Naval Architecture & Ocean Engineering

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Engineering Mathematics 2

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Vector Calculus (2)
: Vector Functions, Vector Fields

Vector Functions

Partial Derivatives

Directional Derivatives

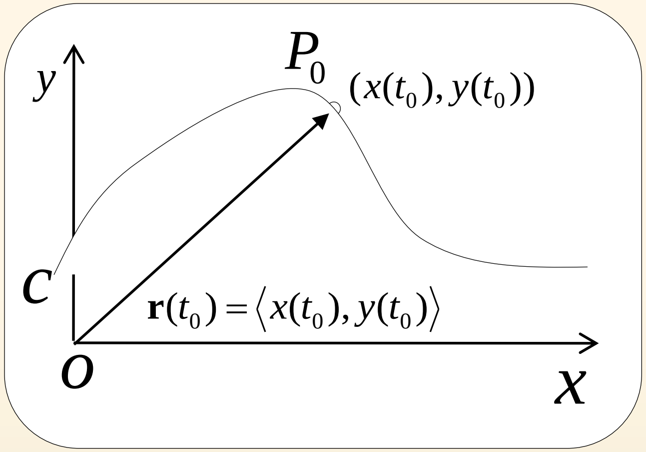
Tangent Planes and Normal Lines

Vector Fields and Fluid Mechanics

Vector Functions

■ Parametric Curve / ■ Vector Function

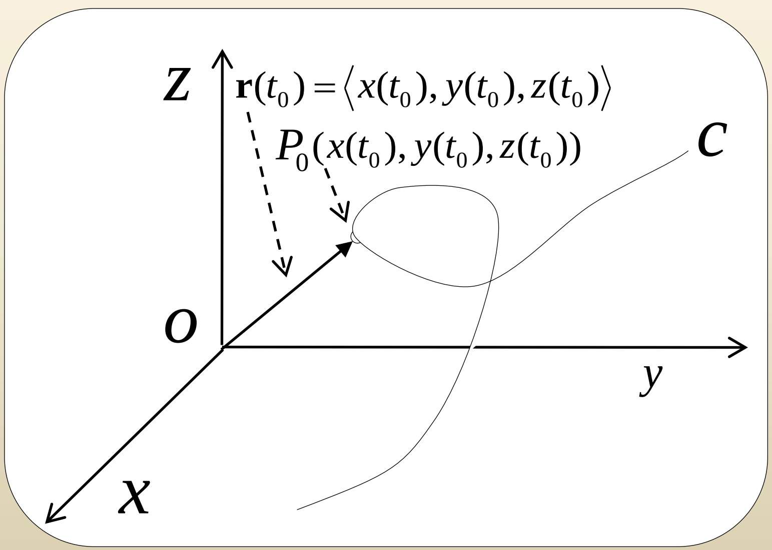
2-D



$$\mathbf{r}(t) = \langle x(t), y(t) \rangle$$
$$= f(t)\mathbf{i} + g(t)\mathbf{j}$$

$$x(t) = f(t)$$
$$y(t) = g(t)$$

3-D



$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$
$$= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

$$x(t) = f(t)$$
$$y(t) = g(t)$$
$$z(t) = h(t)$$

For a given parameter t_0 , $\mathbf{r}(t_0)$ is a position vector of a point P on a curve C

Vector Functions

Definition 9.1 — **Limit of a Vector Function**

If $\lim_{t \to a} f(t)$, $\lim_{t \to a} g(t)$, and $\lim_{t \to a} h(t)$ exist, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

Theorem 9.1 — **Properties of Limits**

If $\lim_{t \to a} \mathbf{r}_1(t) = \mathbf{L}_1$ and $\lim_{t \to a} \mathbf{r}_2(t) = \mathbf{L}_2$, then

$$(i) \quad \lim_{t \to a} c\mathbf{r}_1(t) = c\mathbf{L}_1, \quad c \text{ is a scalar}$$

$$(ii) \quad \lim_{t \to a} \left[\mathbf{r}_1(t) + \mathbf{r}_2(t) \right] = \mathbf{L}_1 + \mathbf{L}_1$$

$$(iii) \quad \lim_{t \to a} \mathbf{r}_1(t) \cdot \mathbf{r}_2(t) = \mathbf{L}_1 \cdot \mathbf{L}_1$$

Vector Functions

Definition 9.2 **Continuity**

A vector function $\mathbf{r}$ is said to be continuous at $t = a$ if

$(i)\, \mathbf{r}(a)$ is defined

$(ii)\, \mathbf{r}(t)$ exists, and

$$(iii)\, \lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

Definition 9.3 **Derivatives of Vector Function**

The derivative of a vector $\mathbf{r}$ is $\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\mathbf{r}(t + \Delta t) - \mathbf{r}(t)\right]$

for all t for which the limit exists

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt}$$

Vector Functions

Theorem 9.2 **Differentiation of Components**

If $\mathbf{r}(t) = \langle f(x), g(x), h(x) \rangle$ where f, g and h are differentiable, then

$$\mathbf{r}'(t) = \langle f'(x), g'(x), h'(x) \rangle$$

■ Proof

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)\right]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)\rangle - \langle f(t), g(t), h(t)\rangle$$

$$= \lim_{\Delta t \to 0}\left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, h\,\frac{f(t+\Delta t) - h(t)}{\Delta t}\right\rangle$$

Vector Functions

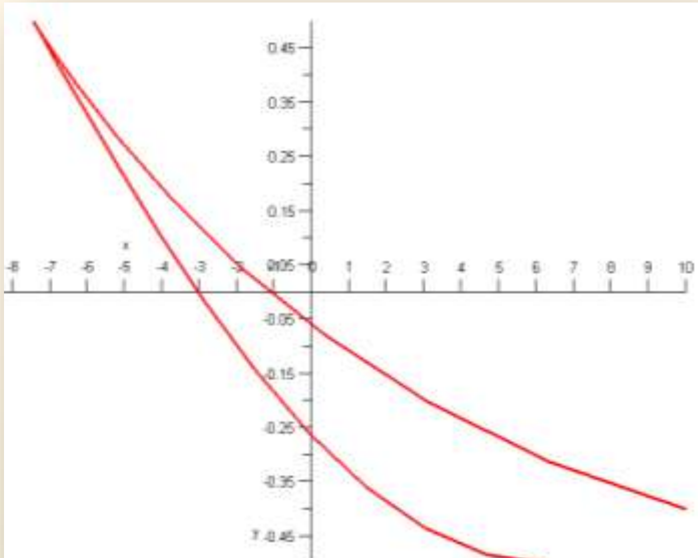
■ Smooth curve

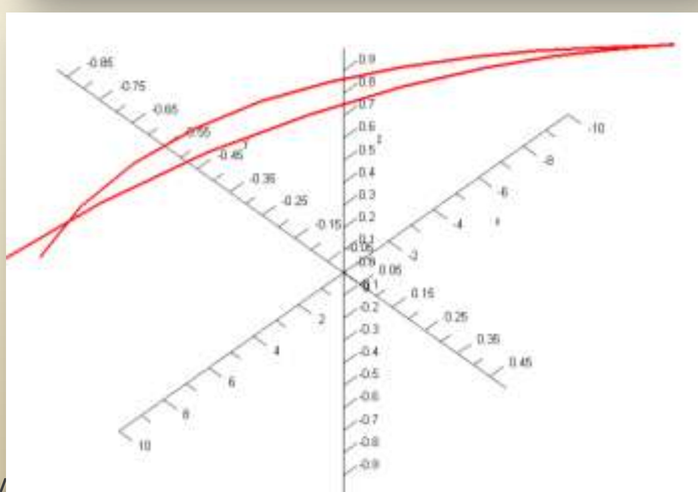
When the component function of a vector function $\mathbf{r}$ have continuous first derivatives and $\mathbf{r}'(t) \neq 0$ for all t in the open interval (a, b) , then $\mathbf{r}$ is said to be a smooth function and the curve C traced by $\mathbf{r}$ is called a smooth curve

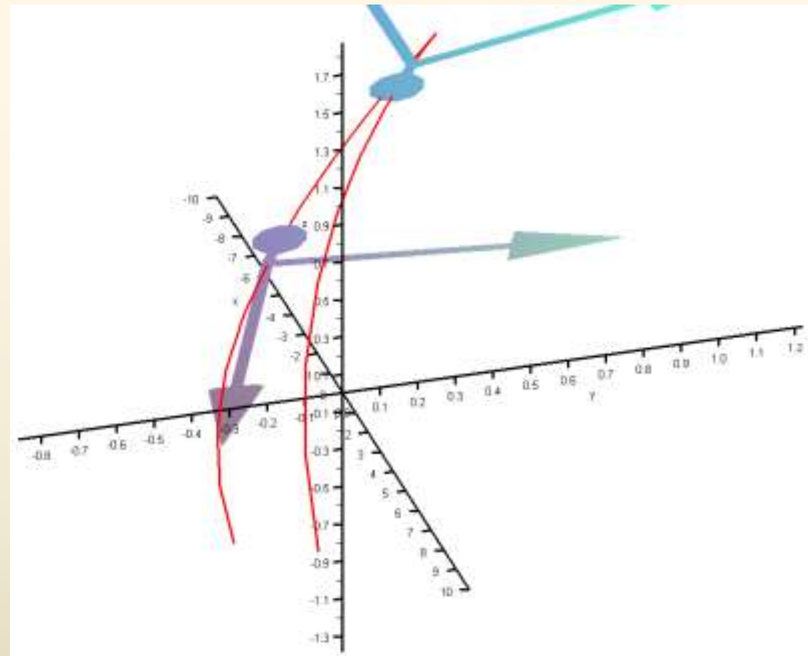
Vector Functions

■ Not Smooth curve ex.)

$$F(t)=\left(2t^3+(4-\frac{3}{2}\pi)t^2-4\pi t+2\pi\right)\mathbf{i}+\left(-\frac{1}{2}\cos 2t\right)\mathbf{j}+(\sin t)\mathbf{k}$$







Vector Functions

Theorem 9.2 **Differentiation of Components**

If $\mathbf{r}(t) = \langle f(x), g(x), h(x) \rangle$ where f, g and h are differentiable, then

$$\mathbf{r}'(t) = \langle f'(x), g'(x), h'(x) \rangle$$

■ High-Order Derivatives

Also obtained by differentiating its components.

Ex) Second-order

$$\mathbf{r}''(t) = \langle f''(x), g''(x), h''(x) \rangle$$

Vector Functions

☑ Example 4
Tangent Vectors

Graph the curve C that is traced by a point P whose position is given by $\mathbf{r}(t)=\cos 2t\mathbf{i}+\sin t\mathbf{j}$, $0 \le t \le 2\pi$. Graph $\mathbf{r}'(0)$ and $\mathbf{r}'(\pi/6)$.

Solution)

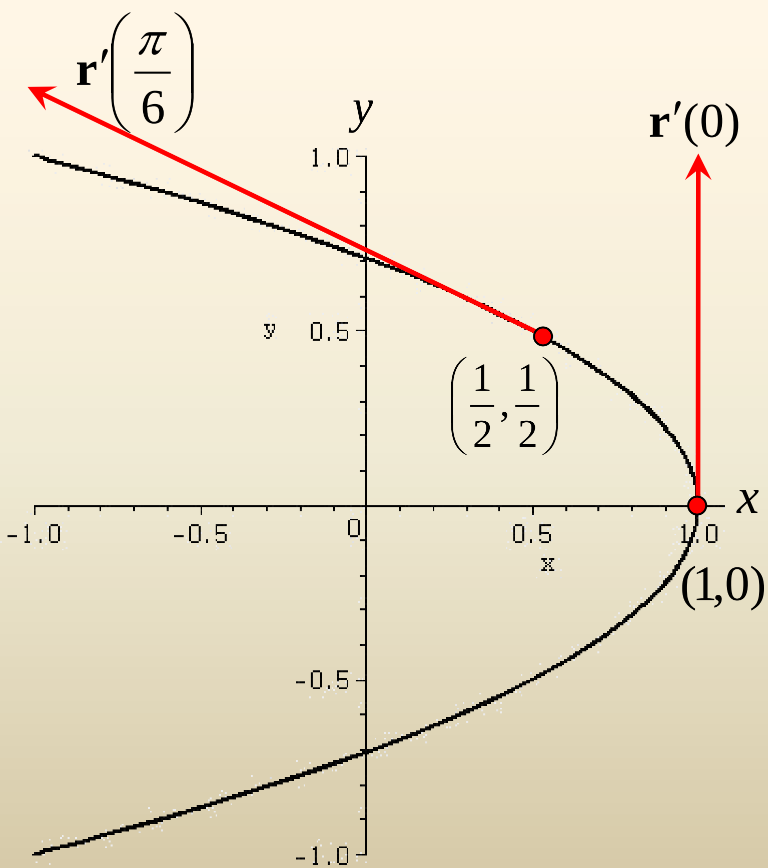
$$\begin{cases} x = \cos 2t = \cos^2 t - \sin^2 t \quad (0 \le t \le 2\pi) \\ \quad = (1-\sin^2 t) - \sin^2 t \\ \quad = 1 - 2\sin^2 t \\ y = \sin t \end{cases}$$

$$x = 1 - 2y^2, \ -1 \le x \le 1$$

$$\mathbf{r}'(t) = -2\sin 2t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{r}'(0) = \mathbf{j}$$

$$\mathbf{r}'\left(\frac{\pi}{6}\right) = -\sqrt{3}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$



Vector Functions

☑ Example 5
Tangent Line

Find parametric equations of the tangent line to the graph of the curve *C* whose parametric equations are $x=t^2$, $y=t^2-t$, $z=-7t$ at $t=3$.

Solution)

$$\mathbf{r}(t)=t^2\mathbf{i}+(t^2-t)\mathbf{j}-7t\mathbf{k}$$

$$\mathbf{r}(3)=9\mathbf{i}+6\mathbf{j}-21\mathbf{k}$$

$$\mathbf{r}'(t)=2t\mathbf{i}+(2t-1)\mathbf{j}-7\mathbf{k}$$

$$\mathbf{r}'(3)=6\mathbf{i}+5\mathbf{j}-7\mathbf{k}$$

$$\mathbf{r}(3)+\mathbf{r}'(3)\cdot t=(9\mathbf{i}+6\mathbf{j}-21\mathbf{k})+(6\mathbf{i}+5\mathbf{j}-7\mathbf{k})\cdot t$$
$$=(9+6t)\mathbf{i}+(6+5t)\mathbf{j}+(-21-7t)\mathbf{k}$$

$$x=9+6t,\ y=6+5t,\ z=-21-7t$$

Vector Functions

☑ Example 6
Derivative of a Vector Function

If $\mathbf{r}(t)=(t^3-2t^2)\mathbf{i}+4t\mathbf{j}+e^{-t}\mathbf{k}$,

then $\mathbf{r}'(t)=(3t^2-4t)\mathbf{i}+4\mathbf{j}-e^{-t}\mathbf{k}$
and $\mathbf{r}''(t)=(6t-4)\mathbf{i}+e^{-t}\mathbf{k}$.

Vector Functions

Theorem 9.3 **Chain Rule**

If $\mathbf{r}$ is a differentiable vector function and $s = u(t)$ is a differentiable scalar function, then, the derivatives of $\mathbf{r}(s)$ with respect to t is

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{r}'(s)u'(t)$$

Vector Functions

☑ Example 7
Chain Rule

If $\mathbf{r}(s)=\cos 2s\mathbf{i}+\sin 2s\mathbf{j}+e^{-3s}\mathbf{k}$, where $s=t^4$, then

$$\frac{d\mathbf{r}}{dt}=\left[-2\sin 2s\mathbf{i}+2\cos 2s\mathbf{j}-3e^{-3s}\mathbf{k}\right]4t^3$$

$$=-8t^3\sin(2t^4)\mathbf{i}+8t^3\cos(2t^4)\,j-12t^3e^{-3t^4}\mathbf{k}.$$

Vector Functions

Theorem 9.4

Rules of Differentiation

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be differentiable vector functions $u(t)$ a differentiable scalar function.

$$(i)\ \frac{d}{dt}[\mathbf{r}_1(t)+\mathbf{r}_2(t)]=\mathbf{r}_1'(t)+\mathbf{r}_2'(t)$$

$$(ii)\ \frac{d}{dt}[u(t)\mathbf{r}_1(t)]=u(t)\mathbf{r}_1'(t)+u'(t)\mathbf{r}_1(t)$$

$$(iii)\ \frac{d}{dt}[\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)]=\mathbf{r}_1'(t)\cdot\mathbf{r}_2(t)+\mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)$$

$$(iv)\ \frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]=\mathbf{r}_1'(t)\times\mathbf{r}_2(t)+\mathbf{r}_1(t)\times\mathbf{r}_2'(t)$$

Differentiation of Cross Product

Vector Functions

Integral of Vector Functions

If f, g and h are integrable, then the indefinite and definite integrals of a vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are defined, respectively, by

$$\int \mathbf{r}(t)dt = \left(\int f(t)dt\right)\mathbf{i} + \left(\int g(t)dt\right)\mathbf{j} + \left(\int h(t)dt\right)\mathbf{k}$$

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}$$

$$\int \mathbf{r}(t)dt = \mathbf{R} + \mathbf{c}, \quad where\ \mathbf{R}'(t) = \mathbf{r}(t)$$

Vector Functions

☑ Example 8
Integral of a Vector Function

If $\mathbf{r}(t)=6t^2\mathbf{i}+4e^{-2t}\mathbf{j}+8\cos 4t\mathbf{k}$ then

$$\begin{aligned}\int \mathbf{r}(t)dt &= \left[\int 6t^2 dt\right]\mathbf{i}+\left[\int 4e^{-2t}dt\right]\mathbf{j}+\left[\int 8\cos 4t\,dt\right]\mathbf{k}\\ &= \left[2t^3+c_1\right]\mathbf{i}+\left[-2e^{-2t}+c_2\right]\mathbf{j}+\left[2\sin 4t+c_3\right]\mathbf{k}\\ &= 2t^3\mathbf{i}-2e^{-2t}\mathbf{j}+2\sin 4t\mathbf{k}+\mathbf{c},\end{aligned}$$

where $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.

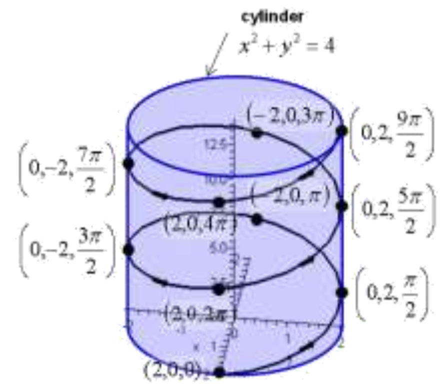
Vector Functions

☑ Example 9

Consider the helix of Example 1. Find the length of the curve from $\mathbf{r}(0)$ to an arbitrary point $\mathbf{r}(t)$. Find the parametric equations of the helix.

9.1 Example 1.

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ \ t \ge 0.$$



Solution)

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1^2}$$

$$= \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$$

$$\|\mathbf{r}'(t)\| = \sqrt{5}$$

$$s = \int_0^t \|\mathbf{r}'(u)\| du = \int_0^t \sqrt{5} du = \sqrt{5}t$$

$$\mathbf{r}(s) = 2\cos\frac{s}{\sqrt{5}}\mathbf{i} + 2\sin\frac{s}{\sqrt{5}}\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$$

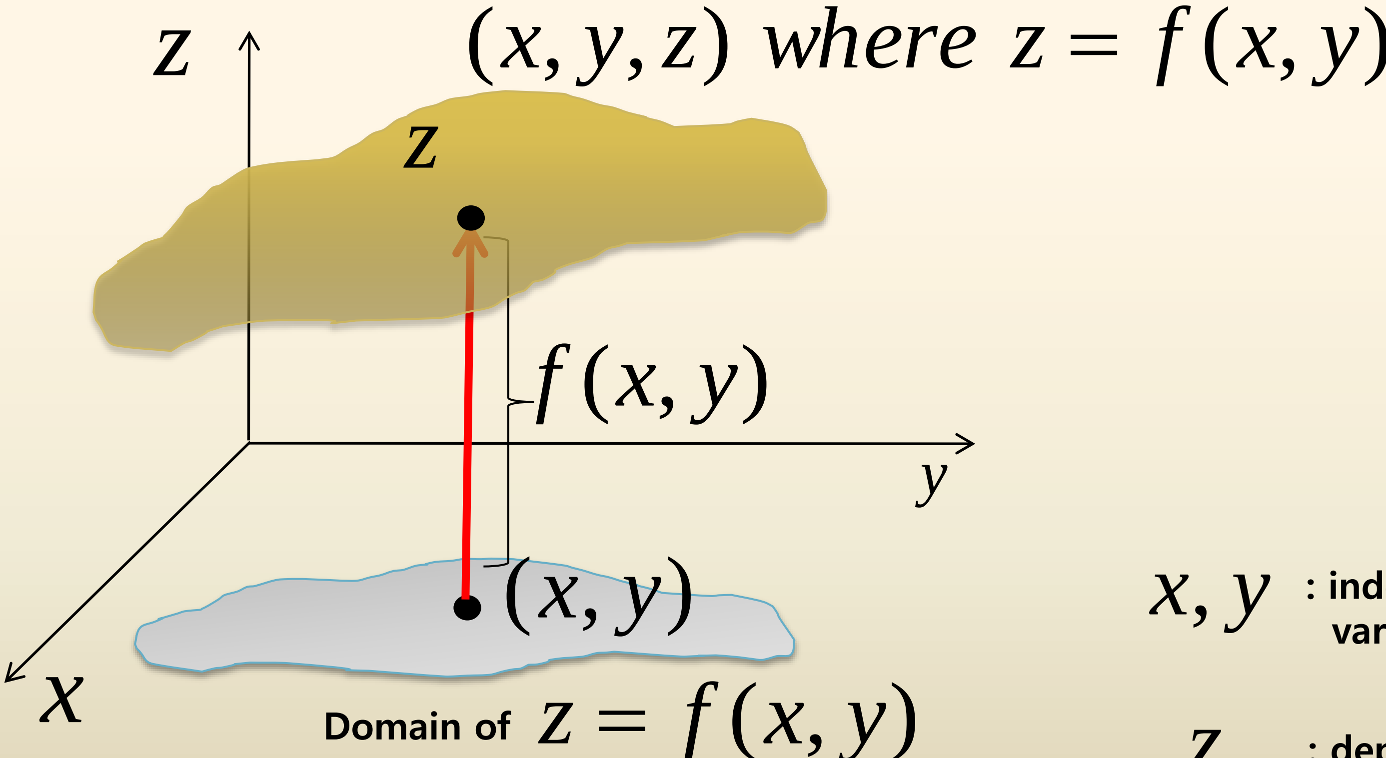
$$f(s) = 2\cos\frac{s}{\sqrt{5}}$$

$$g(s) = 2\sin\frac{s}{\sqrt{5}}$$

$$h(s) = \frac{s}{\sqrt{5}}$$

Partial Derivatives

Function of two variables $z = f(x, y)$



x, y : independent variables

z : dependent variable

Partial Derivatives

Ordinary Derivatives

$$y = f(x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Partial Derivatives

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

$$= \frac{\partial f}{\partial x} = z_x = f_x$$

partial derivative with respect to x

treating y **as a constant**

$$\frac{\partial z}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

$$= \frac{\partial f}{\partial y} = z_y = f_y$$

partial derivative with respect to y

treating x **as a constant**

Partial Derivatives

☑ Example 3
Partial Derivatives

If $z=4x^3y^2-4x^2+y^6+1$, find $\partial z/\partial x$ and $\partial z/\partial y$.

$$\frac{\partial z}{\partial x} = 12x^2y^2 - 8x$$

$$\frac{\partial z}{\partial y} = 8x^3y^2 + 6y^5$$

Partial Derivatives

☑ Example 4
Partial Derivatives

If $F(x,y,t)=e^{-3\pi t}\cos 4x\sin 6y$, then the partial derivatives with respect to x, y, and t are, in turn,

$$F_x(x,y,t) = -4e^{-3\pi t}\sin 4x \sin 6y$$

$$F_y(x,y,t) = 6e^{-3\pi t}\cos 4x \cos 6y$$

$$F_t(x,y,t) = -3\pi e^{-3\pi t}\cos 4x \sin 6y.$$

Partial Derivatives

Chain Rule

$$y = f(u)\ , u = g(x)$$

$$\rightarrow y = f(g(x))$$

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx}$$

Theorem 9.5 **Chain Rule**

If $z = f(u,v)$ is differentiable and $u = g(x,y)$ and $v = h(x,y)$ have continuous first partial derivatives, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$z = f(u,v) = f(g(x,y), h(x,y))$$

Partial Derivatives

☑ Example 5
Chain Rule

If $z=u^2-v^3$ and $u=e^{2x-3y}$, $v=\sin(x^2-y^2)$, find $\partial z/\partial x$ and $\partial z/\partial y$.

Solution)

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=2u\cdot\left(2e^{2x-3y}\right)-3v^2\left[2x\cos(x^2-y^2)\right]$$

$$=4ue^{2x-3y}-6xv^2\cos(x^2-y^2)$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$=2u\cdot(-3e^{2x-3y})-3v^2\left[(-2y)\cos(x^2-y^2)\right]$$

$$=-6ue^{2x-3y}+6yv^2\cos(x^2-y^2)$$

Partial Derivatives

Chain Rule

$$z = f(u,v)\ ,\ u = g(x,y)\ ,\ v = h(x,y)$$

$$z = f(u,v) = f(g(x,y), h(x,y))$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

---

special case

$$z = f(u,v)\ ,\ u = g(t)\ ,\ v = h(t)$$

$$z = f(u,v) = f(g(t), h(t))$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$$

Partial Derivatives

Chain Rule

example)

If $r(x,y,z) = x^2+y^5z^3$

and $x(u,v)=uve^{2s}$, $y(u,v)=u^2-v^2s$, $z(u,v)=\sin(uvs^2)$,

find $\partial r/\partial s$.

$$\frac{\partial r}{\partial s} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial s}$$

$$= 2x(2uve^{2s}) + 5y^{4z^3}(-v^2) + 3y^5z^2\left(2uvs\cos(uvs^2)\right)$$

Partial Derivatives

special case

$$z = f(u,v)\ ,\ \ u = g(t)\ \ , v = h(t)$$

$$z = f(u,v) = f(g(t), h(t))$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$$

---

generalization $z = f(u_1, u_2, \ldots, u_n)$

$$u_1 = g_1(t)$$

$$u_2 = g_2(t)$$

$$\vdots$$

$$u_n = g_n(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u_1}\frac{du_1}{dt} + \frac{\partial z}{\partial u_2}\frac{du_2}{dt} + \cdots + \frac{\partial z}{\partial u_n}\frac{du_n}{dt}$$

Partial Derivatives

Chain Rule $z = f(u,v)$, $u = g(x,y)$, $v = h(x,y)$

$$z = f(u,v) = f(g(t),h(t))$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

---

generalization $z = f(u_1, u_2, \ldots, u_n)$

$$u_1 = g_1(x_1, x_2, \ldots, x_k)$$
$$u_2 = g_2(x_1, x_2, \ldots, x_k)$$
$$\vdots$$
$$u_n = g_n(x_1, x_2, \ldots, x_k)$$

$$\frac{dz}{dx_i} = \frac{\partial z}{\partial u_1}\frac{\partial u_1}{\partial x_i} + \frac{\partial z}{\partial u_2}\frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial z}{\partial u_n}\frac{\partial u_n}{\partial x_i}$$

Gradient of a Function

Introduce a new vector based on partial differentiation

Gradients of function

Differentiable function

$$z = f(x, y)$$

$$w = F(x, y, z)$$

Apply Vector differential operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

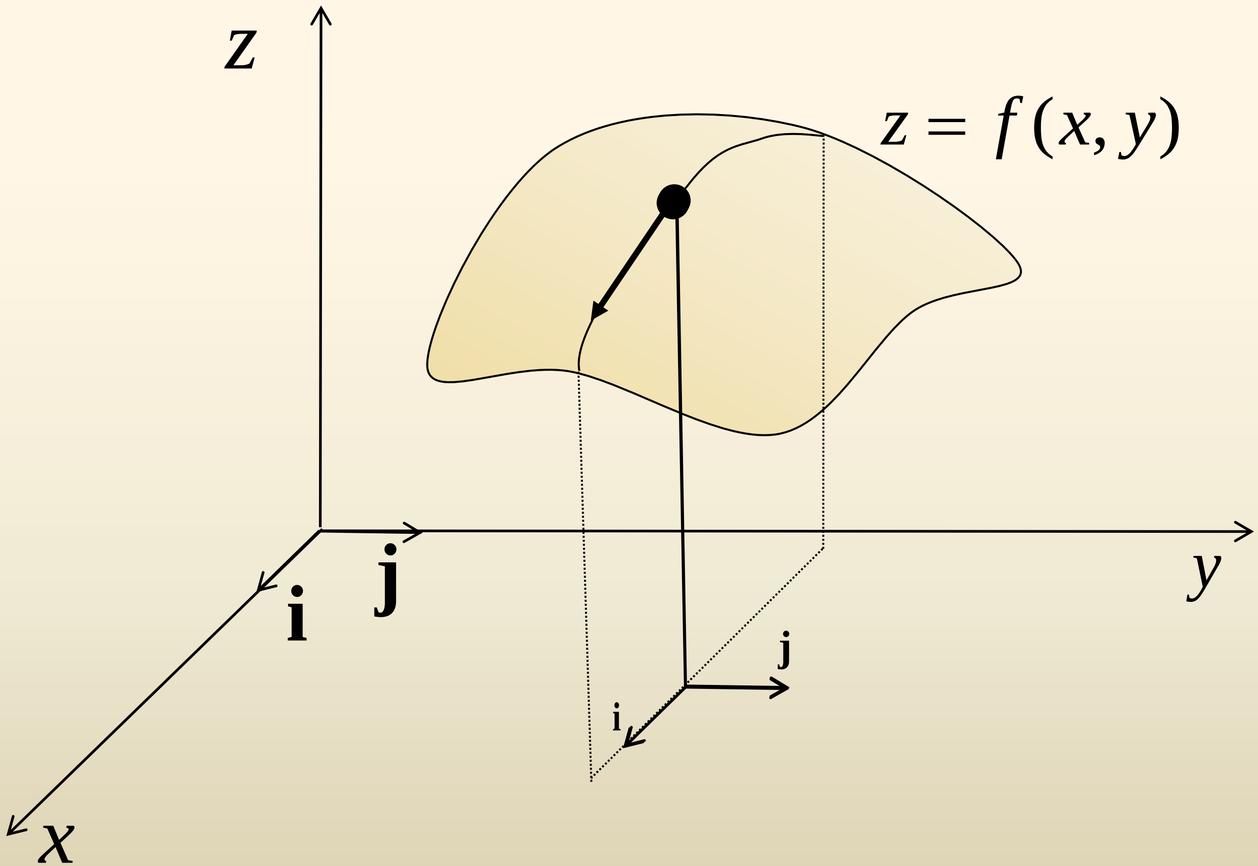
$$\nabla F(x, y, z) = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}$$

