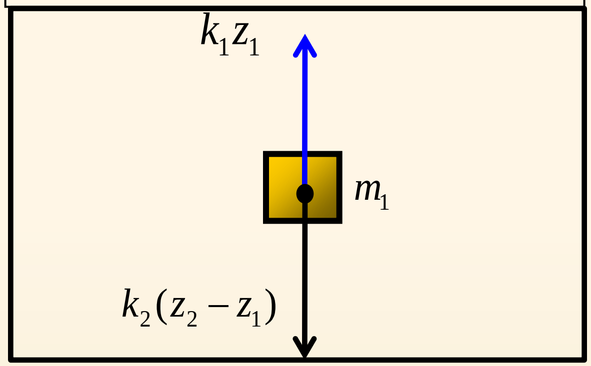
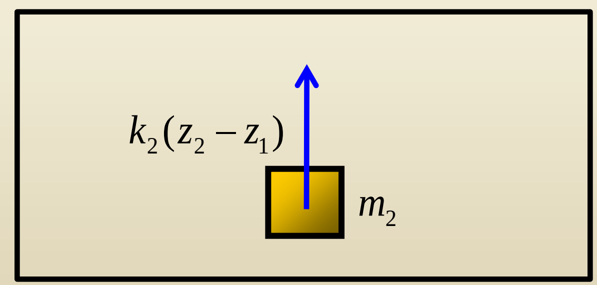
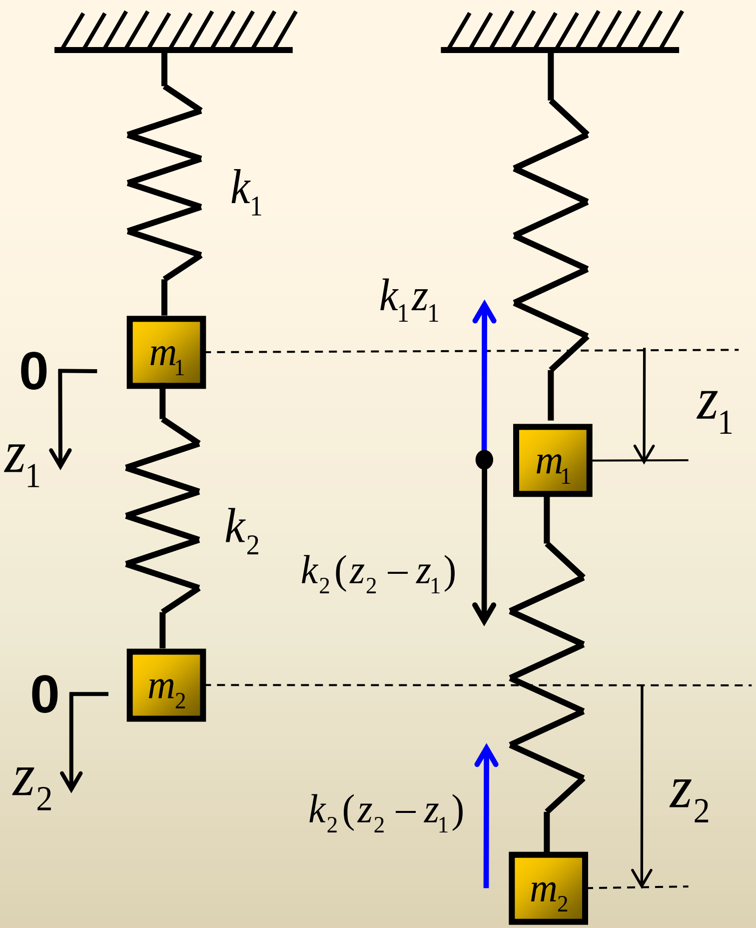




$$m_2 \frac{d^2 z_2}{dt^2} = -k_2(z_2 - z_1)$$

Systems of Linear Differential Equations

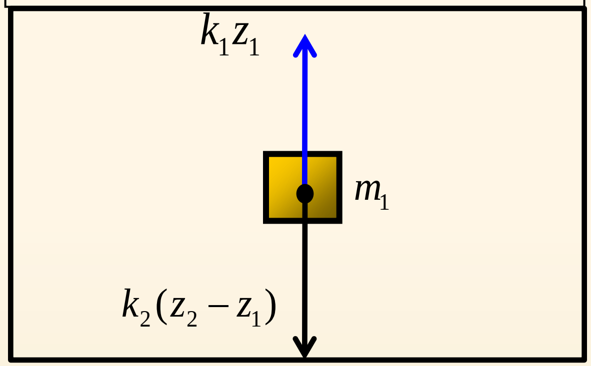
☑ Coupled Spring/Mass System

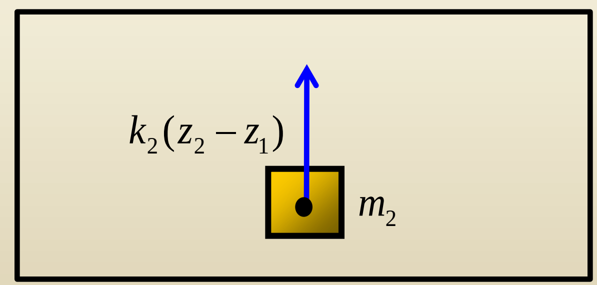


(a) equilibrium (b) Motion

From Newton's 2nd raw,

Free body diagram for m_1

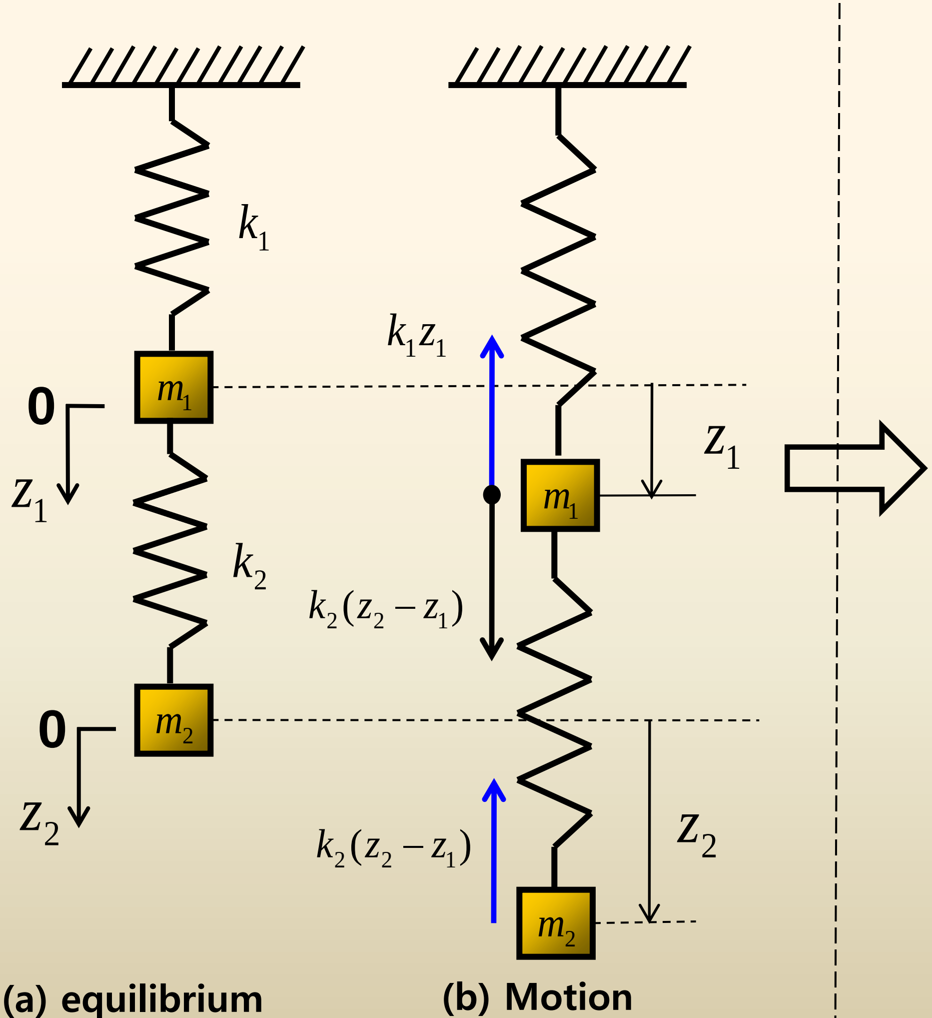




$$m_2 \frac{d^2 z_2}{dt^2} = -k_2(z_2 - z_1)$$

Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



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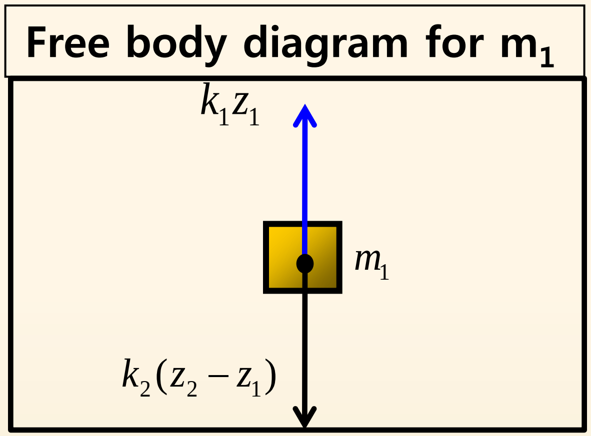
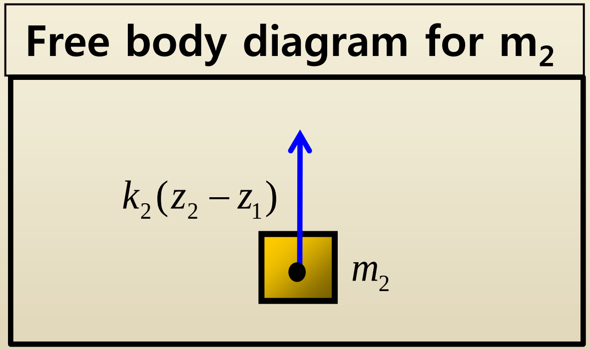
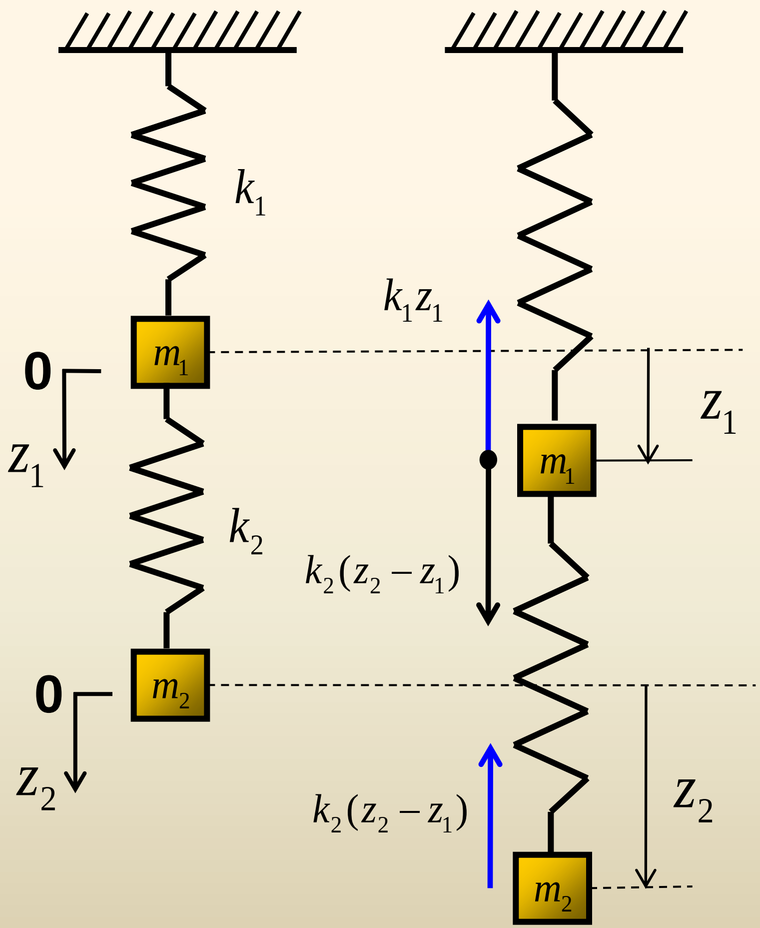




$$m_2 \frac{d^2 z_2}{dt^2} = -k_2(z_2 - z_1)$$

Systems of Linear Differential Equations

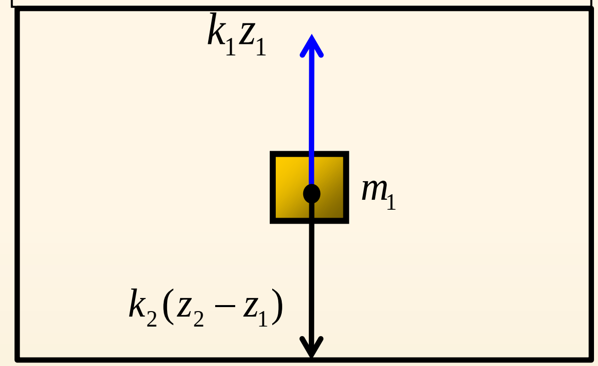
☑ Coupled Spring/Mass System



(a) equilibrium **(b) Motion**

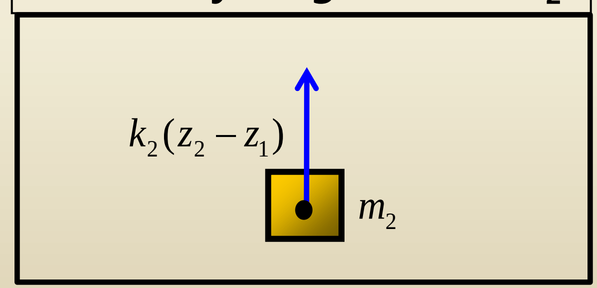
From Newton's 2nd raw,

Free body diagram for m_1



$$m_1 \frac{d^2 z_1}{dt^2} = -k_1 z_1 + k_2 (z_2 - z_1)$$

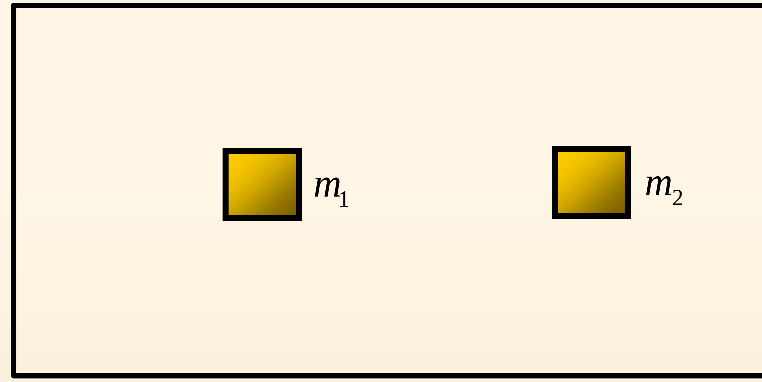
Free body diagram for m_2



$$m_2 \frac{d^2 z_2}{dt^2} = -k_2 (z_2 - z_1)$$

Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



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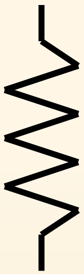

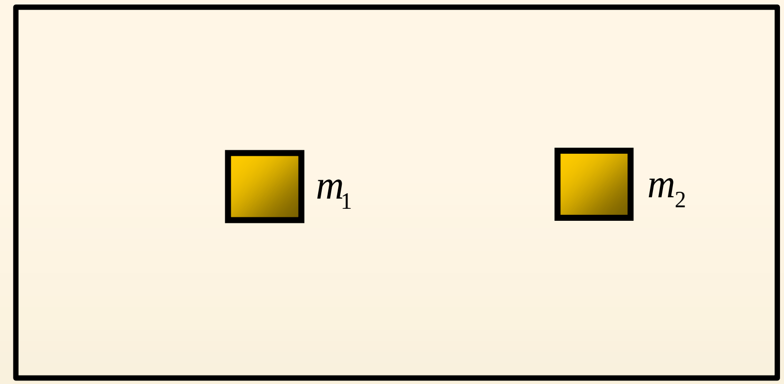

Systems of Linear Differential Equations

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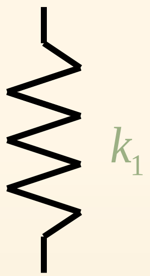

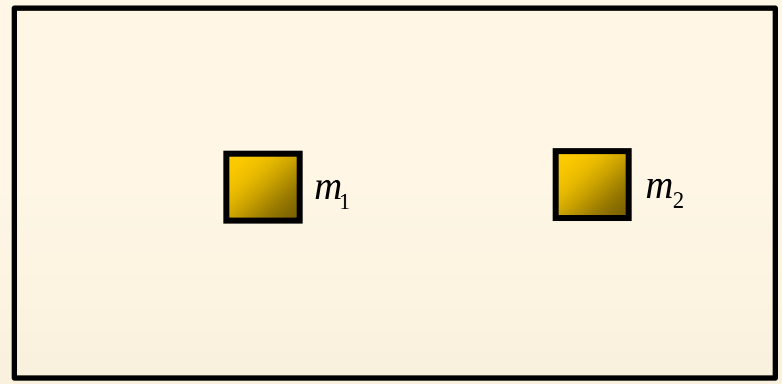

Systems of Linear Differential Equations

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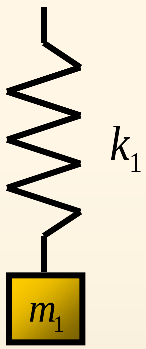

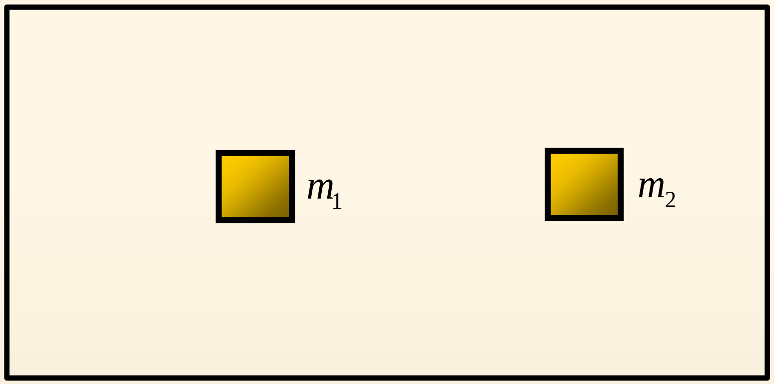

Systems of Linear Differential Equations

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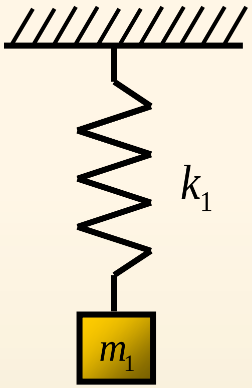
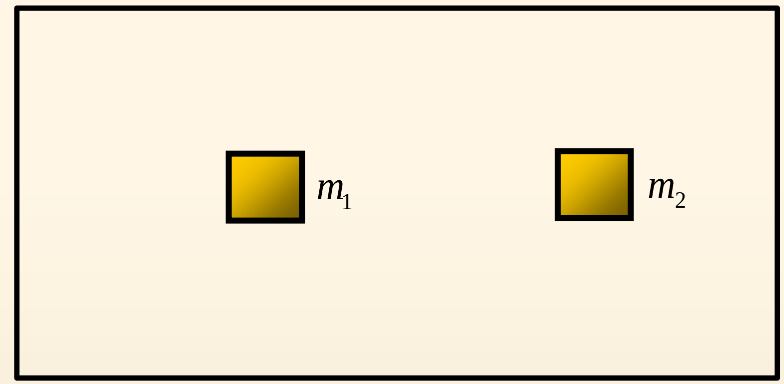
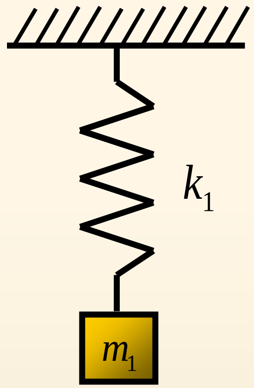




Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



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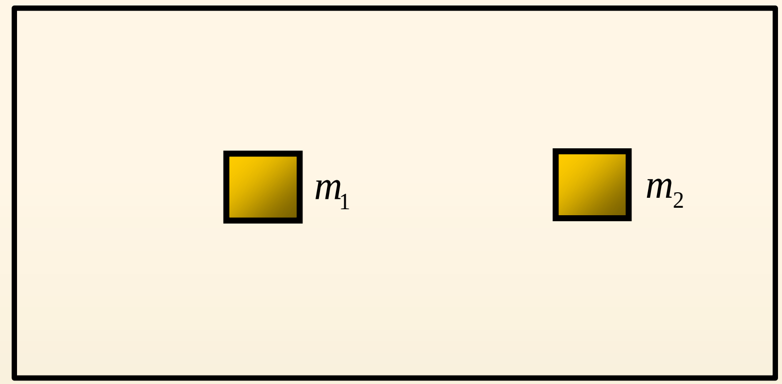

Systems of Linear Differential Equations

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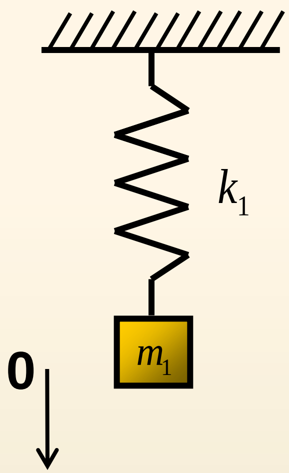

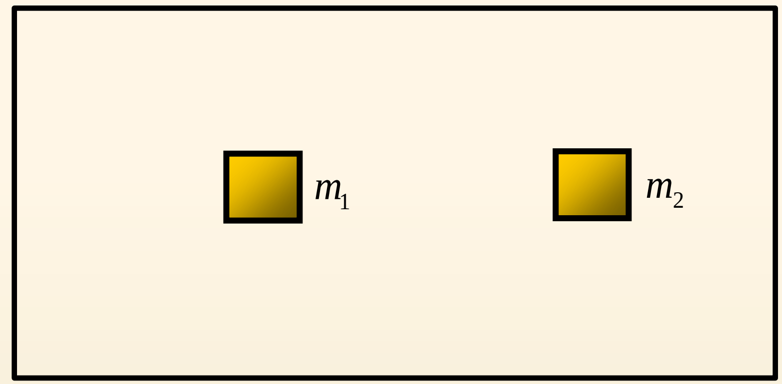

Systems of Linear Differential Equations

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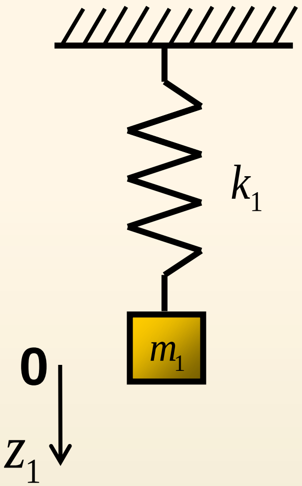

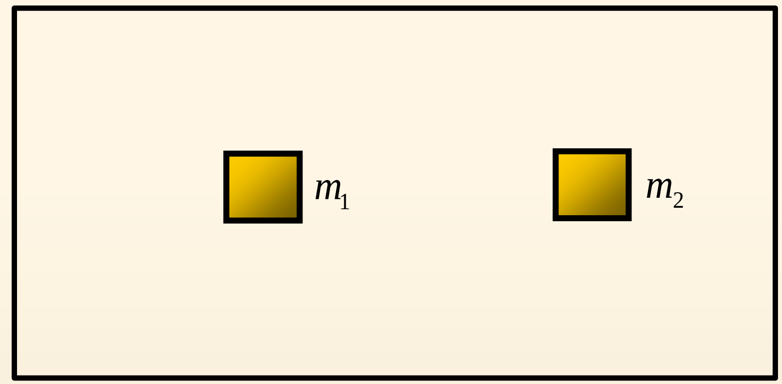

Systems of Linear Differential Equations

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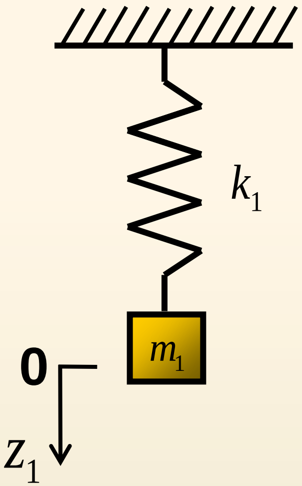

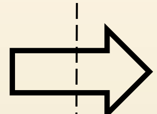

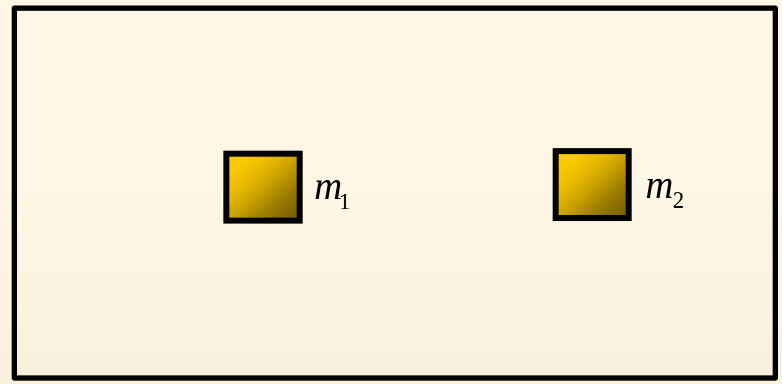

Systems of Linear Differential Equations

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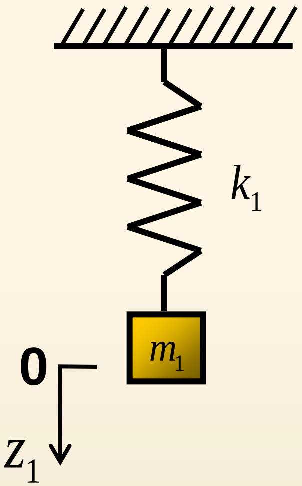

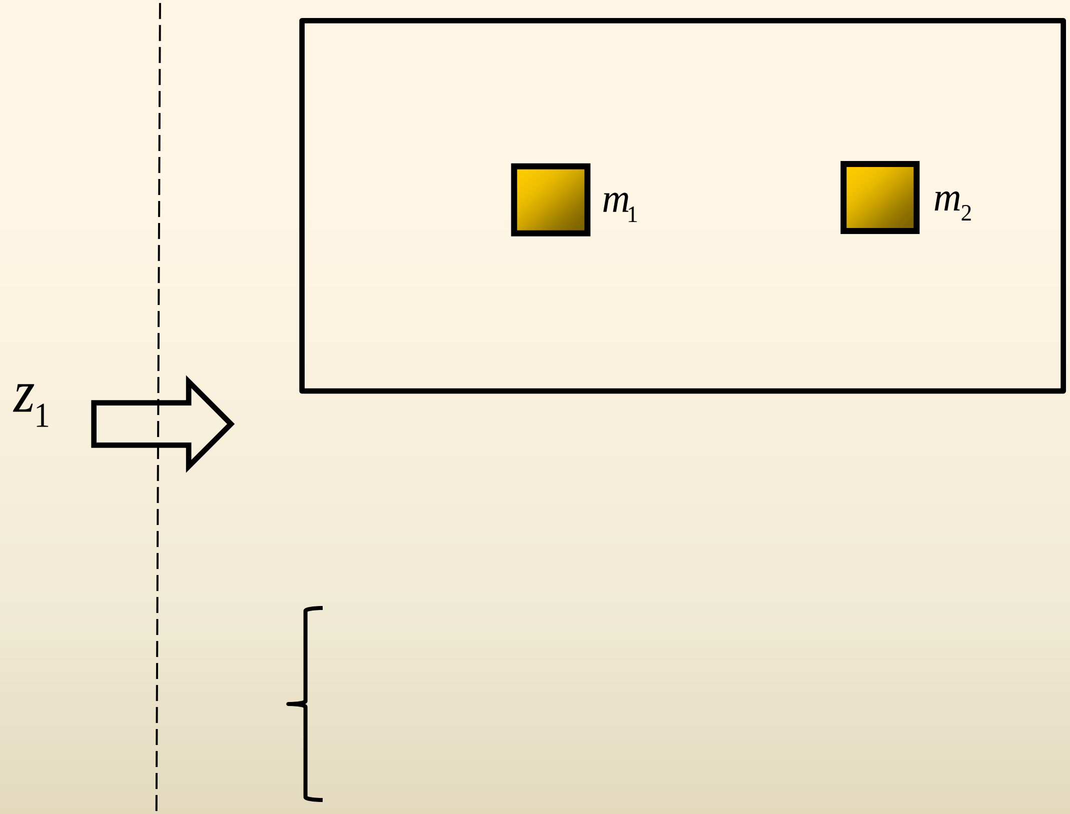

Systems of Linear Differential Equations

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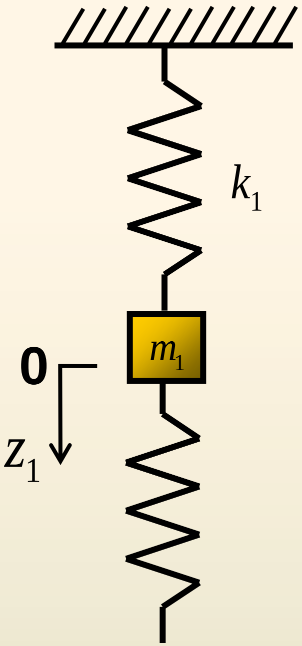

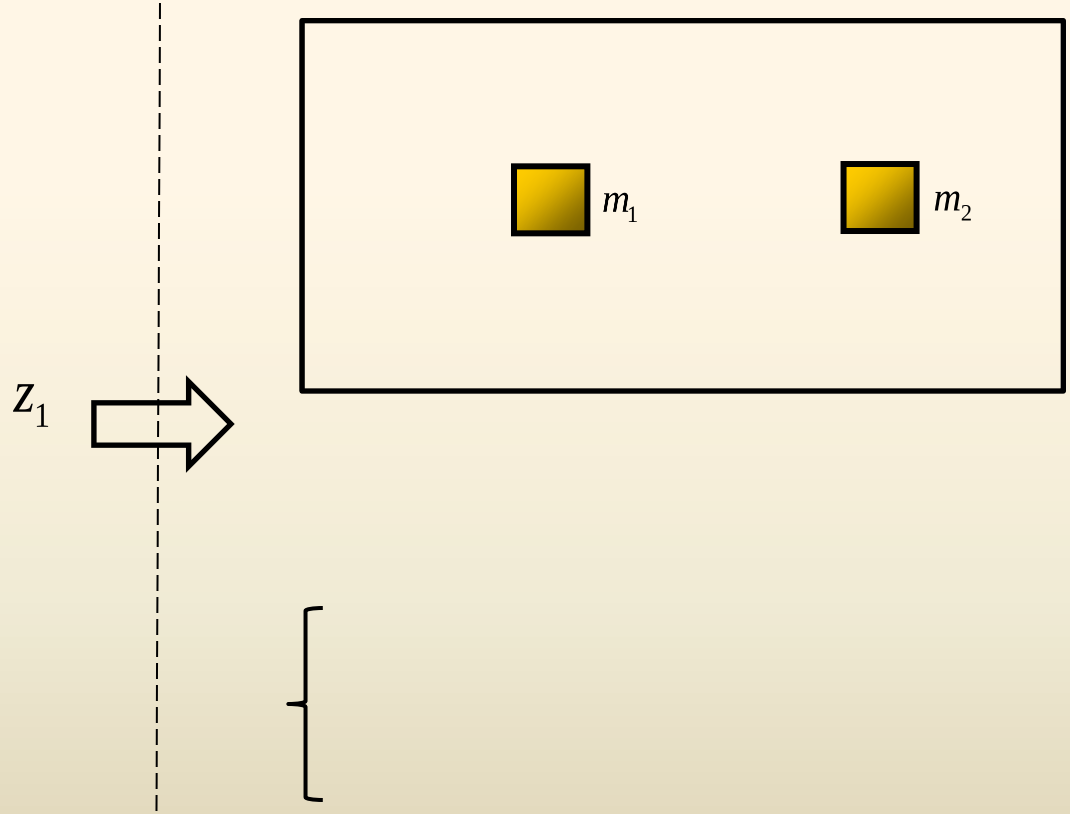

Systems of Linear Differential Equations

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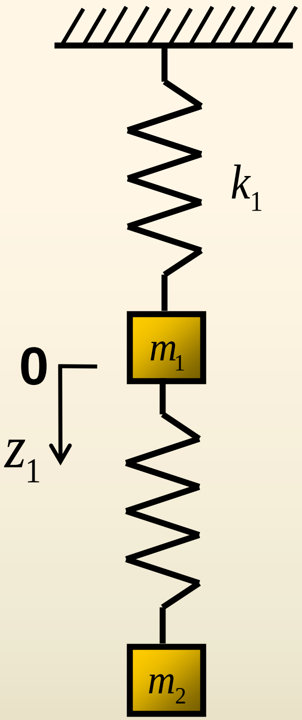

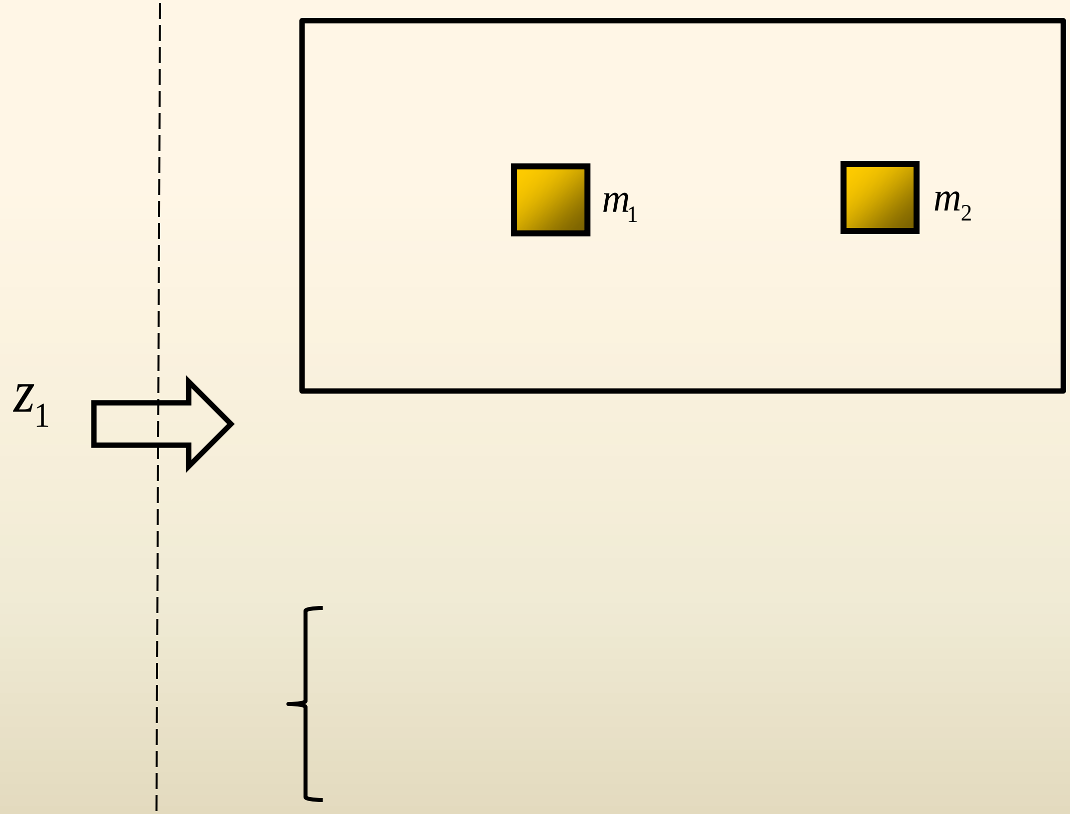

Systems of Linear Differential Equations

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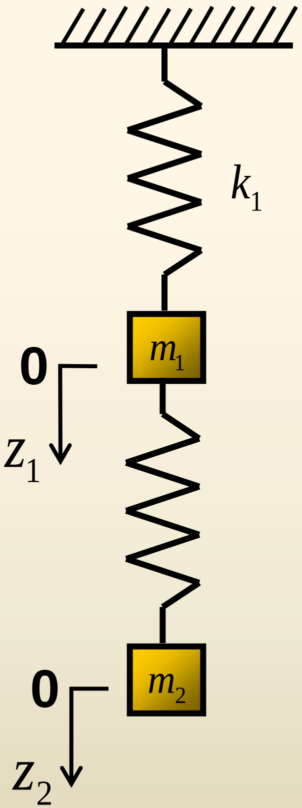

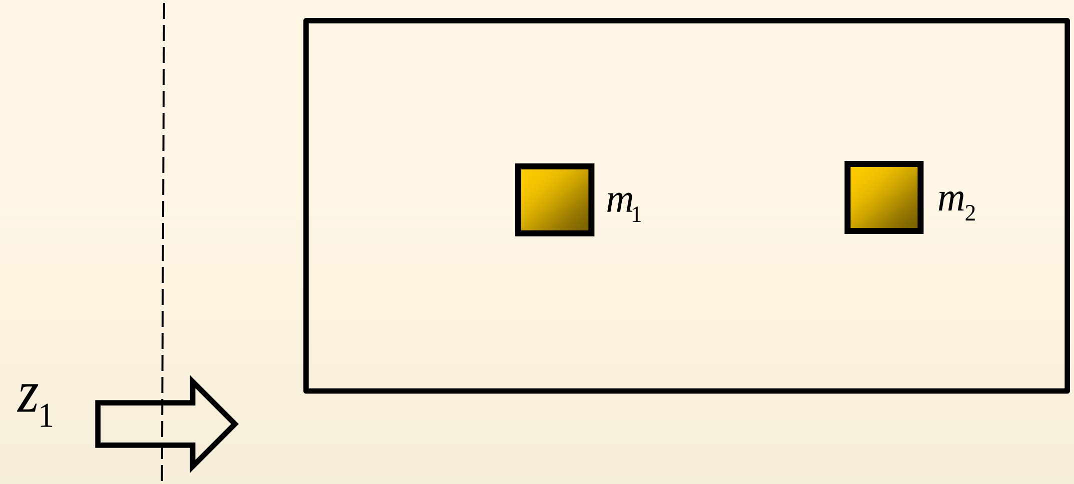

Systems of Linear Differential Equations

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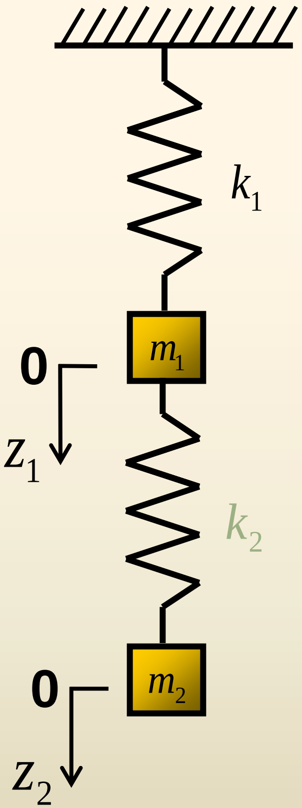

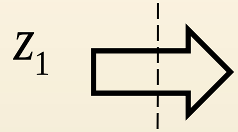

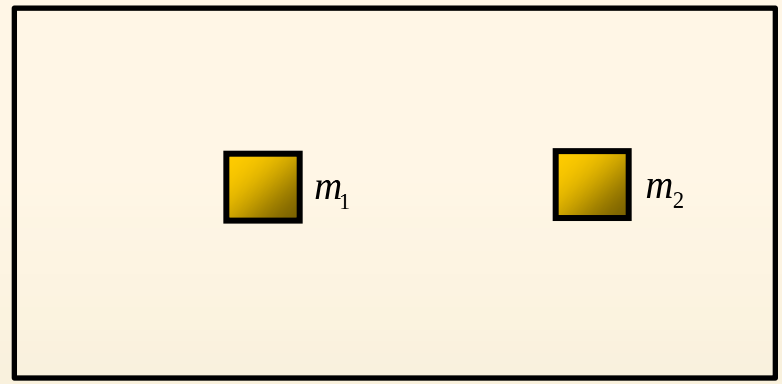

Systems of Linear Differential Equations

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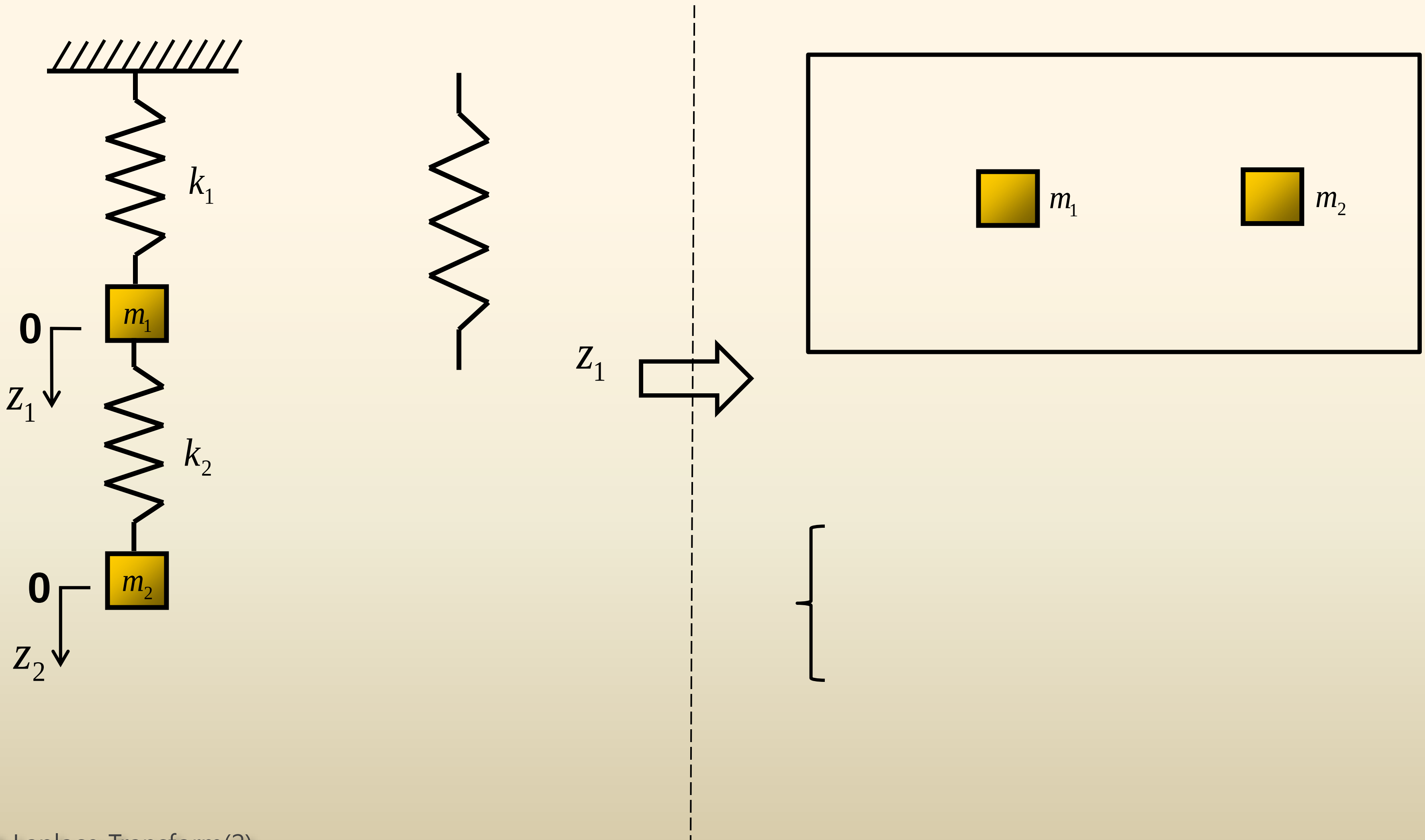