∇ **: "del" or "nabla"**

∇f **: "grad** f **"**

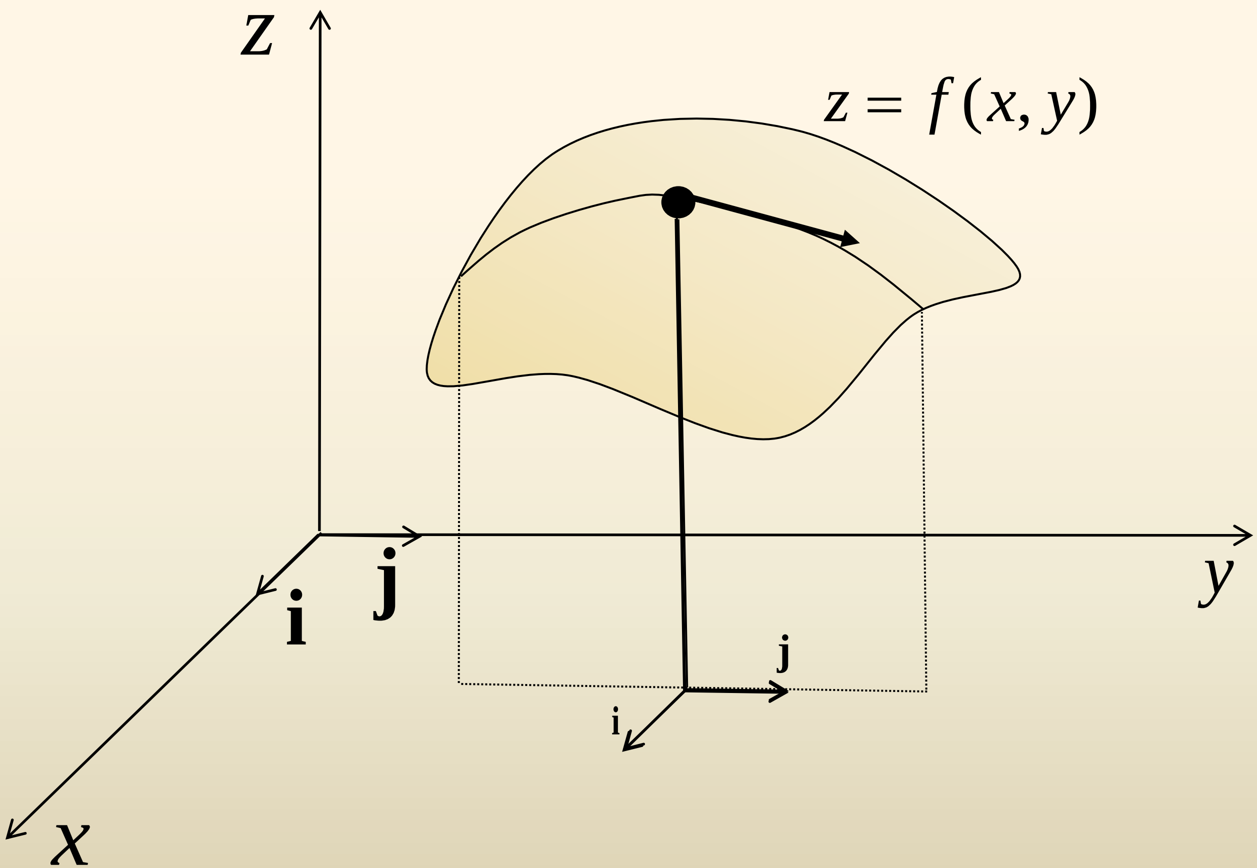
Directional Derivatives

$\frac{\partial f}{\partial x}$:Rate of change of f in the $\mathbf{i}$ -direction



Directional Derivatives

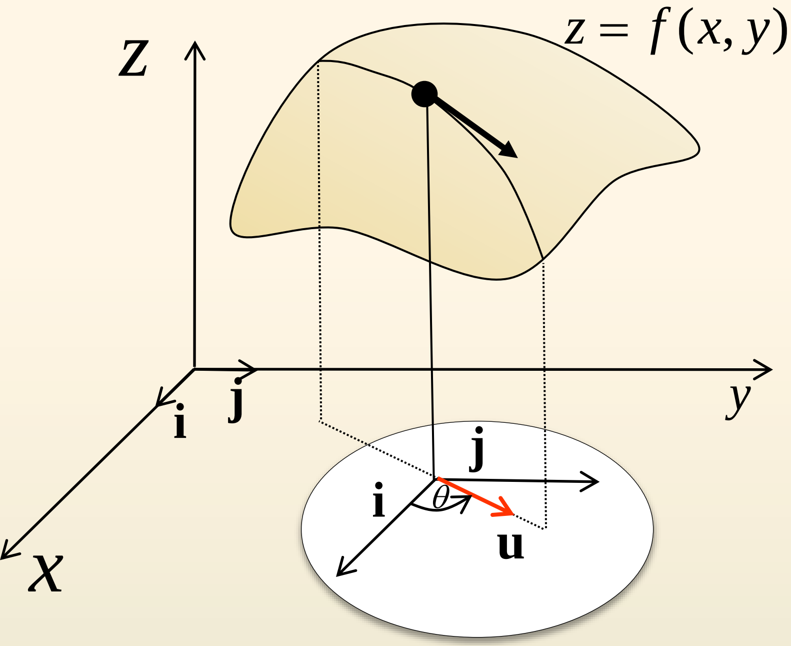
$\frac{\partial f}{\partial y}$:Rate of change of f in the $\mathbf{j}$ -direction



Directional Derivatives

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

The rate of change of f in the direction given by the vector $\mathbf{u}$: $D_{\mathbf{u}} f$



The rate of change of f in the direction of $\mathbf{x}$: $\frac{\partial f}{\partial x}$

The component of $\frac{\partial f}{\partial x}$ in the direction of $\mathbf{u}$: $\frac{\partial f}{\partial x}\cos\theta$

The rate of change of f in the direction of $\mathbf{y}$: $\frac{\partial f}{\partial y}$

The component of $\frac{\partial f}{\partial y}$ in the direction of $\mathbf{u}$: $\frac{\partial f}{\partial y}\cos(\frac{\pi}{2}-\theta)$

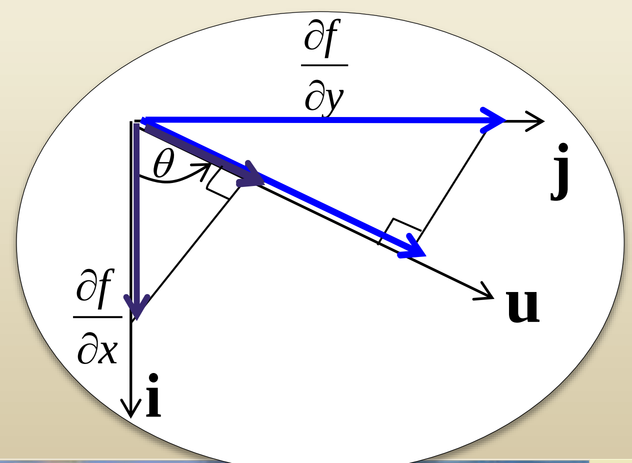
$= \frac{\partial f}{\partial y}\sin\theta$

The rate of change of f in the direction given by the vector $\mathbf{u}$: $D_{\mathbf{u}} f$

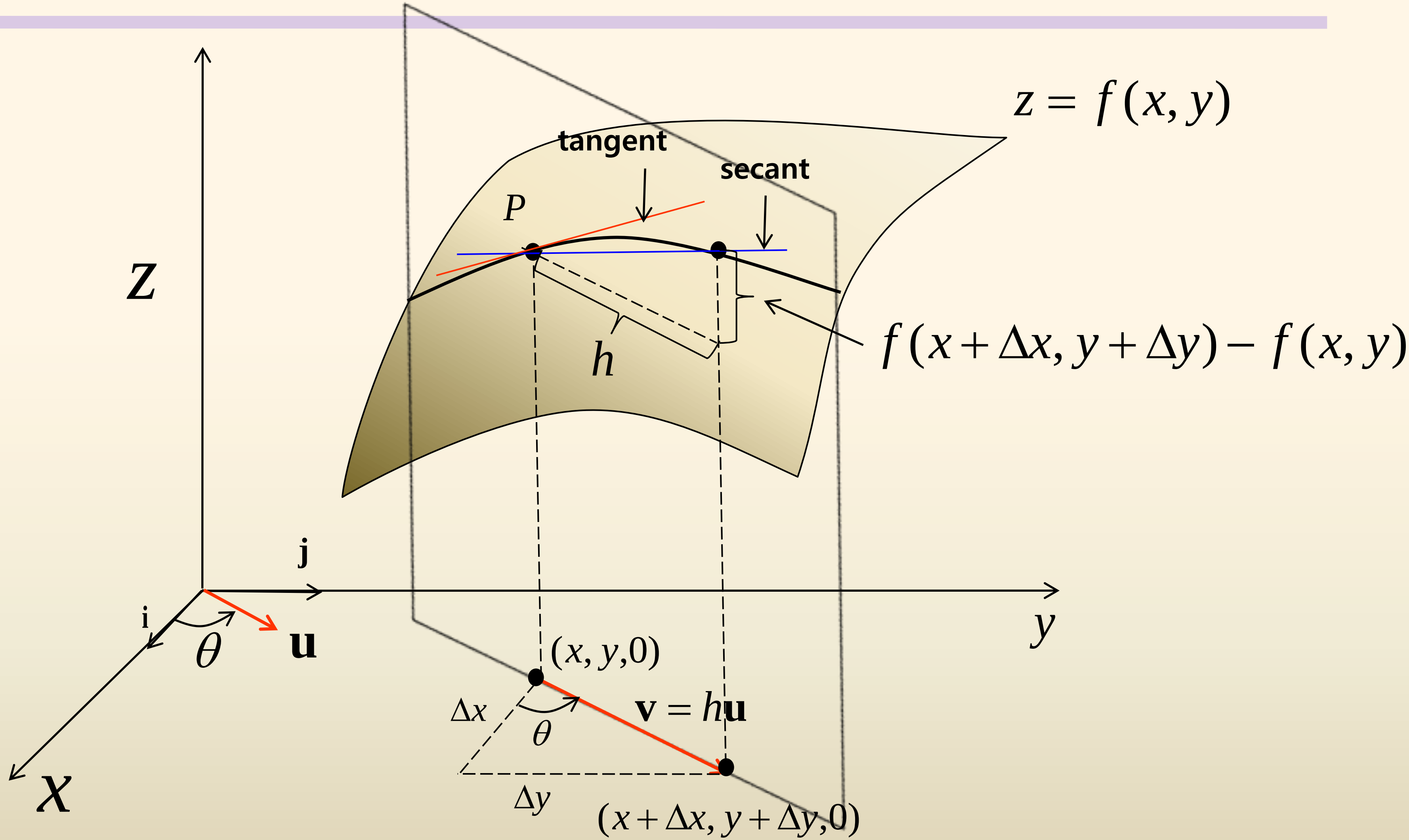
= +

$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$

$$= (\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}) \bullet (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = \nabla f \cdot \mathbf{u}$$



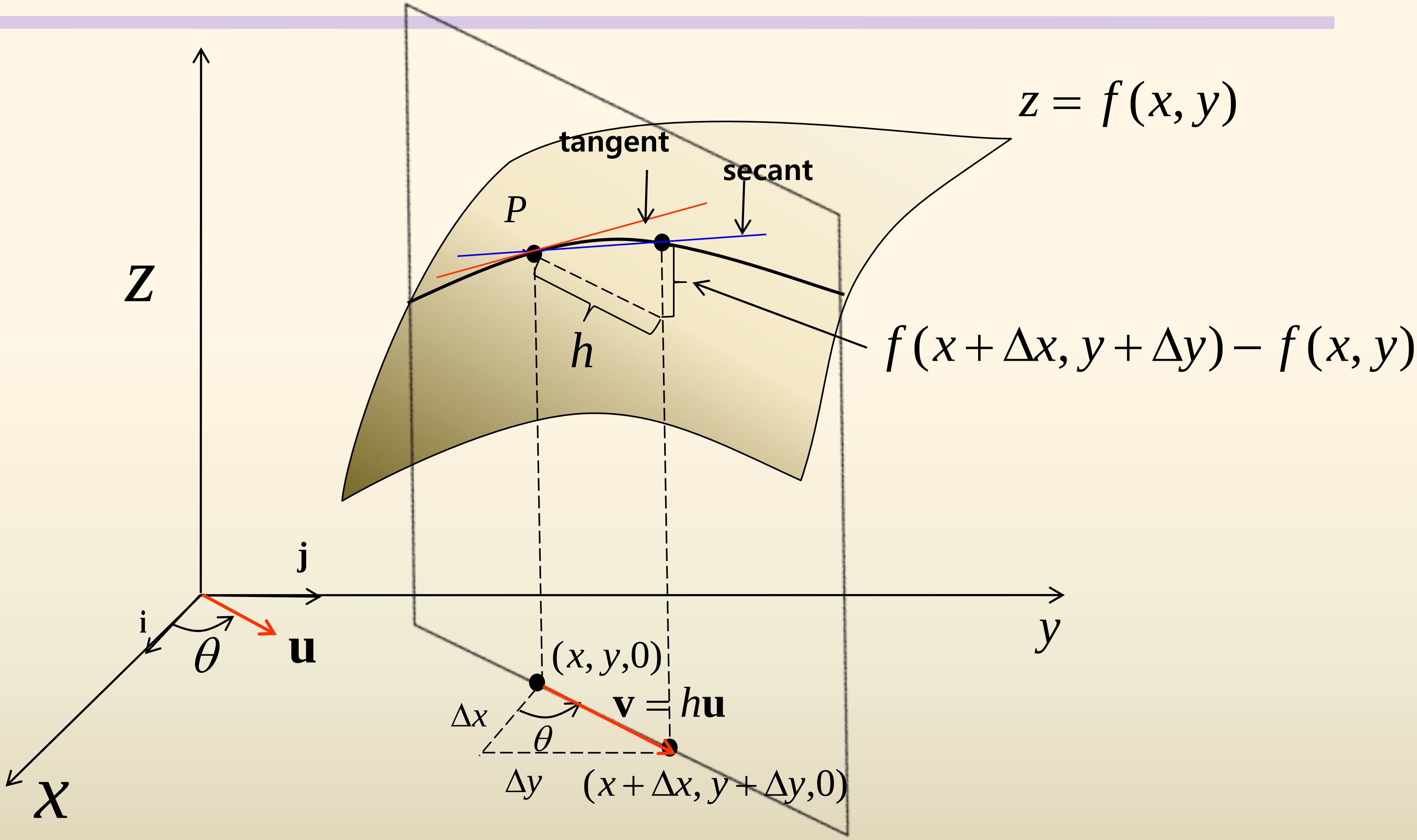
Directional Derivatives



Slope of indicated secant line is

$$\frac{f(x+\Delta x, y+\Delta y)-f(x,y)}{h}=\frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y)}{h}$$

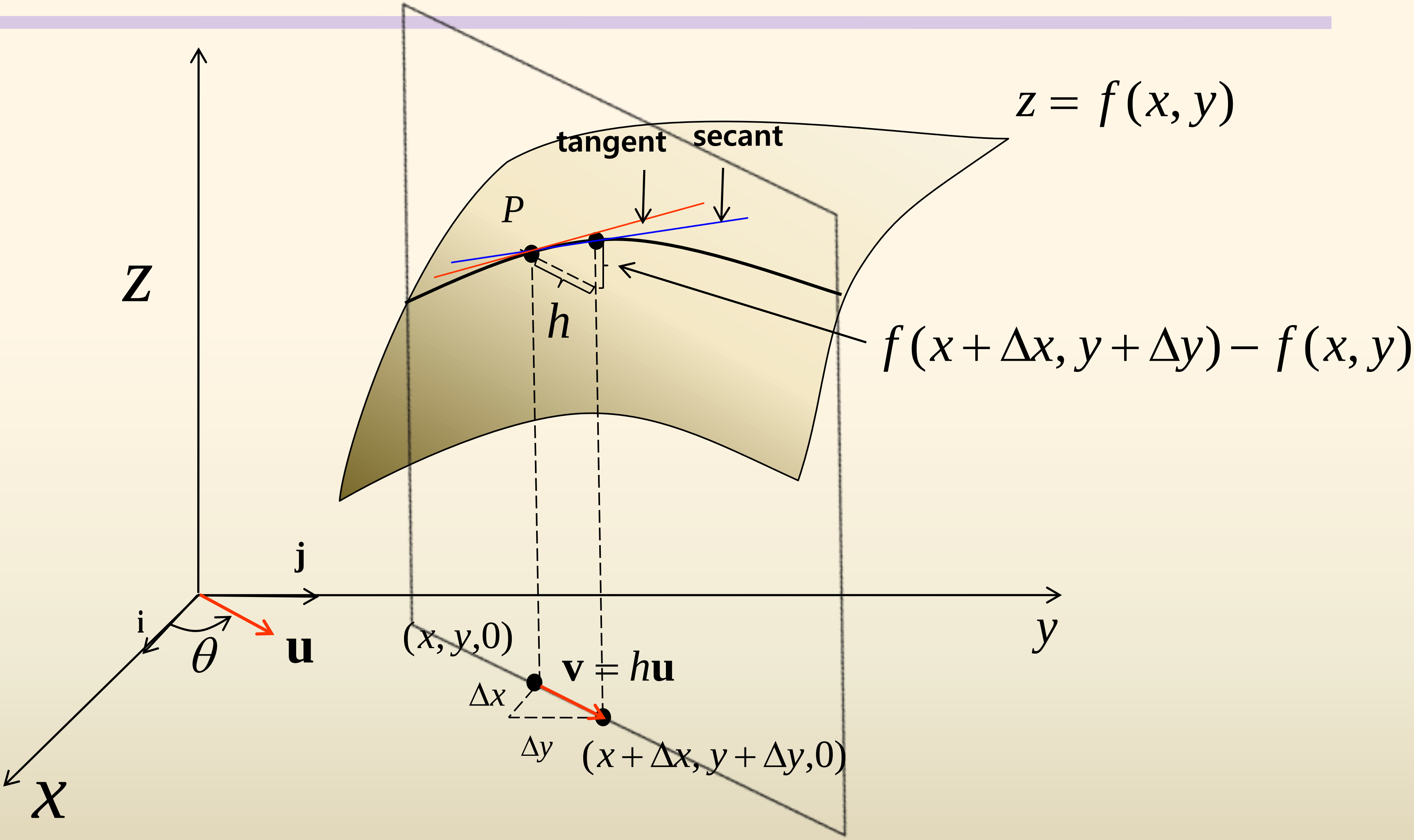
Directional Derivatives



Slope of indicated secant line is

$$\frac{f(x+\Delta x, y+\Delta y)-f(x,y)}{h}=\frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y)}{h}$$

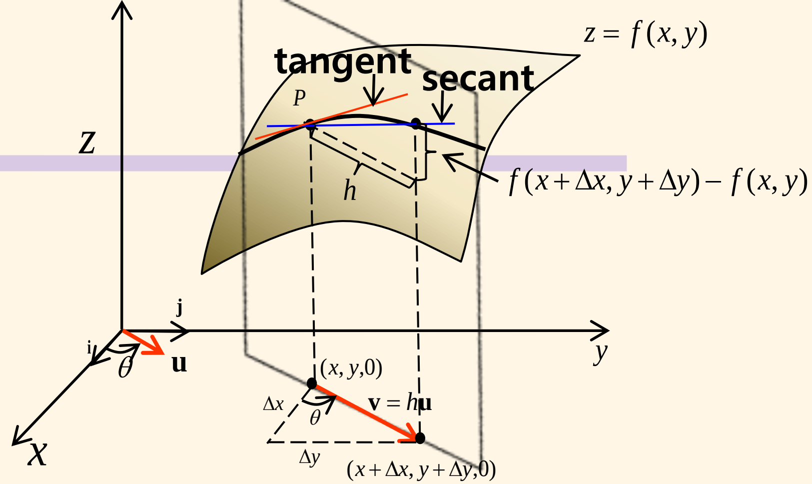
Directional Derivatives



As $h \to 0$ we expect the slope of secant line would be tangent line

$$\frac{f(x+\Delta x, y+\Delta y) - f(x, y)}{h} = \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x, y)}{h}$$

Directional Derivatives



Slope of secant line

$$\frac{f(x+\Delta x, y+\Delta y)-f(x,y)}{h}=\frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y)}{h}$$

Definition 9.5 **Directional Derivative**

The directional derivative of $z=f(x,y)$ **in the direction of a unit vector** $\mathbf{u}=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}$ **is**

$$D_{\mathbf{u}}f(x,y)=\lim_{h\to 0}\frac{f(x+h\cos\theta, y+h\sin\theta)-f(x,y)}{h}$$

provided the limit exists.

$$\theta=0 \text{ implies } D_{\mathbf{i}}f(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}=\frac{\partial z}{\partial x}$$

$$\theta=\frac{\pi}{2} \text{ implies } D_{\mathbf{j}}f(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}=\frac{\partial z}{\partial y}$$

Directional Derivatives

Theorem 9.6 **Computing a Directional Derivative**

If $z = f(x, y)$ is differentiable function of x and y and $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ then,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Proof)

$$g(t) = f(x + t\cos\theta, y + t\sin\theta)$$

by definition of a derivative

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x, y)}{h} = D_{\mathbf{u}} f(x, y)$$

by the Chain Rule,

$$g'(t) = f_1(x + t\cos\theta, y + t\sin\theta)\frac{d}{dt}(x + t\cos\theta) + f_2(x + t\cos\theta, y + t\sin\theta)\frac{d}{dt}(y + t\sin\theta)$$

$$= f_1(x + t\cos\theta, y + t\sin\theta)\cos\theta + f_2(x + t\cos\theta, y + t\sin\theta)\sin\theta$$

$$\therefore g'(0) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

Directional Derivatives

Theorem 9.6 **Computing a Directional Derivative**

If $z = f(x, y)$ is differentiable function of x and y and $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ then,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Proof)

$$g(t) = f(x + t\cos\theta, y + t\sin\theta)$$

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$\begin{aligned} D_{\mathbf{u}} f(x, y) = g'(0) &= f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \\ &= [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \\ &= \nabla f(x, y) \cdot \mathbf{u} \end{aligned}$$

Directional Derivatives

☑ Example 2
Gradient at a Point

If $F(x,y,z)=xy^2+3x^2-z^3$, find $\nabla F(x,y,z)$ at (2,-1,4).

Solution)

$$\nabla F(x,y,z) = \frac{\partial}{\partial x}(xy^2+3x^2-z^3)\mathbf{i} + \frac{\partial}{dy}(xy^2+3x^2-z^3)\mathbf{j} + \frac{\partial}{dz}(xy^2+3x^2-z^3)\mathbf{k}$$

$$= (y^2+6x)\mathbf{i} + 2xy\mathbf{j} - 3z^2\mathbf{k}$$

$$\nabla F(2,-1,4) = 13\mathbf{i} - 4\mathbf{j} - 48\mathbf{k}$$

Directional Derivatives

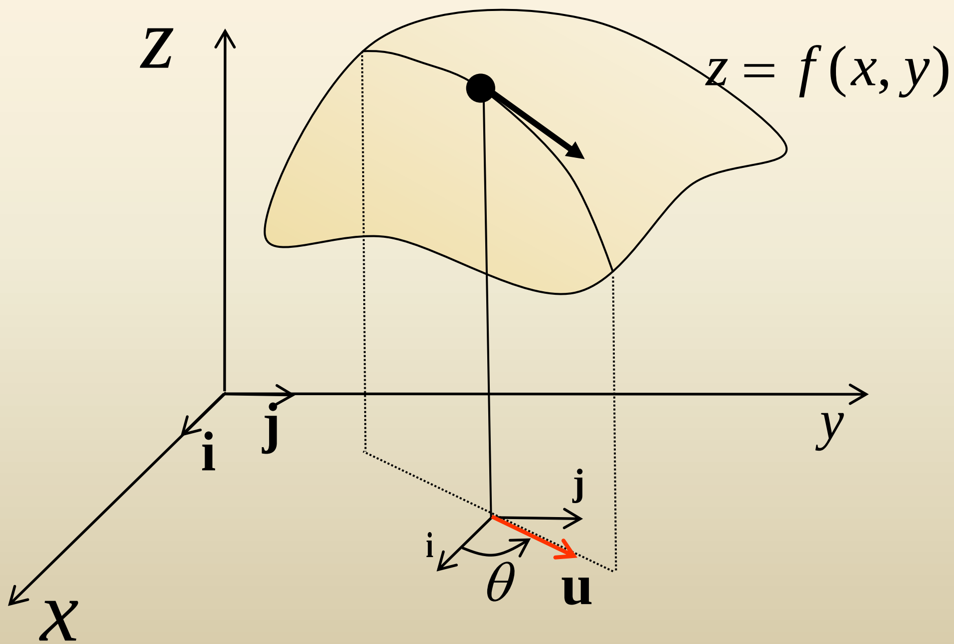
$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Theorem 9.6 **Computing a Directional Derivative**

If $z = f(x,y)$ **is differentiable function of x and y and** $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$
then,

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y)\cdot\mathbf{u}$$

$$D_{\mathbf{u}} f(x,y) = [f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}]\cdot(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$
$$= \nabla f(x,y)\cdot\mathbf{u}$$



Directional Derivatives

☑ Example 3
Directional Derivative

Find the directional derivative of $f(x,y)=2x^2y^3+6xy$ at (1,1) in the direction of a unit vector whose angle with the positive x-axis is π/6.

Solution)

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

$$= (4xy^3 + 6y)\mathbf{i} + (6x^2y^2 + 6x)\mathbf{j}$$

$$\nabla f(1,1) = 10\mathbf{i} + 12\mathbf{j}$$

$$\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} = \cos\frac{\pi}{6}\mathbf{i} + \sin\frac{\pi}{6}\mathbf{j}$$

$$D_{\mathbf{u}} f(1,1) = \nabla f(1,1)\cdot\mathbf{u}$$

$$= (10\mathbf{i} + 12\mathbf{j})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right)$$

$$= 5\sqrt{3} + 6$$

Directional Derivatives

☑ Example 4
Directional Derivative

Consider the plane that is perpendicular to the xy-plane and passes through the points $P(2,1)$ and $Q(3,2)$. What is the slope of the tangent line to the curve on intersection of this plane with the surface $f(x,y)=4x^2+y^2$ at $(2,1,17)$ in the direction of Q?

Solution)

$$f(x,y) = 4x^2 + y^2$$

$$\nabla f(x,y) = 8x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla f(2,1) = 16\mathbf{i} + 2\mathbf{j}$$

$$\overrightarrow{PQ} = \mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$D_{\mathbf{u}} f(2,1) = \nabla f(2,1) \bullet \mathbf{u}$$

$$= (16\mathbf{i} + 2\mathbf{j}) \bullet \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \right)$$

$$= 9\sqrt{2}$$

Directional Derivatives

☑ Example 5
Directional Derivative

Find the directional derivative of $F(x,y,z)=xy^2-4x^2y+z^2$ at (1,-1,2) in the direction of $6\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

Solution)

$$f(x,y,z) = xy^2 - 4x^2y + z^2$$

$$\nabla f(x,y,z) = \frac{\partial}{\partial x}(xy^2 - 4x^2y + z)\mathbf{i}$$

$$+\frac{\partial}{\partial y}(xy^2 - 4x^2y + z)\mathbf{j}$$

$$+\frac{\partial}{\partial z}(xy^2 - 4x^2y + z)\mathbf{k}$$

$$= (y^2 - 8xy)\mathbf{i} + (2xy - 4x^2)\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(1,-1,2) = 9\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$$

$$\|6\mathbf{i}+2\mathbf{j}+3\mathbf{k}\| = 7$$

$$\mathbf{u} = \frac{1}{7}\cdot(6\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = \frac{6}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$$

$$D_{\mathbf{u}}F(1,-1,2) = (9\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}) \bullet \left(\frac{6}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}\right)$$

$$= \frac{54}{7}$$

Directional Derivatives

∇f **Points in the direction of maximum increase of** f **at P**

The rate of change of f **in the direction given by the vector** $\mathbf{u}$ **:**

$$D_{\mathbf{u}} f(x,y) = [f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = \nabla f(x,y) \cdot \mathbf{u}$$

$$D_{\mathbf{u}} f = |\nabla f||\mathbf{u}|\cos\phi = |\nabla f|\cos\phi, \quad \phi\text{:angle between } \nabla f \text{ and } \mathbf{u}$$

$$-1 \le \cos\phi \le 1$$

The maximum value of $D_{\mathbf{u}} f$ ⇨ $D_{\mathbf{u}} f = |\nabla f|$ **, When** $\cos\phi = 1,\ \phi = 0$

⇩

$\mathbf{u}$ **has the same direction of** ∇f

⇨ ∇f **is the direction of maximum increase of** f **at P**

$-\nabla f$ **is the direction of maximum decrease of** f **at P**

Directional Derivatives

☑ Example 6
Max/Min of Directional Derivative

In Example 5 the maximum value of the directional derivative at F at $(1,-1,2)$ is

$$\|\nabla F(1,-1,2)\| = \sqrt{133}.$$

The minimum value of $D_{\mathbf{u}}F(1,-1,2)$ is then $-\sqrt{133}.$

Directional Derivatives

☑ Example 7
Direction of Steepest Ascent

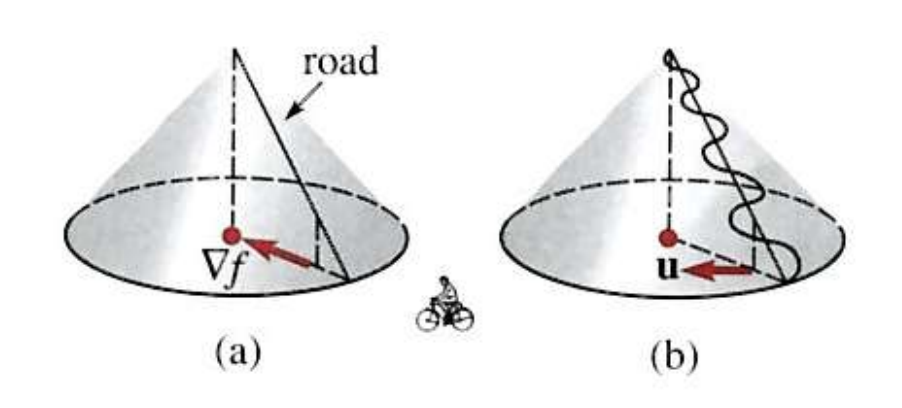
Each year in Los Angeles there is a bicycle race up to the top of a hill by a road known to be the steepest in the city. To understand why a bicyclist with a modicum of sanity will zigzag up the road, let us suppose the graph of

$$f(x,y) = 4 - \frac{2}{3}\sqrt{x^2+y^2},\ 0 \le z \le 4,$$

shown in Figure (a) is a mathematical model of the hill. The gradient of f is

$$\nabla f(x,y) = \frac{2}{3}\left[\frac{-x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{-y}{\sqrt{x^2+y^2}}\mathbf{j}\right] = \frac{2/3}{\sqrt{x^2+y^2}}\mathbf{r}$$

where $\mathbf{r}=-x\mathbf{i}-y\mathbf{j}$ is a vector pointing to the center of the circular base.



Thus the steepest ascent up the hill is a straight road whose projection in the *xy*-plane is a radius of the circular base. Since $D_{\mathbf{u}}f=\text{comp}_{\mathbf{u}}\nabla f$, a bicyclist will zigzag, or seek a direction $\mathbf{u}$ other than ∇f, in order to reduce this component.

Directional Derivatives

☑ Example 8
Direction to Cool Off Fastest

The temperature in a rectangular box is approximated by

$$T(x,y,z) = xyz(1-x)(2-y)(3-z)$$

$$0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3.$$

If a mosquito is located at (½,1,1), in which direction should it fly to cool off as rapidly as possible?

Solution)

$$\nabla T(x,y,z) = \frac{\partial T(x,y,z)}{\partial x}\mathbf{i} + \frac{\partial T(x,y,z)}{\partial y}\mathbf{j} + \frac{\partial T(x,y,z)}{\partial x}\mathbf{k}$$

$$= -yz(2-y)(3-z)(1-2x)\mathbf{i}$$

$$+ xz(1-x)(3-z)(2-2y)\mathbf{j}$$

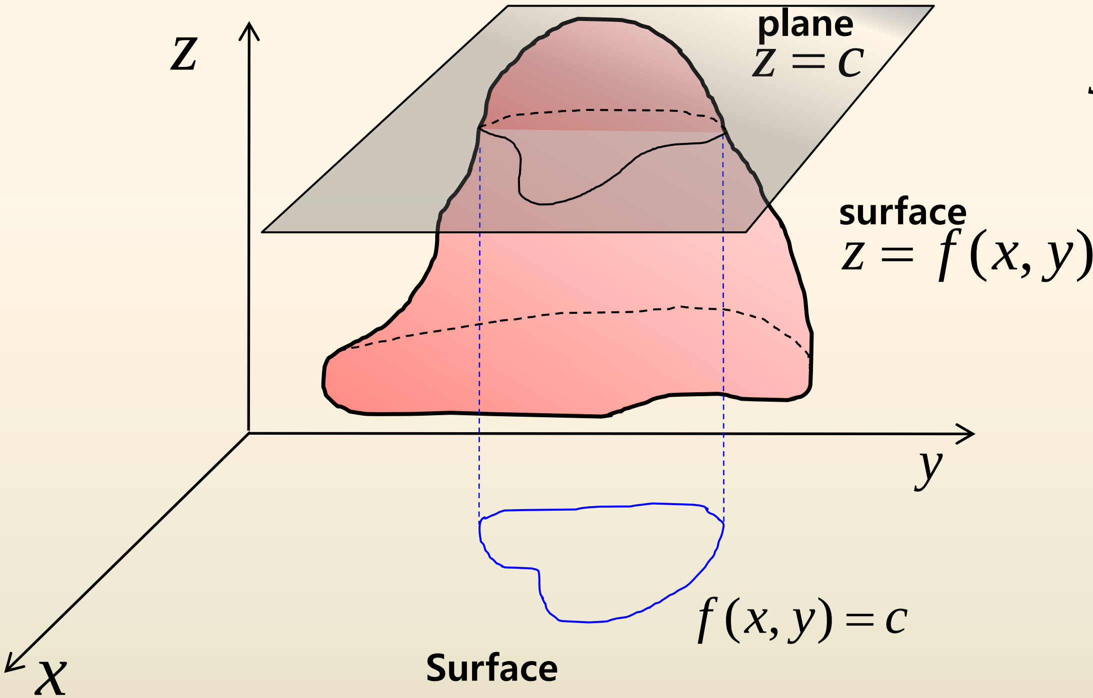
$$+ xy(1-x)(2-y)(3-2z)\mathbf{k}$$

$$\nabla T\left(\frac{1}{2},1,1\right) = \frac{1}{4}\mathbf{k}$$

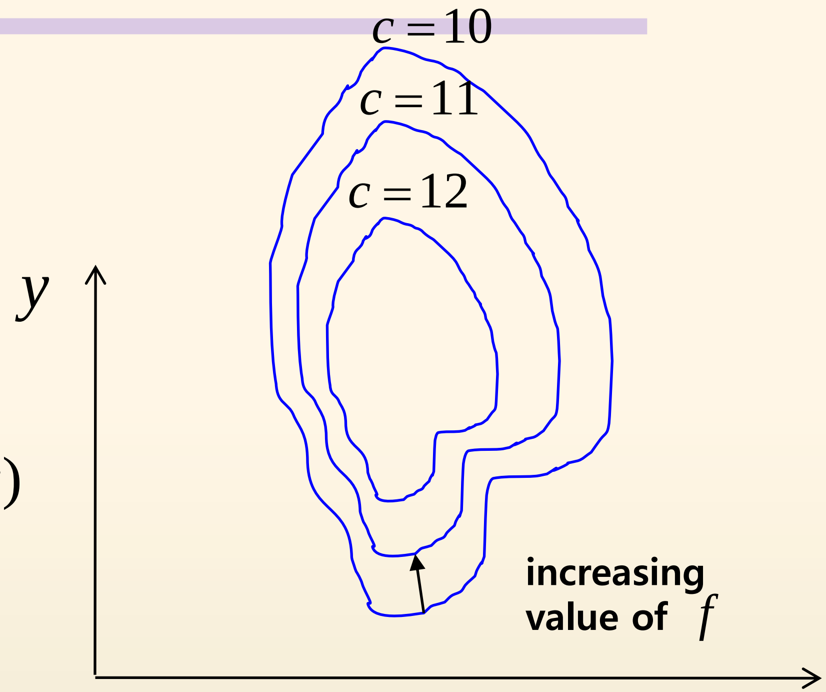
To cool off most rapidly, the mosquito should fly in the direction of -$1/4k$; that is, it should dive for the floor of the box, where the temperature is $T(x,y,0)=0$.

Directional Derivatives

Level Curves



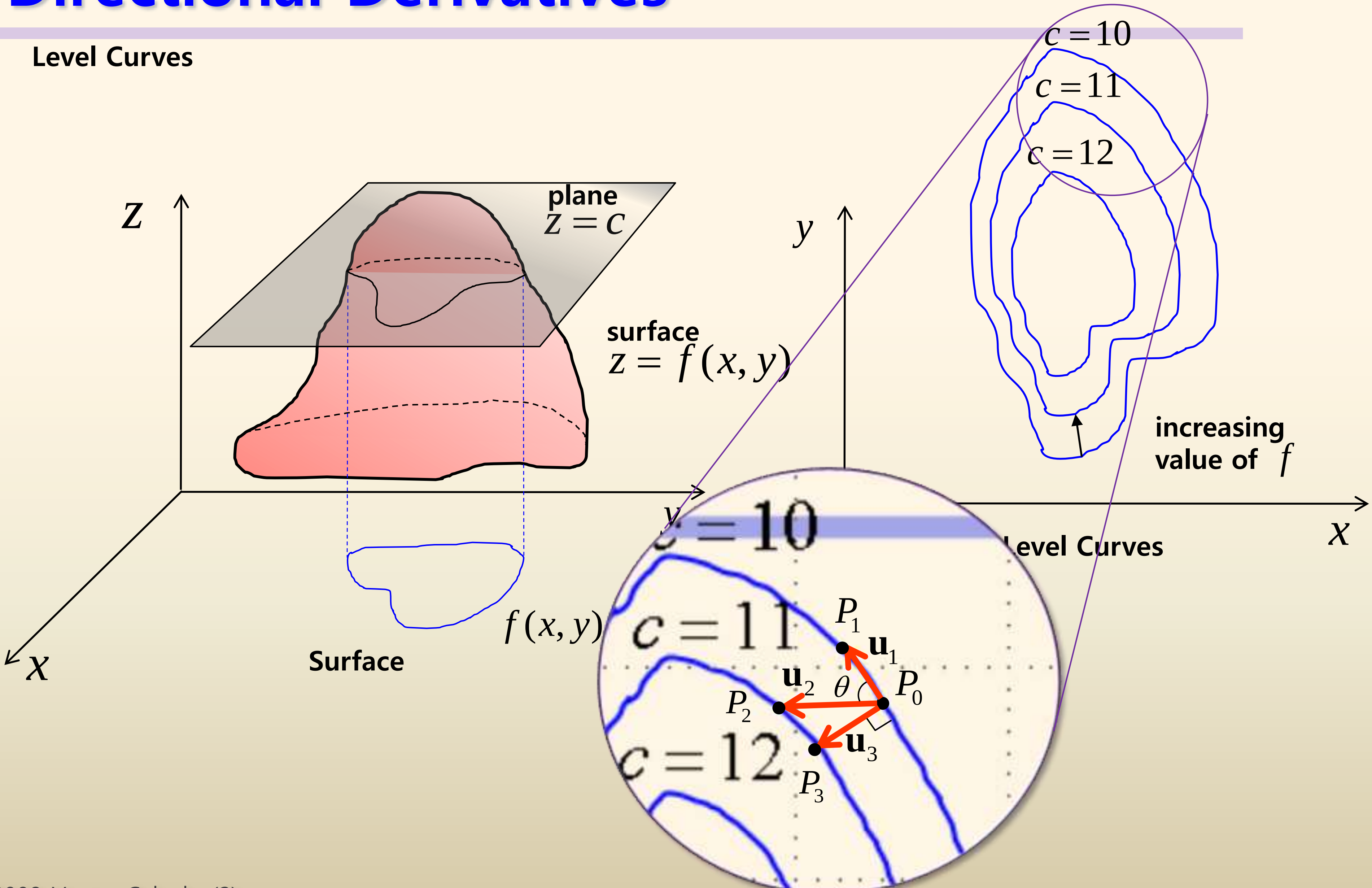
Surface



Level Curves

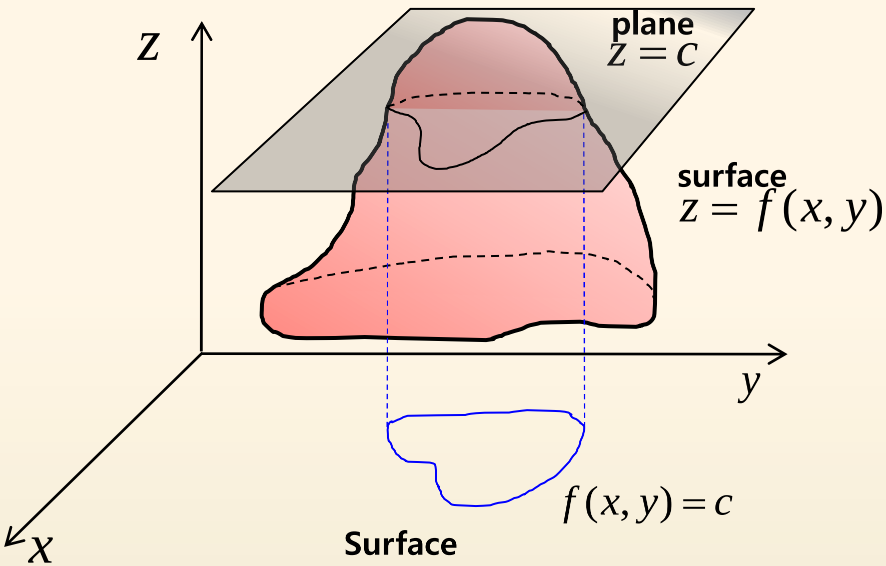
Directional Derivatives

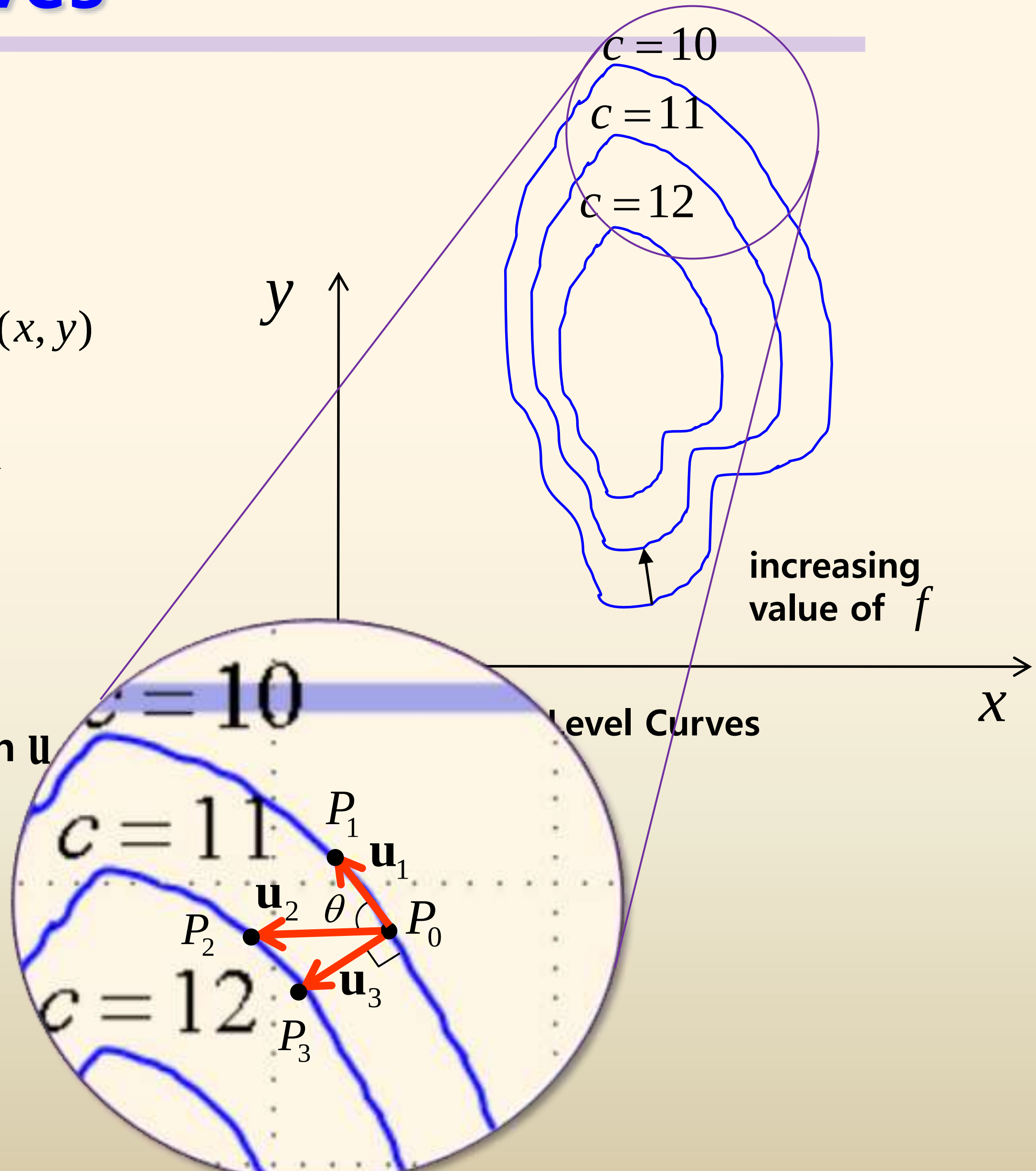
Level Curves



Directional Derivatives

Level Curves



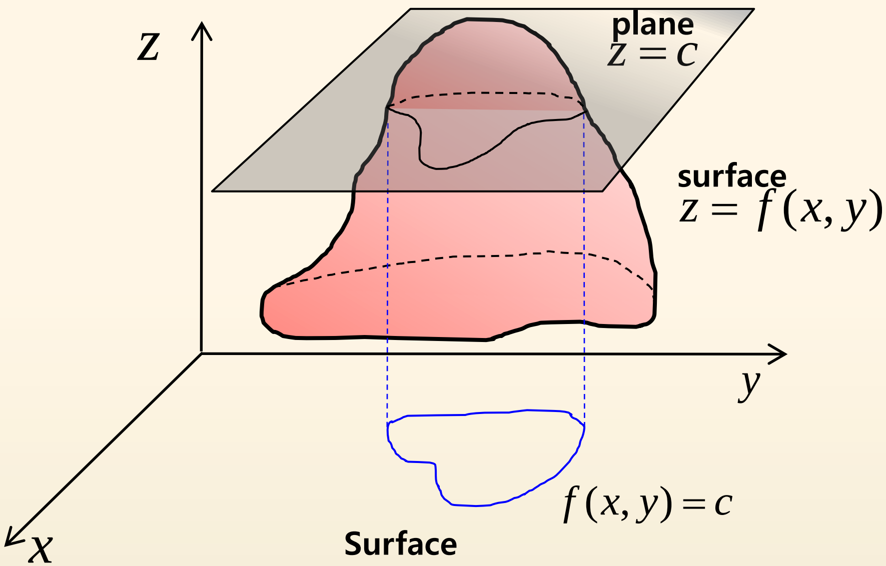


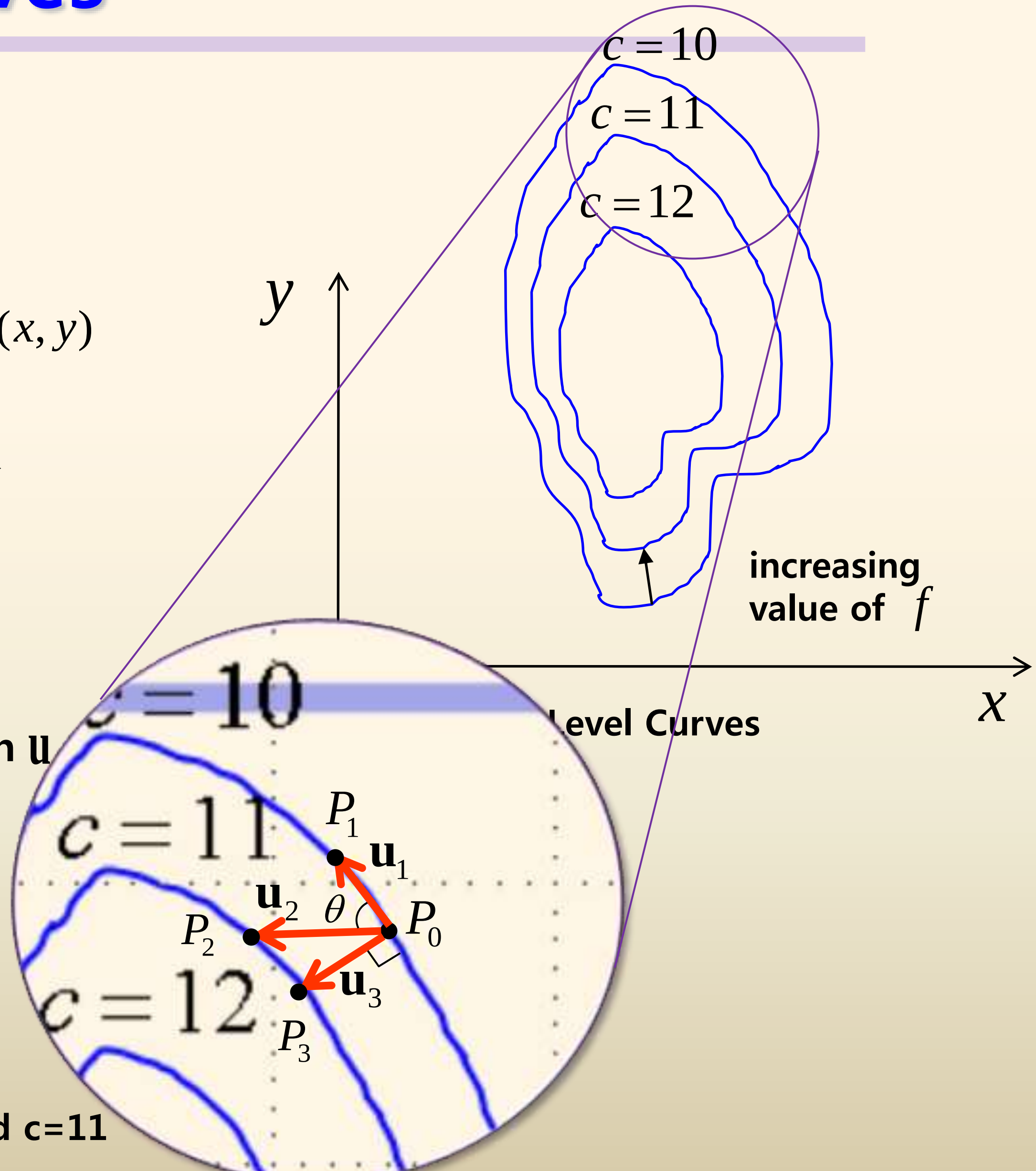
The rate of change of f **in the direction** $\mathbf{u}$
given by the vector : $D_{\mathbf{u}} f(x,y)$

$$D_{\mathbf{u}_1} f(x,y) = 0$$

$$\because \frac{f(P_1)-f(P_0)}{\overline{P_1P_0}} = \frac{10-10}{\overline{P_1P_0}} = 0$$

Directional Derivatives





The rate of change of f in the direction u given by the vector : $D_{\mathbf{u}} f(x, y)$

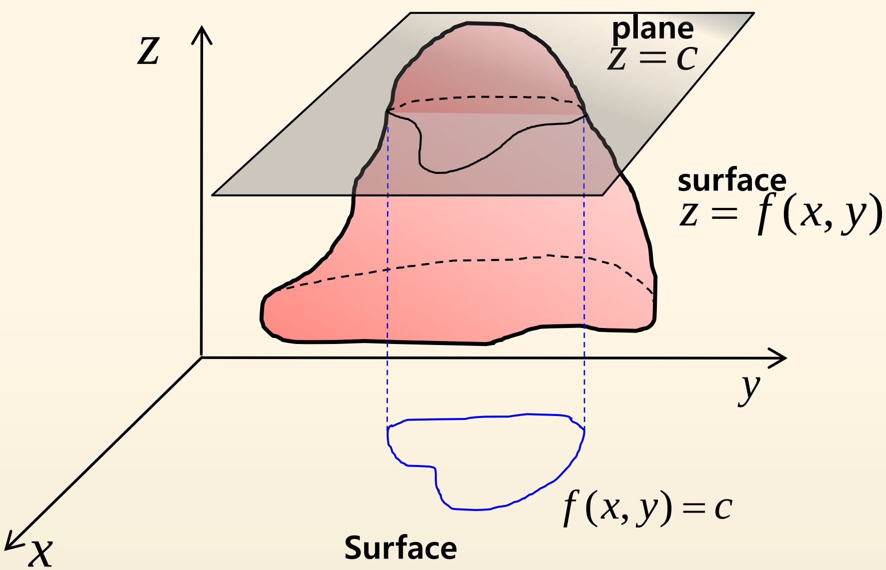
$$D_{\mathbf{u}_2} f(x, y) \text{ Vs. } D_{\mathbf{u}_3} f(x, y)$$

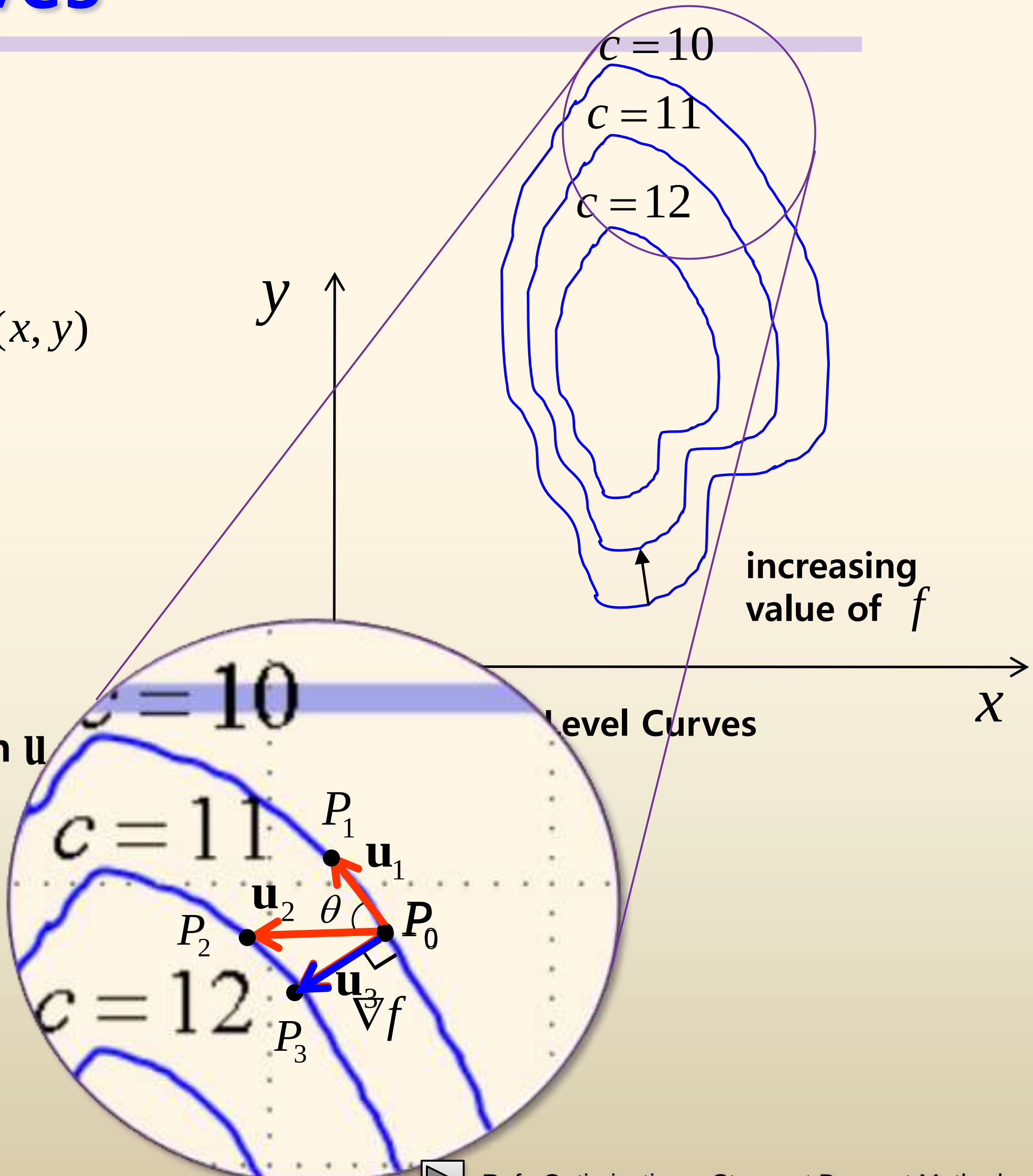
$$\frac{f(P_2) - f(P_0)}{\overline{P_2P_0}} < \frac{f(P_3) - f(P_0)}{\overline{P_3P_0}}$$

$\because \overline{P_3P_0}$ **is the shortest path between c=10 and c=11**

Directional Derivatives

Level Curves





The rate of change of f in the direction u given by the vector : $D_{\mathbf{u}} f(x, y)$

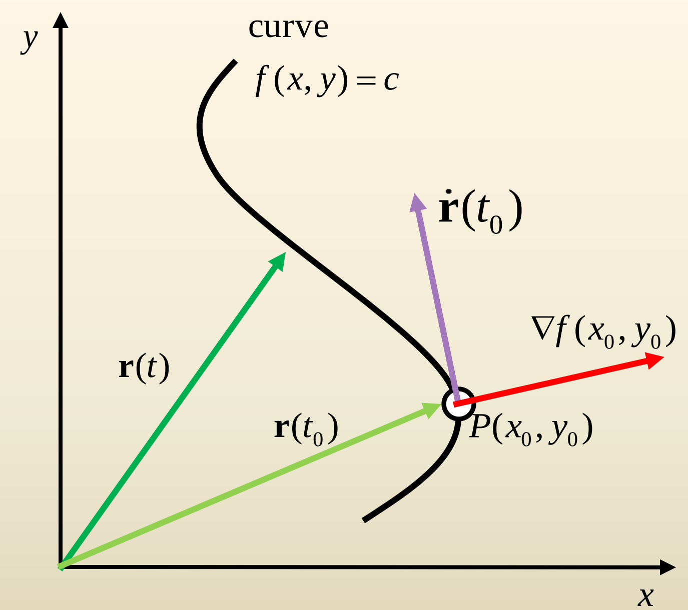
∇f is the direction of maximum increase of f at P_0

Gradient as Curve Normal Vector

$f\left(x(t), y(t)\right) = c$ **: A curve passes through a specified point** $P(x_0, y_0)$

$\mathbf{r} = \left(x(t), y(t)\right)$ **: A position vector of a point on the curve.**

$$f\left(x(t), y(t)\right) = c \quad \Rightarrow \frac{df\left(x(t), y(t)\right)}{dt} = 0$$



$$\text{LHS}: \frac{df\left(x(t), y(t)\right)}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \bullet \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right)$$

$$= \nabla f(x, y) \qquad = \dot{\mathbf{r}}(t)$$ **Tangent vector**

$$\therefore \nabla f(x, y) \bullet \dot{\mathbf{r}}(\mathbf{t}) = 0$$

$\nabla f(x, y)$ **is normal to the curve at the point P.**

Gradient as Curve Normal Vector

☑ Example 1
Gradient at a Point

Find the level curve of $f(x,y)=-x^2+y^2$ passing through (2,3). Graph the gradient at the point.