Systems of Linear Differential Equations

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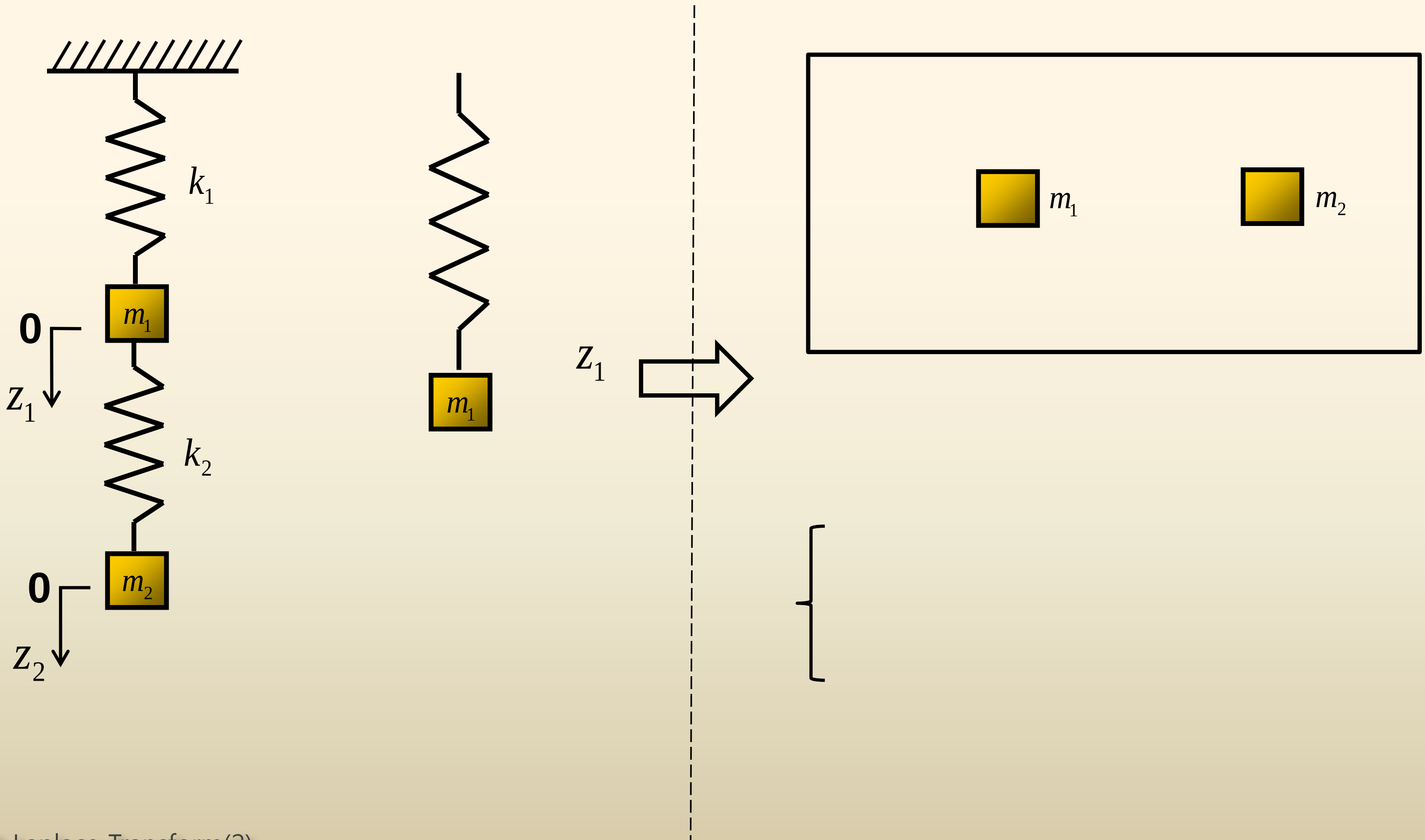

Systems of Linear Differential Equations

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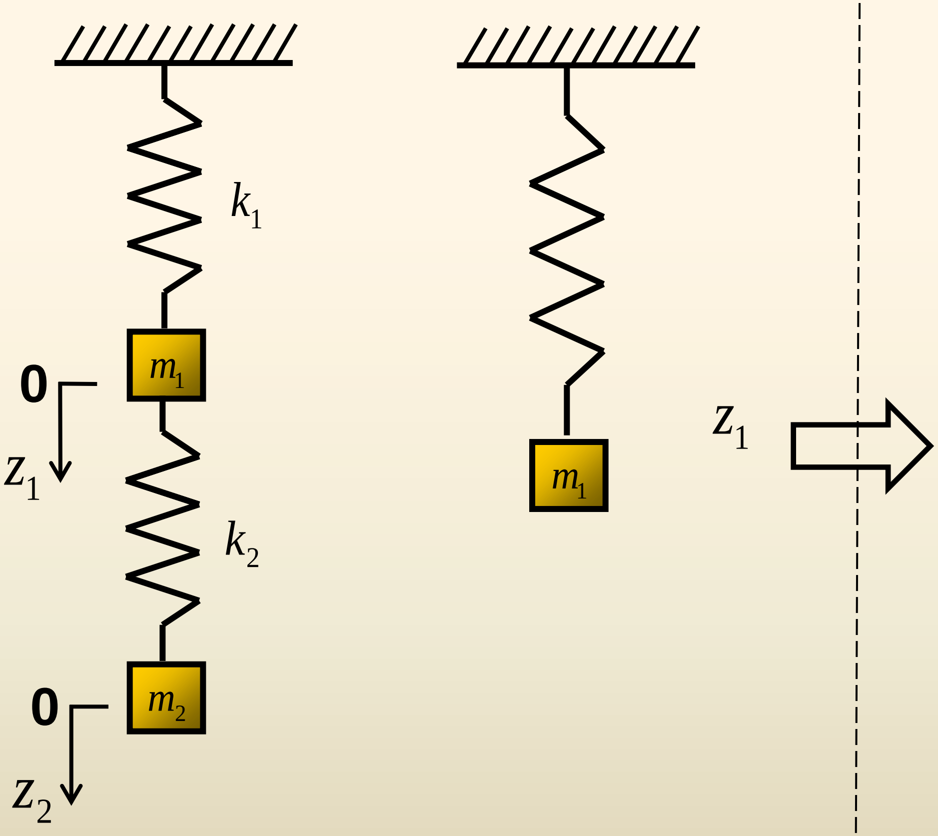

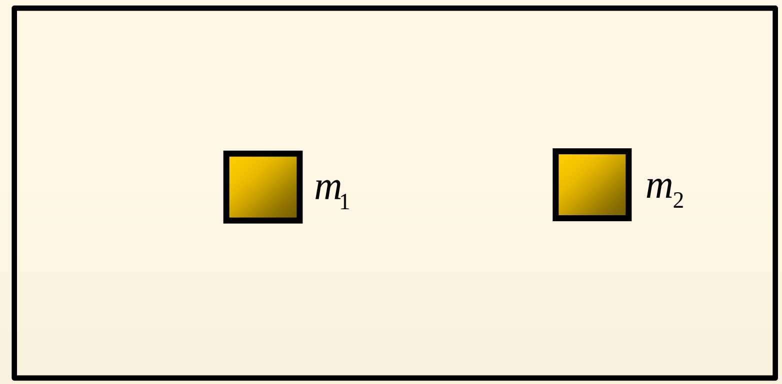

Systems of Linear Differential Equations

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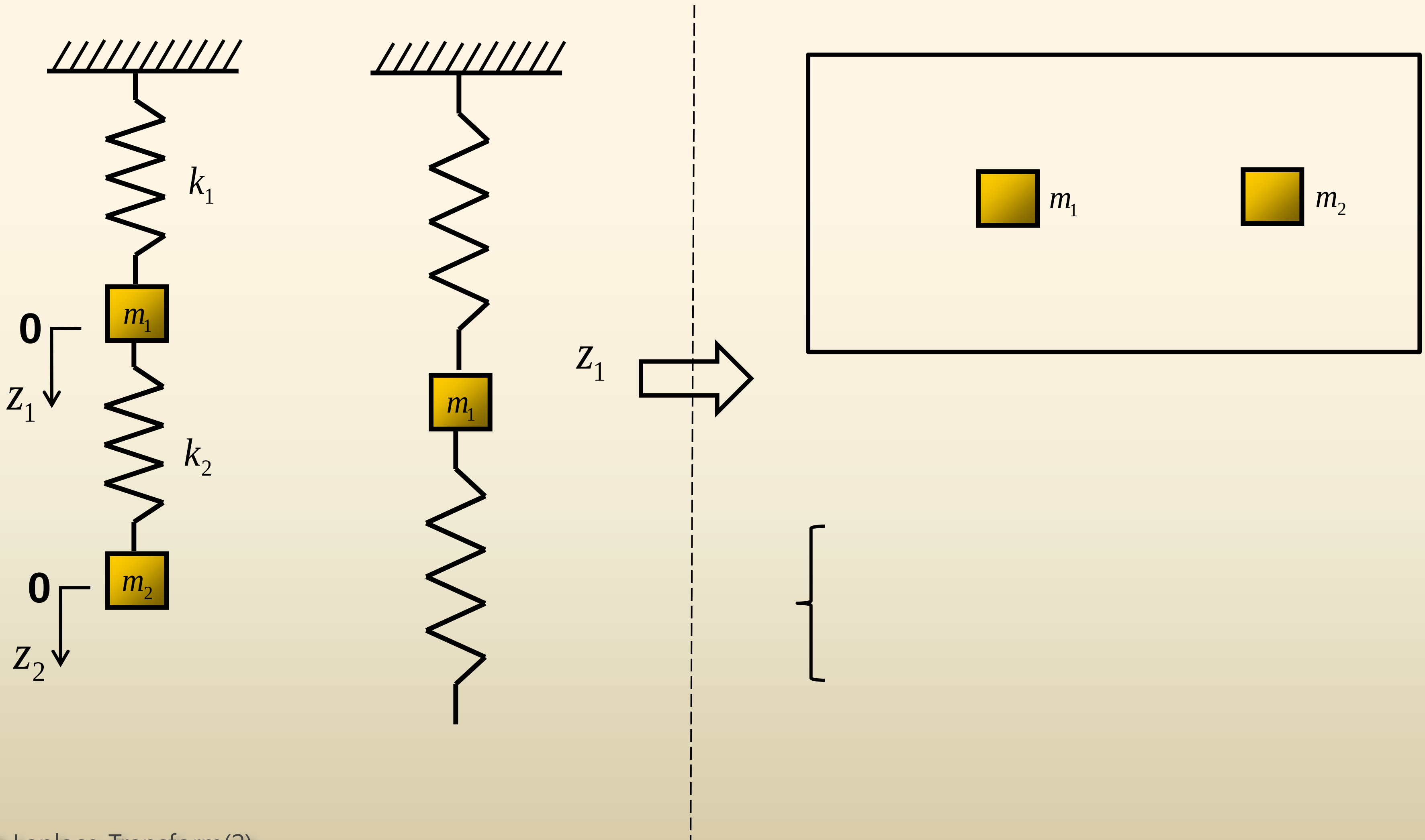

Systems of Linear Differential Equations

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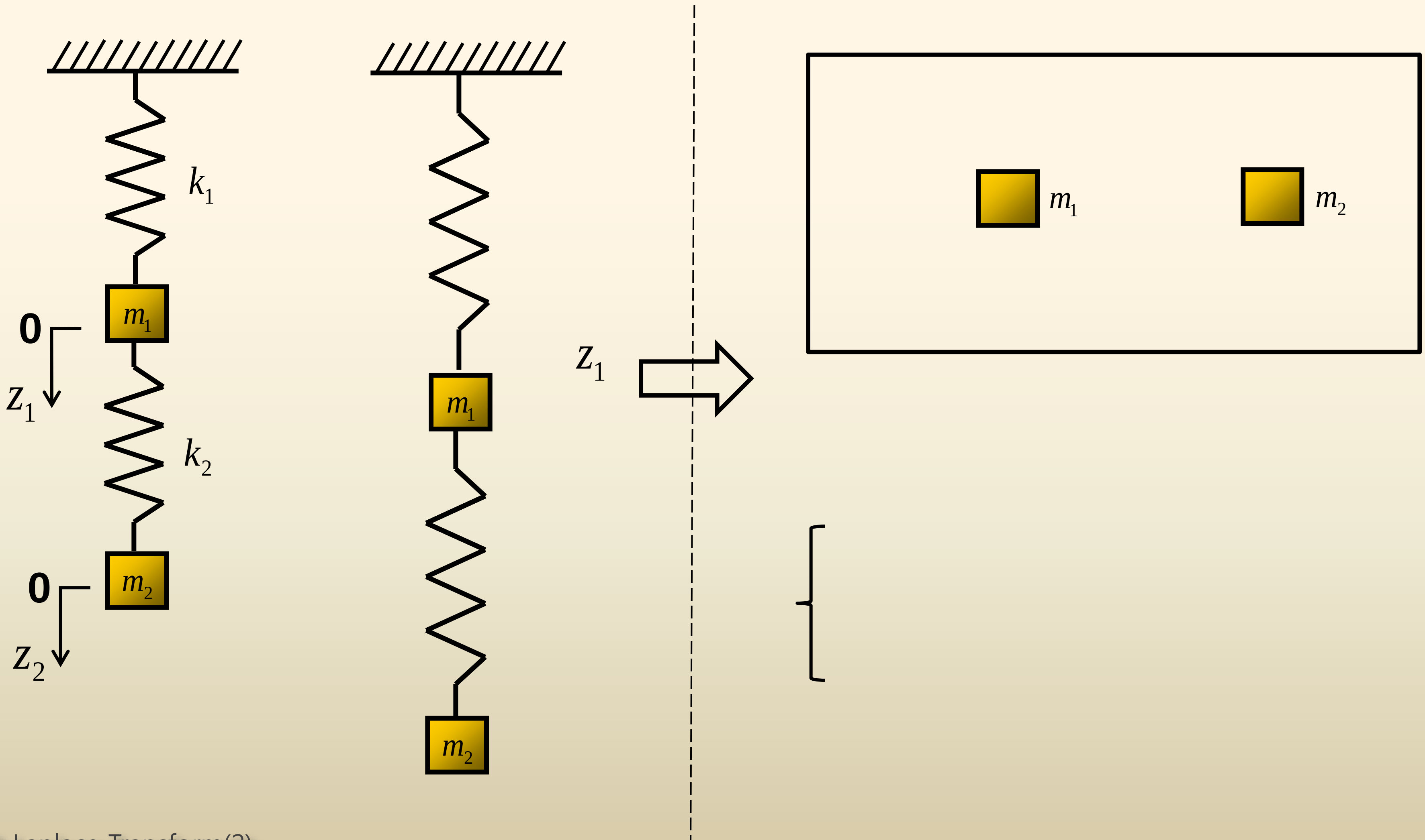

Systems of Linear Differential Equations

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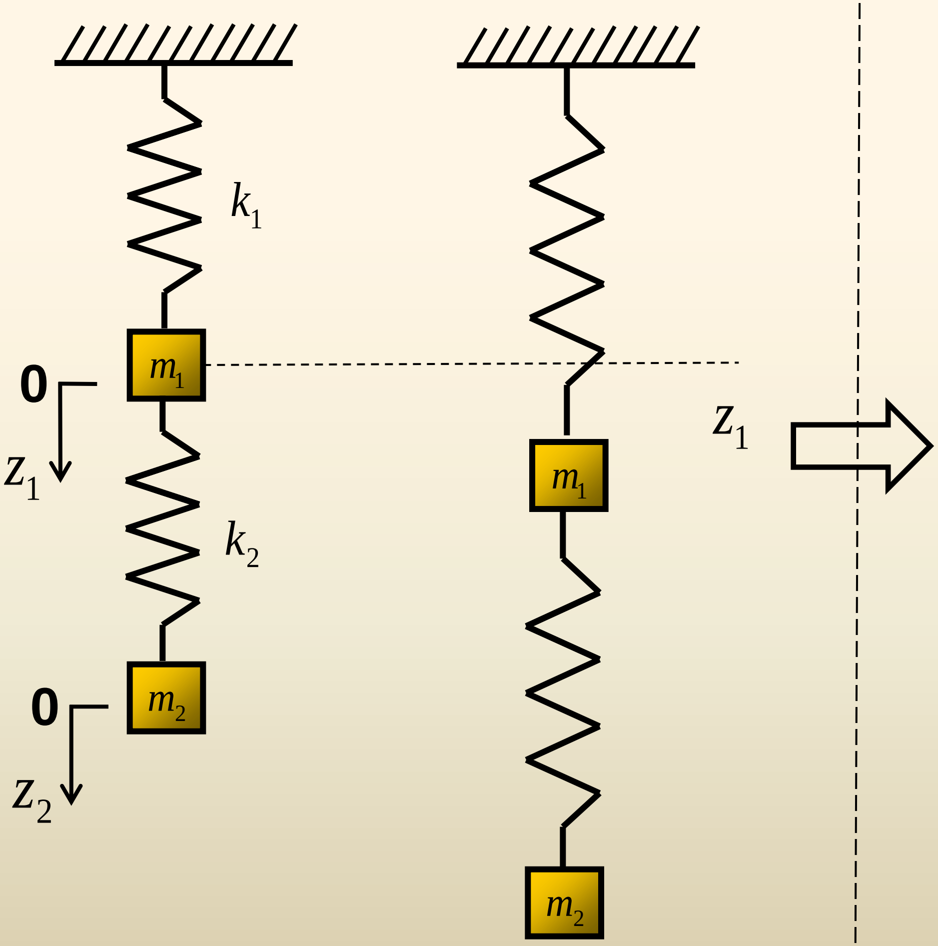

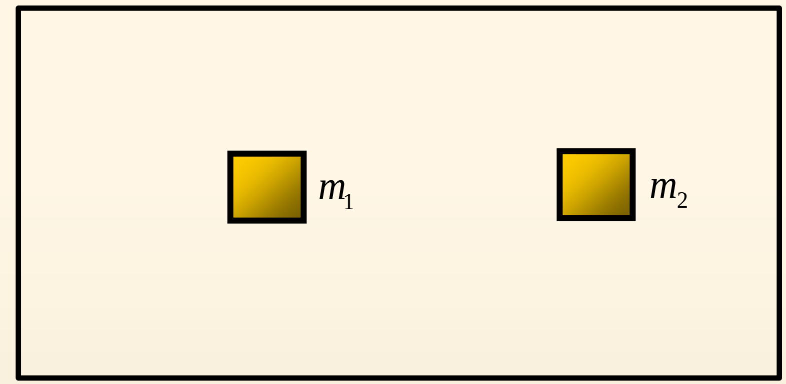

Systems of Linear Differential Equations

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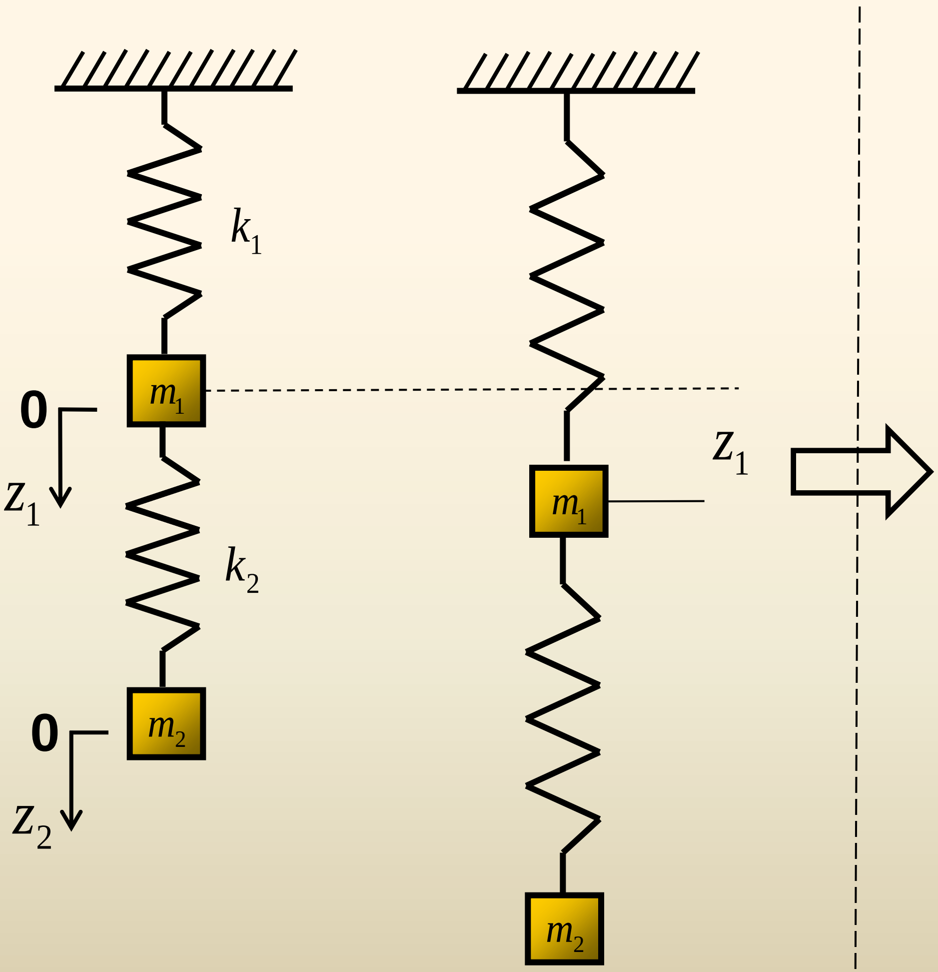