Solution)

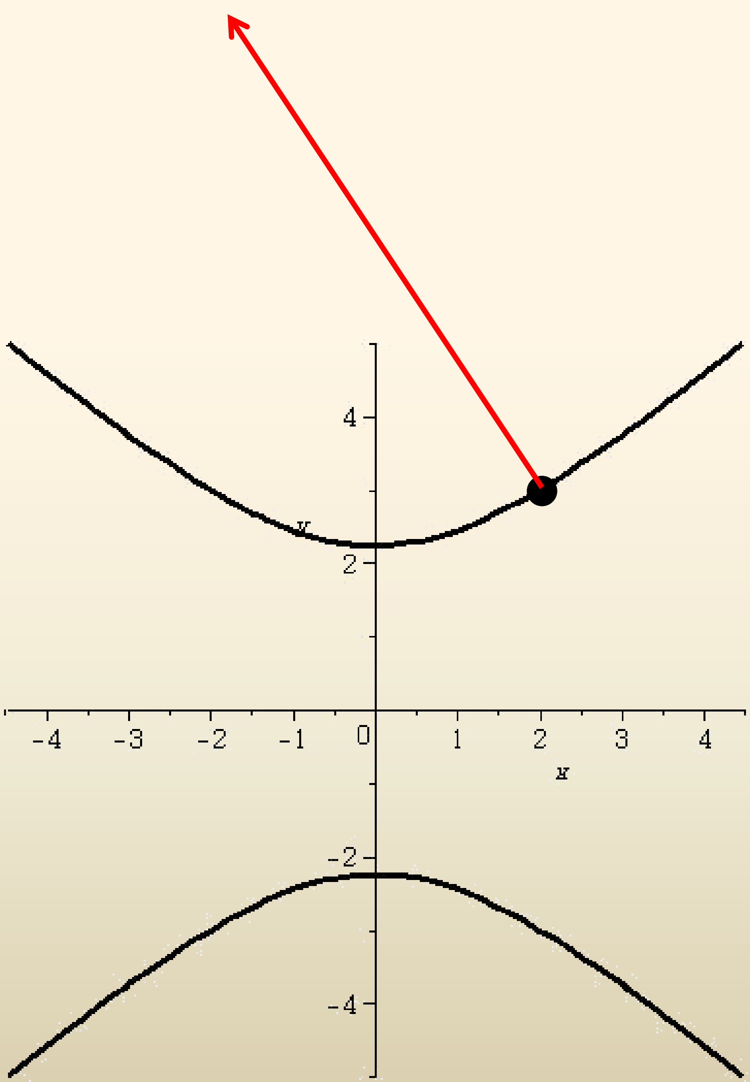
$$f(2,3) = -2^2 + 3^2 = 5$$

$$\text{Level curve : } -x^2 + y^2 = 5$$

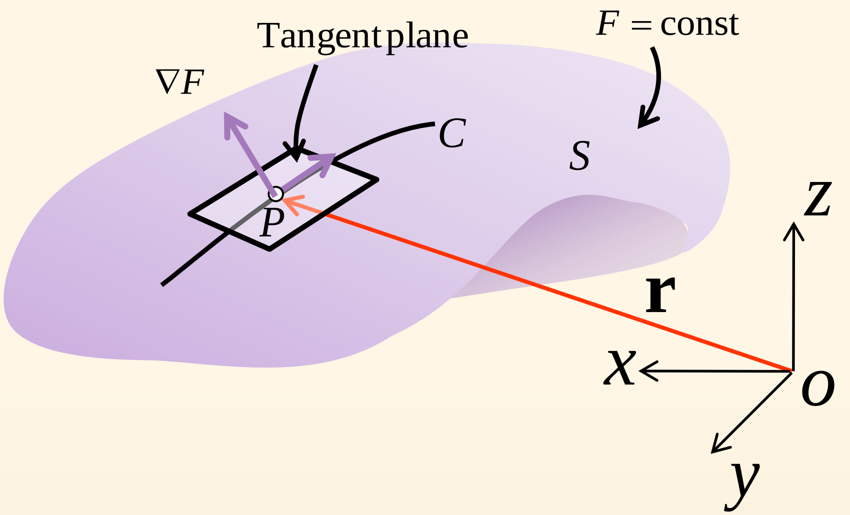
$$\nabla f(x,y) = -2x\mathbf{i} + 2y\mathbf{j}$$

$$\nabla f(2,3) = -2\cdot 2\mathbf{i} + 2\cdot 3\mathbf{j}$$

$$= -4\mathbf{i} + 6\mathbf{j}$$



Gradient as Surface Normal Vector



surface $S : F(x, y, z) = c$

curve C on the surface :

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$$F\big(x(t), y(t), z(t)\big) = c$$

$$\frac{dF\big(x(t), y(t), z(t)\big)}{dt} = 0$$

$$\text{LHS}: \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = \left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}\right) \bullet \left(\dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}\right)$$

$$= \nabla F\big(x(t), y(t), z(t)\big) \qquad = \dot{\mathbf{r}}(t)$$

$$\text{R.H.S.}: 0$$

Tangent vector to curve C at the point P on the surface

The tangent vectors of all curves on S passing through P will generally form a plane, called tangent plane of S at P

$$\therefore \nabla F \bullet \dot{\mathbf{r}} = 0$$

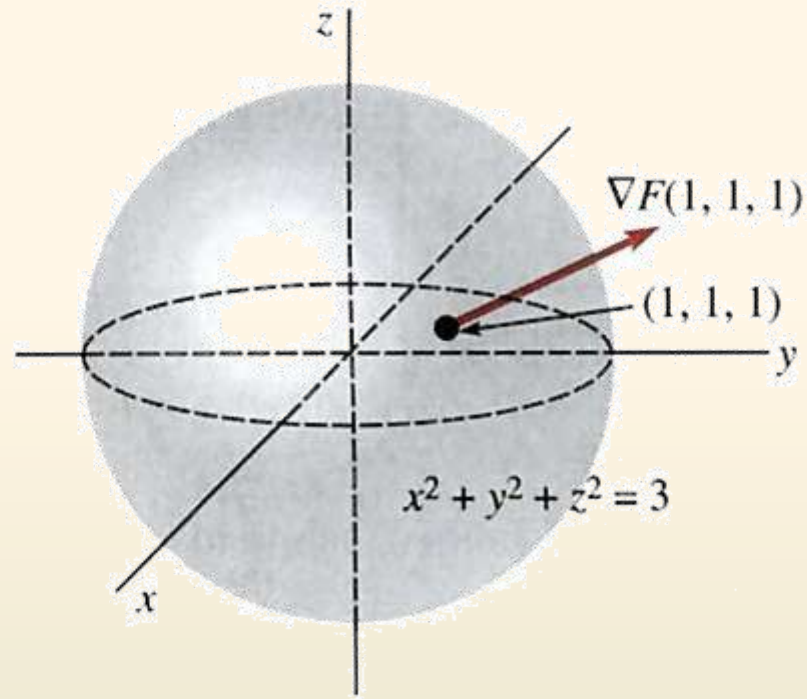
∇F **is normal to the Surface at the point P.**

Gradient as Surface Normal Vector

☑ Example 2

Gradient at a Point

Find the level surface of $F(x,y,z)=x^2+y^2+z^2$ passing through $(1,1,1)$. Graph the gradient at the point.



Solution)

$$F(1,1,1)=1^2+1^2+1^2=3$$

Level surface : $x^2+y^2+z^2=3$

$$\nabla F(x,y,z)=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$$

$$\nabla F(1,1,1)=2\cdot 1\mathbf{i}+2\cdot 1\mathbf{j}+2\cdot 1\mathbf{k}$$

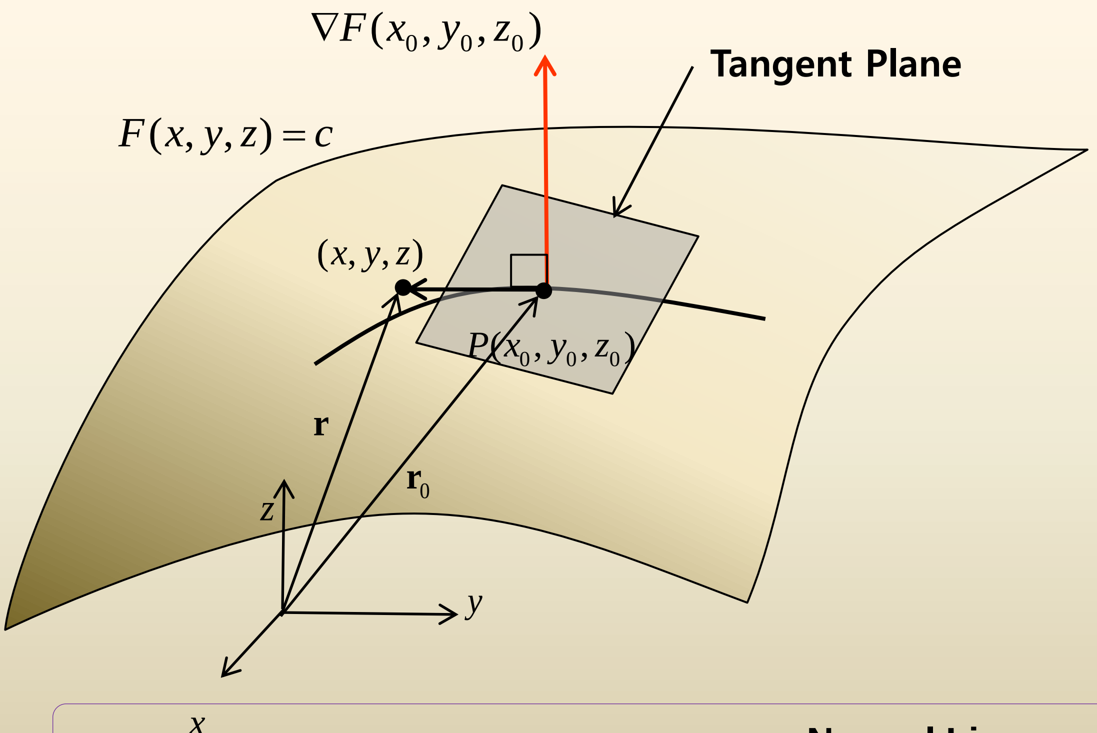
Tangent Planes and Normal Lines

Tangent Plane

Definition 9.6 **Tangent Plane**

Let $P(x_0, y_0, z_0)$ **be a point on the curve of** $F(x, y, z) = c$ **where** ∇F **is not 0.**

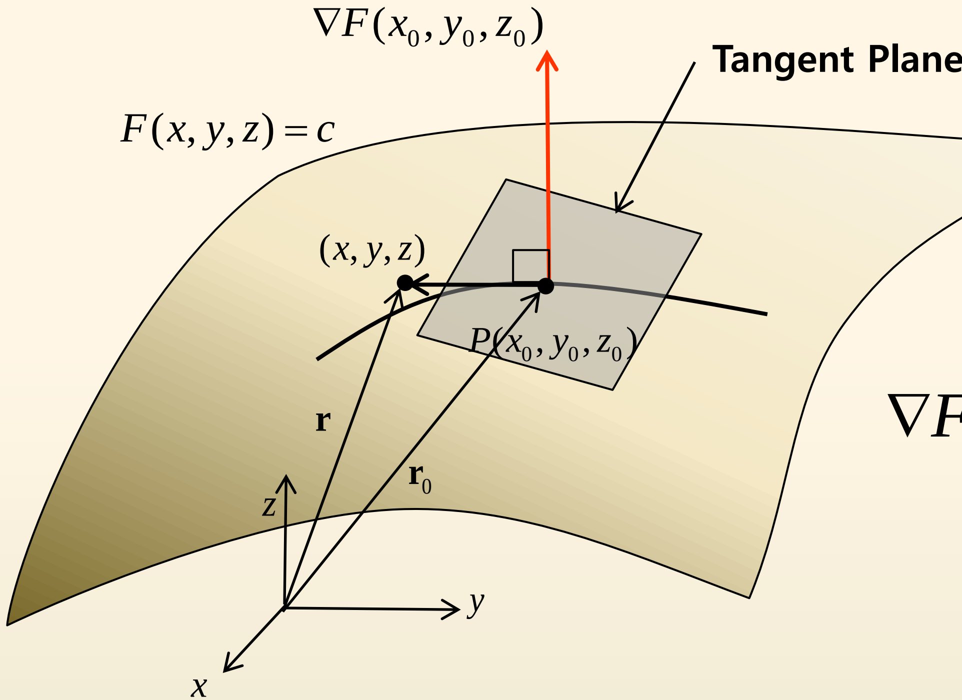
The tangent plane at P is that plane through P that is perpendicular to ∇F **evaluated at P**



Normal Line

The line containing $P(x_0, y_0, z_0)$ **that is parallel to** $\nabla F(x_0, y_0, z_0)$ **is called normal line**

Tangent Planes and Normal Lines



$$\nabla F(x_0, y_0, z_0) \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Theorem 9.7

Equation of Tangent Plane

Let $P(x_0, y_0, z_0)$ **be a point on the graph of** $F(x, y, z) = c$ **where** ∇F **is not 0. Then an equation of the tangent plane at P is**

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Tangent Planes and Normal Lines

☑ Example 3
Equation of Tangent Plane

Find an equation of the tangent plane to the graph of $x^2-4y^2+z^2=16$ at $(2,1,4)$.

Solution)

$$F(x,y,z)=x^2-4y^2+z^2$$

$$F(2,1,4)=2^2-4\cdot 1^2+4^2=16$$

$$F_x(x,y,z)=2x,\ F_x(2,1,4)=2$$

$$F_y(x,y,z)=-8y,\ F_y(2,1,4)=-8$$

$$F_z(x,y,z)=2z,\ F_z(2,1,4)=8$$

equation of the tangent plane :

$$F_x(x_0,y_0,z_0)(x-x_0)$$
$$+F_y(x_0,y_0,z_0)(y-y_0)$$
$$+F_z(x_0,y_0,z_0)(z-z_0)=0$$

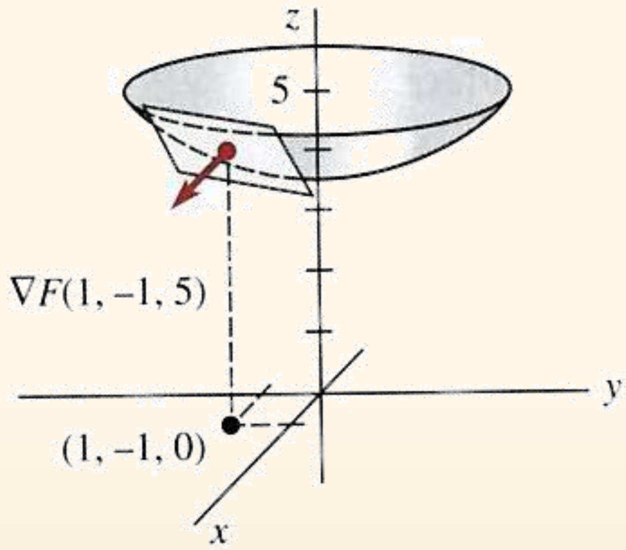
$$\therefore 4(x-2)-8(y-1)+8(z-4)=0$$

$$x-2y+2z=8$$

Tangent Planes and Normal Lines

☑ **Example 4**
Equation of Tangent Plane

Find an equation of the tangent plane to the graph of $z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4$ at (1,-1,5).



Solution)

$$F(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - z + 4$$

$$F(1,-1,5) = \frac{1}{2}\cdot 1^2 + \frac{1}{2}\cdot(-1)^2 - 5 + 4 = 0$$

$$\nabla F(x,y,z) = x\mathbf{i} + y\mathbf{j} - \mathbf{k}$$

$$\nabla F(1,-1,5) = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

equation of the tangent plane :

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$(x+1)-(y-1)-(z-5)=0$$

$$\therefore -x+y+z=7$$

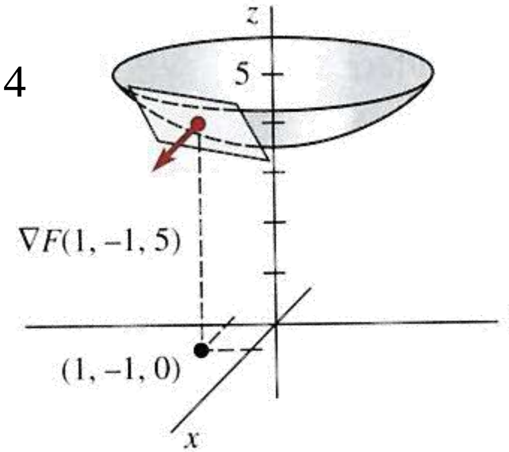
Tangent Planes and Normal Lines

☑ Example 5
Normal Line to a Surface

Find parametric equation for the normal line to the surface in Example 4 at (1,-1,5).

Example 4.

$$z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4$$



Solution)

$$F(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - z + 4$$

$$\nabla F(x,y,z) = x\mathbf{i} + y\mathbf{j} - \mathbf{k}$$

$$\begin{cases} x = 1 + t \\ y = -1 - t \\ z = 5 - t \end{cases}$$

Tangent Planes and Normal Lines

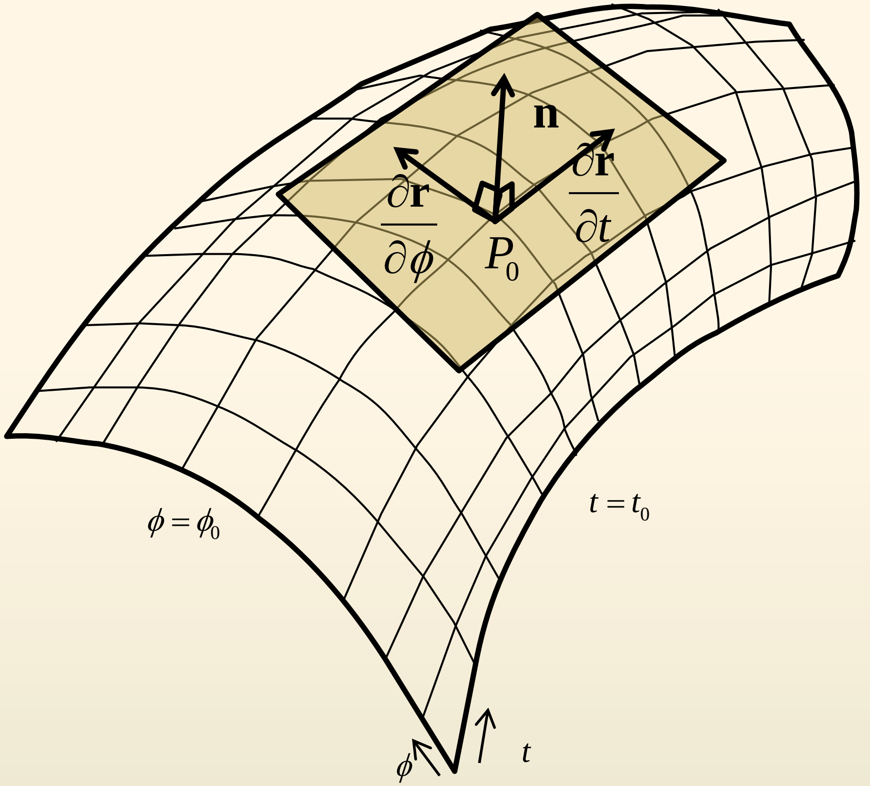
The tangent plane at $\mathbf{r} = \mathbf{r}(u_0, v_0)$ contains

two tangent vectors $\dfrac{\partial \mathbf{r}}{\partial u}, \dfrac{\partial \mathbf{r}}{\partial v}$

So the normal to the surface is a multiple of

their vector product.

$$\mathbf{n} = \pm\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) / \left|\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}\right|$$



Tangent Planes and Normal Lines

$$\mathbf{n} = \pm\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \Big/ \left|\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v}\right|$$

Ex) On the paraboloid of revolution given by

$$\mathbf{r} = \mathbf{r}(t,\phi) = 2at\cos\phi\mathbf{i} + 2at\sin\phi\mathbf{j} + at^2\mathbf{k}$$

We have

$$\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial \phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2a\cos\phi & 2a\sin\phi & 2at \\ -2at\sin\phi & 2at\cos\phi & 0 \end{vmatrix}$$

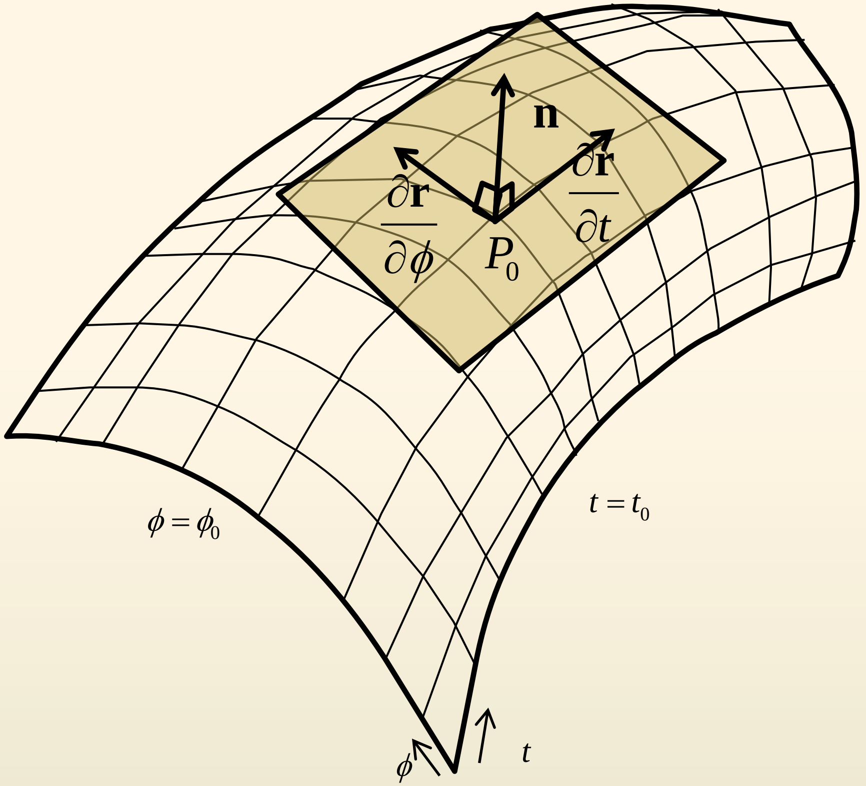
$$= -4a^2t^2\cos\phi\mathbf{i} - 4a^2t^2\sin\phi\mathbf{j} + 4a^2t\mathbf{k}$$

Then

$$\left|\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial \phi}\right|^2 = 16a^4t^2(1+t^2)$$

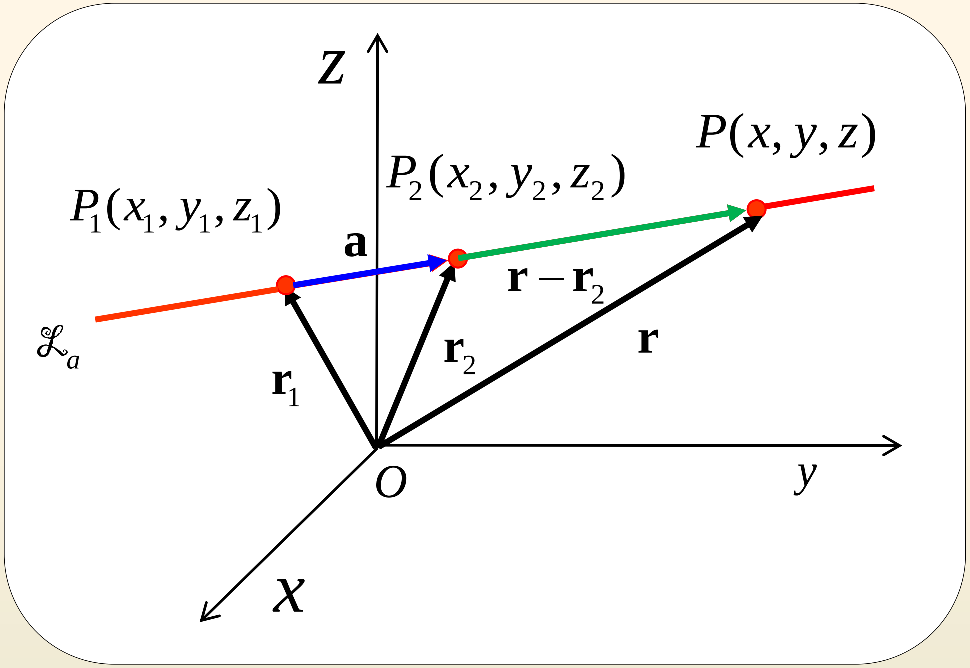
Hence

$$\mathbf{n} = \frac{1}{\sqrt{1+t^2}}(-t\cos\phi, -t\sin\phi, 1)$$



Lines and Planes in 3-Space

■ Lines: Vector Equation



To find an equation of the line through P_1 and P_2 , assume P is any point on the line.

$$\mathbf{r}-\mathbf{r}_2 = t(\mathbf{r}_2-\mathbf{r}_1)$$

$$\mathbf{a} = \mathbf{r}_2-\mathbf{r}_1$$

Vector equation for the line $\mathscr{L}_a$ is

$$\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$$

direction vector

$\mathbf{r} = r_1 + t\mathbf{a}$, $\mathbf{r} = r_1 + t(-\mathbf{a})$, $\mathbf{r} = r_1 + t(k\mathbf{a})$, Are also equation of $\mathscr{L}_a$

Lines and Planes in 3-Space

■ Example 1 Vector Equation of a Line

Find a vector equation for the line through (2,-1,8) and (5,6,-3)

Sol) **Let** $\mathbf{a} = \langle 2-5, -1-6, 8-(-3) \rangle = \langle -3, 7, 11 \rangle$

Then, The three possible vector equations for the line :

$$\langle x, y, z \rangle = \langle 2, -1, 8 \rangle + t\langle -3, -7, 11 \rangle$$

$$\langle x, y, z \rangle = \langle 5, 6, -3 \rangle + t\langle -3, -7, 11 \rangle$$

$$\langle x, y, z \rangle = \langle 5, 6, -3 \rangle + t\langle 3, 7, -11 \rangle$$