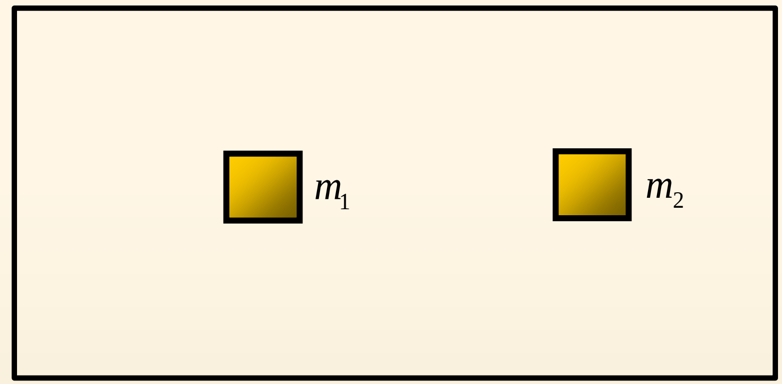

Systems of Linear Differential Equations

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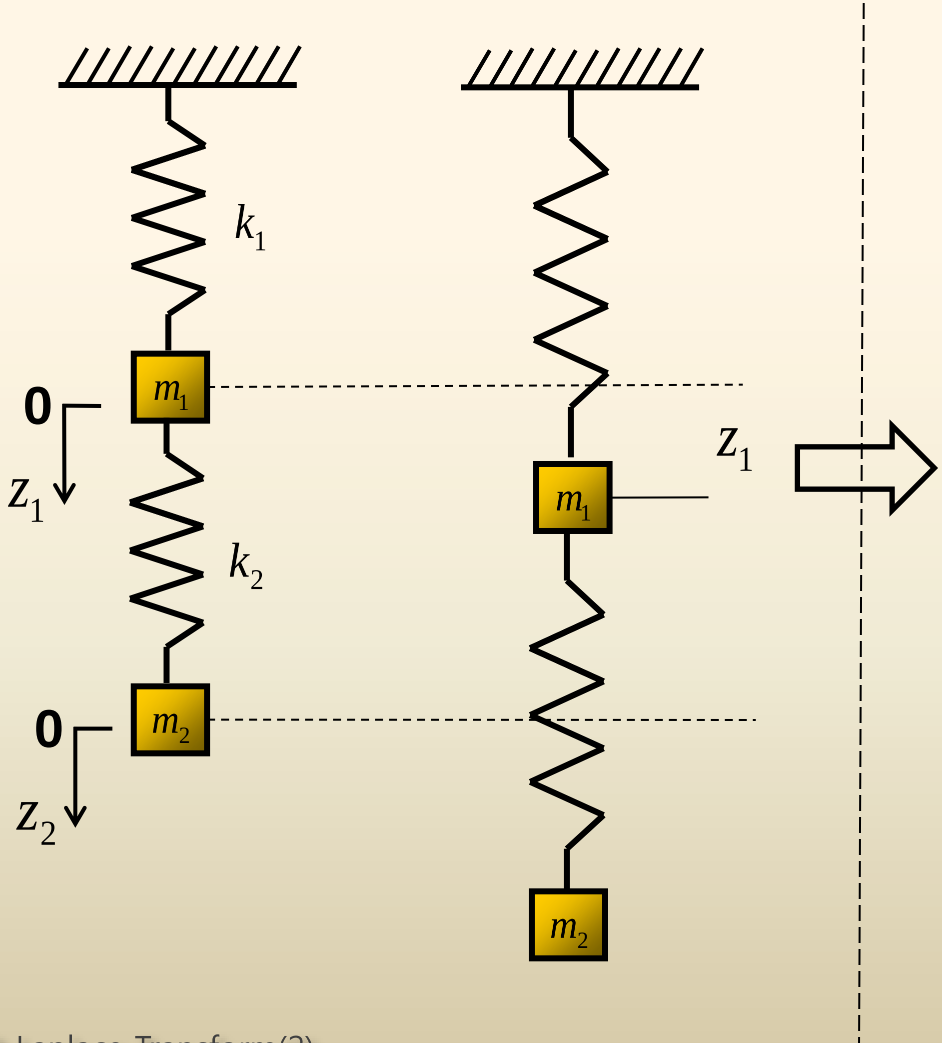

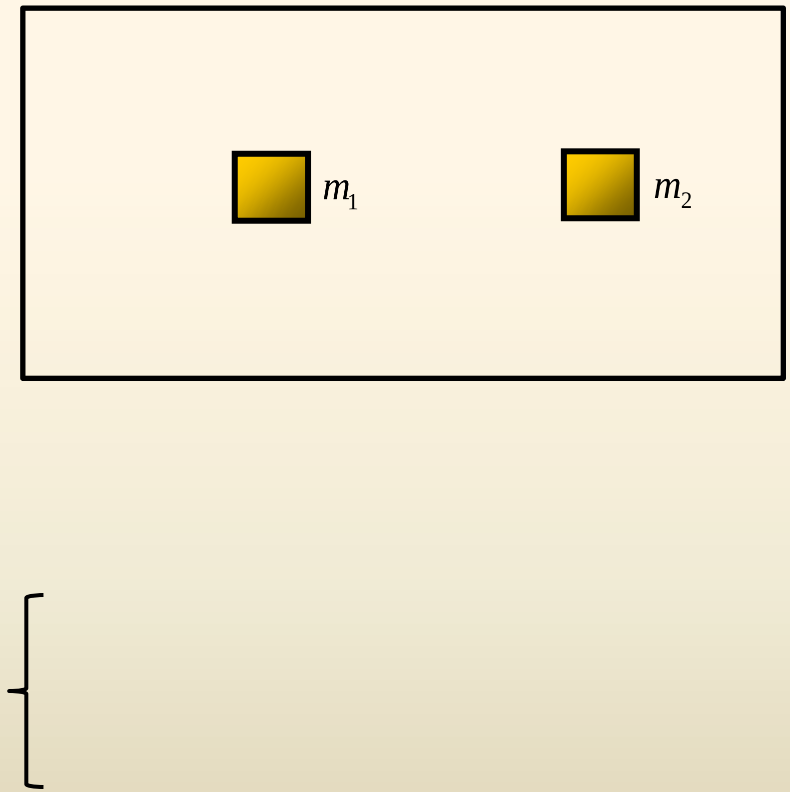

Systems of Linear Differential Equations

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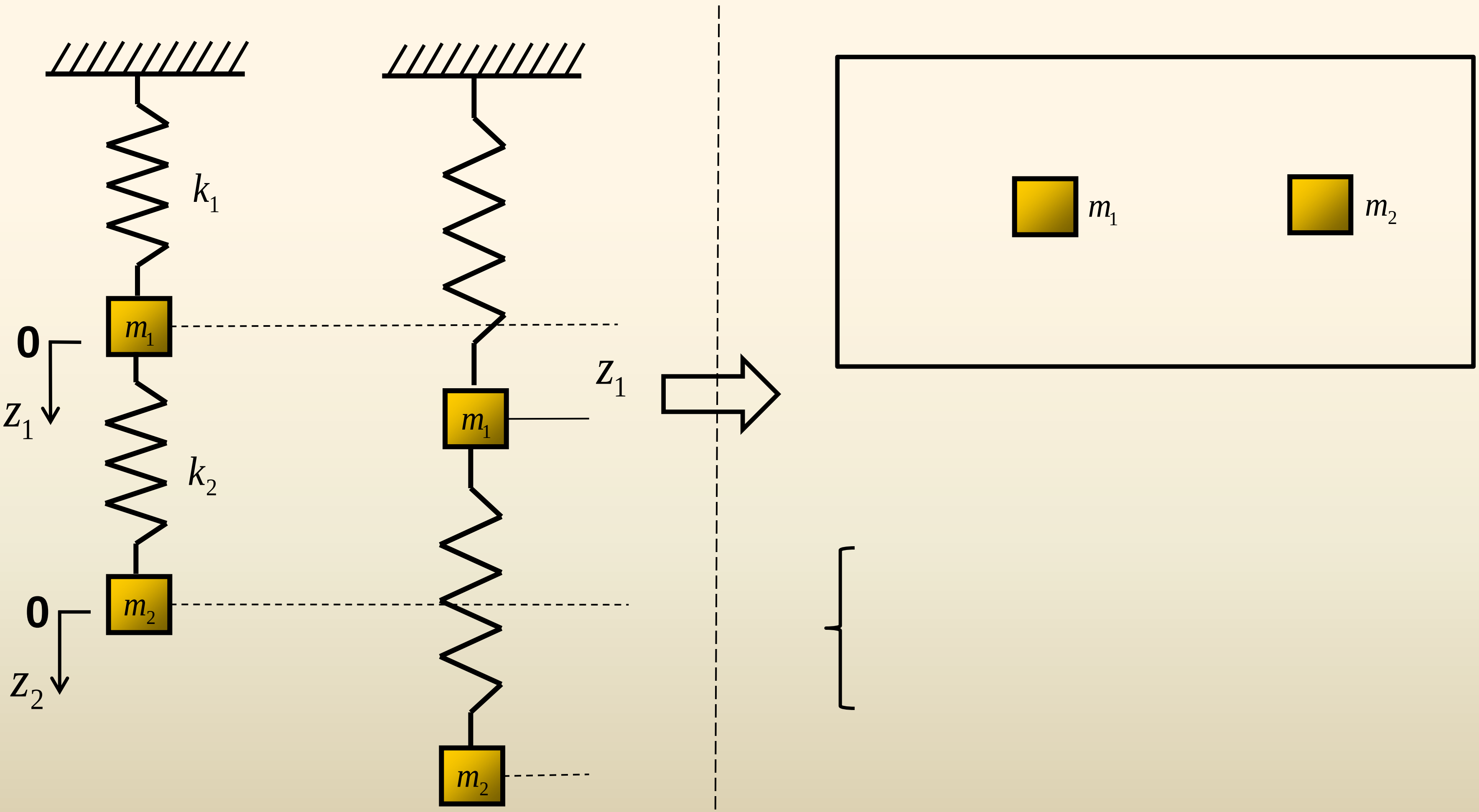

Systems of Linear Differential Equations

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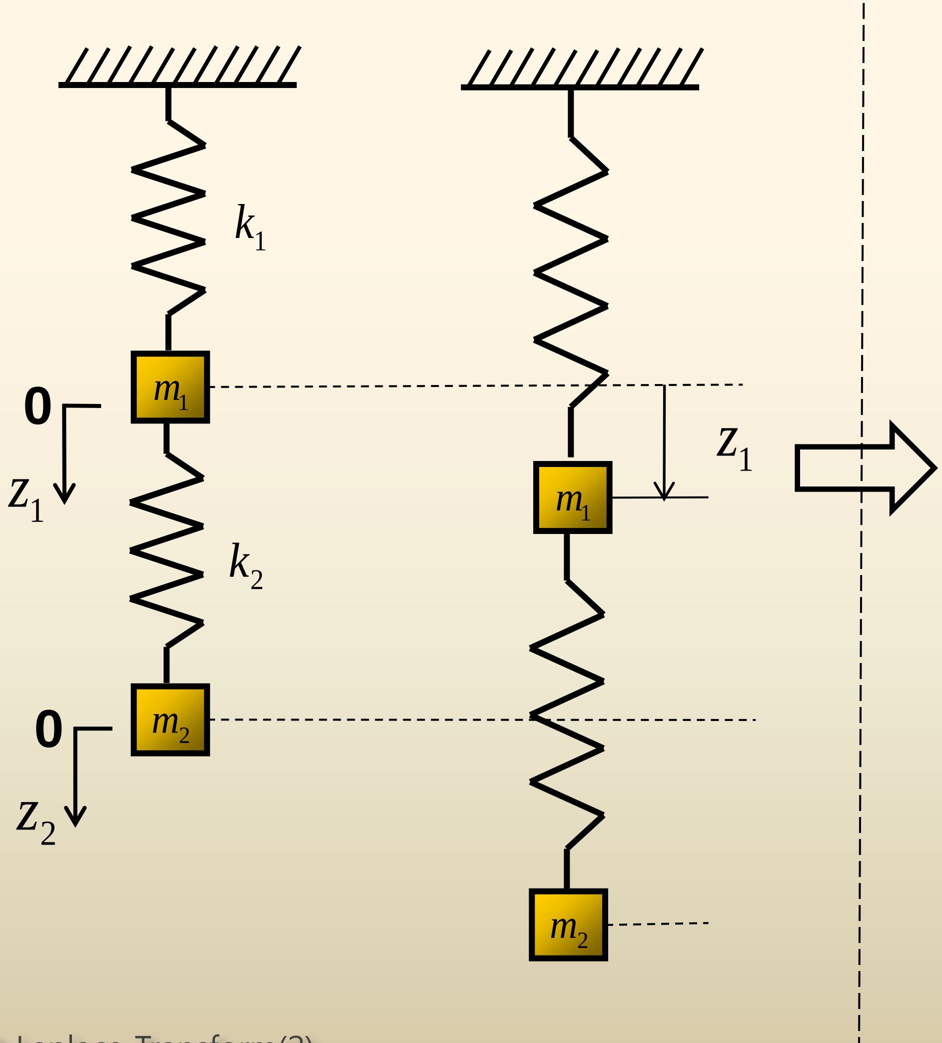

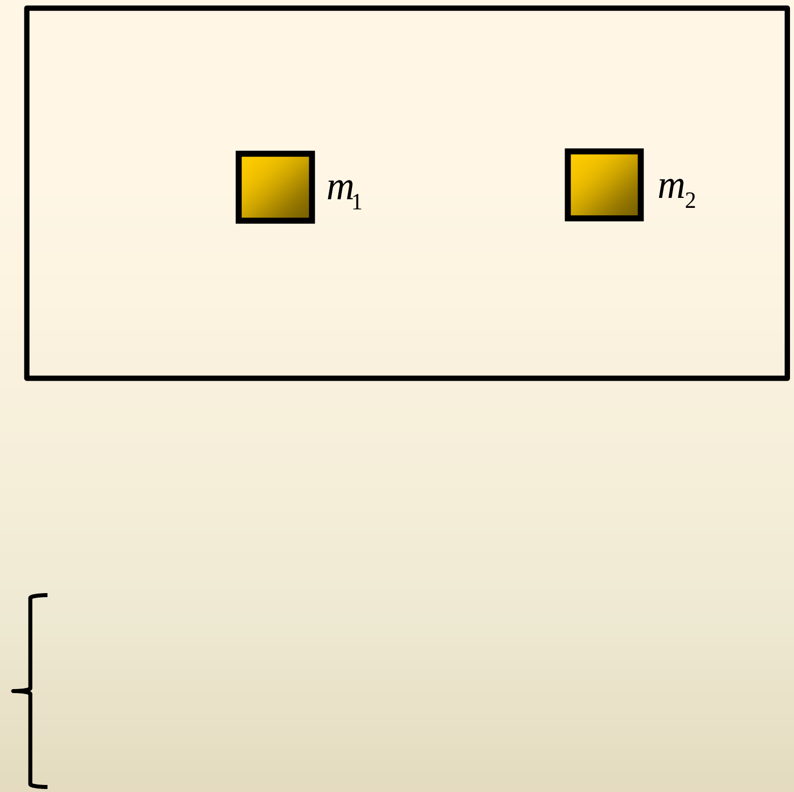

Systems of Linear Differential Equations

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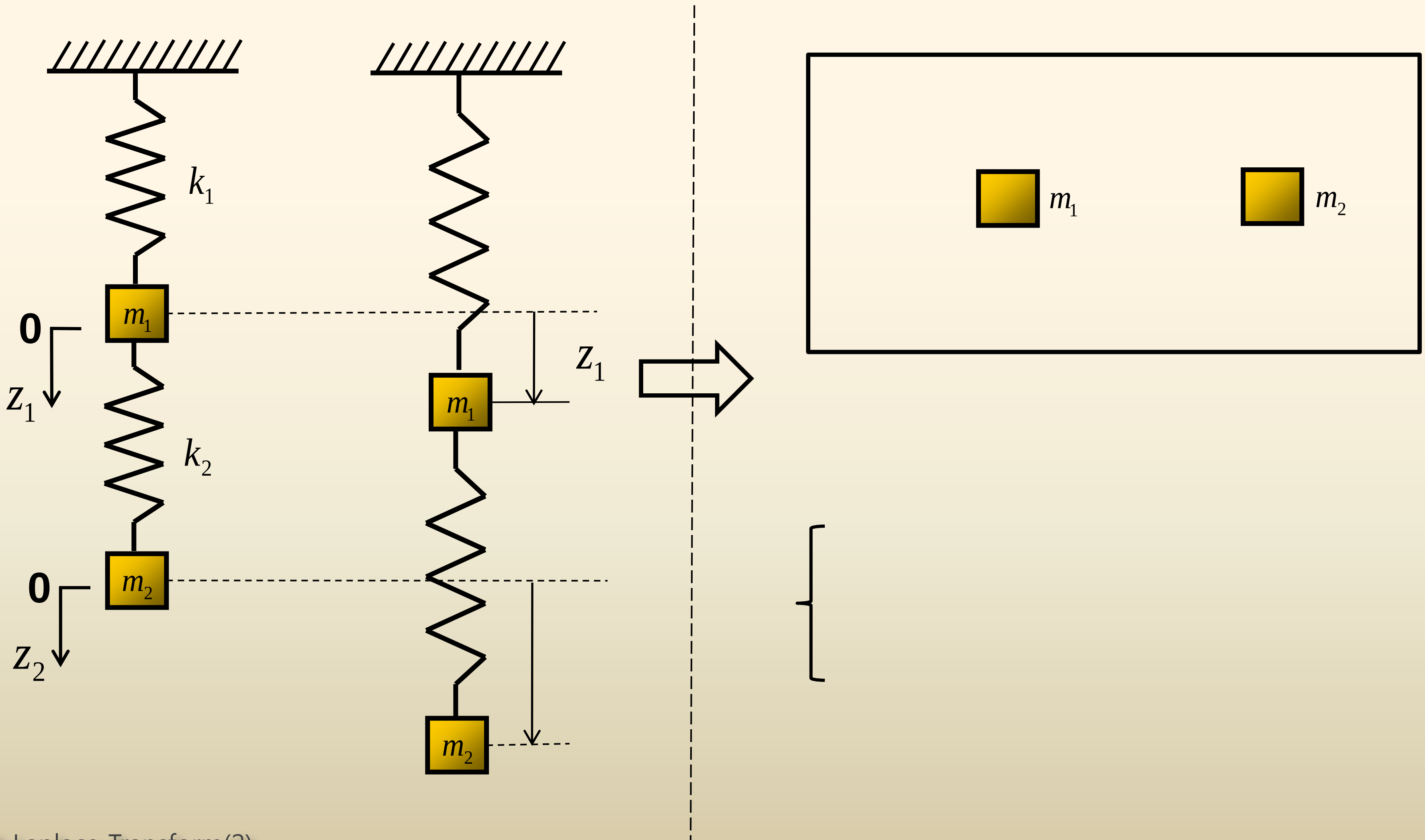

Systems of Linear Differential Equations

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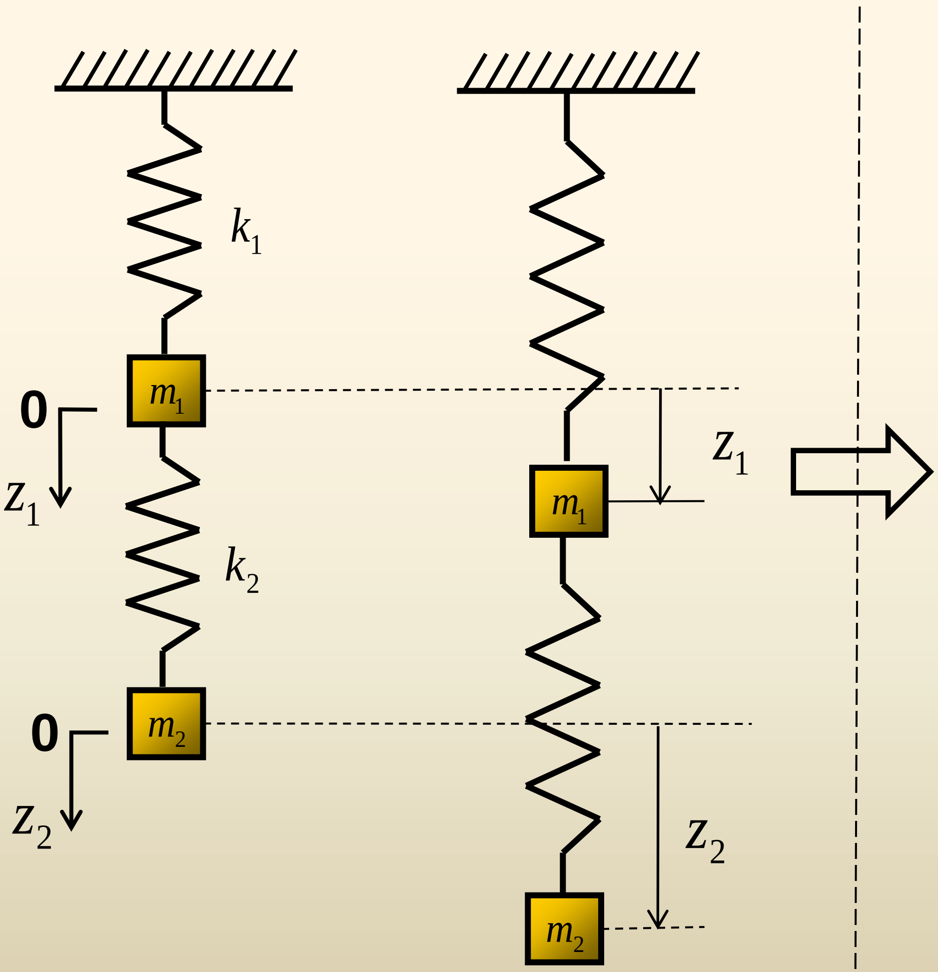