Lines and Planes in 3-Space

■ Parametric Equations

$$\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$$

Write this equation by component,

$$\begin{aligned}\langle x, y, z \rangle &= \langle x_2 + t(x_2 - x_1), y_2 + t(y_2 - y_1), z_2 + t(z_2 - z_1) \rangle \\ &= \langle x_2 + a_1 t, y_2 + a_2 t, z_2 + a_3 t \rangle\end{aligned}$$

$$x = x_2 + a_1 t, \quad y = y_2 + a_2 t, \quad z = z_2 + a_3 t$$

This is called parametric equations

Lines and Planes in 3-Space

■ Example 2 Parametric Equations of a Line

Find a parametric equations for the line in Example 1

$$\langle x, y, z\rangle = \langle 2, -1, 8\rangle + t\langle -3, -7, 11\rangle$$: Line in Example 1

Sol)

$$\langle x, y, z\rangle = \langle 2-3t, -1-7t, 8+11t\rangle$$

$$x = 2-3t,\ \ y = -1-7t,\ \ z = 8+11t$$

Lines and Planes in 3-Space

■ Example 3 Vector Parallel to a Line

Find a vector **a** that is parallel to the line $\mathscr{L}_a$ whose parametric equations are

$$x = 4 + 9t,\ y = -14 + 5t,\ z = 1 - 3t$$

Sol) The coefficients of the parameter in each equation are the components of a vector that is parallel to the line.

$$\therefore \mathbf{a} = 9\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$$

Lines and Planes in 3-Space

■ Symmetric Equations

From $x = x_2 + a_1 t,\ \ y = y_2 + a_2 t,\ \ z = z_2 + a_3 t$

$$t = \frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

$$\frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

Are said to be **symmetric equations** for the line through P_1 and P_2

Lines and Planes in 3-Space

■ Example 4 Symmetric Equations of a Line

Find a symmetric equations for the line through (4,10,-6) and (7,9,2)

Sol)

$$\frac{x-x_2}{a_1}=\frac{y-y_2}{a_2}=\frac{z-z_2}{a_3}$$ **Symmetric equation**

$$a_1=4-7=-3,\quad a_2=10-9=1,\quad a_3=-6-2=-8$$

$$\frac{x-7}{3}=\frac{y-9}{-1}=\frac{z-2}{8}$$

■ Example 5 Symmetric Equations of a Line

Find a symmetric equations for the line through (5,3,1) and (2,1,1)

Sol) $$\frac{x-x_2}{a_1}=\frac{y-y_2}{a_2}=\frac{z-z_2}{a_3}$$ **Symmetric equation**

Comes from $x=x_2+a_1t,\ \ y=y_2+a_2t,\ \ z=z_2+a_3t$

$$a_1=5-2=3,\ \ a_2=3-1=2,\ \ a_3=1-1=0$$

$$x=x_2+a_1t,\ \ y=y_2+a_2t,\ \ z=z_2+a_3t \ \Rightarrow\ \frac{x-5}{3}=t,\ \ \frac{y-3}{2}=t,\ \ z=1$$

$$a_1=3,\ a_2=2,\ a_3=0$$

$$(x_2,y_2,z_2)=(5,3,1)$$

$$\frac{x-5}{3}=\frac{y-3}{2},\ \ z=1$$

The symmetric equations describe a line in the plane z=1

Lines and Planes in 3-Space

■ Example 6 Line Parallel to a Vector

Write vector, parametric, and symmetric equations for the line through (4,6,-3) and parallel to $\mathbf{a} = 5\mathbf{i} - 10\mathbf{j} + 2\mathbf{k}$

Sol) $\langle x, y, z \rangle = \langle x_2 + a_1 t, y_2 + a_2 t, z_2 + a_3 t \rangle$ **Vector equation**

$x = x_2 + a_1 t,\ \ y = y_2 + a_2 t,\ \ z = z_2 + a_3 t$ **Parametric equation**

$$\frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$ **Symmetric equation**

$$Vector: \langle x, y, z \rangle = \langle 4, 6, -3 \rangle + t \langle 5, -10, 2 \rangle$$

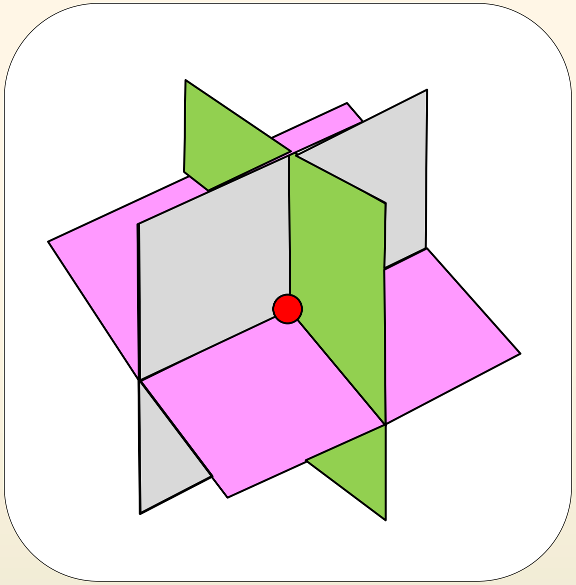
$a_1 = 5,\ a_2 = -10,\ a_3 = 2$

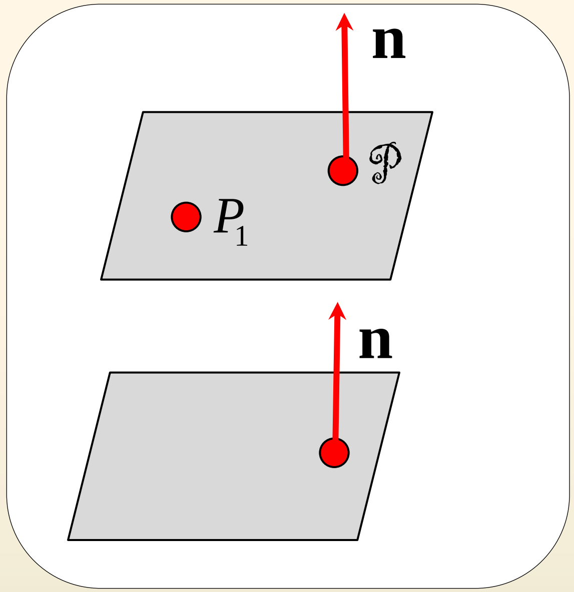
$$Parametric: x = 4 + 5t,\ \ y = 6 - 10t,\ \ z = -3 + 2t$$

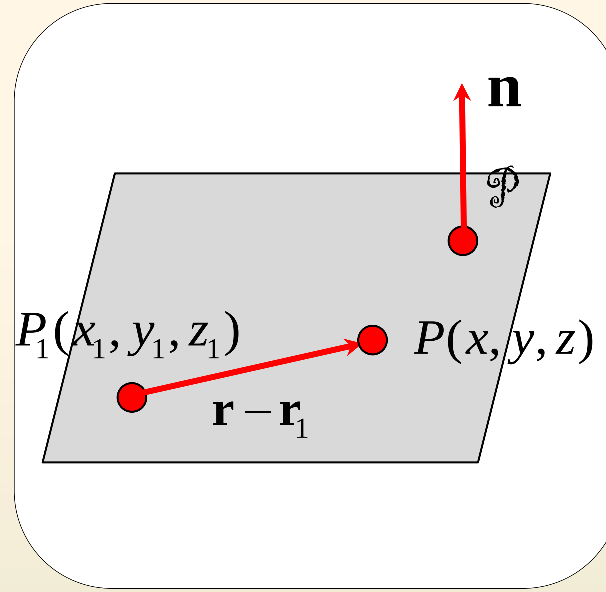
$$Symmetric: \frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

Lines and Planes in 3-Space

■ Planes: Vector Equation







Through a given point there pass an infinite number of planes

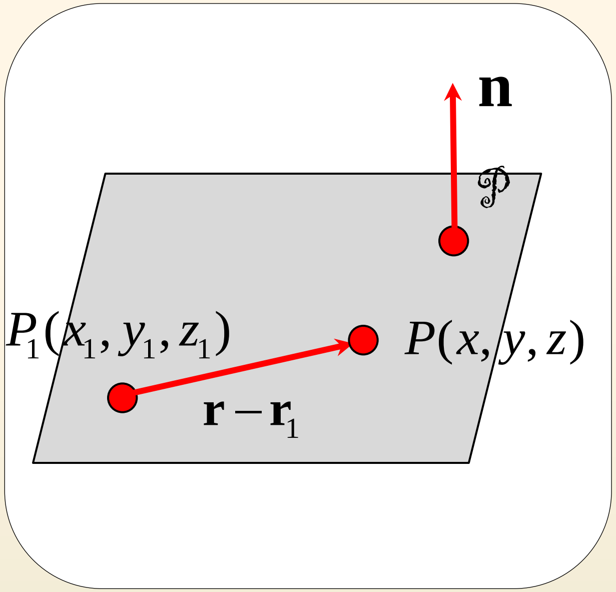
If point and normal are specified, only one plane is determined.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

"Vector equation of plane"

Lines and Planes in 3-Space

■ Cartesian Equation



$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

If normal Vector is $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

"Cartesian Equation" of plane containing $P_1(x_1, y_1, z_1)$

Lines and Planes in 3-Space

■ Example 7 Plane Perpendicular to a Vector

Find an equation of the plane that contains the point (4,-1,3) and is perpendicular to the vector $\mathbf{n} = 2\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}$

Sol)

$$2(x-4)+8(y+1)-5(z-3)=0$$

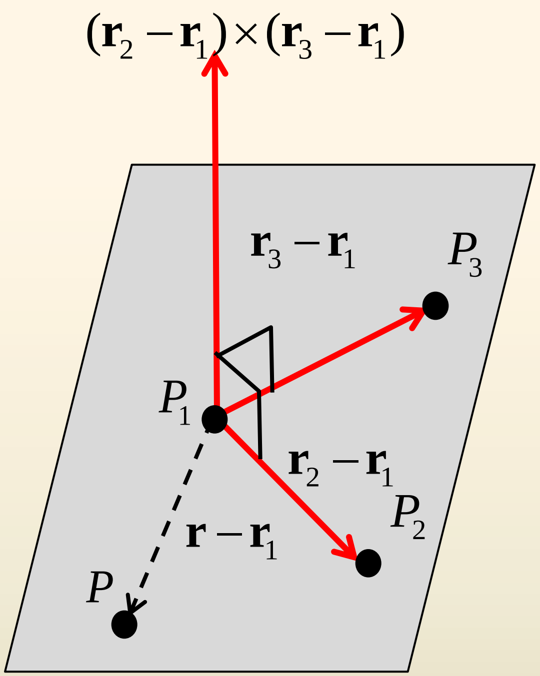
$$or\ \ 2x+8y-5z+15=0$$

Theorem 7.3 **Plane with Normal Vector**

The graph of any equation $ax+by+cz+d=0, a, b, c$ **not all zero, is a plane with the normal vector** $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Lines and Planes in 3-Space

■ Three noncollinear points



$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$ "Vector Equation"

$$\mathbf{n} = (\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1)$$

$$[(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1)] \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

Three noncollinear points P_1, P_2 *and* P_3 **also determine a plane**

■ Example 9 Three Points That Determine a Plane

Find a equation of the plane that contains (1,0,-1),(3,1,4) and (2,-2,0)

Sol)

$$\left.\begin{matrix}(1,0,-1)\\(3,1,4)\end{matrix}\right\}\mathbf{u}=2\mathbf{i}+\mathbf{j}+5\mathbf{k} \qquad \left.\begin{matrix}(3,1,4)\\(2,-2,0)\end{matrix}\right\}\mathbf{v}=\mathbf{i}+3\mathbf{j}+4\mathbf{k}$$

$$\left.\begin{matrix}(x,y,z)\\(2,-2,0)\end{matrix}\right\}\mathbf{w}=(x-2)\mathbf{i}+(y+2)\mathbf{j}+z\mathbf{k}$$

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 5\\ 1 & 3 & 4\end{vmatrix}=-11\mathbf{i}-3\mathbf{j}+5\mathbf{k}$$

$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=0$ yields

$$-11(x-2)-3(y+2)+5z=0 \quad or \quad -11x-3y+5z+16=0$$

Lines and Planes in 3-Space

■ Graphs

To graph a plane, we should try to find

(i) the x-,y-, and z-intercept

If necessary,

(ii) The trace of the plane in each coordinate plane

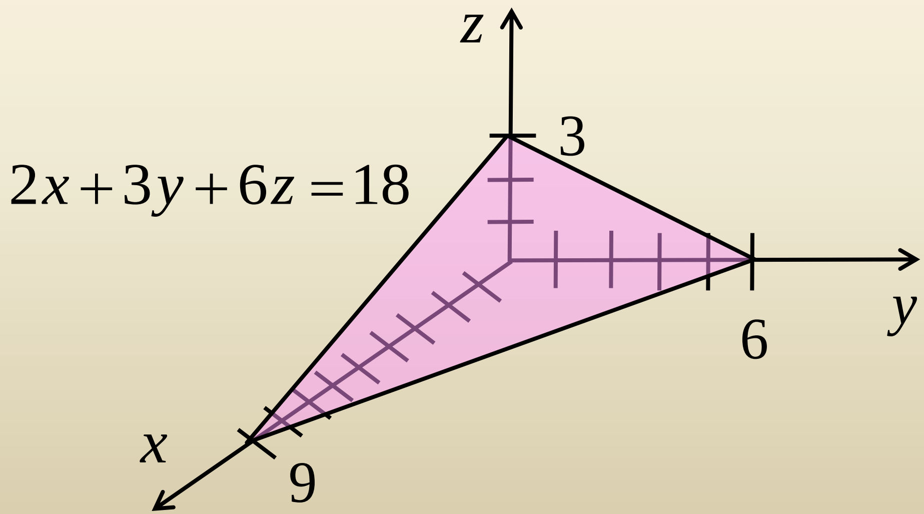
Lines and Planes in 3-Space

■ Example 10 Graph of a Plane

Graph the equation $2x + 3y + 6z = 18$

Sol)

$$y = 0,\ z = 0\ gives\ x = 9 \qquad\qquad x - \text{intercept} = 9$$
$$x = 0,\ z = 0\ gives\ y = 6 \quad \Rightarrow \quad y - \text{intercept} = 6$$
$$x = 0,\ y = 0\ gives\ z = 3 \qquad\qquad z - \text{intercept} = 3$$



Lines and Planes in 3-Space

■ Example 11 Graph of a Plane

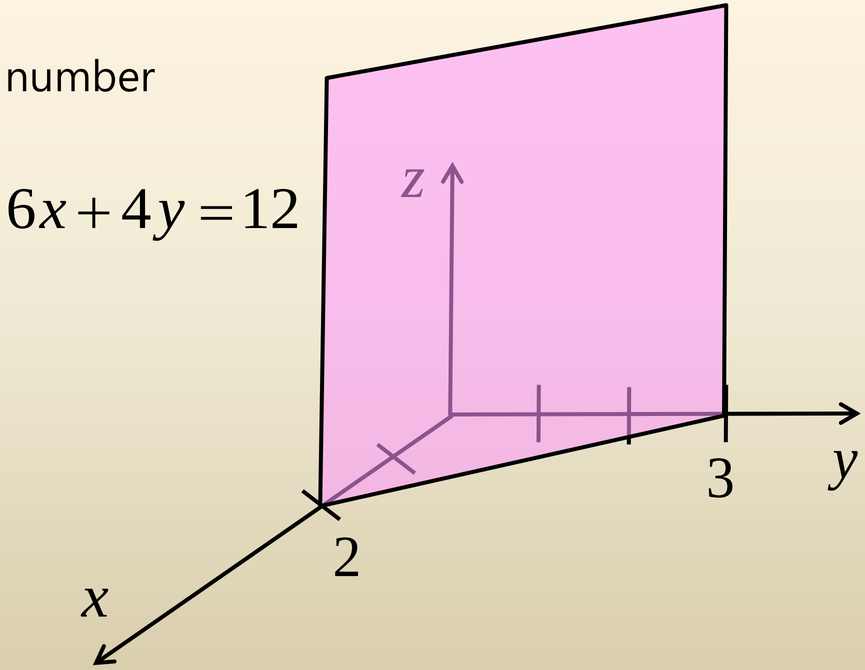
Graph the equation $6x+4y=12$

Sol) $x-\text{intercept}=2$

$y-\text{intercept}=3$

In three dimensions, it is the trace of a plane in the xy-coordinate plane.

Since z is not specified, it can be any real number

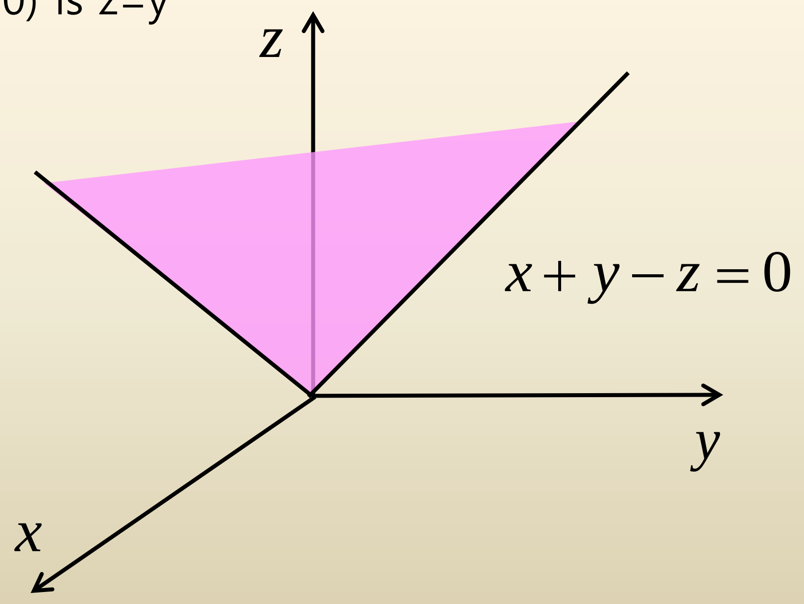


■ Example 12 Graph of a Plane

Graph the equation $x + y - z = 0$

Sol) Trace in xz-plane(y=0) is z=x

Trace in yz-plane(x=0) is z=y



■ Example 13 Line of Intersection of Two Planes

Find parametric equations for the line of intersection of

$$2x-3y+4z=1$$

$$x-y-z=5$$

Sol) In a system of 2 equations and 3 unknowns, we choose one variable arbitrary, $z=t$

Then,

$$x=14+7t,$$

$$y=9+6t$$

$$z=t$$

Lines and Planes in 3-Space

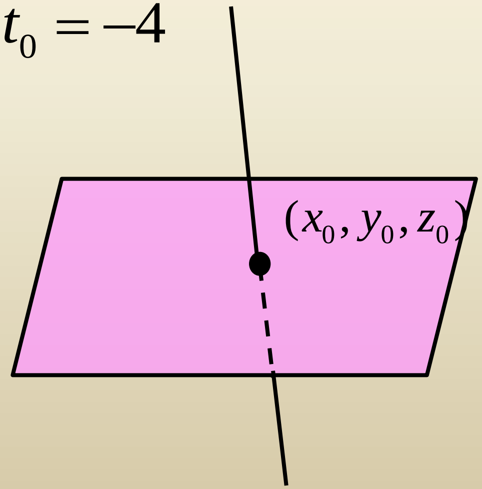
■ Example 14 Point of Intersection of a Line and Plane

Find the point of intersection of the plane $3x - 2y + z = -5$
and the line $x = 1 + t, y = -2 + 2t, z = 4t$

Sol) For some number t_0, plane and line are intersect in a point (x_0, y_0, z_0)

So, $2(1+t_0) - 2(-2+2t_0) + 4t_0 = -5 \quad or \quad t_0 = -4$

$x_0 = -3, \ y_0 = -10, \ z_0 = -16$

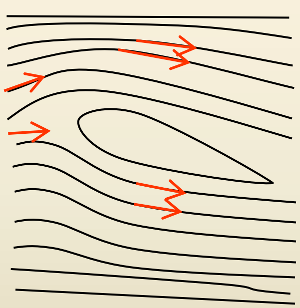


Vector Fields

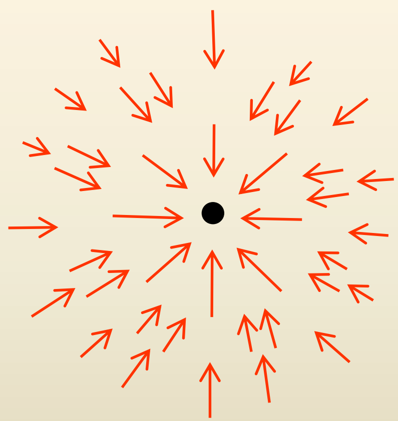
Vector Fields : Vector Functions of two or three variables

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

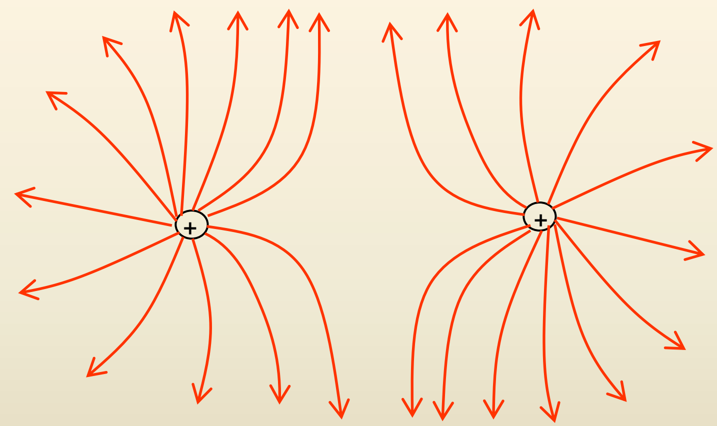
$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$



Air flow around an airplane wing



Magnitude of attractive force



Force around two equal positive charge

Vector Fields

Example

Two-dimensional Vector Field

Graph the two-dimensional vector field

$$\mathbf{F}(x,y)=-y\mathbf{i}+x\mathbf{j}.$$

Solution)

$$\mathbf{F}(1,1) = -\mathbf{i} + \mathbf{j}$$

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$$

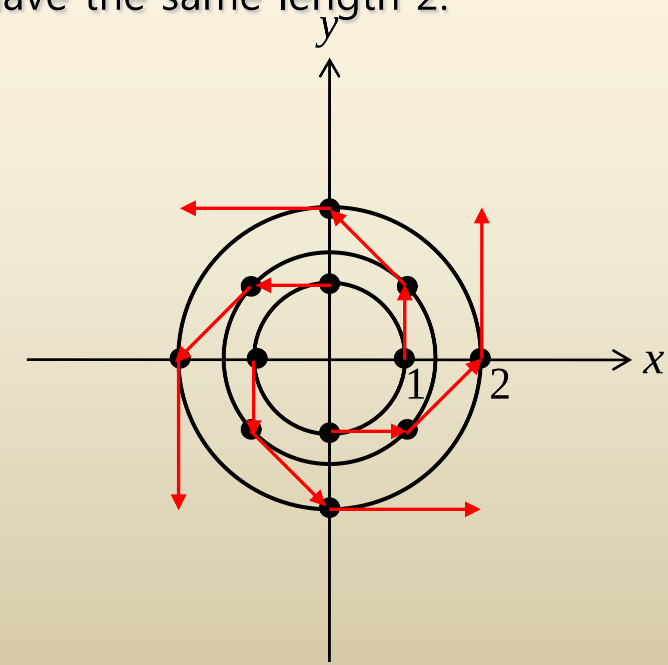
$$\|\mathbf{F}(x, y)\| = \sqrt{y^2 + x^2} = \sqrt{x^2 + y^2}$$

Vectors of the same length k must lie along the curve defined by $\sqrt{x^2 + y^2} = k$.

$\sqrt{x^2 + y^2} = 1$: At the points (1,0), (0,1), (-1,0), (0,-1), the corresponding vectors **j, -i, -j, i** have the same length 1.

$\sqrt{x^2 + y^2} = 2$: At the points (1,1), (-1,1), (-1,-1), (1,-1), the corresponding vectors –i+**j, -i-j, i-j, i+j** have the same length $\sqrt{2}$.

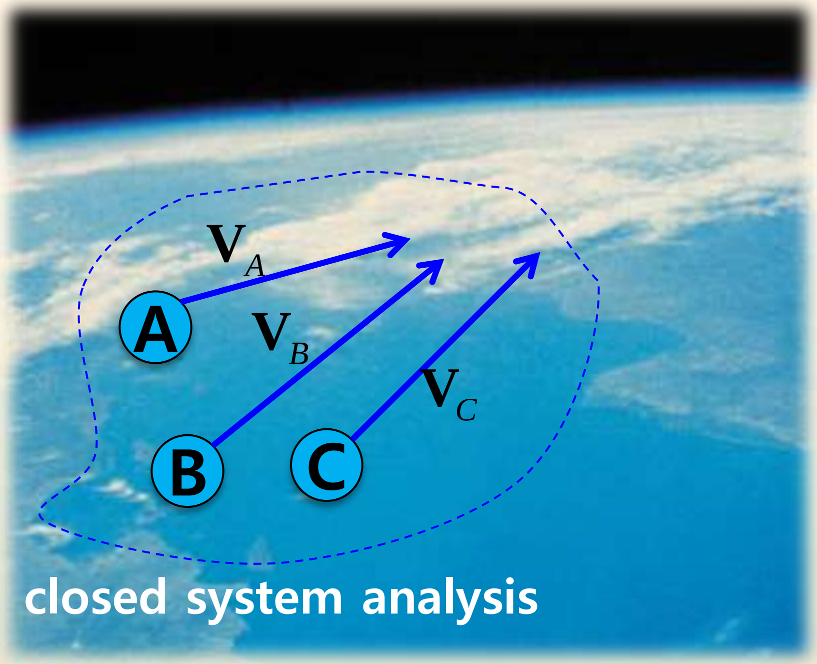
$\sqrt{x^2 + y^2} = 4$: At the points (2,0), (0,2), (-2,0), (0,-2), the corresponding vectors 2**j, -2i, -2j, 2 i** have the same length 2.



Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid motion

•Lagrangian Description



Keeping track of

the position vector $\mathbf{x}_A(t), \mathbf{x}_B(t), \mathbf{x}_C(t), \ldots$

and the velocity vector $\mathbf{V}_A(t), \mathbf{V}_B(t), \mathbf{V}_C(t), \ldots$

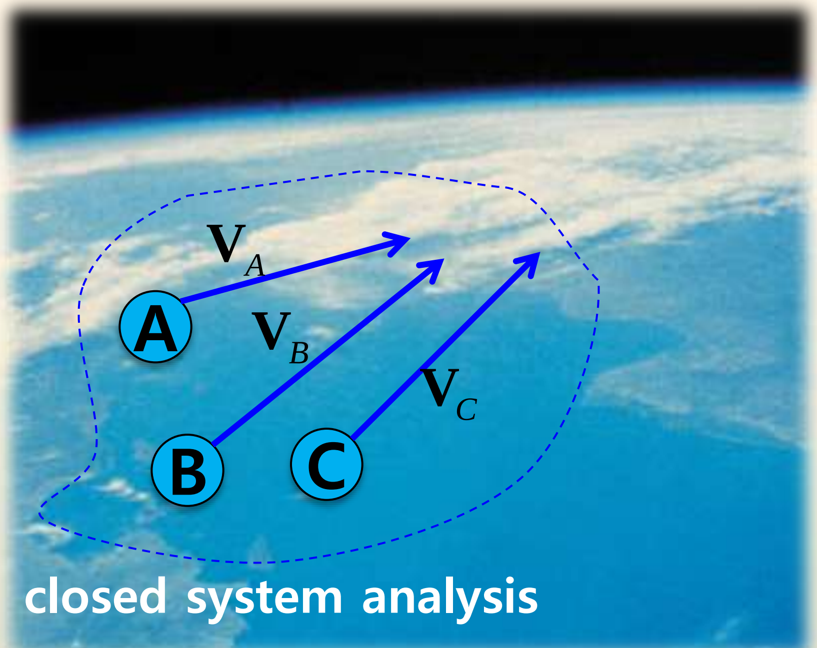
of each fluid particle, A B C

*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p90~95

Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid <u>motion</u>

•Lagrangian Description



Keeping track of
the position vector $\mathbf{x}_A(t), \mathbf{x}_B(t), \mathbf{x}_C(t), \ldots$
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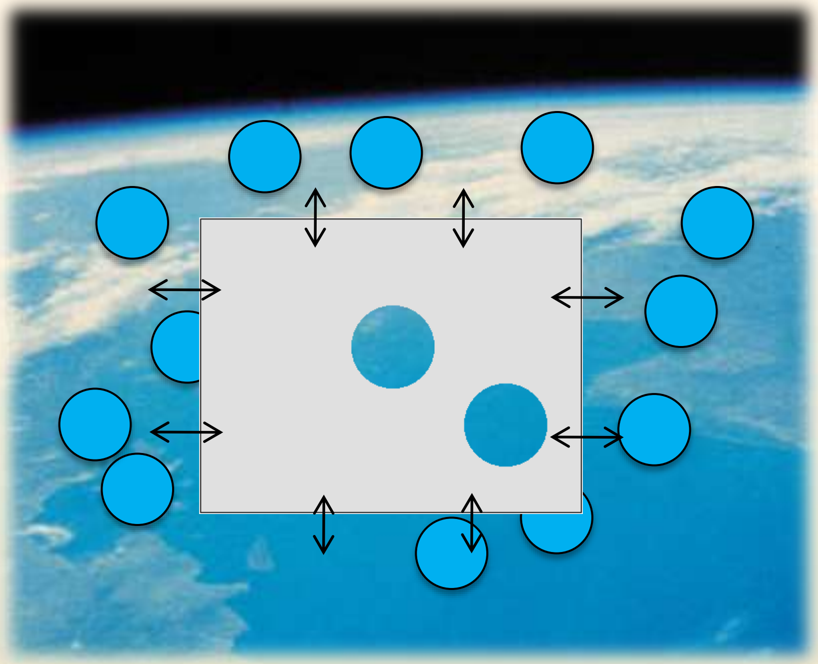
of each fluid particle, (A) (B) (C)

•Eulerian Description



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Vector Fields and Fluid Mechanics

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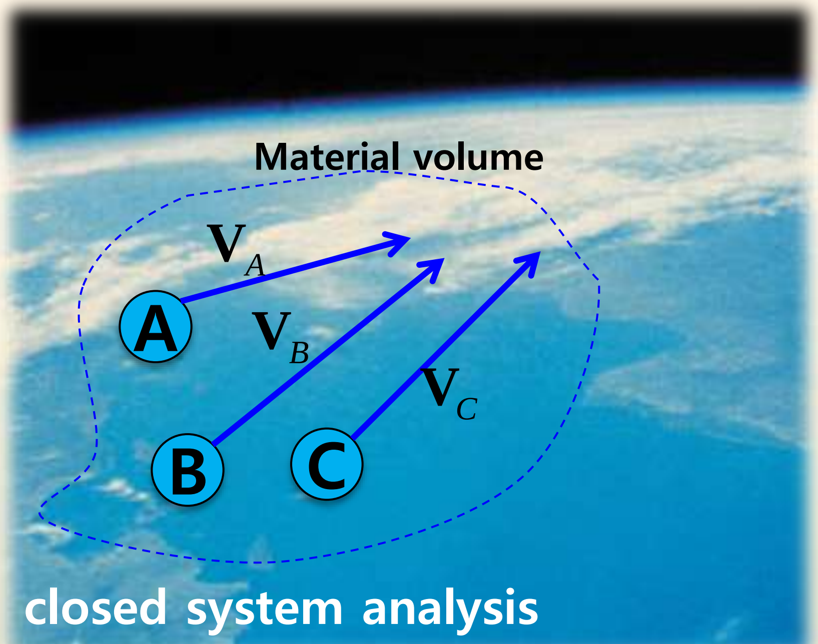

Keeping track of

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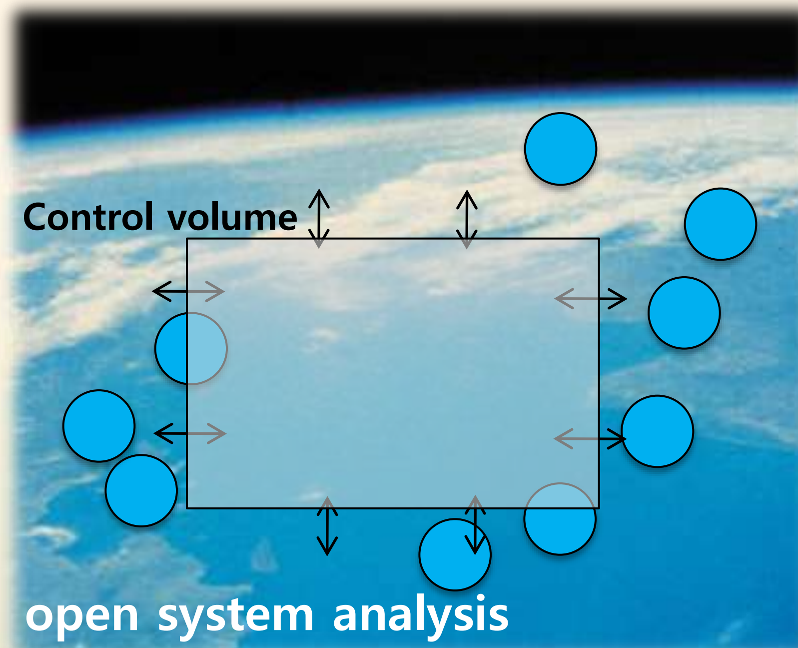

Define <u>Field Variables</u>

The field variables at a particular location and A particular time is...

<u>the value of the variable</u> for whichever **fluid particle happen to occupy that location at that time**

*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p90~95

Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid <u>motion</u>

•**Eulerian Description**

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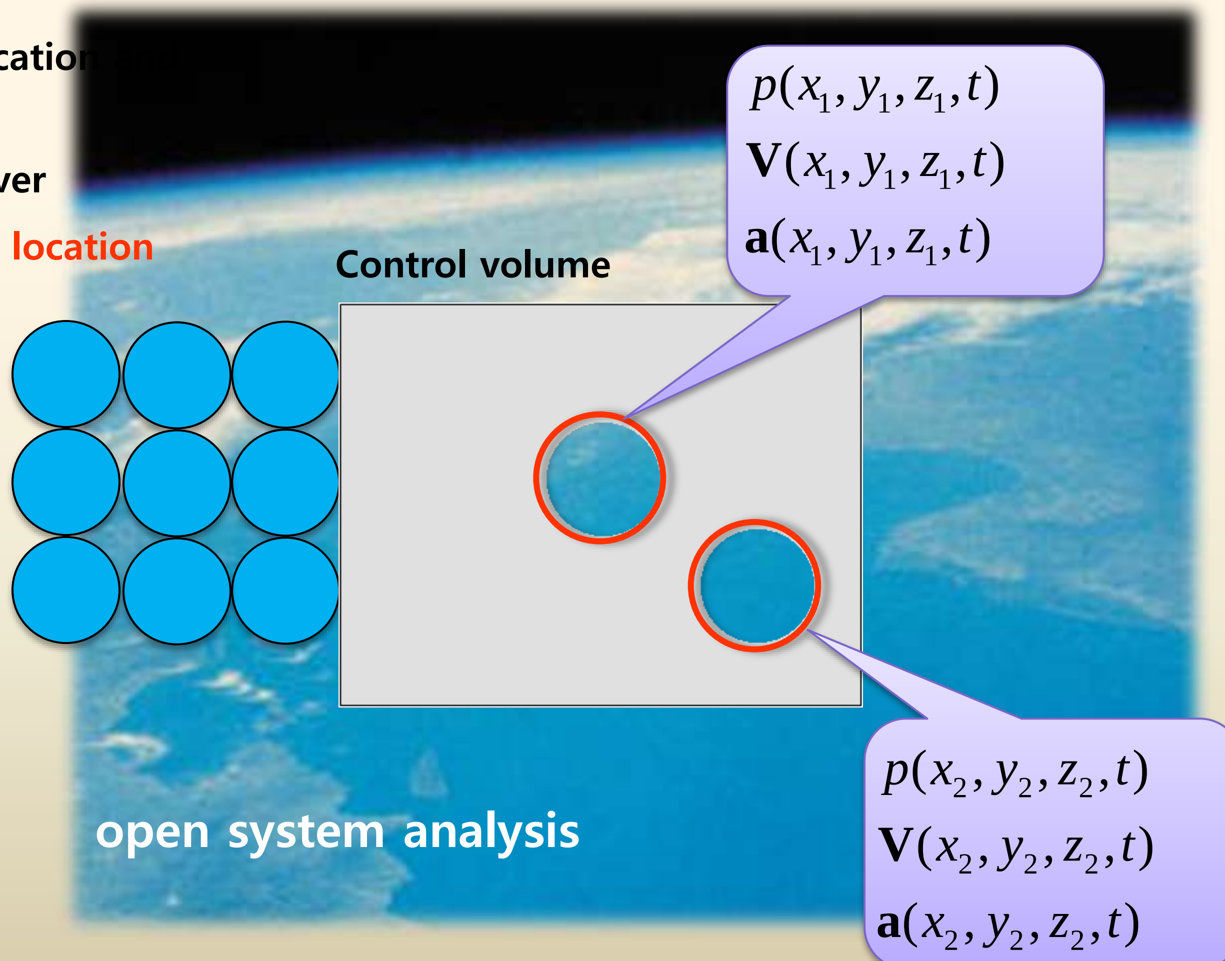

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Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid <u>motion</u>

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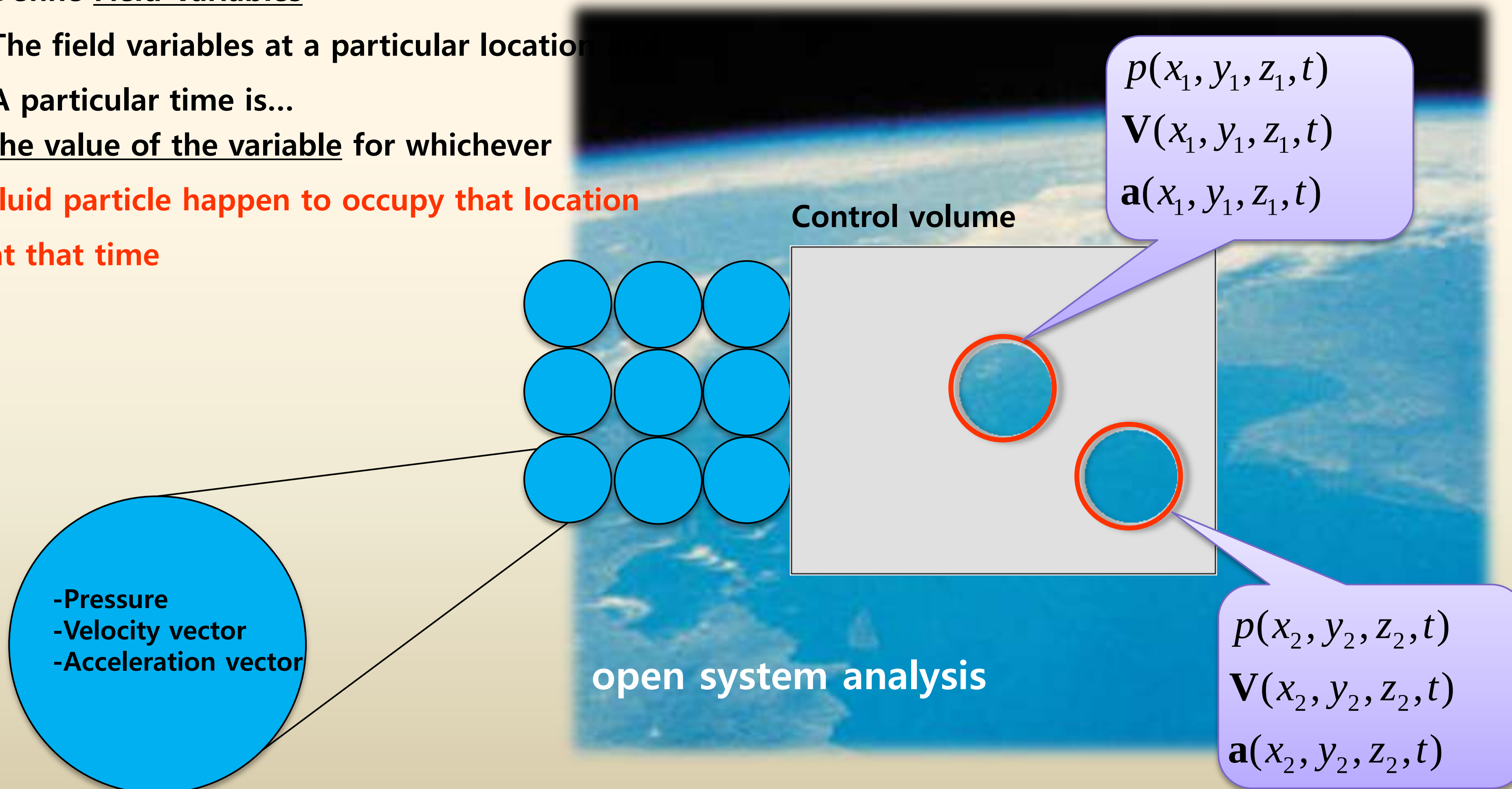

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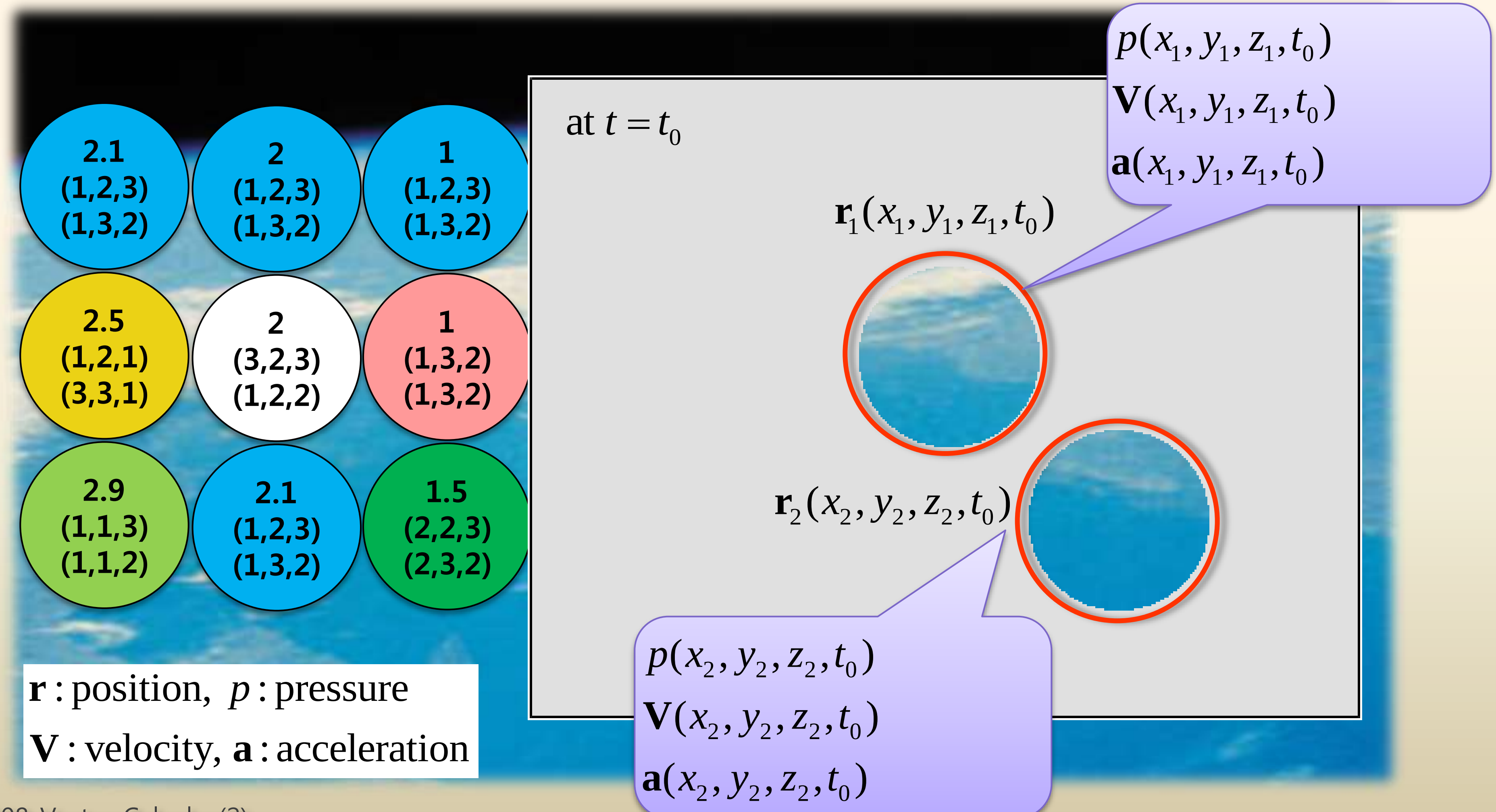
Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid motion

- **Eulerian Description**

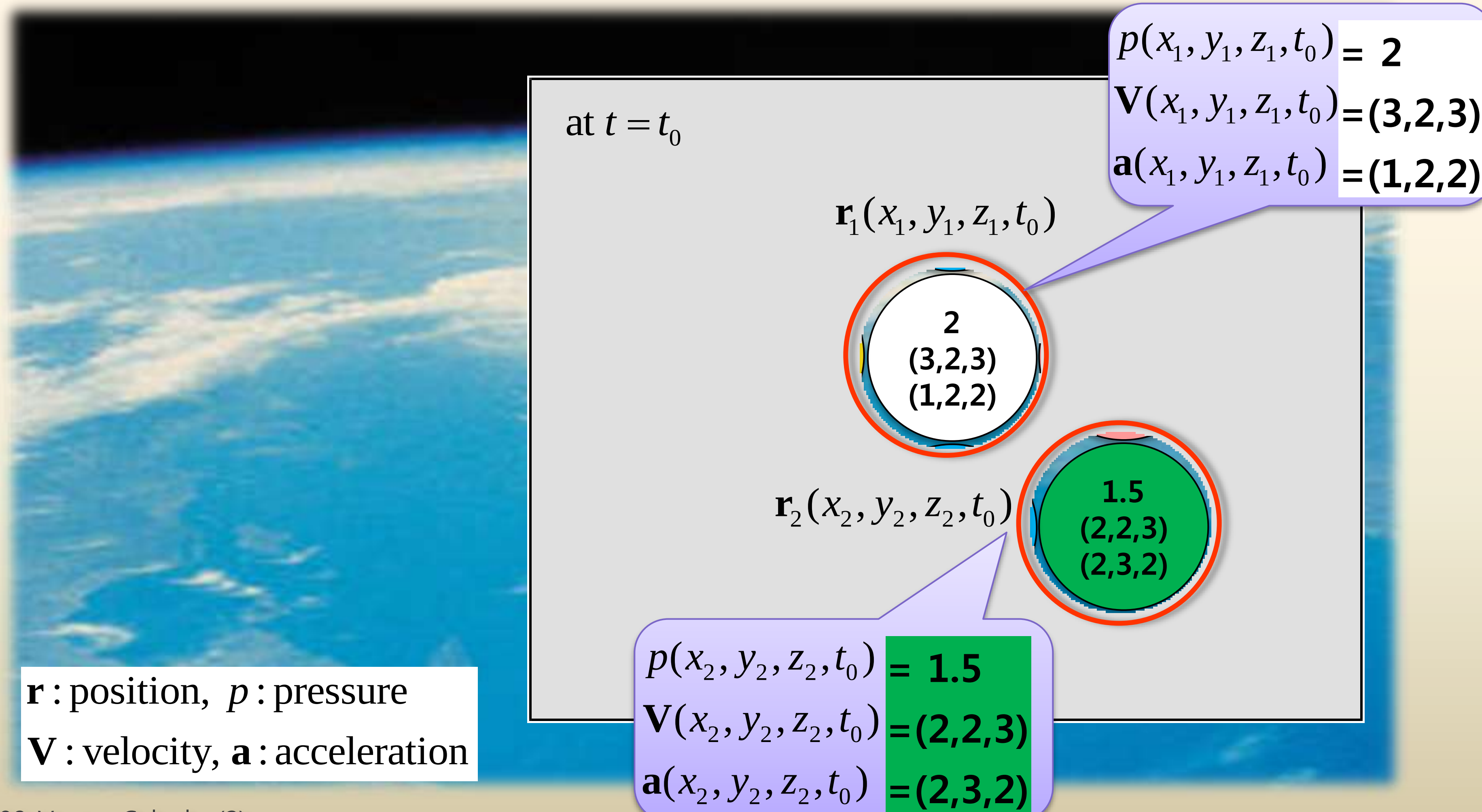


*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p90~95

Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid motion

- **Eulerian Description**



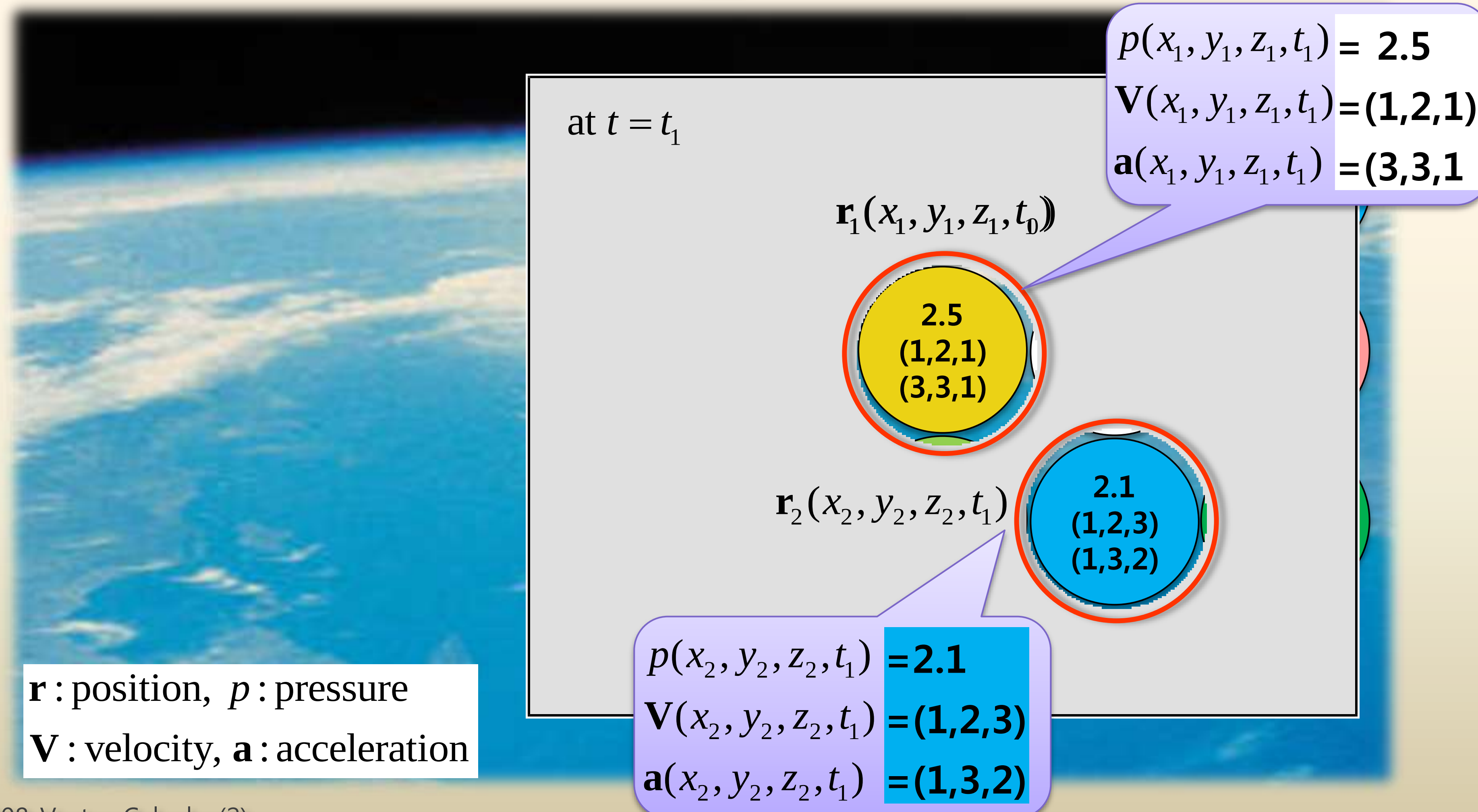
*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p90~95

Vector Fields and Fluid Mechanics

Fluid Kinematics* : how fluids flow and how to describe fluid <u>motion</u>

- **Eulerian Description**

Ref. Lagrangian & Eulerian Description



*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p90~95

■ Example 4-1*　A Steady Two-Dimensional Velocity Field

A steady, incompressible, two-dimensional velocity filed is given by

$$\mathbf{V} = (u, v) = (0.5 + 0.8x)\mathbf{i} + (1.5 - 0.8y)\mathbf{j}$$

A stagnation point is defined as a point in the flow field where the velocity is identically zero.

(a) Determine if there are any stagnation points in this flow filed and, if so, where?

(b) Sketch velocity vectors at several locations in the domain between x=-2 to 2 and y=0 to 5

Sol) (a)

$$\left.\begin{aligned} u &= 0.5 + 0.8x = 0 \;\rightarrow\; x = -0.625 \\ v &= 1.5 - 0.8y = 0 \;\rightarrow\; y = 1.875 \end{aligned}\right\} \text{Stagnation point}$$

*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p91

Vector Fields and Fluid Mechanics

■ Example 4-1* A Steady Two-Dimensional Velocity Field

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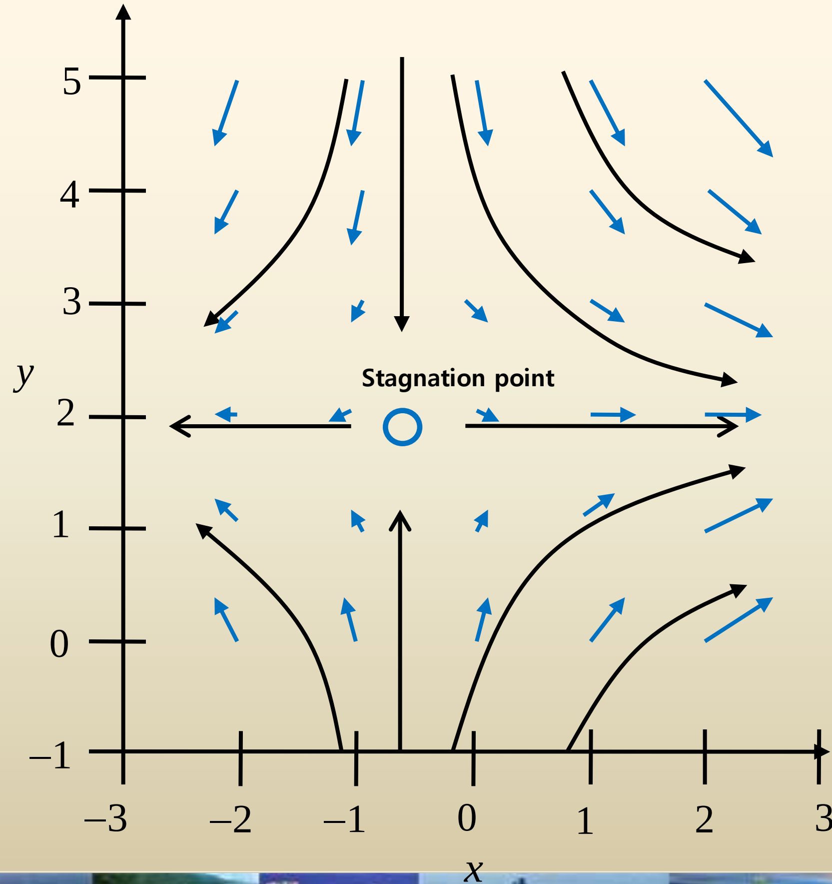
Sol) (b)



*Cimbala J.M., Cengel Y.A., Essential of Fluid Mechanics : Fundamental and Application, McGraw Hill, 2008, p91

Curl of a Vector Fields

Definition. **Curl (introduced by Maxwell*)**

Let $\mathbf{v}(x, y, z) = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ **be a differentiable vector function of Cartesian coordinates** x, y, z**.**

Then the curl of the vector function $\mathbf{v}$ **or of the vector field given by** $\mathbf{v}$ **is defined by the "symbolic" determinant**

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)\mathbf{k}.$$

Curl of a Vector Fields

$$\operatorname{curl}\mathbf{v}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

Let $\mathrm{v}=[yz,3zx,z]=yz\mathrm{i}+3zx\mathrm{i}+z\mathrm{k}$ with right-handed x,y,z.

Find *curl* v.

Solution)

$$\operatorname{curl}\mathbf{v}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

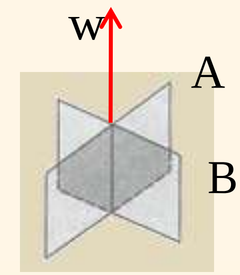
$$=\left(\frac{\partial z}{\partial y}-\frac{\partial 3zx}{\partial z}\right)\mathbf{i}+\left(\frac{\partial yz}{\partial z}-\frac{\partial z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial 3zx}{\partial x}-\frac{\partial yz}{\partial y}\right)\mathbf{k}$$

$$=-3x\mathbf{i}+y\mathbf{j}+2z\mathbf{k}$$

Curl of a Vector Fields

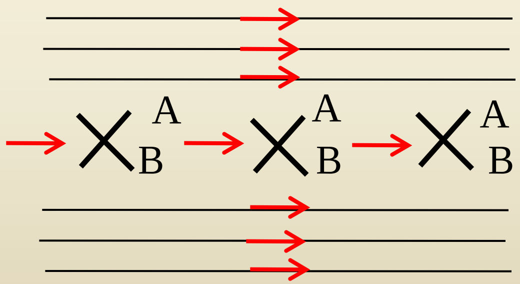
Paddle device

If a paddle device is inserted in a flowing fluid, then the curl of the velocity field $\mathbf{v}$ is a measure of the tendency of the fluid to turn the device about its vertical axis w.