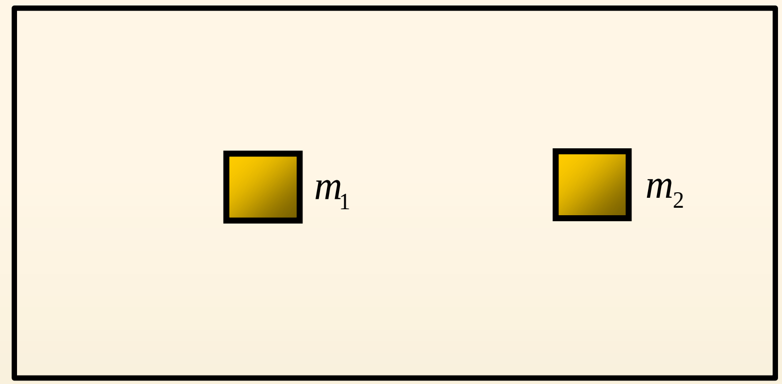

Systems of Linear Differential Equations

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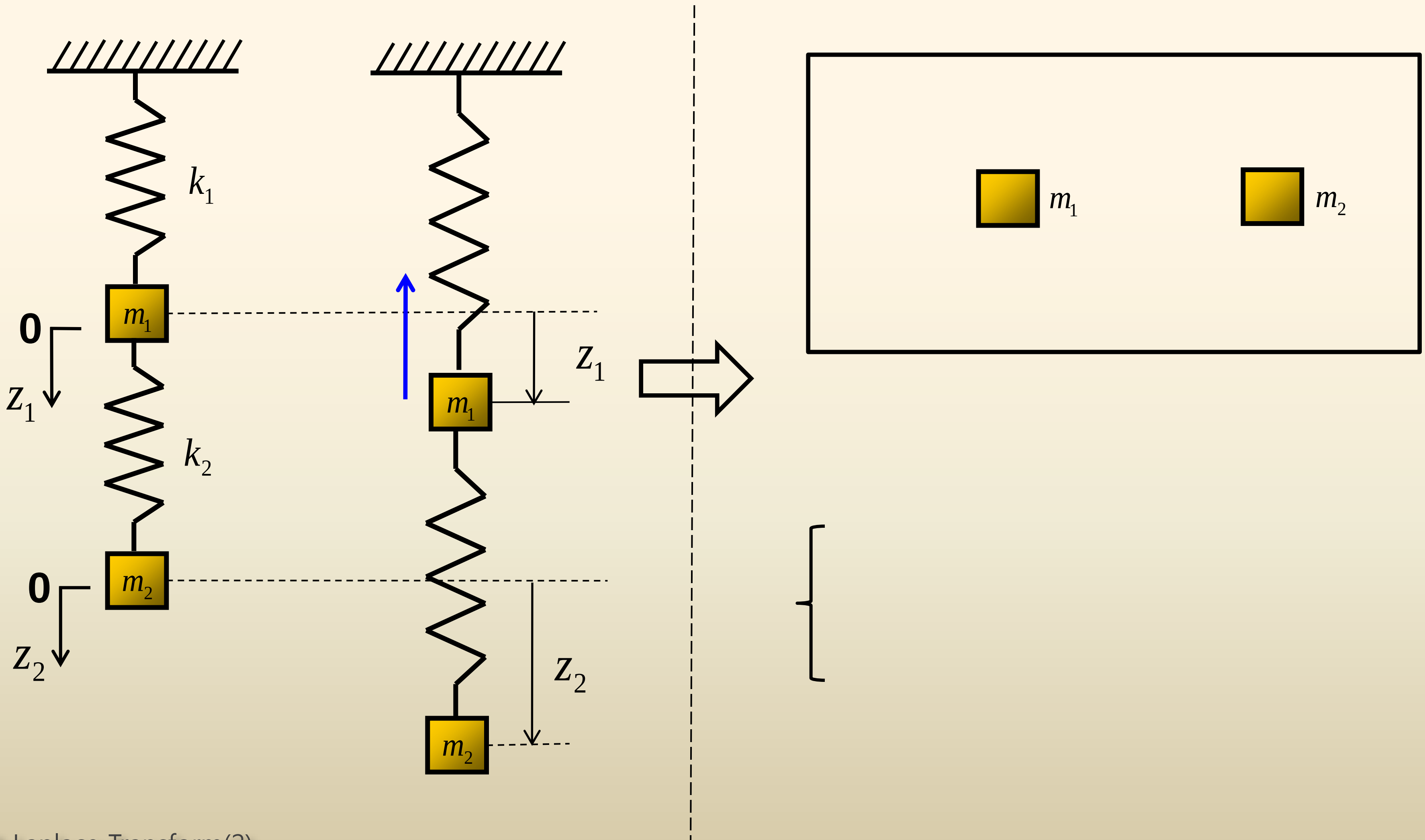

Systems of Linear Differential Equations

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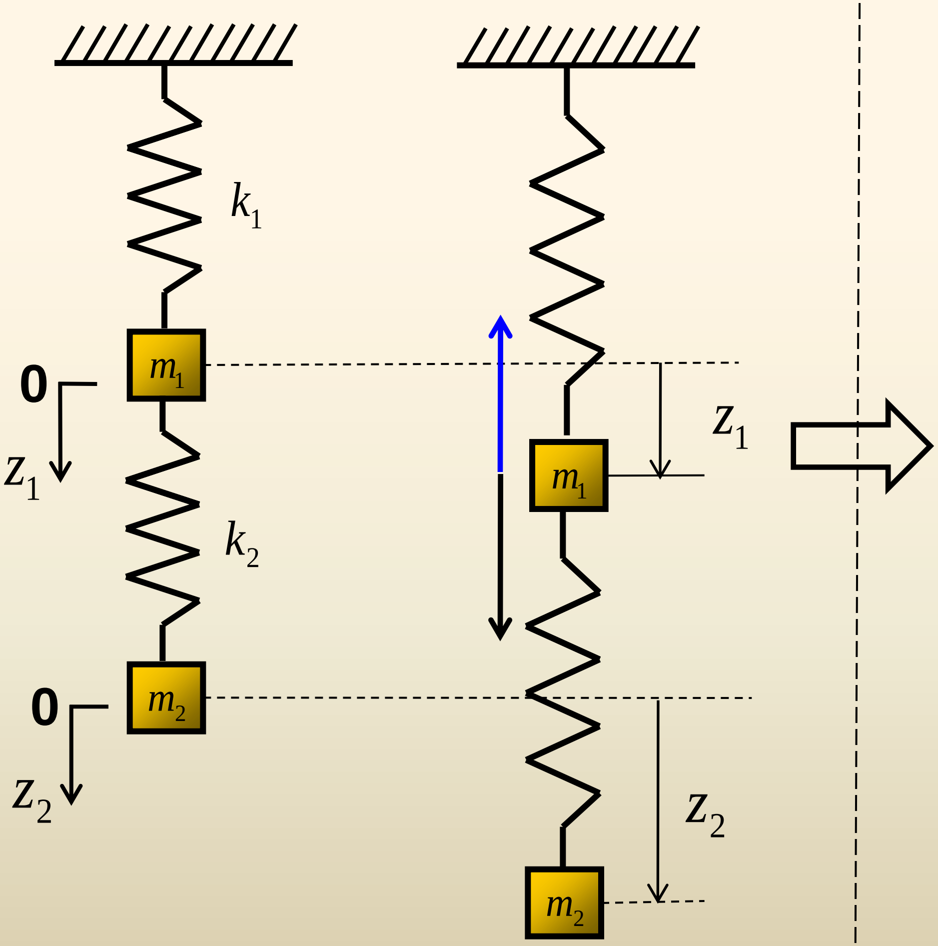

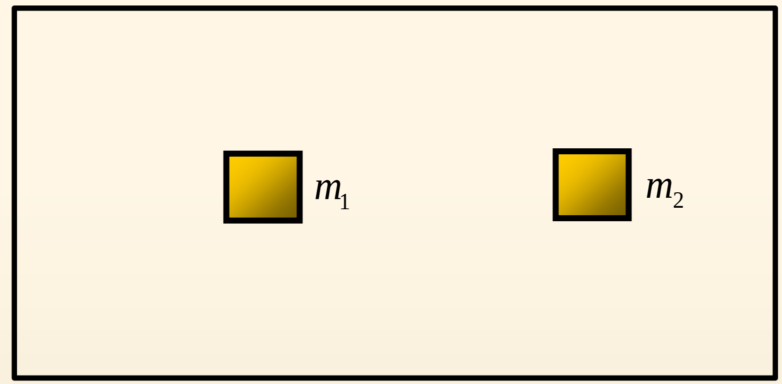

Systems of Linear Differential Equations

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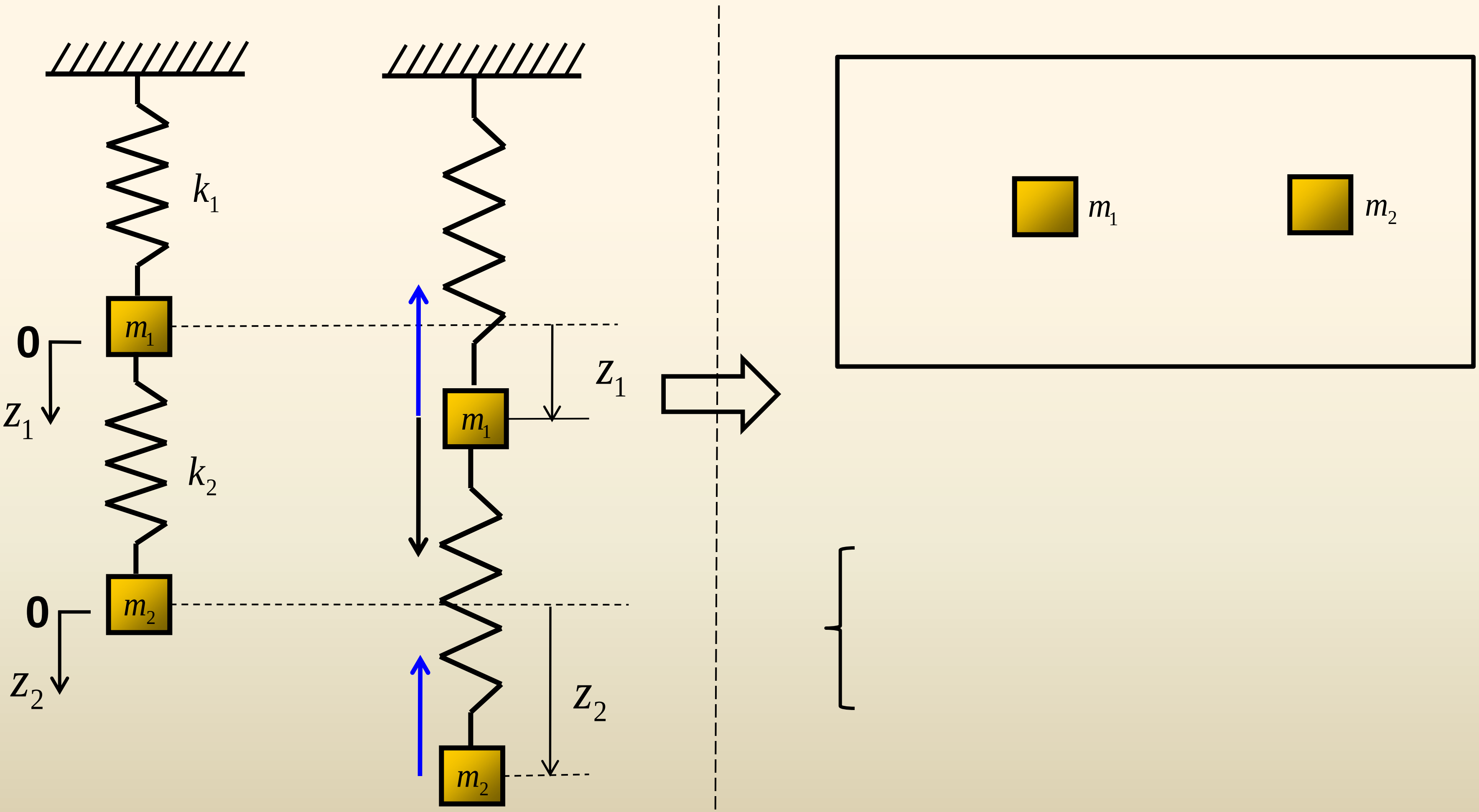

Systems of Linear Differential Equations

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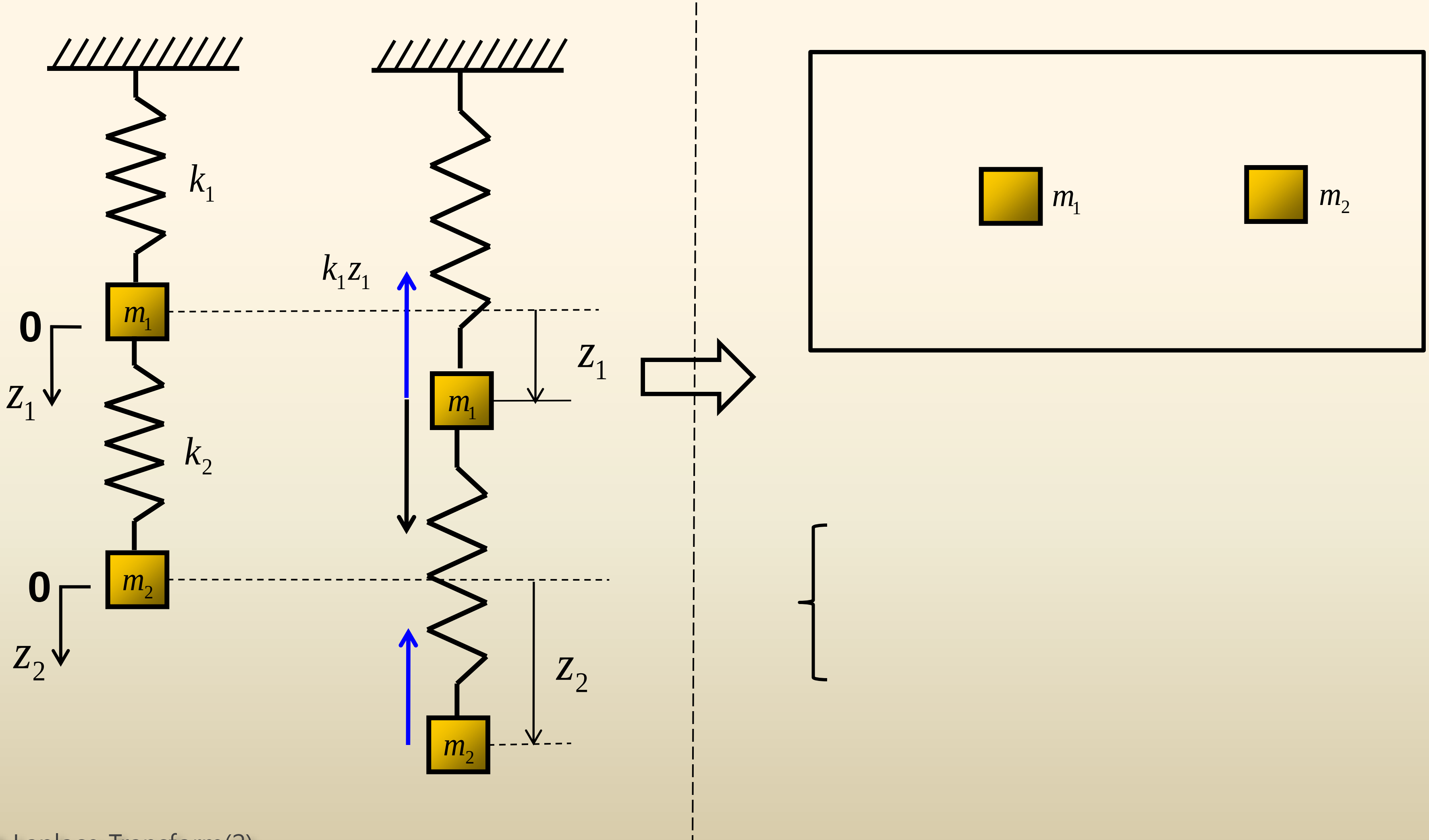

Systems of Linear Differential Equations

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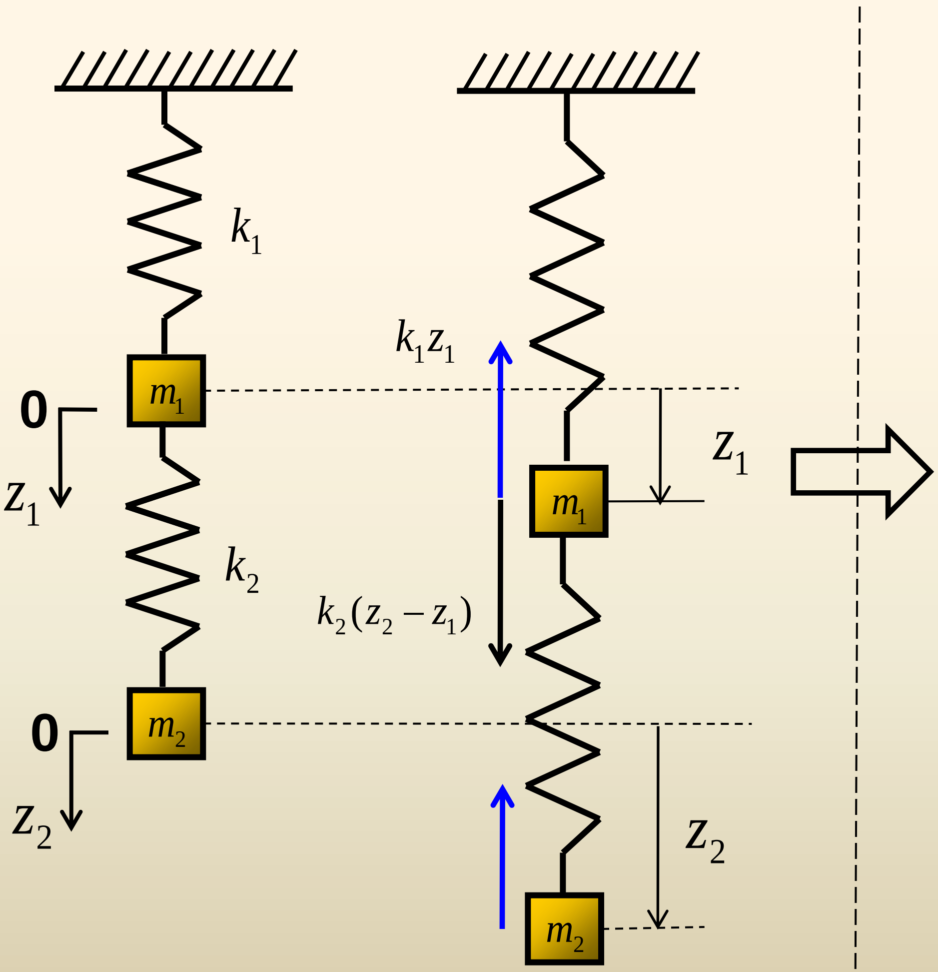

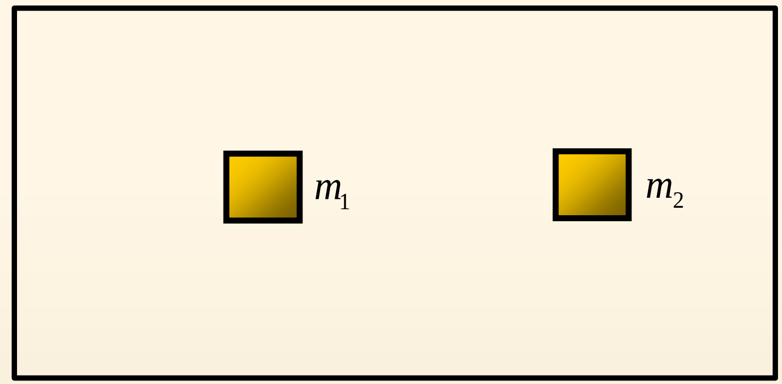

Systems of Linear Differential Equations

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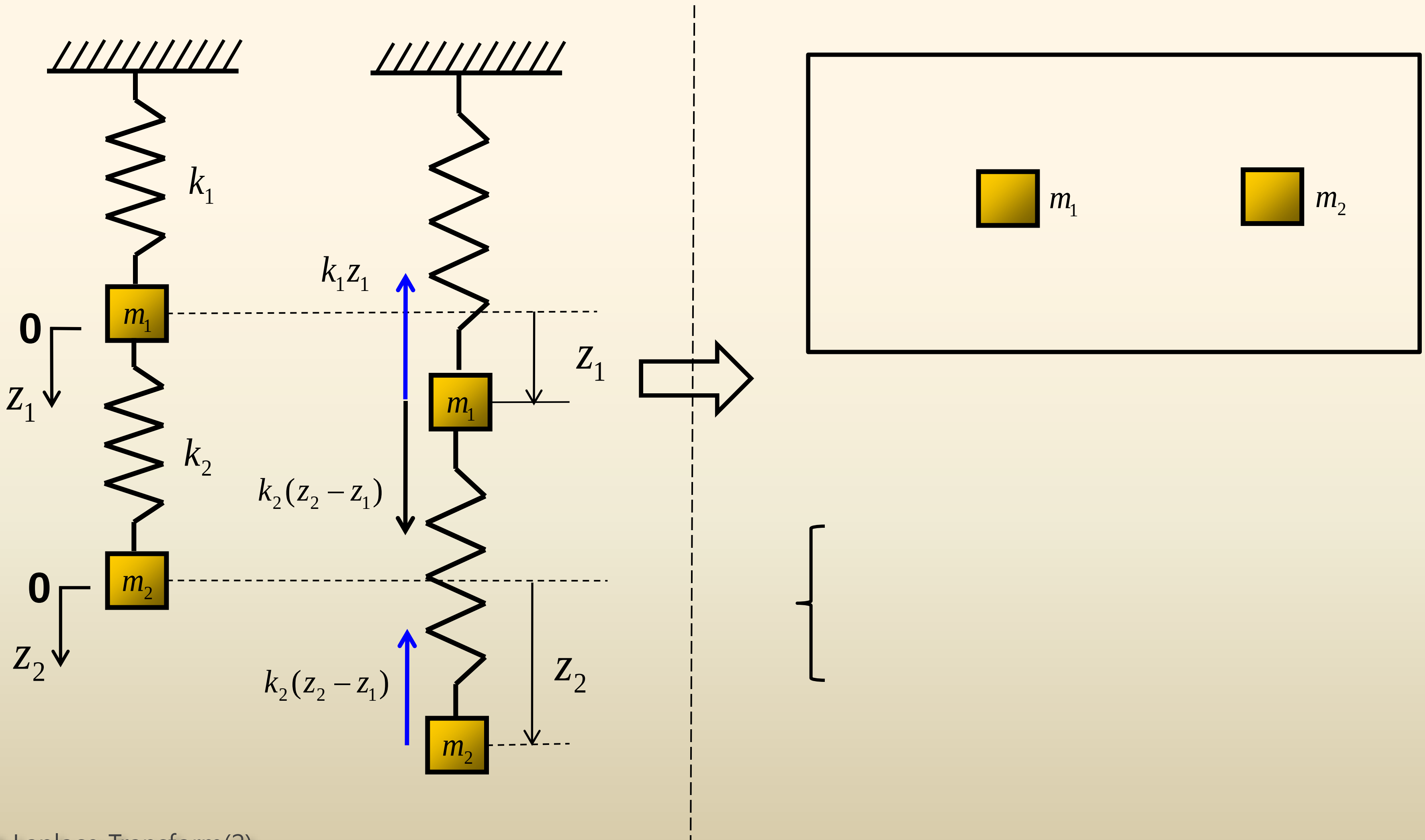

Systems of Linear Differential Equations

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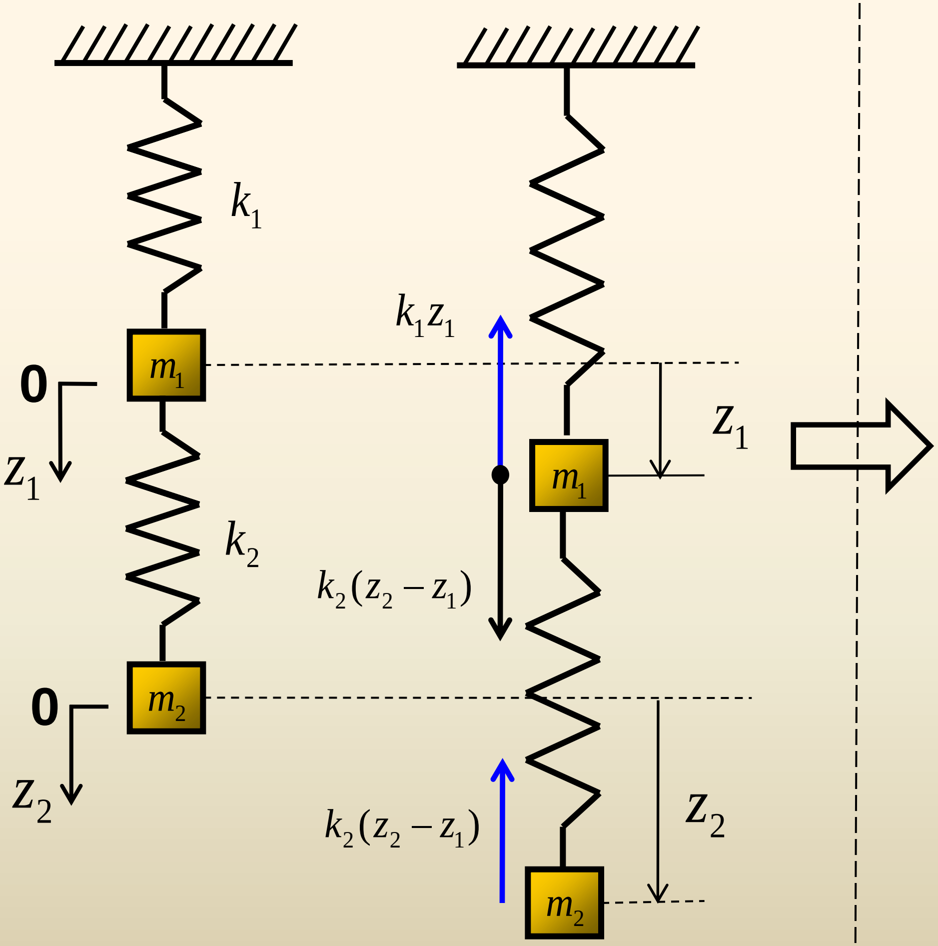

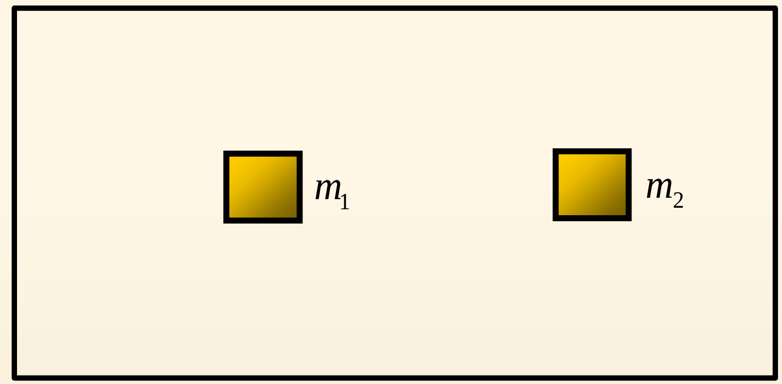

Systems of Linear Differential Equations

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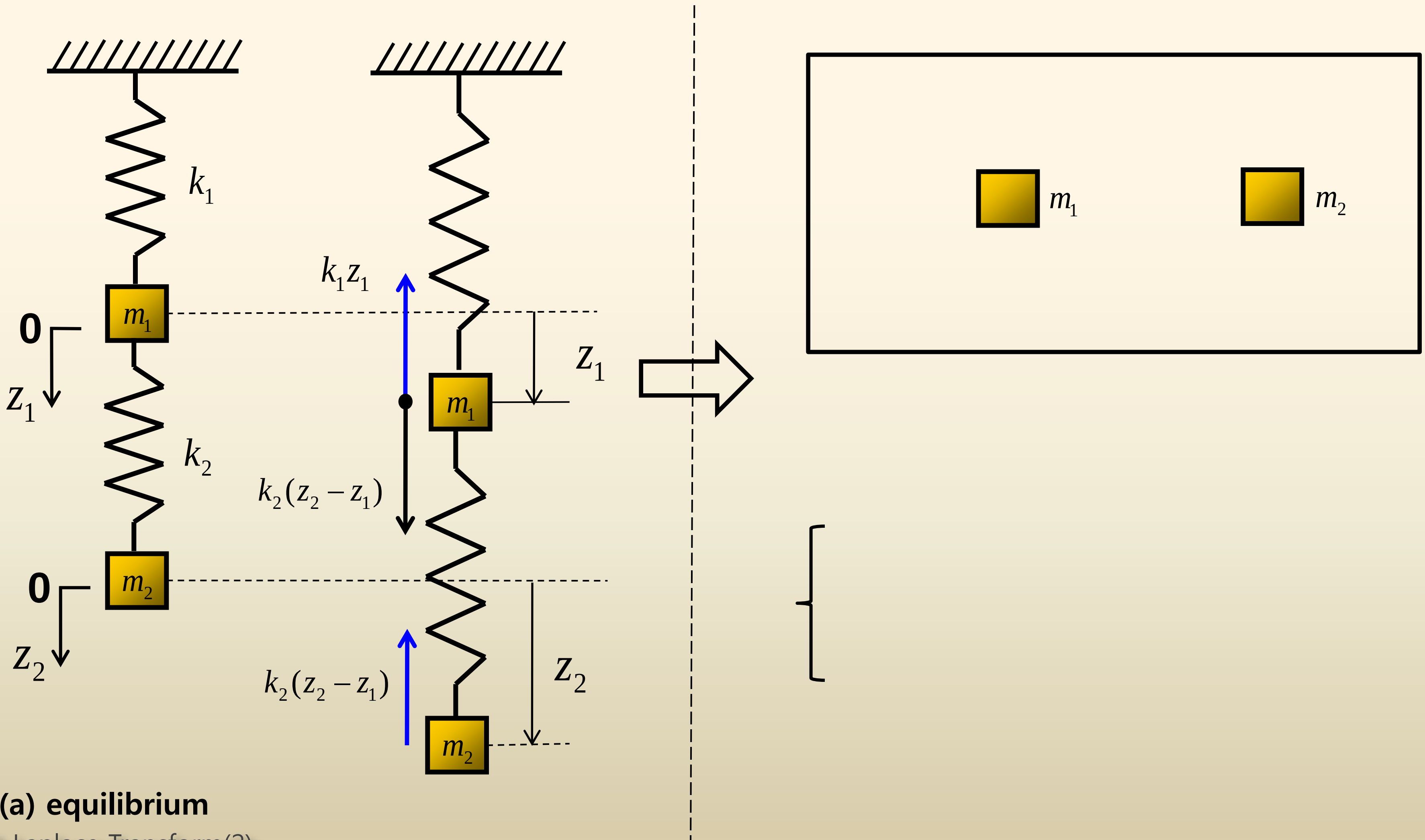
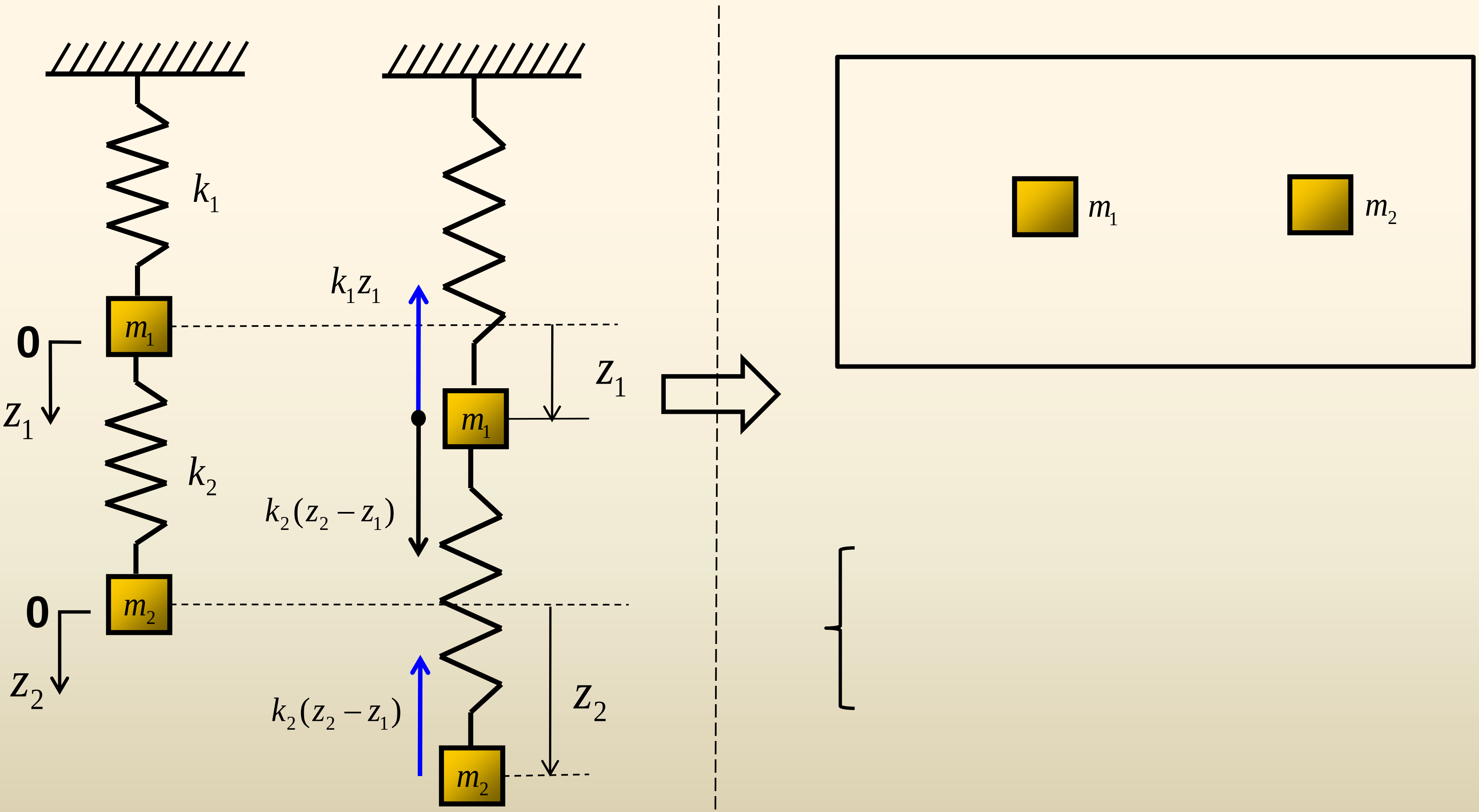


(a) equilibrium

Systems of Linear Differential Equations

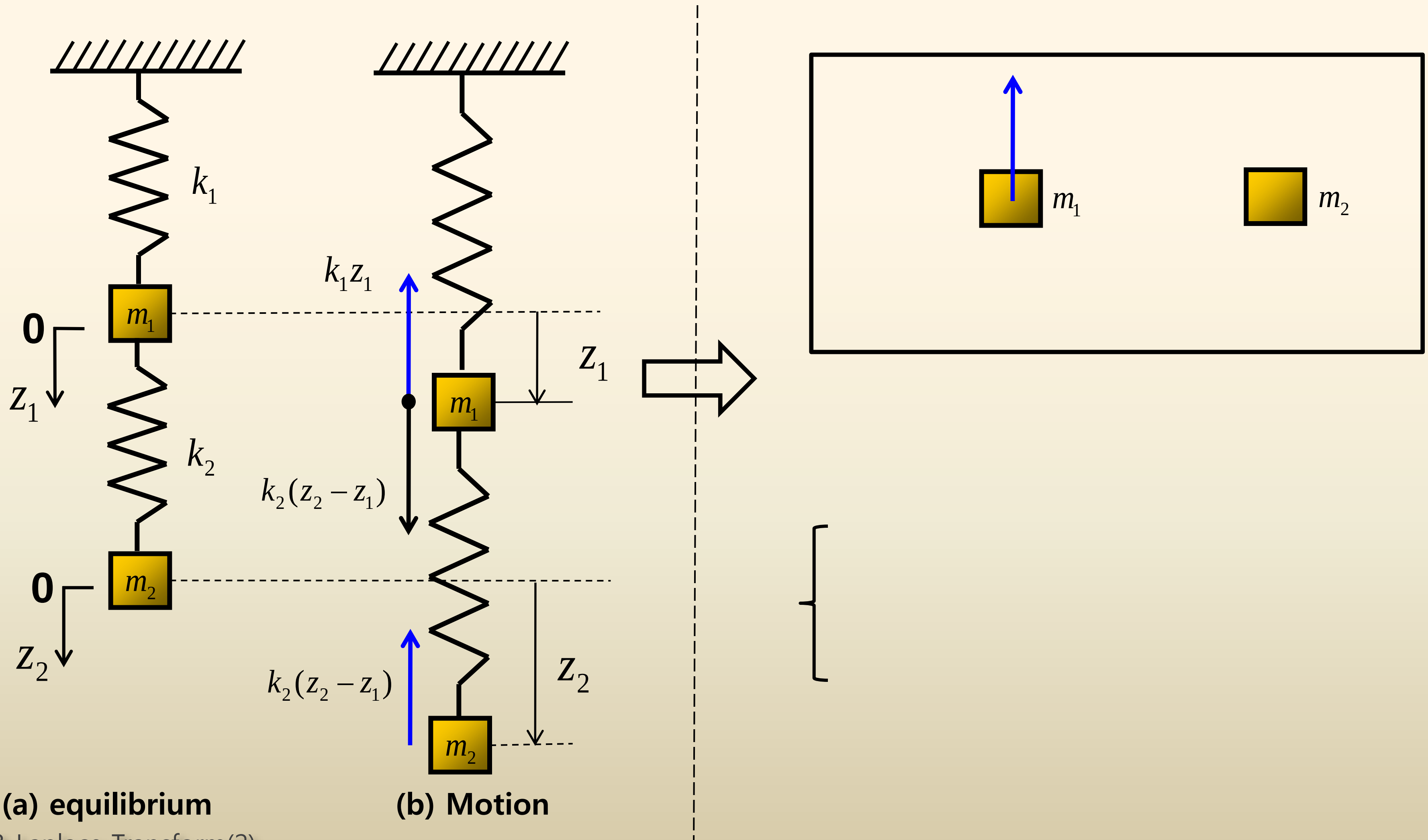
☑ Coupled Spring/Mass System



(a) equilibrium (b) Motion

Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

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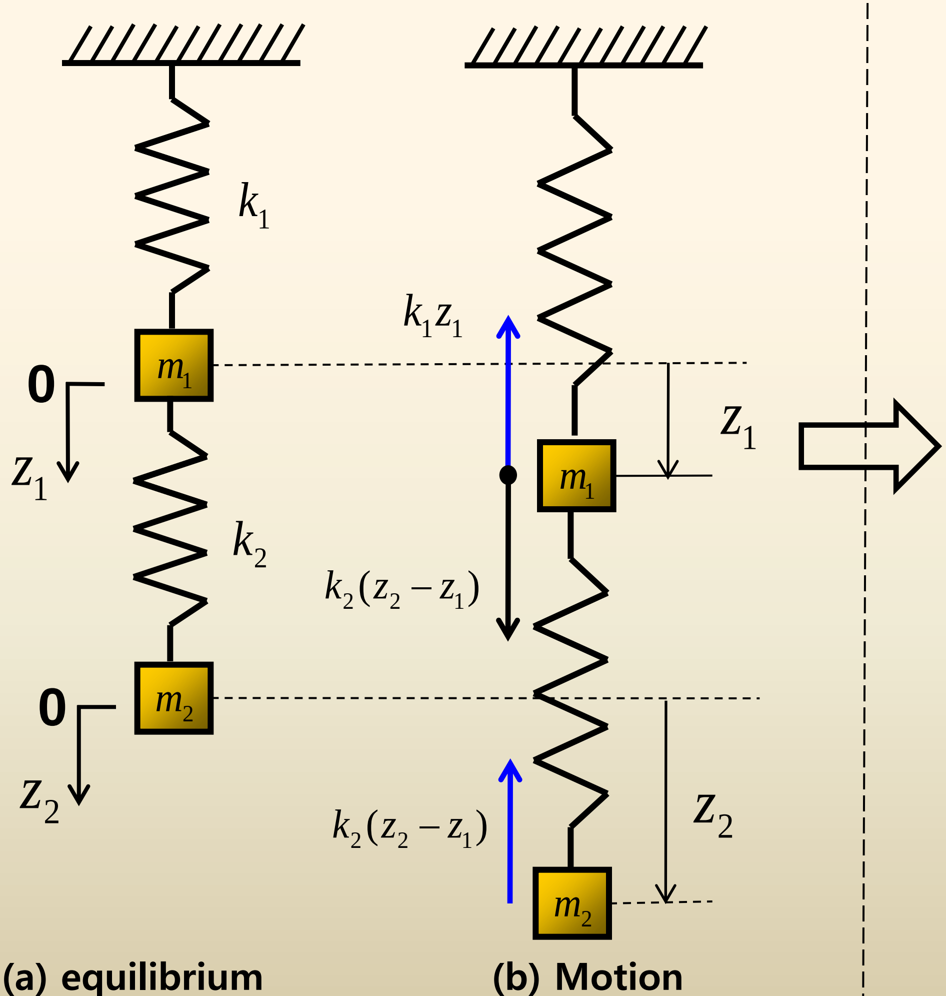
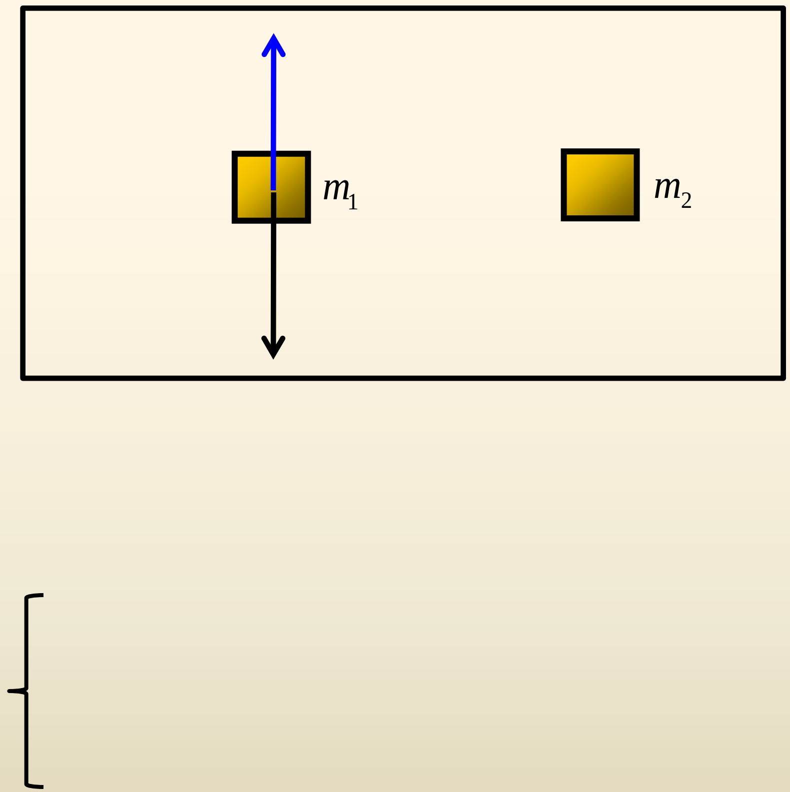
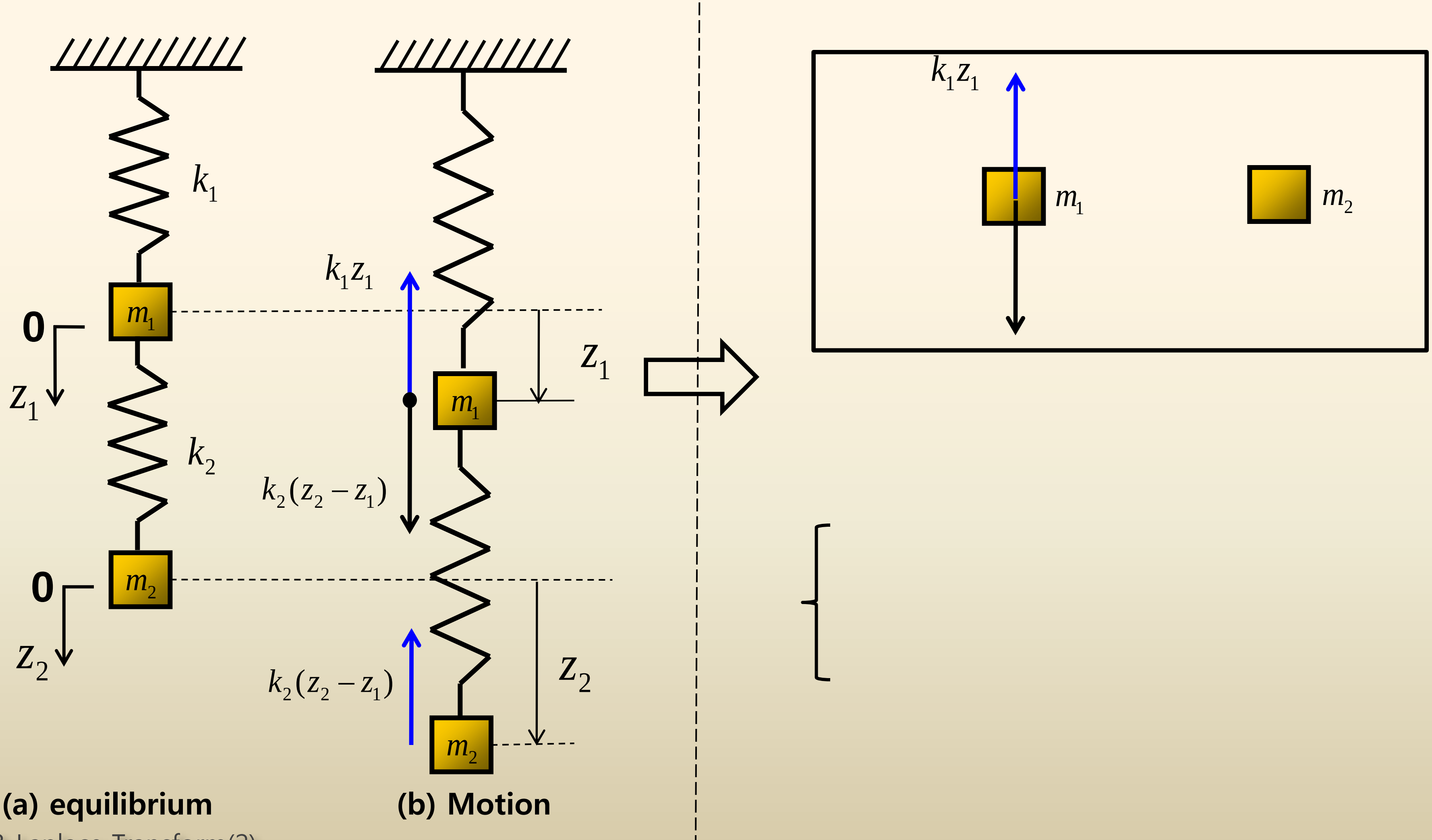