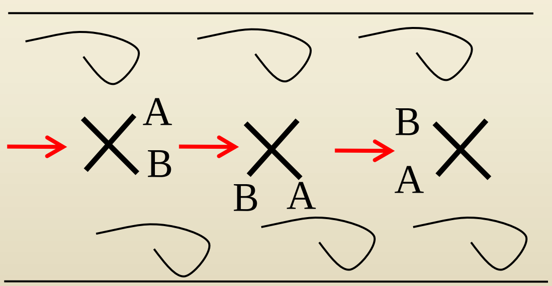
Paddle device

Physical Interpretations



Irrotational flow

(curl $\mathbf{F} = 0$)



Rotational flow

(curl $\mathbf{F} \neq 0$)

Curl of a Vector Fields

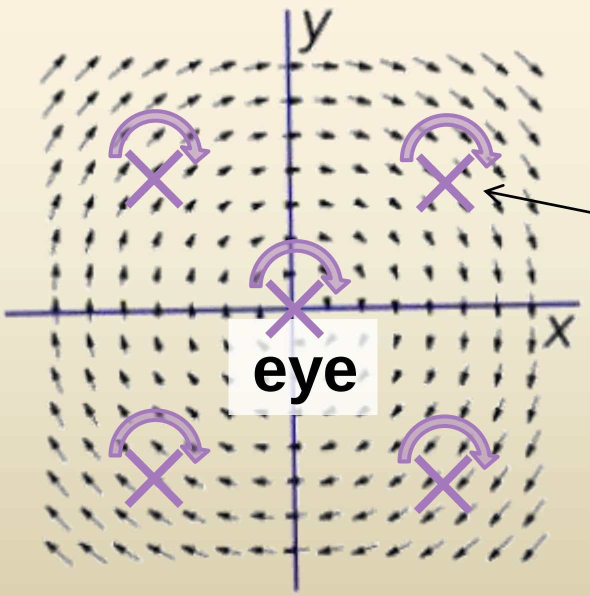
revolution

In a tornado the winds are rotating about the eye, and a vector field showing wind velocities would have a non-zero curl at the eye, and possibly elsewhere.

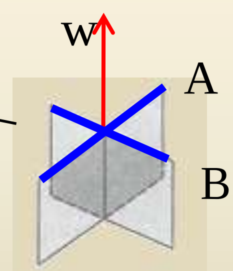
rotation

$$\mathbf{V}(x,y,z) = y\mathbf{i} + x\mathbf{j}$$

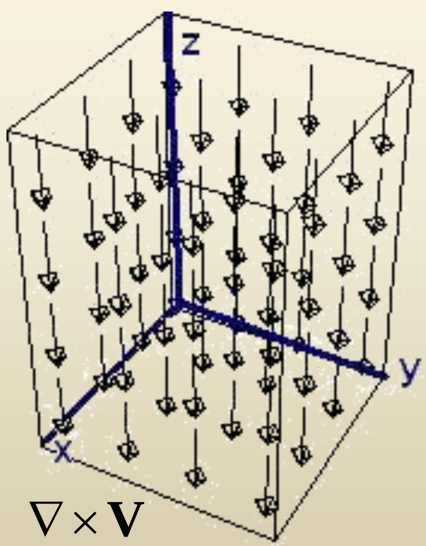
$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 2\mathbf{k} = -2\mathbf{k}$$



Velocity Vector

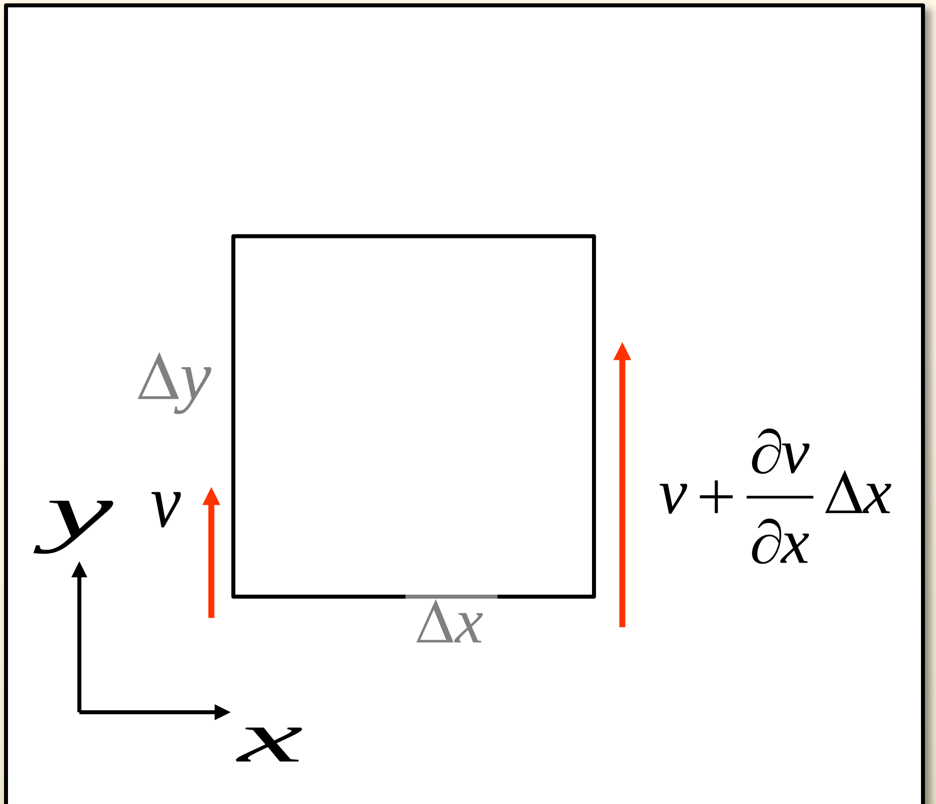


Paddle device



$\nabla \times \mathbf{V}$

Curl and Rotation in Fluids



✓ 유체 입자의 왼쪽 아래의 ***y***방향 속도를 ***v***라고 할 때, ***x***축으로 △***x*** 만큼 떨어진 지점에서의 ***y***방향 속도

(Tayler Series Expansion)

$$v=v(x,y,t)$$

↓ ***y,t*** 가 고정이라면,

$$v=v(x)$$

↓ Tayler Series Expansion

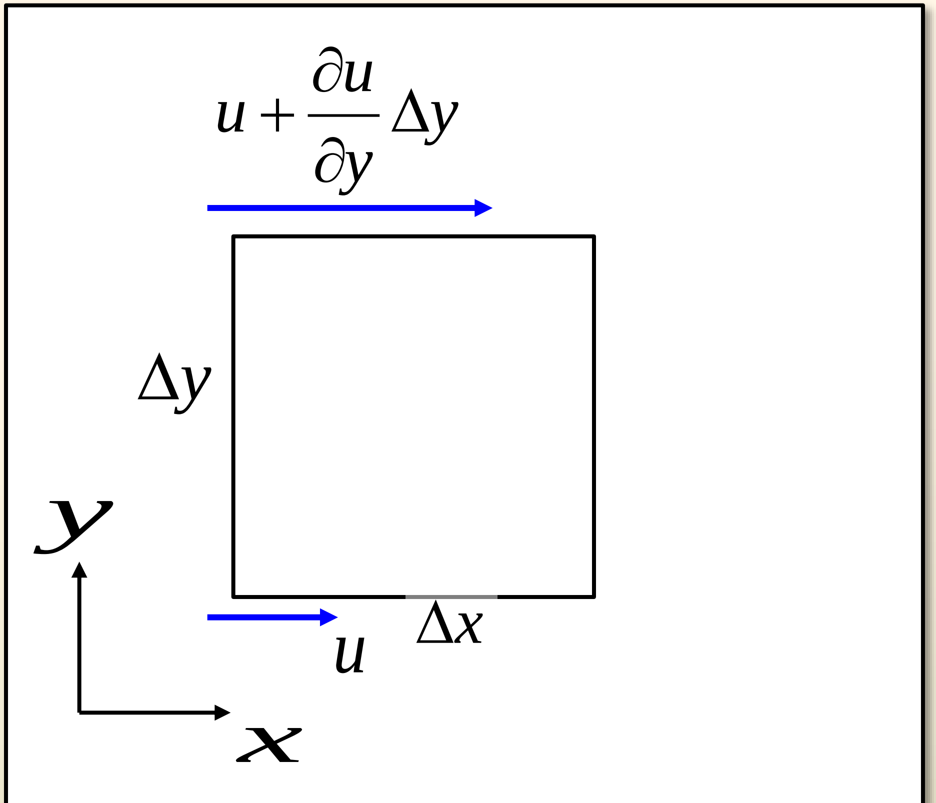
$$v(x+\Delta x)=v(x)+\frac{\partial v}{\partial x}\Delta x+\frac{\partial^2 v}{\partial x^2}(\Delta x)^2+\dots$$

↓ **△*x*가 작다고 가정하면,**

$$v(x+\Delta x)=v(x)+\frac{\partial v}{\partial x}\Delta x$$

(반시계 방향)

Curl and Rotation in Fluids



✓ 유체 입자의 왼쪽 아래의 ***x***방향 속도를 ***u***라고 할 때, ***y***축으로 △***y*** 만큼 떨어진 지점에서의 ***x***방향 속도
(Tayler Series Expansion)

$$u=u(x,y,t)$$

↓ ***x,t*** 가 고정이라면,

$$u=u(y)$$

↓ Tayler Series Expansion

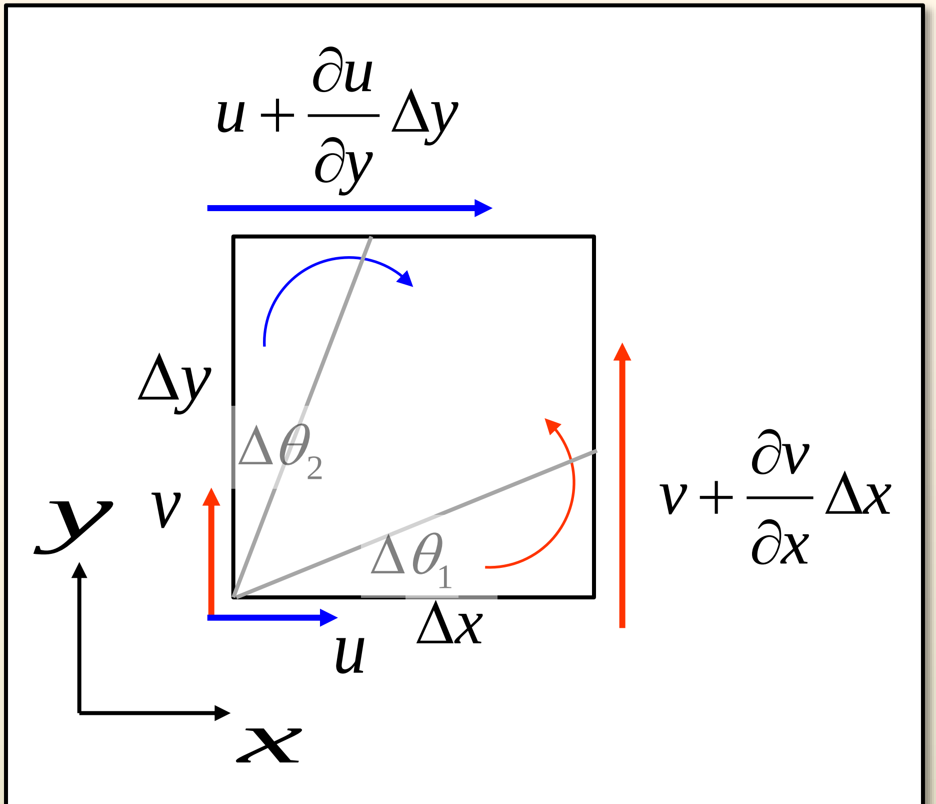
$$u(y+\Delta y)=u(y)+\frac{\partial u}{\partial y}\Delta y+\frac{\partial^2 u}{\partial y^2}(\Delta y)^2+\ldots$$

↓ **△*y*가 작다고 가정하면,**

$$u(y+\Delta y)=u(y)+\frac{\partial u}{\partial y}\Delta y$$

(시계 방향)

Curl and Rotation in Fluids



✓ 각속도

(속도 차만큼 변위 발생)

$$\omega_{z1} = \frac{\Delta\theta_1}{\Delta t} \approx \frac{\tan\Delta\theta_1}{\Delta t} = \frac{\Delta v \Delta t}{\Delta x}\cdot\frac{1}{\Delta t}$$

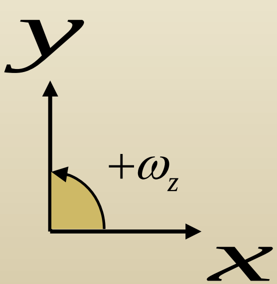
$$= \frac{v + \frac{\partial v}{\partial x}\Delta x - v}{\Delta x} = \frac{\partial v}{\partial x}$$

(반시계 방향)

$$\omega_{z2} = -\frac{\Delta\theta_2}{\Delta t} \approx -\frac{\tan\Delta\theta_2}{\Delta t} = -\frac{\Delta u \Delta t}{\Delta y}\cdot\frac{1}{\Delta t}$$

$$= -\frac{u + \frac{\partial u}{\partial y}\Delta y - u}{\Delta y} = -\frac{\partial u}{\partial y}$$

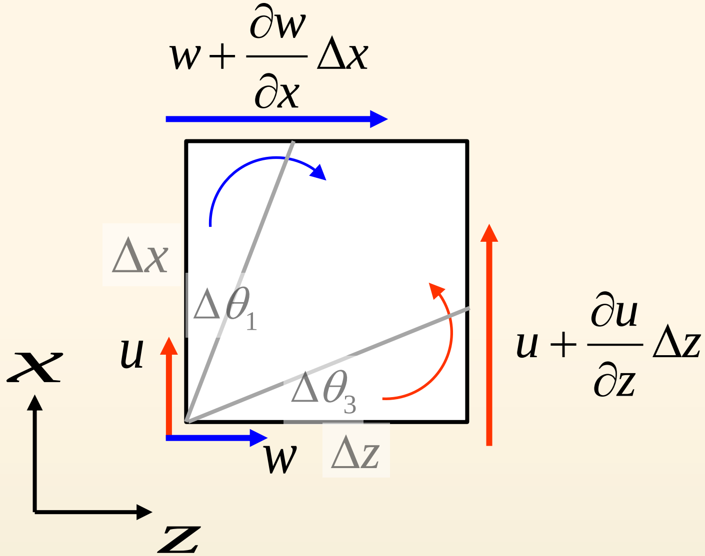
(시계 방향)

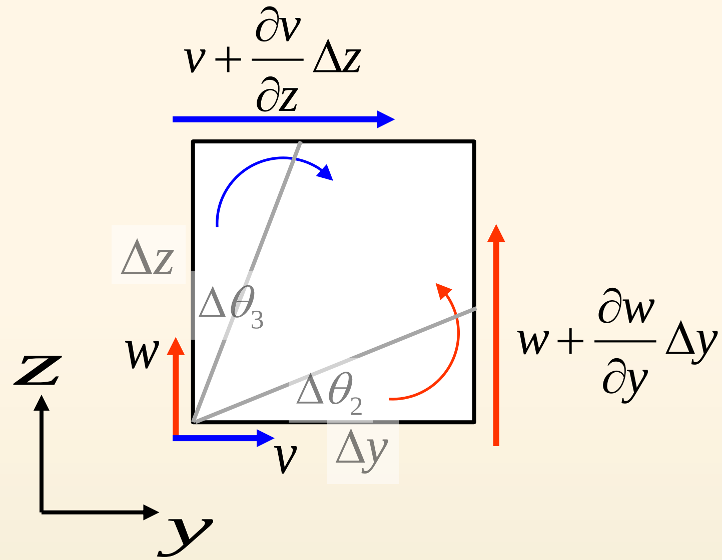


✓ z축에 대한 각속도는 두 각속도의 평균으로 정의하면

$$\omega_z = \frac{1}{2}(\omega_{z1} + \omega_{z2}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Curl and Rotation in Fluids



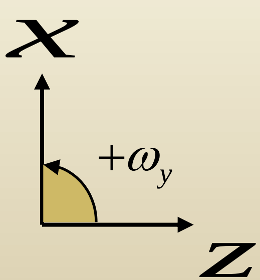


✓ y축에 대한 각속도

(반시계 방향) $\omega_{y1}=\frac{\partial u}{\partial z}$

(시계 방향) $\omega_{y2}=-\frac{\partial w}{\partial x}$

$$\therefore \omega_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

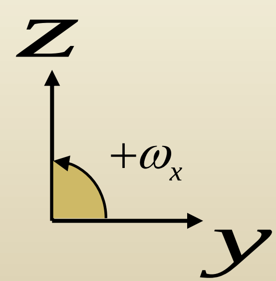


✓ x축에 대한 각속도

(반시계 방향) $\omega_{x1}=\frac{\partial w}{\partial y}$

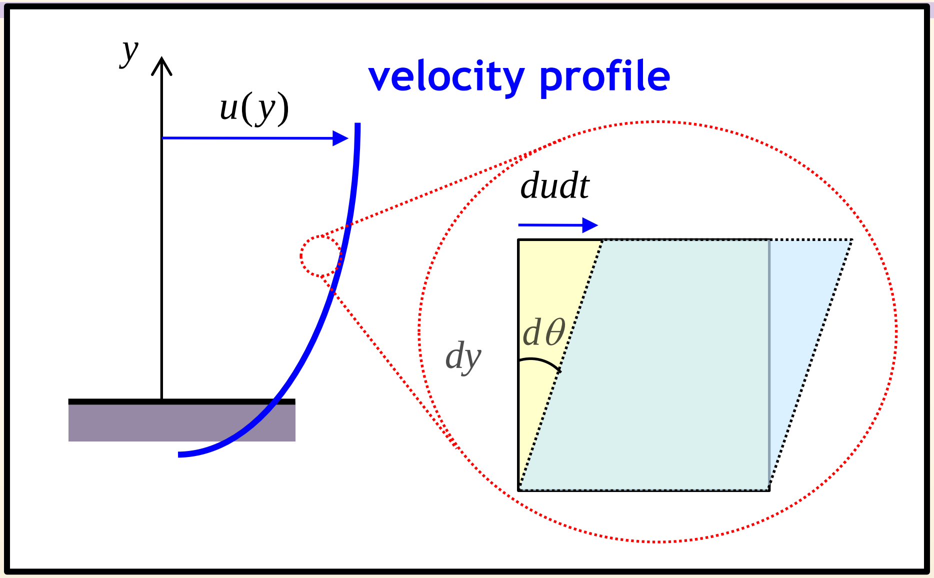
(시계 방향) $\omega_{x2}=-\frac{\partial v}{\partial z}$

$$\therefore \omega_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$$



1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264

뉴턴 유체1) (Newtonian Fluid)

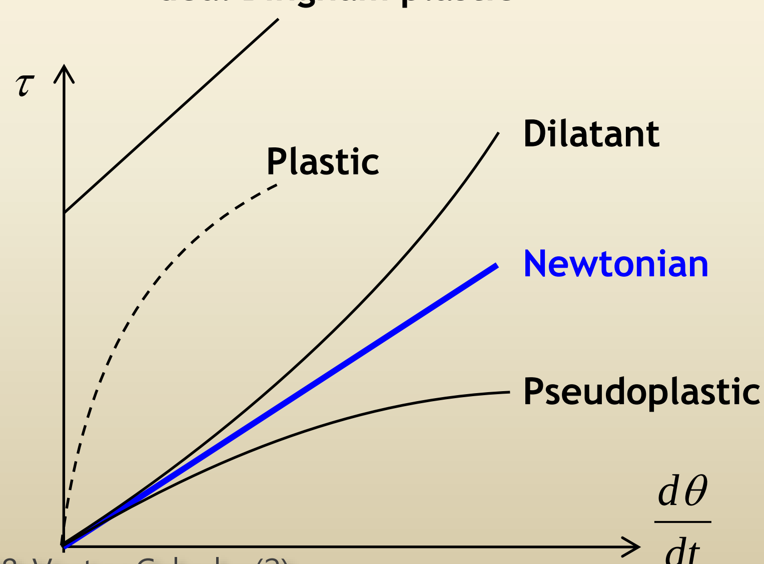


✓ 미소 구간에서의 전단변형율

$$d\theta \approx \tan d\theta = \frac{dudt}{dy} = \frac{du}{dy}dt$$

✓ 전단변형율의 시간변화율은 속도 구배와 같음

$$\frac{d\theta}{dt} = \frac{du}{dy} \quad \text{---①}$$



✓ 뉴턴 유체(Newtonian fluid)
: 전단응력이 전단변형율의 시간변화율에 비례

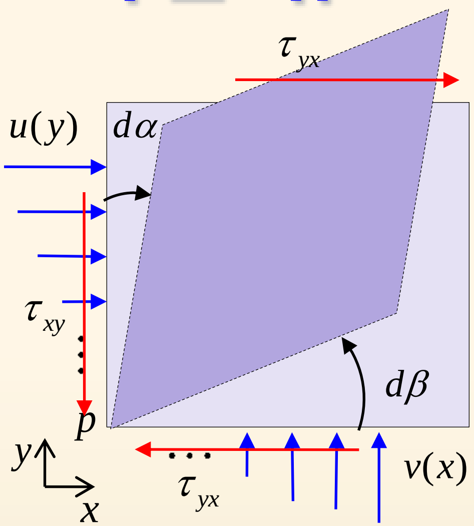
$$\tau \propto \frac{d\theta}{dt} \quad \text{---②}$$

✓ 뉴턴 유체(Newtonian fluid)의 특징
: ①,②에 의해, 전단응력은 속도구배에 비례함
(비례 상수 μ : 점성 계수)

$$\tau \propto \frac{du}{dy} \Rightarrow \tau = \mu\frac{du}{dy}$$

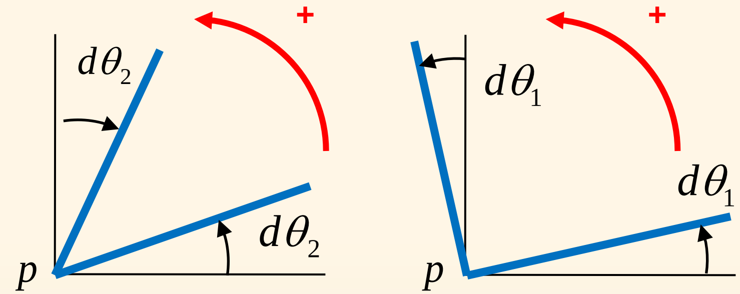
Curl and Rotation in Fluids - 다른 유도 방법

1) Cengel & Cimbala, Fluid Mechanics Mc Graw Hill,2005, pp46~50,pp426-431
2) kundu,P.K., Fluid Mechanics, Academic Press, 2008, pp.100~105
3) 김찬중, "공학도를 위한 길잡이 유체공학입문" , 문운당, 2002, pp258-264



회전과 전단변형으로 나타낼 수 있음

전단변형(shear strain) 회전 (rotation)



$$-d\alpha = -d\theta_2 + d\theta_1 \quad \cdots ①$$

$$d\beta = d\theta_2 + d\theta_1 \quad \cdots ②$$

Given : $d\alpha, d\beta$

Find : 회전각속도 ω_z

(1) ②와 ①을 더하면, $d\beta - d\alpha = 2d\theta_1$

$$d\theta_1 = \frac{1}{2}(d\beta - d\alpha)$$

➡ z축에 대한 회전 각도

(2) ***dt*****로 나누면,**

$$\frac{d\theta_1}{dt} = \frac{1}{2}\left(\frac{d\beta}{dt} - \frac{d\alpha}{dt}\right)$$

➡ z축에 대한 회전 각속도

(3) 전단변형율의 시간 변화율은 속도 구배와 같음

$$\omega_z = \frac{1}{2}\left(\frac{dv}{dx} - \frac{du}{dy}\right)$$

(5) 동일한 방법에 의해 다음도 성립함

➡ *y*축에 대한 회전각속도 : $\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

➡ *x*축에 대한 회전각속도 : $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$

Curl and Rotation in Fluids

✓ curl의 정의

$$\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

$$= 2\left[\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}\right]$$

$$= 2\omega_x\mathbf{i} + 2\omega_y\mathbf{j} + 2\omega_z\mathbf{k} = 2\boldsymbol{\omega}$$

✓ x축에 대한 각속도

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

✓ y축에 대한 각속도

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

✓ z축에 대한 각속도

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

=> curl v 크기는 는 유체 입자의 회전 각속도x2에 해당

Curl and Rotation in Fluids

✓ curl의 정의

$$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

✓ Irrotational Flow : $\nabla \times \mathbf{v} = 0 \left(\dfrac{\partial w}{\partial y} = \dfrac{\partial v}{\partial z}, \dfrac{\partial u}{\partial z} = \dfrac{\partial w}{\partial x}, \dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial y}\right)$

See also
Streeter V.L., Fluid Mechanics, McGraw-Hill, 1948, p19, '9. Irrotational Flow, Velocity Potential'

✓ $\nabla \times \mathbf{v} = 0$ 일 때, $\mathbf{v} = \nabla \Phi$ 인 Scalar Function Φ 가 존재함 ⇨ **Velocity Potential**

Curl and Rotation in Fluids

For any scalar function $\phi = \phi(x, y, z)$, $\text{curl}(\text{grad}\,\phi) = \mathbf{0}$ is always true.

$$\text{curl}(\text{grad}\,\phi) = \text{curl}\left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)$$

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)\mathbf{k} = 0$$

$\text{Velocity Vector} : \mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and $\text{curl}\,\mathbf{V} = \mathbf{0}$

$$\left.\begin{aligned} \text{curl}(\text{grad}\,\phi) &= \mathbf{0} \\ \text{curl}\,\mathbf{V} &= \mathbf{0} \end{aligned}\right\} \mathbf{V} = \text{grad}\,\phi$$

$$\phi = \phi(x, y, z) : \text{Velocity Potential}$$

$\mathbf{V}$ → ϕ

(Vector, 3 unknown components) (1 unknown)

We can reduce the unknowns from 3 to 1.