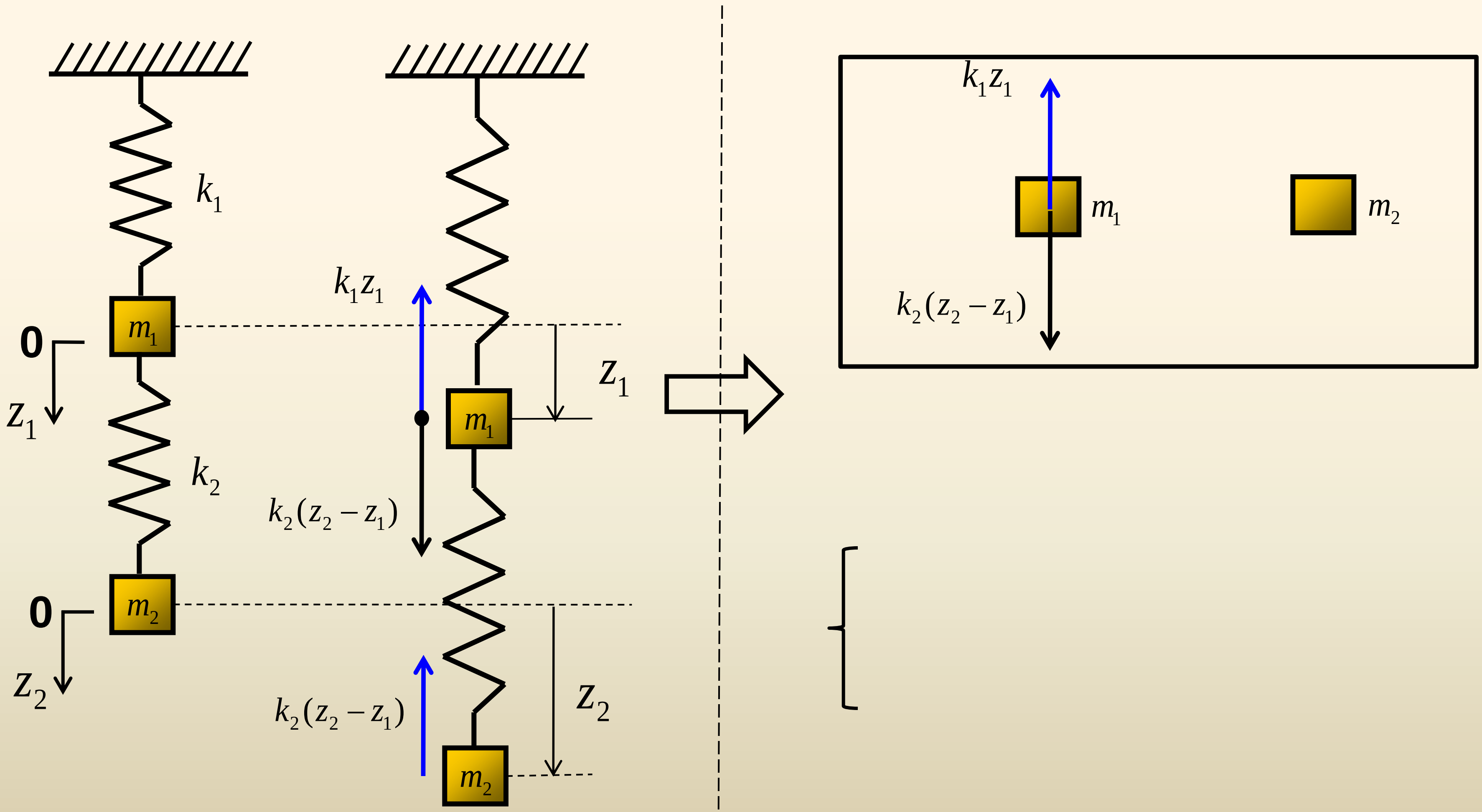
Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

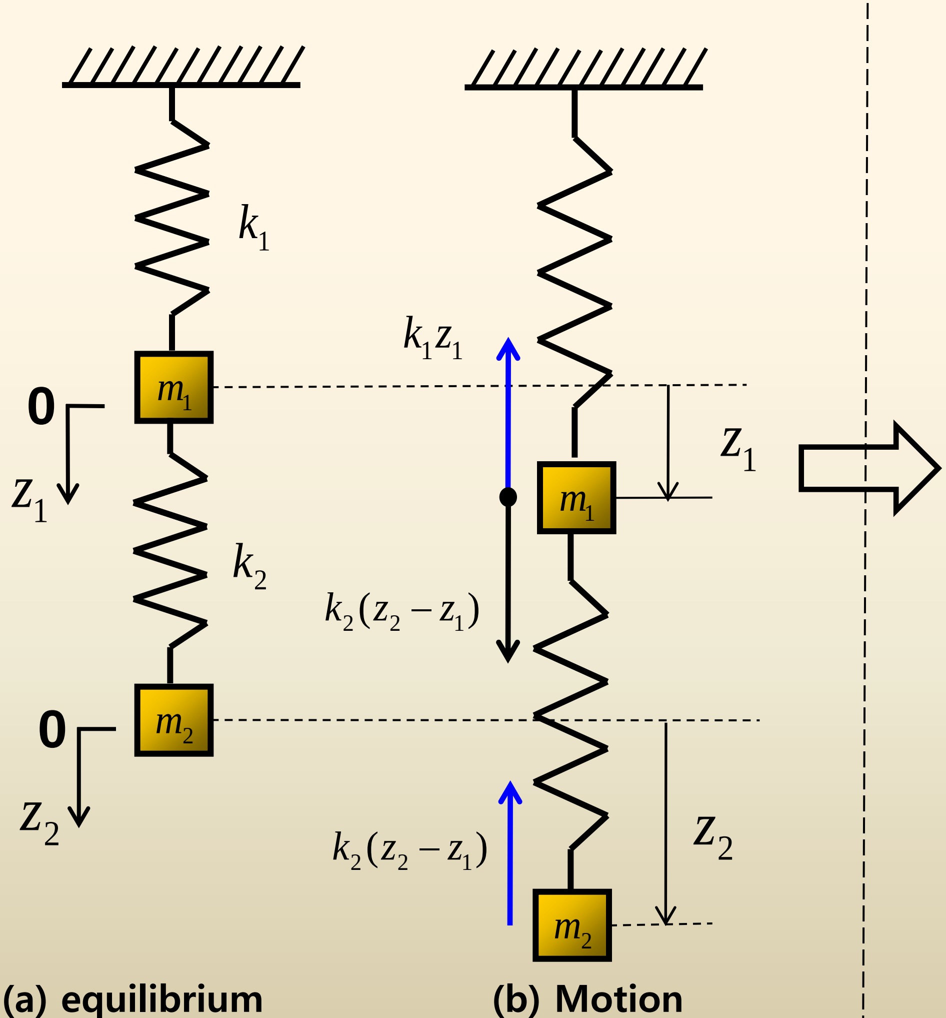
☑ Coupled Spring/Mass System

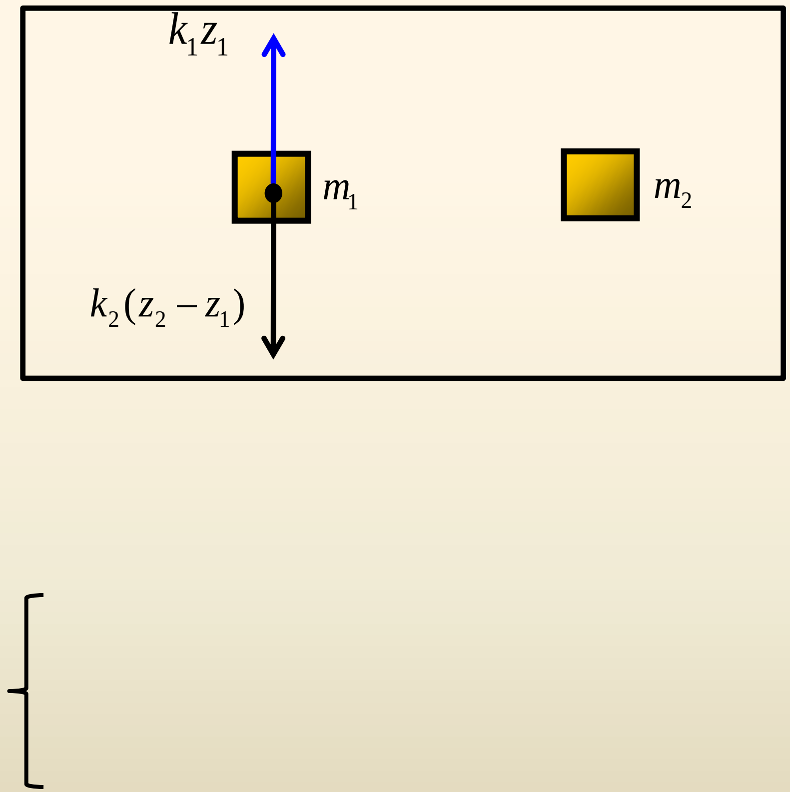


(a) equilibrium **(b) Motion**

Systems of Linear Differential Equations

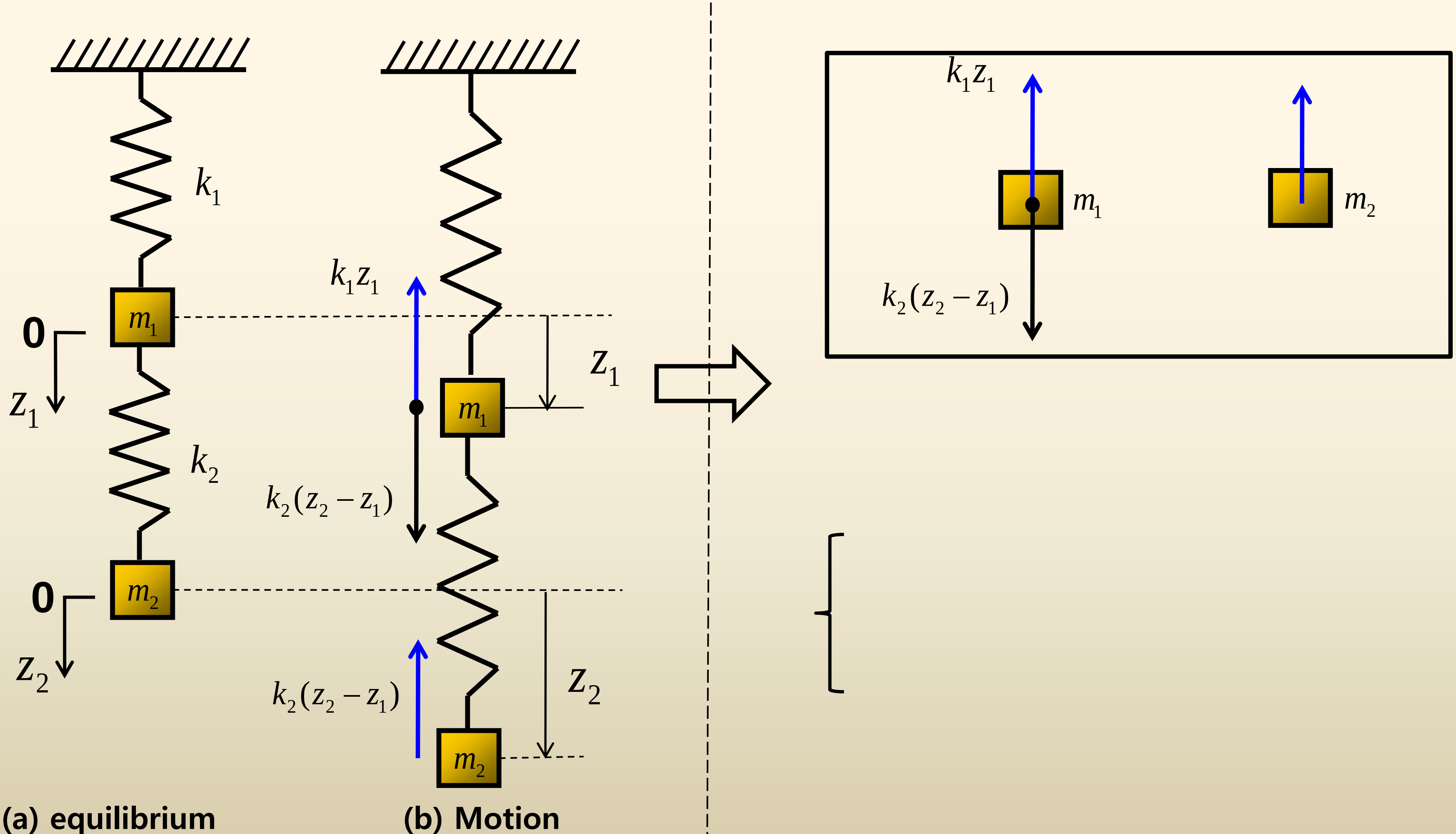
☑ Coupled Spring/Mass System





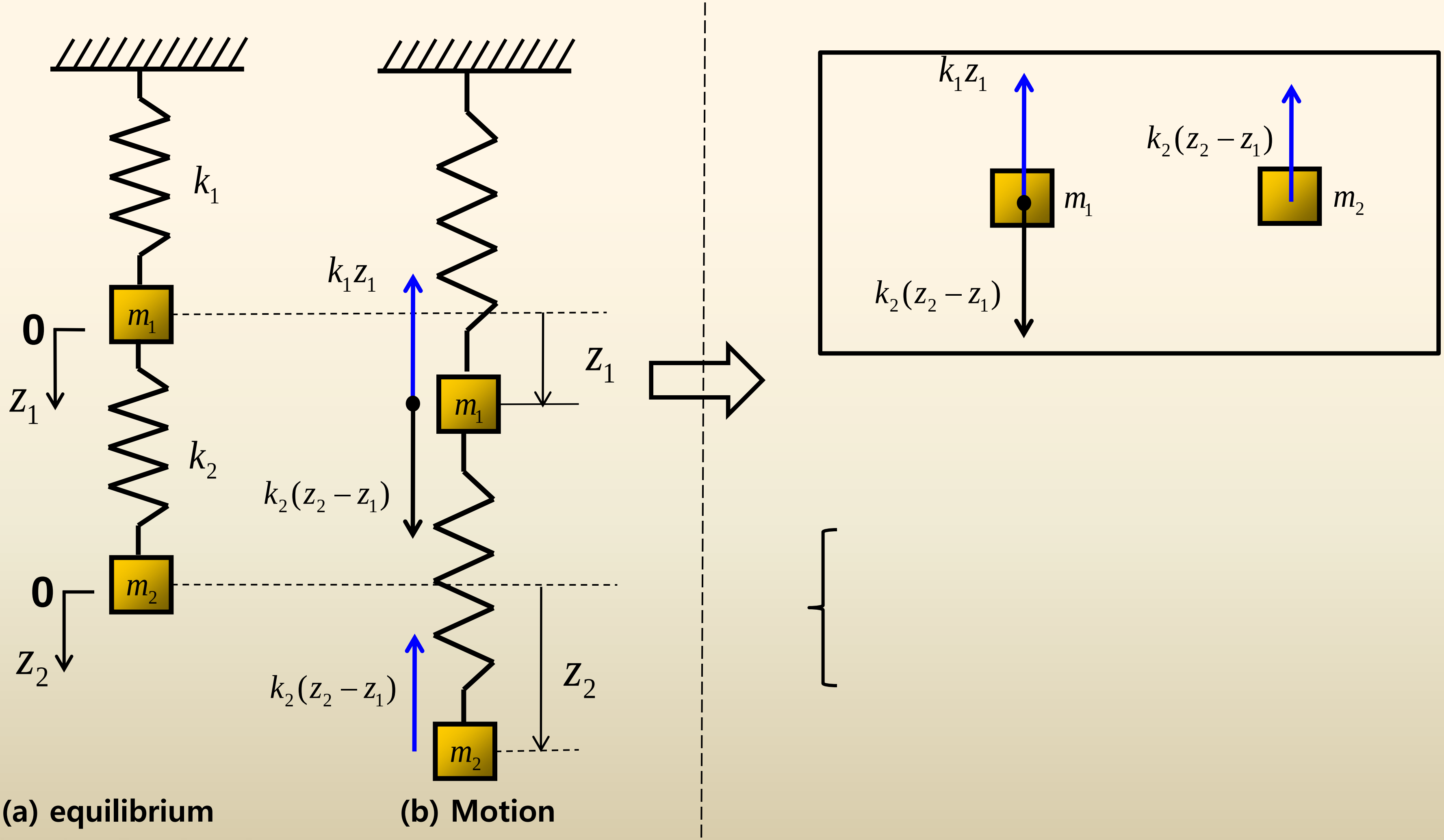
Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Systems of Linear Differential Equations

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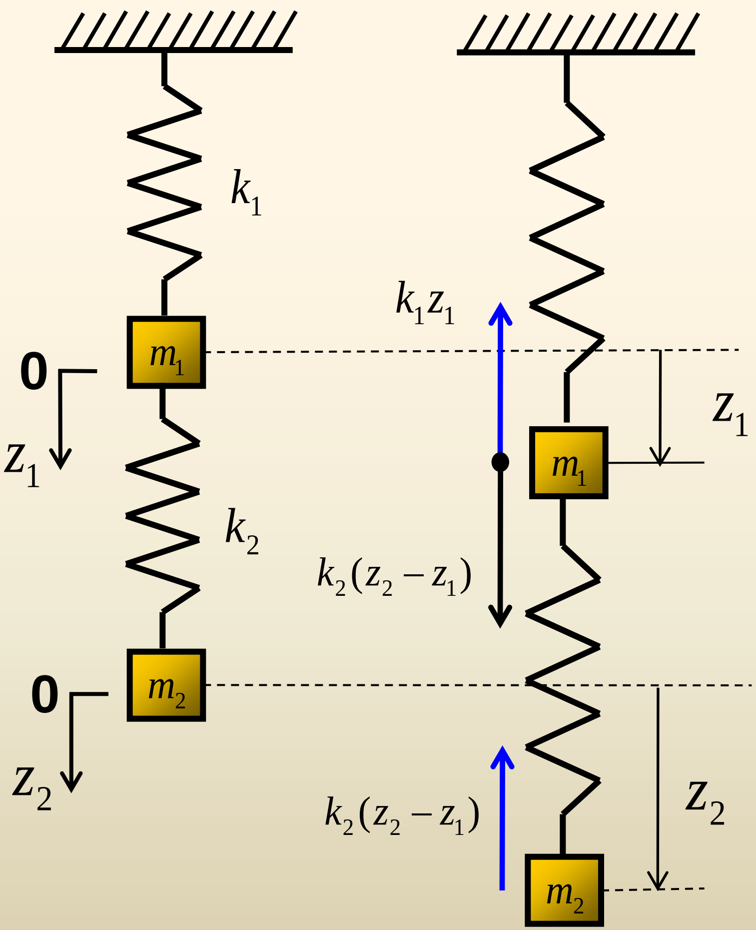
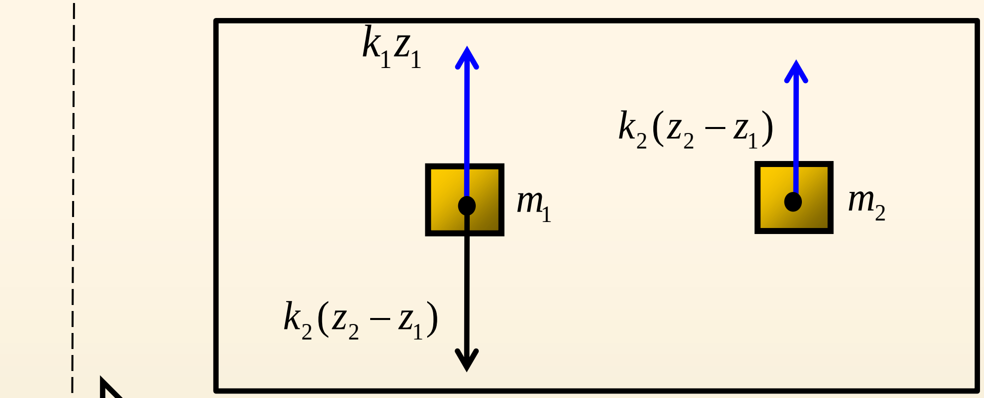
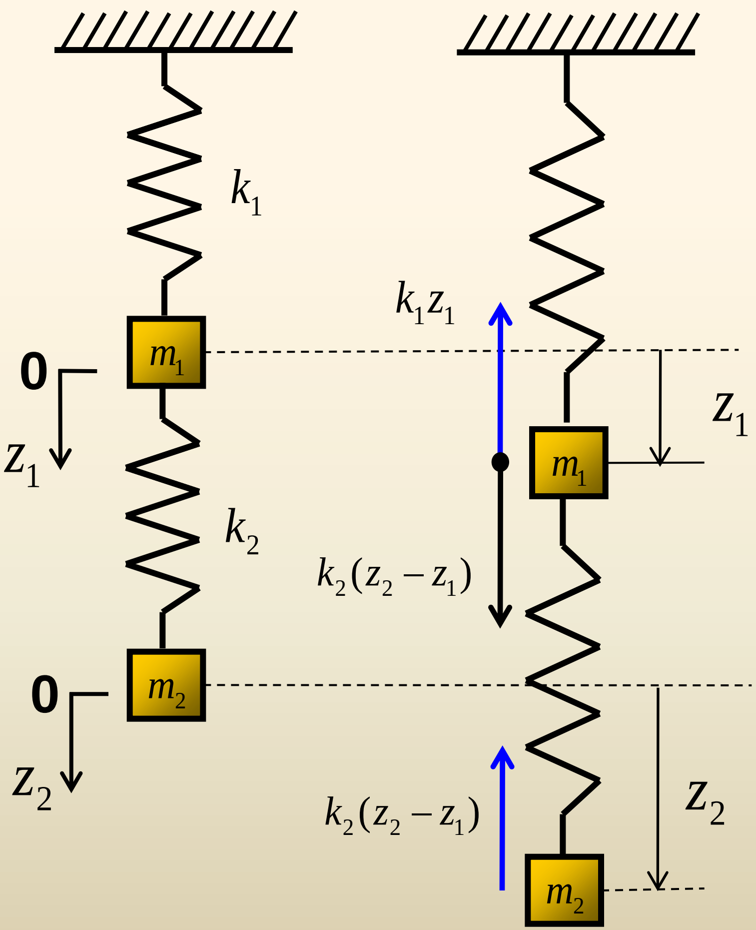


(a) equilibrium (b) Motion

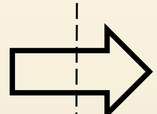


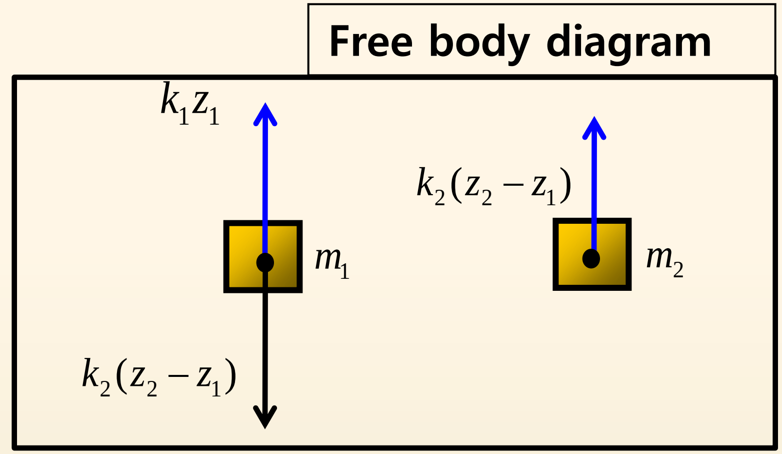
Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



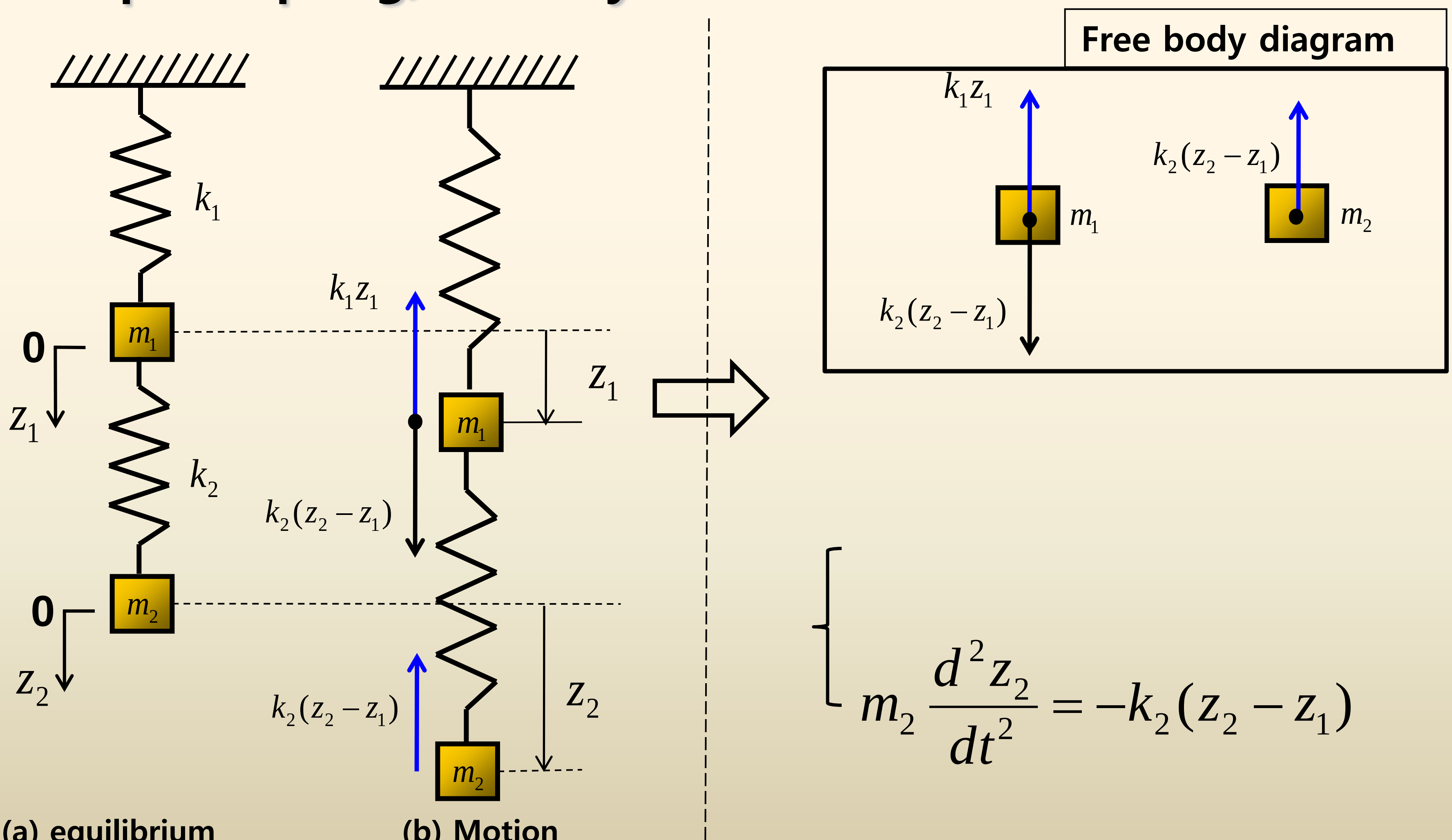
(a) equilibrium (b) Motion





Systems of Linear Differential Equations

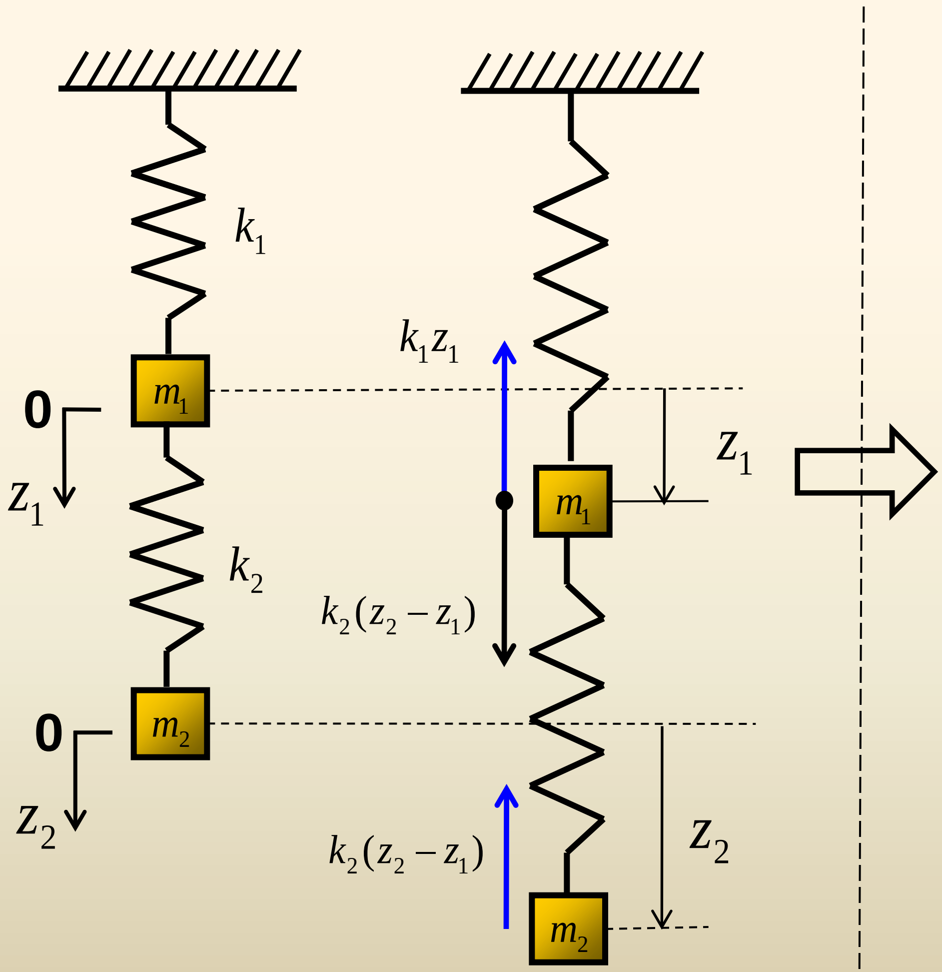
☑ Coupled Spring/Mass System



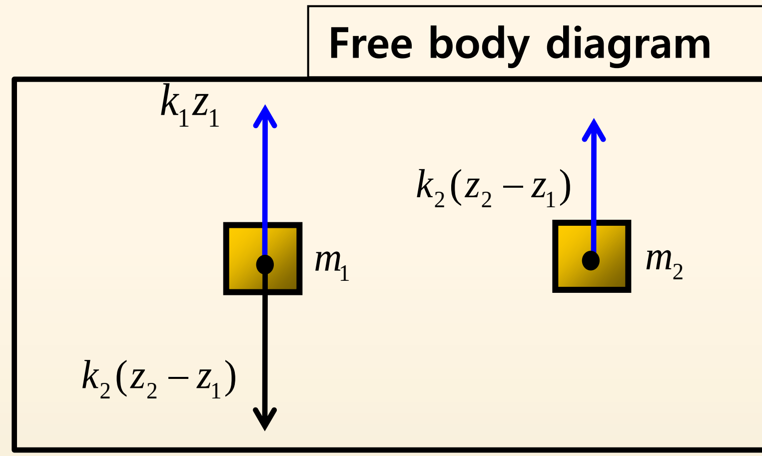
$$\begin{cases} \\ m_2 \dfrac{d^2 z_2}{dt^2} = -k_2(z_2 - z_1) \end{cases}$$

Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



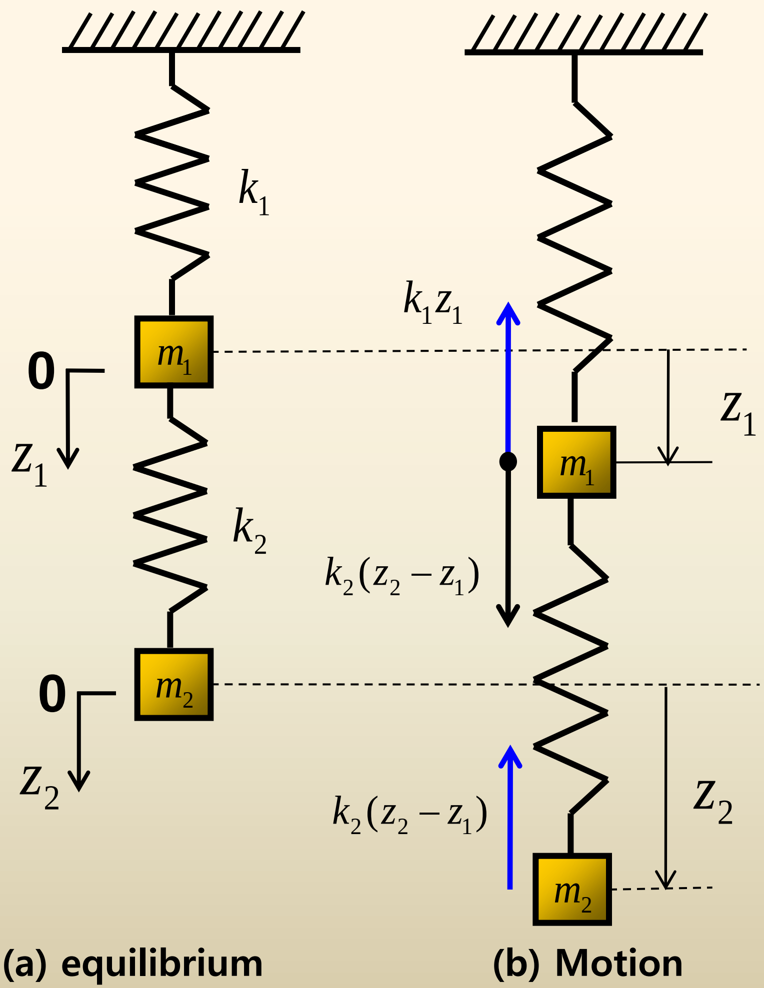
(a) equilibrium (b) Motion



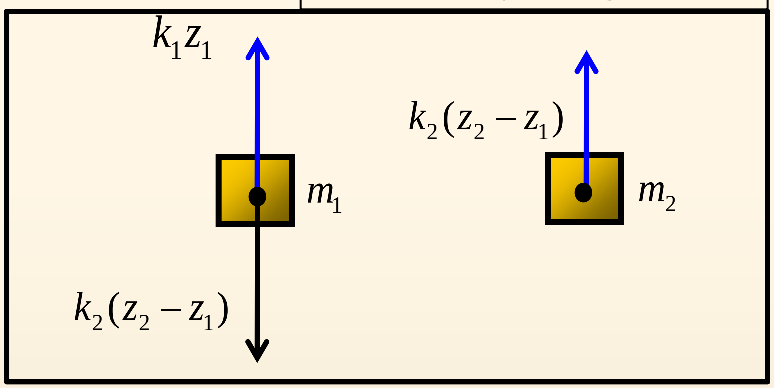
$$\begin{cases} m_1 \dfrac{d^2 z_1}{dt^2} = -k_1 z_1 + k_2(z_2 - z_1) \\ m_2 \dfrac{d^2 z_2}{dt^2} = -k_2(z_2 - z_1) \end{cases}$$

Systems of Linear Differential Equations

☑ Coupled Spring/Mass System



Free body diagram



System of linear 2nd order equations

$$\begin{cases} m_1 \dfrac{d^2 z_1}{dt^2} = -k_1 z_1 + k_2(z_2 - z_1) \\ m_2 \dfrac{d^2 z_2}{dt^2} = -k_2(z_2 - z_1) \end{cases}$$

Systems of Linear Differential Equations

☑ Example 1 Example 4 of Section 3.11 Revisited

Use the Laplace transform to solve

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$4x_1 + x_2'' + 4x_2 = 0$$

Subject to

$$x_1(0) = 0,\ x_1'(0) = 1$$
$$x_2(0) = 0,\ x_2'(0) = -1$$

$$s^2 X_1(s) - s x_1(0) - x_1'(0) + 10 X_1(s) - 4 X_2(s) = 0$$
$$-4 X_1(s) + s^2 X_2(s) - s x_2(0) - x_2'(0) + 4 X_2(s) = 0$$

$$\begin{aligned} (s^2+10) X_1(s) - \qquad\quad 4X_2(s) &= 1 \\ -4X_1(s) + (s^2+4) X_2(s) &= -1 \end{aligned} \quad \cdots(1)$$

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{1/5}{s^2+2} + \frac{6/5}{s^2+12}$$

$$X_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

Systems of Linear Differential Equations

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Use the Laplace transform to solve

$$\begin{aligned} x_1'' + 10x_1 - 4x_2 &= 0 \\ 4x_1 + x_2'' + 4x_2 &= 0 \end{aligned} \qquad \textbf{Subject to} \qquad \begin{aligned} x_1(0) = 0,\ x_1'(0) = 1 \\ x_2(0) = 0,\ x_2'(0) = -1 \end{aligned}$$

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{1/5}{s^2+2} + \frac{6/5}{s^2+12}$$

$$x_1(t) = -\frac{1}{5\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} + \frac{6}{5\sqrt{12}}\mathscr{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$

$$= -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t$$

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Use the Laplace transform to solve

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$4x_1 + x_2'' + 4x_2 = 0$$

Subject to

$$x_1(0) = 0,\ x_1'(0) = 1$$
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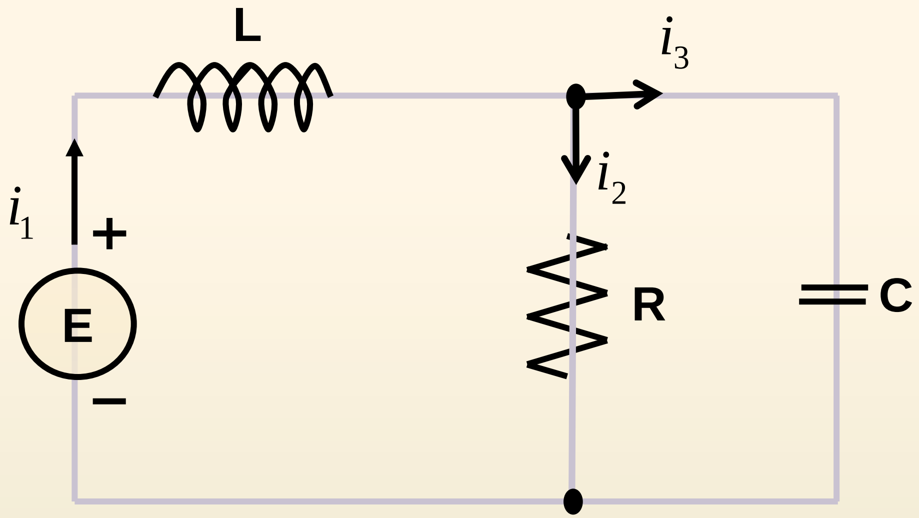
$$X_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

$$x_2(t) = -\frac{2}{5\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} - \frac{3}{5\sqrt{12}}\mathscr{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\}$$

$$= -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t$$

Systems of Linear Differential Equations

☑ Networks



$$L\frac{di_1}{dt} + Ri_2 = E(t)$$

$$RC\frac{di_2}{dt} + i_2 - i_1 = 0$$

Systems of Linear Differential Equations

☑ Example 2 An Electrical Network

Solve the system under the condition $E(t) = 60[\mathrm{V}],\ L = 1[h],\ R = 50[\Omega]$

$C = 10^{-4}[\mathrm{f}]$

$$L\frac{di_1}{dt} + Ri_2 = E(t)$$

$$RC\frac{di_2}{dt} + i_2 - i_1 = 0$$

Subject to $i_1(0) = 0,\ i_2(0) = 0$

$$\frac{di_1}{dt} + 50i_2 = 60 \qquad sI_1(s) + 50I_2(s) = \frac{60}{s}$$

$$50(10^{-4})\frac{di_2}{dt} + i_2 - i_1 = 0 \qquad -200I_1(s) + (s+200)I_2(s) = 0$$

$$I_1(s) = \frac{60s + 12{,}000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{60}{(s+100)^2}$$

$$I_2(s) = \frac{12{,}000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{120}{(s+100)^2}$$

Systems of Linear Differential Equations

☑ Example 2 An Electrical Network

Solve the system under the condition $E(t)=60[\mathrm{V}],\ L=1[h],\ R=50[\Omega]$

$C=10^{-4}[\mathrm{f}]$

$$L\frac{di_1}{dt}+Ri_2=E(t)$$

$$RC\frac{di_2}{dt}+i_2-i_1=0$$

Subject to $i_1(0)=0,\ i_2(0)=0$

$$i_1(t)=\frac{6}{5}-\frac{6}{5}e^{-100t}-60te^{-100t}$$

$$i_2(t)=\frac{6}{5}-\frac{6}{5}e^{-100t}-120te^{-100t}$$

Reference slides

Properties of Convolution

Property of Convolution

Commutative law of convolution

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau$$

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replace

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$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau$$

replace $\tau = t - \tau'$

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$$\tau$$

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Differentiate with τ

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replace $\tau = t - \tau'$

Differentiate with τ $\qquad 1 = -\frac{d\tau'}{d\tau}$

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Differentiate with τ $\quad 1 = -\dfrac{d\tau'}{d\tau} \Rightarrow d\tau = -d\tau'$

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τ'

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if Varies 0 ~ t , τ' **varies t ~ 0**

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if τ **Varies 0 ~ t ,** τ' **varies t ~ 0**

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

$$=\int_t^0 f(t-\tau')g(\tau')(-d\tau')$$

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$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

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$$\begin{aligned}(f*g)(t)&=\int_0^t f(\tau)g(t-\tau)d\tau\\&=\int_t^0 f(t-\tau')g(\tau')(-d\tau')\\&=-\int_t^0 f(t-\tau')g(\tau')d\tau'\end{aligned}$$

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Property of Convolution

Associative law of convolution

$$\{(f*g)*v\}(t)=\int_0^t (f*g)(\tau)v(t-\tau)d\tau$$

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

$$(f*g)(\tau)=\int_0^\tau f(p)g(\tau-p)dp$$

$$\tau-p=k,\ \ 1=\frac{dk}{d\tau},\ \ d\tau=dk$$

$$if\ \ \tau\ is\ 0\rightarrow t,\ \ k\,is\ -p\rightarrow t-p$$

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$$=\int_0^t\int_0^\tau f(p)g(\tau-p)dp\,v(t-\tau)d\tau$$

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$$=\int_0^t\int_0^\tau f(p)g(\tau-p)v(t-\tau)dp\,d\tau$$

$$=\int_0^t\int_{-p}^{t-p} f(p)g(k)v(t-(p+k))dk\,dp$$

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

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$$=\int_0^t\int_0^{t-p} f(p)g(k)v(t-p-k))dk\,dp$$

$$=\int_0^t f(p)\int_0^{t-p} g(k)v(t-p-k))dk\,dp$$

$$=\int_0^t f(p)\cdot(g*v)(t-p)dp$$

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

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$$=\int_0^t f(p)\int_0^{t-p} g(k)v(t-p-k))dk\,dp$$

$$=\int_0^t f(p)\cdot(g*v)(t-p)dp$$

$$=\{f*(g*v)\}(t)$$

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$$

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Distributive law of convolution

$$f*(g_1+g_2)(t)=\int_0^t f(\tau)\cdot(g_1+g_2)(t-\tau)d\tau$$

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$$=\int_0^t f(\tau)\cdot g_1(t-\tau)+f(\tau)\cdot g_2(t-\tau)d\tau$$

$$=\int_0^t f(\tau)\cdot g_1(t-\tau)d\tau+\int_0^t f(\tau)\cdot g_2(t-\tau)d\tau$$

Property of Convolution

Distributive law of convolution

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$$=\int_0^t f(\tau)\cdot g_1(t-\tau)+f(\tau)\cdot g_2(t-\tau)d\tau$$

$$=\int_0^t f(\tau)\cdot g_1(t-\tau)d\tau+\int_0^t f(\tau)\cdot g_2(t-\tau)d\tau$$

$$=f*g_1(t)+f*g_2(t)$$