Divergence of a Vector Fields

Definition. **Divergence**

Let $\mathbf{v}(x,y,z)$ be a <u>differentiable vector function</u>, where x,y,z are Cartesian coordinates. And let v_1, v_2, v_3 be components of $\mathbf{v}$. Then the function

$$\operatorname{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \quad \cdots(1)$$

is called the divergence of $\mathbf{v}$ or the divergence of the vector field defined by $\mathbf{v}$.

$$\operatorname{div}\mathbf{v} = \nabla \bullet \mathbf{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \bullet [v_1, v_2, v_3]$$

$$= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \bullet (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \quad = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Divergence of a Vector Fields

divergence of v : $\mathrm{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \quad \cdots(1)$

$$\text{If } \mathbf{v} = \mathrm{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

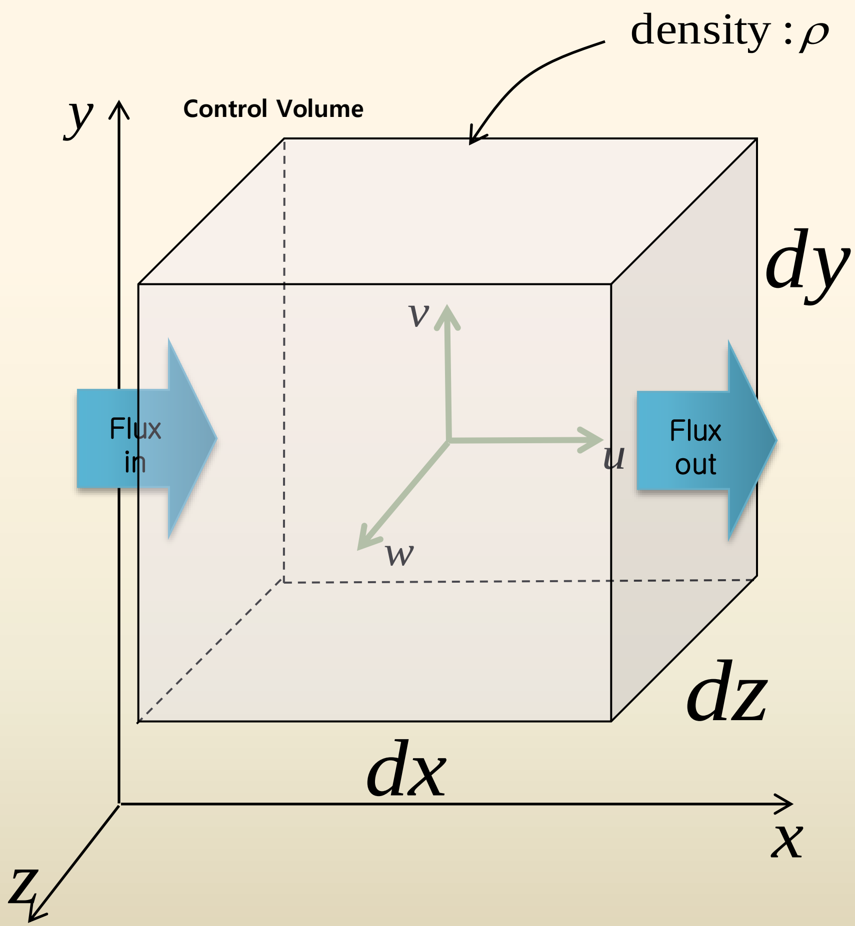
$$\mathrm{div}\,\mathbf{v} = \mathrm{div}(\mathrm{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f.$$

See also
Streeter V.L., Fluid Mechanics, McGraw-Hill, 1948,
p26, '11. The Laplace Equation'

Laplace operator

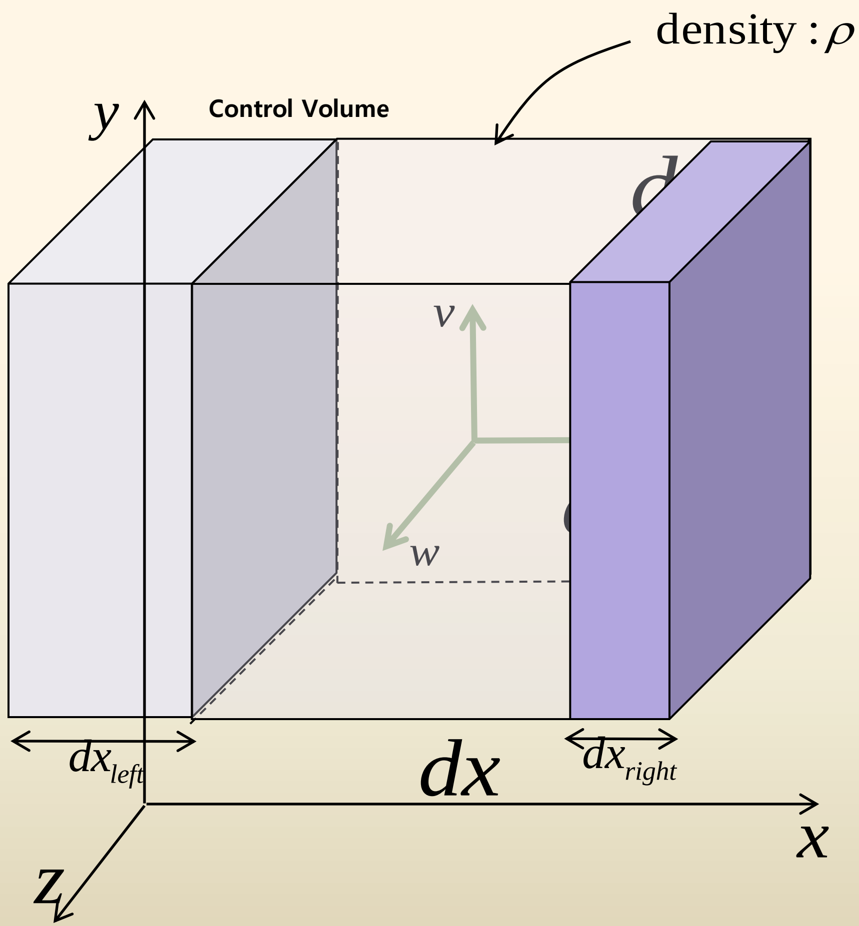
Divergence of a Vector Fields : Continuity Eqn.

at time t



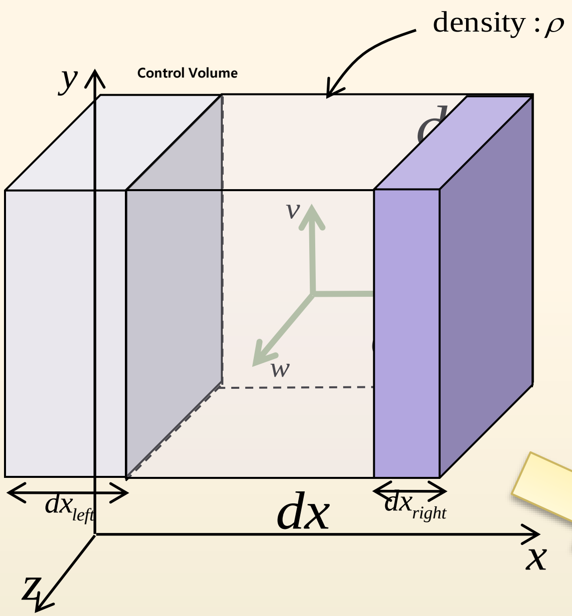
Divergence of a Vector Fields : Continuity Eqn.

at time $t + \Delta t$



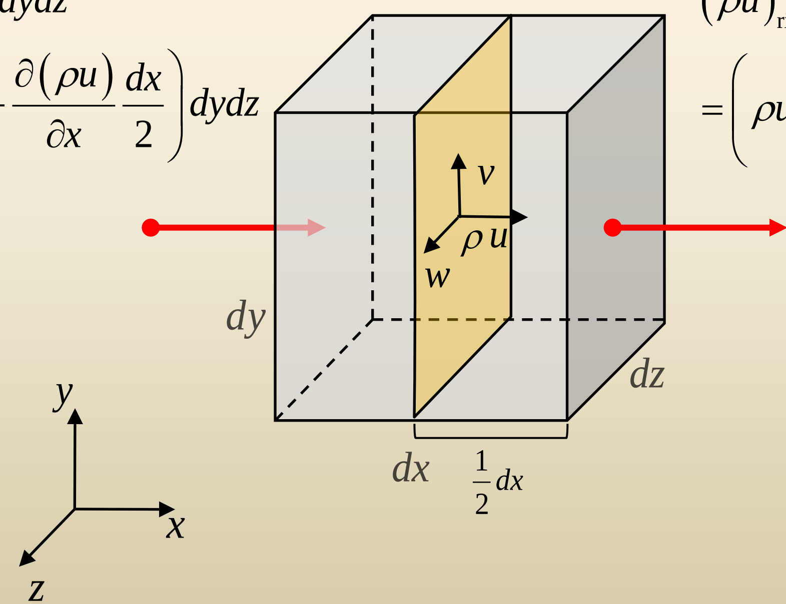
Divergence of a Vector Fields : Continuity Eqn.

at time $t+\Delta t$

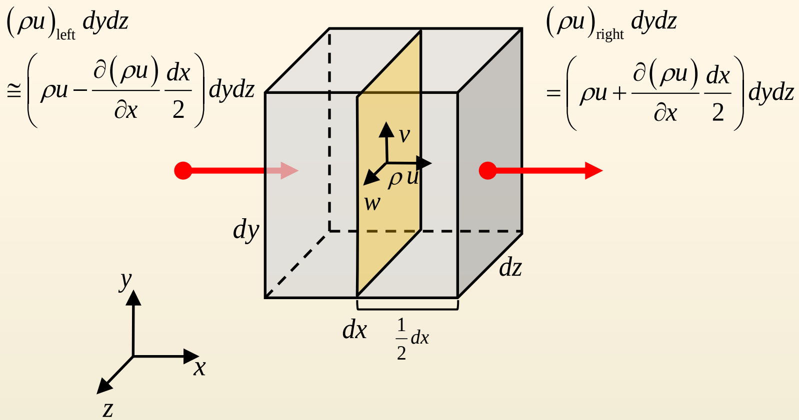


$$(\rho u)_{\text{left}}\, dydz \cong \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dydz$$

$$(\rho u)_{\text{right}}\, dydz = \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dydz$$



Divergence of a Vector Fields : Continuity Eqn.



✓ 오른쪽 면을 통해 검사체적으로부터 빠져나간 유체의 부피

$$dV_{right} = (dydz)dx_{right} = (dydz)u_{right}dt$$

✓ 오른쪽 면을 통해 검사체적으로부터 빠져나간 유체의 질량

$$\rho_{\text{right}} dV_{\text{right}} = \rho_{\text{right}}(dydz)u_{\text{right}}dt = (\rho u)_{\text{right}} dydz \cdot dt$$

✓ 단위 시간당 오른쪽 면을 통해 빠져나간 유체의 질량

$$\rho_{\text{right}} \frac{dV_{\text{right}}}{dt} = \rho_{right}(dydz)u_{\text{right}} = (\rho u)_{\text{right}} dydz$$

✓ $(\rho u)_{right}$ 를 Tayler Series로 전개하면,

$$(\rho u)_{right} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \cdots$$

$$\cong \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}$$ (1차항까지만 선택)

✓ 단위 시간당 오른쪽 면을 통해 빠져나간 유체의 질량

$$(\rho u)_{\text{right}} dydz = \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right)dydz$$

Divergence of a Vector Fields : Continuity Eqn.

Flow of a Compressible Fluid. Physical meaning of Divergence

✓ 단위 시간당 각 면을 통해 빠져나간 유체의 질량

Right(+x) : $(\rho u)_{\text{right}}\, dydz \cong \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} \right) dydz$

Top(+y) : $(\rho v)_{\text{top}}\, dxdz \cong \left(\rho v + \frac{\partial(\rho v)}{\partial z}\frac{dy}{2} \right) dxdz$

Front(+z) : $(\rho w)_{\text{front}}\, dxdy \cong \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2} \right) dxdy$

✓ 단위 시간당 각 면을 통해 들어온 유체의 질량

left(-x) : $(\rho u)_{\text{left}}\, dydz \cong \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} \right) dydz$

Bottom(-y) : $(\rho v)_{\text{bottom}}\, dxdz \cong \left(\rho v - \frac{\partial(\rho v)}{\partial z}\frac{dy}{2} \right) dxdz$

Rear(-z) : $(\rho w)_{\text{rear}}\, dxdy \cong \left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2} \right) dxdy$

⇓

✓ 단위 시간당 검사 체적을 통과한 유체의 유입량 (들어온 양을 +로 봄)

$$\sum \dot{m} = \sum_{in} \dot{m} - \sum_{out} \dot{m} = \left[\left((\rho u) - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} \right) dydz + \left((\rho v) - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2} \right) dxdz + \left((\rho w) - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2} \right) dxdy \right]$$

$$- \left[\left((\rho u) + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} \right) dydz + \left((\rho v) + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2} \right) dxdz + \left((\rho w) + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2} \right) dxdy \right]$$

$$= -\frac{\partial(\rho u)}{\partial x} dxdydz - \frac{\partial(\rho v)}{\partial y} dxdydz - \frac{\partial(\rho w)}{\partial z} dxdydz$$

Divergence of a Vector Fields : Continuity Eqn.

Mass conservation

✓ 단위 시간당 검사 체적을 통과한 유체의 유입량 (들어온 양을 +로 봄)

$$= -\frac{\partial(\rho u)}{\partial x}dxdydz - \frac{\partial(\rho v)}{\partial y}dxdydz - \frac{\partial(\rho w)}{\partial z}dxdydz \text{ ---------} ①$$

✓ (검사체적 내부의 질량 변화율) $= \frac{\partial \rho}{\partial t}dxdydz$ --------- ②

① = ②

$$\frac{\partial \rho}{\partial t}dxdydz = -\frac{\partial(\rho u)}{\partial x}dxdydz - \frac{\partial(\rho v)}{\partial y}dxdydz - \frac{\partial(\rho w)}{\partial z}dxdydz$$

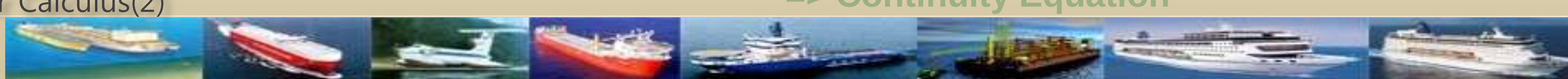
$dxdydz$ 으로 양변을 나누면,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{V}) = 0}$$

=> Continuity Equation

See also
Streeter V.L., Fluid Mechanics,
McGraw-Hill, 1948,
p14, '7. Equation of Continuity'



Mass Conservation

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{V}) = 0$$

비압축성 유체(Incompressible fluid)라고 가정하면,

$$(\rho = const)$$

$$\nabla \bullet \mathbf{V} = 0 \quad \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)$$

⇨ **Divergence**

비회전 유동(Irrotational flow)이라고 가정하면,

$$(\mathbf{V} = \nabla \Phi)$$

⇨ **Gradient**

$$\nabla \bullet \nabla \Phi = \nabla^2 \Phi = 0$$

$$\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \right)$$

⇨ **Laplace Equation**

Further Applications of the Divergence Theorem :Potential Theory, Harmonic Function

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}P(x,y,z) + \frac{\partial}{\partial y}Q(x,y,z) + \frac{\partial}{\partial z}R(x,y,z)$$

- **Potential theory : the theory of solutions of Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

- **Harmonic function : A solution of Laplace's equation with continuous 2nd order partial derivatives**

A basic property of solutions of Laplace's equation

$$\iiint_T \operatorname{div}\mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dA$$

If $\mathbf{F} = \operatorname{grad} f,\ \operatorname{div}\mathbf{F} = \operatorname{div}(\operatorname{grad} f) = \nabla^2 f$

And, $\mathbf{F} \bullet \mathbf{n} = \mathbf{n} \bullet \mathbf{F} = \mathbf{n} \bullet (\operatorname{grad} f) = \frac{\partial f}{\partial n}$

Thus the formula in the divergence theorem becomes

$$\iiint_T \nabla^2 f\, dV = \iint_S \frac{\partial f}{\partial n}\, dA$$

Further Applications of the Divergence Theorem :Potential Theory, Harmonic Function

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}P(x,y,z) + \frac{\partial}{\partial y}Q(x,y,z) + \frac{\partial}{\partial z}R(x,y,z)$$

- **Potential theory : the theory of solutions of Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

- **Harmonic function : A solution of Laplace's equation with continuous 2nd order partial derivatives**

$$\iiint_T \nabla^2 f dV = \iint_S \frac{\partial f}{\partial n} dA$$

$$if,\ \nabla^2 f = 0 \ \ then \ \ \iiint_T \nabla^2 f dV = 0$$

$$\because \iint_S \frac{\partial f}{\partial n} dA = 0$$

(Theorem 1) A Basic Property of harmonic Functions

Let $f(x,y,z)$ be a harmonic function in some domain D in space. Let S be any piecewise smooth closed orientable surface in D whose entire region it encloses belongs to D. Then the integral of the normal derivatives of f taken over S is zero.

Further Applications of the Divergence Theorem :Potential Theory, Harmonic Function

- **Potential theory : the theory of solutions of Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

- **Harmonic function : A solution of Laplace's equation with continuous 2nd order partial derivatives**

$$\iiint_T \nabla^2 f dV = \iint_S \boxed{\frac{\partial f}{\partial n}} dA$$

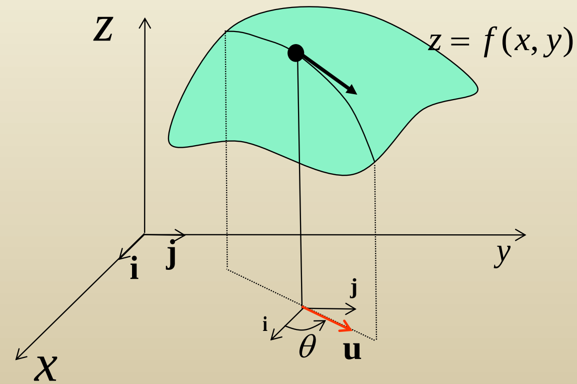
Theorem 9.6 **Computing a Directional Derivative**

If $z = f(x, y)$ is differentiable function of x and y and $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ then,

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Recall, Directional Derivative

$$\boxed{D_{\mathbf{u}} f(x, y)} = [f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}] \cdot (\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$
$$= \nabla f(x, y) \cdot \mathbf{u}$$



Further Applications of the Divergence Theorem : Green's Theorems

$$\mathbf{a}\bullet\mathbf{b}=(a_1b_1+a_2b_2+a_3b_3)=\mathbf{b}\bullet\mathbf{a}$$

Let $\mathbf{F}=f\ \mathrm{grad}\ g$

$$\mathrm{div}\ \mathbf{F}=\mathrm{div}(f\ \mathrm{grad}\ g)=\mathrm{div}\left(\left[f\frac{\partial g}{\partial x},f\frac{\partial g}{\partial y},f\frac{\partial g}{\partial z}\right]\right)$$

$$=\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$=\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+f\frac{\partial^2 g}{\partial x^2}\right)+\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+f\frac{\partial^2 g}{\partial y^2}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}+f\frac{\partial^2 g}{\partial z^2}\right)$$

$$=f\nabla^2 g+\mathrm{grad}\ f\bullet\mathrm{grad}\ g$$

$$\mathbf{F}\bullet\mathbf{n}=\mathbf{n}\bullet\mathbf{F}=\mathbf{n}\bullet(f\ \mathrm{grad\ g})=f(\mathbf{n}\bullet\mathrm{grad\ g})=f\frac{\partial g}{\partial n}$$

(1) Green's first formula

$$\iiint_T\left(f\nabla^2 g+\mathrm{grad}\ f\bullet\mathrm{grad}\ g\right)dV=\iint_S f\frac{\partial g}{\partial n}dA$$

Further Applications of the Divergence Theorem : Green's Theorems

Let $\mathbf{F} = f \operatorname{grad} g$

$$\iiint_T \left(f\nabla^2 g + \operatorname{grad} f \bullet \operatorname{grad} g \right) dV = \iint_S f \frac{\partial g}{\partial n} dA \dashrightarrow ①$$

Let $\mathbf{F} = g \operatorname{grad} f$

$$\iiint_T \left(g\nabla^2 f + \operatorname{grad} g \bullet \operatorname{grad} f \right) dV = \iint_S g \frac{\partial f}{\partial n} dA \dashrightarrow ②$$

① - ② :

(2) Green's second formula

$$\iiint_T \left(f\nabla^2 g - g\nabla^2 f \right) dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dA$$

Further Applications of the Divergence Theorem
: Green's Theorems

(2) Green's second formula

$$\iiint_T \left(f\nabla^2 g - g\nabla^2 f \right) dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA$$

If g, f are solution of Laplace equation, then

$$\iiint_T \left(f\nabla^2 g - g\nabla^2 f \right) dV = 0$$

$$\therefore \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA = 0$$

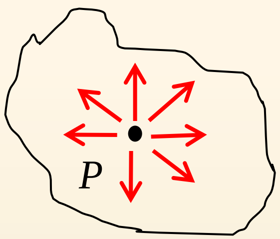
$$\therefore f\frac{\partial g}{\partial n} = g\frac{\partial f}{\partial n}$$

Ref. Froude-Krylov Force & Diffraction Force

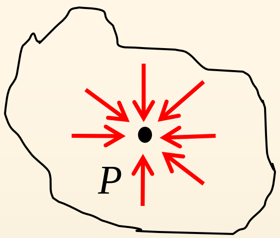
Source & Sink

$\nabla \cdot \mathbf{F} = 0$:incompressible flow

$\nabla \cdot \mathbf{F} \neq 0$:compressible flow



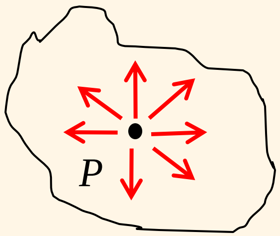
Source
:net outward flow
$(\text{div}\ \mathbf{F}(P) > 0)$



Sink
:net inward flow
$(\text{div}\ \mathbf{F}(P) < 0)$

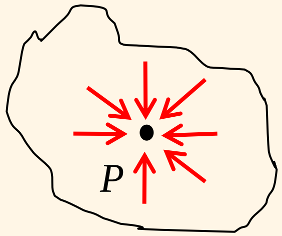
Generate a body shape by using Source and Sink

Source & Sink



Source
:net outward flow

$(\operatorname{div} \mathbf{F}(P) > 0)$



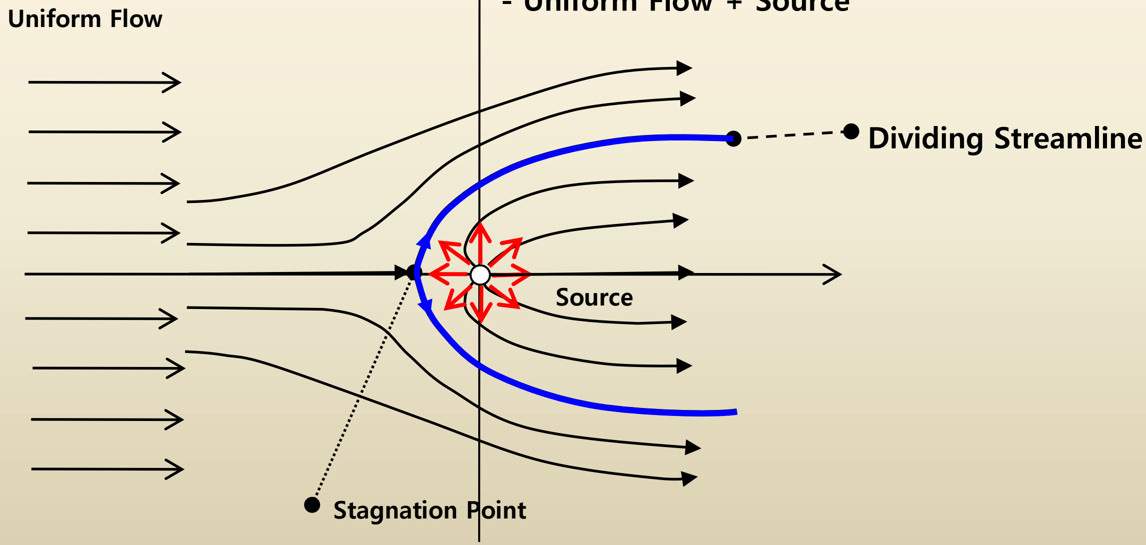
Sink
:net inward flow

$(\operatorname{div} \mathbf{F}(P) < 0)$

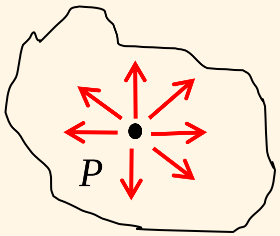
Generate a body-like shape by using Source and Sink

Half Body

- Uniform Flow + Source

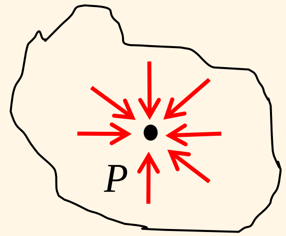


Source & Sink



Source
:net outward flow

$(\operatorname{div} \mathbf{F}(P) > 0)$

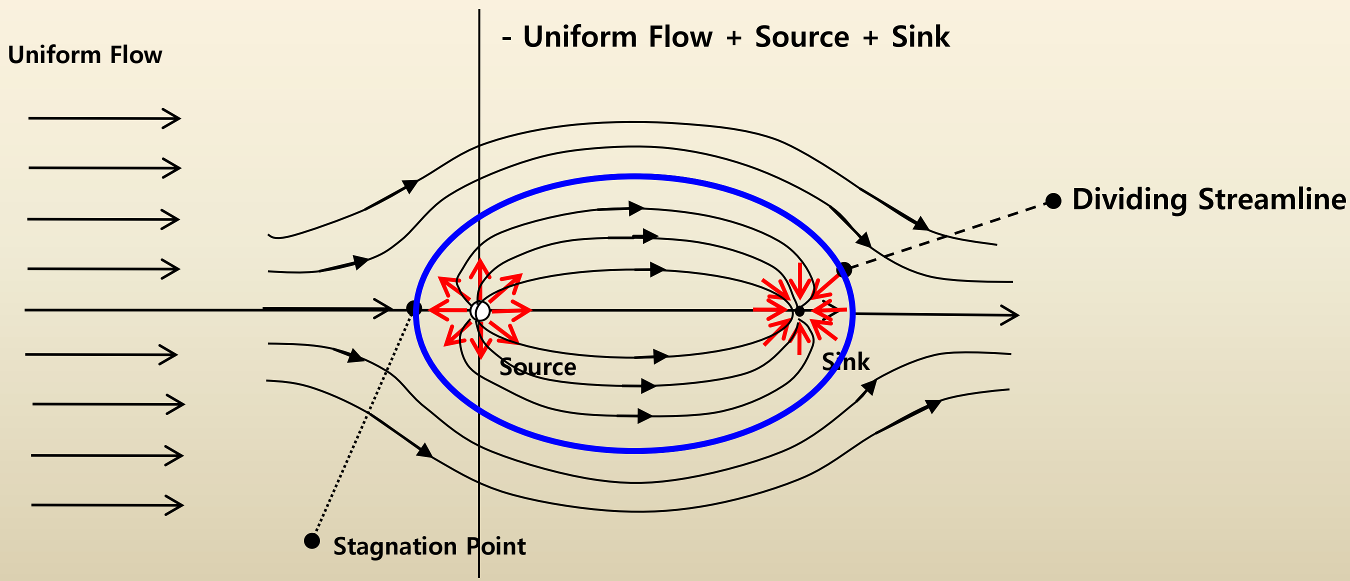


Sink
:net inward flow

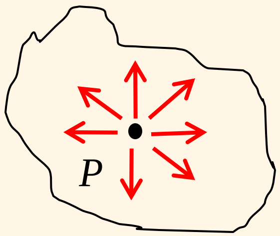
$(\operatorname{div} \mathbf{F}(P) < 0)$

Generate a body-like shape by using Source and Sink

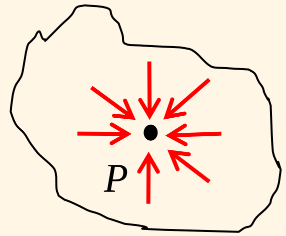
Rankine Ovoid



Source & Sink



Source
:net outward flow
$(\text{div } \mathbf{F}(P) > 0)$



Sink
:net inward flow
$(\text{div } \mathbf{F}(P) < 0)$

Generate a body-like shape by using Source and Sink

✓ **Singularity Distribution Method[2)] (2-D)**

Laplace Equation을 만족함 $\nabla^2 \Phi = 0$

물체 표면에 특이점 (source, doublet, vortex)을 분포시켜 수학적으로 물체 경계면을 생성시키고, 이들 특이점들의 강도(Strength)를 구하여 전체 유장의 velocity potential을 구하는 방법

Velocity Potential → **Pressure** → **Surface Force to Hull**

$$\rho \frac{\partial \Phi}{\partial t} + P + \frac{1}{2}\rho |\nabla \Phi|^2 + \rho g\, z = C \qquad \mathbf{F}_{Fluid} = \iint_{S_B} P\mathbf{n}dS$$

Reference slides

Differentiation of Cross Product

증명

$$\mathbf{r}_1(t)=\left\langle x_1(t), y_1(t), z_1(t)\right\rangle$$

$$\mathbf{r}_1'(t)=\left\langle x_1'(t), y_1'(t), z_1'(t)\right\rangle$$

$$\mathbf{r}_2(t)=\left\langle x_2(t), y_2(t), z_2(t)\right\rangle$$

$$\mathbf{r}_2(t)=\left\langle x_2'(t), y_2'(t), z_2'(t)\right\rangle$$

$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]=\mathbf{r}_1'(t)\times\mathbf{r}_2(t)+\mathbf{r}_1(t)\times\mathbf{r}_2'(t)$$

$L.H.S.:$

$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]=\frac{d}{dt}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1(t) & y_1(t) & z_1(t)\\ x_2(t) & y_2(t) & z_2(t)\end{vmatrix}$$

$$=\frac{d}{dt}\left\{\left[y_1(t)z_2(t)-y_2(t)z_1(t)\right]\mathbf{i}+\left[z_1(t)x_2(t)-z_2(t)x_1(t)\right]\mathbf{j}+\left[x_1(t)y_2(t)-x_2(t)y_1(t)\right]\mathbf{k}\right\}$$

$$=\left\{\frac{d}{dt}\left[y_1(t)z_2(t)-y_2(t)z_1(t)\right]\mathbf{i}+\frac{d}{dt}\left[z_1(t)x_2(t)-z_2(t)x_1(t)\right]\mathbf{j}+\frac{d}{dt}\left[x_1(t)y_2(t)-x_2(t)y_1(t)\right]\mathbf{k}\right\}$$

$$=\left[y_1'(t)z_2(t)+y_1(t)z_2'(t)-y_2'(t)z_1(t)-y_2(t)z_1'(t)\right]\mathbf{i}$$

$$+\left[z_1'(t)x_2(t)+z_1(t)x_2'(t)-z_2'(t)x_1(t)-z_2(t)x_1'(t)\right]\mathbf{j}$$

$$+\left[x_1'(t)y_2(t)+x_1(t)y_2'(t)-x_2(t)y_1'(t)-x_2(t)y_1'(t)\right]\mathbf{k}$$

증명

$$\mathbf{r}_1(t)=\langle x_1(t), y_1(t), z_1(t)\rangle$$

$$\mathbf{r}_1'(t)=\langle x_1'(t), y_1'(t), z_1'(t)\rangle$$

$$\mathbf{r}_2(t)=\langle x_2(t), y_2(t), z_2(t)\rangle$$

$$\mathbf{r}_2(t)=\langle x_2'(t), y_2'(t), z_2'(t)\rangle$$

$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)]=\mathbf{r}_1'(t)\times\mathbf{r}_2(t)+\mathbf{r}_1(t)\times\mathbf{r}_2'(t)$$

$R.H.S.:$

$$\mathbf{r}_1'(t)\times\mathbf{r}_2(t)+\mathbf{r}_1(t)\times\mathbf{r}_2'(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1'(t) & y_1'(t) & z_1'(t)\\ x_2(t) & y_2(t) & z_2(t)\end{vmatrix}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x_1(t) & y_1(t) & z_1(t)\\ x_2'(t) & y_2'(t) & z_2'(t)\end{vmatrix}$$

$$=\left\{\left[y_1'(t)z_2(t)-y_2(t)z_1'(t)\right]\mathbf{i}+\left[z_1'(t)x_2(t)-z_2(t)x_1'(t)\right]\mathbf{j}+\left[x_1'(t)y_2(t)-x_2(t)y_1'(t)\right]\mathbf{k}\right\}$$
$$+\left\{\left[y_1(t)z_2'(t)-y_2'(t)z_1(t)\right]\mathbf{i}+\left[z_1(t)x_2'(t)-z_2'(t)x_1(t)\right]\mathbf{j}+\left[x_1(t)y_2'(t)-x_2'(t)y_1(t)\right]\mathbf{k}\right\}$$

$$=\left[y_1'(t)z_2(t)+y_1(t)z_2'(t)-y_2'(t)z_1(t)-y_2(t)z_1'(t)\right]\mathbf{i}$$
$$+\left[z_1'(t)x_2(t)+z_1(t)x_2'(t)-z_2'(t)x_1(t)-z_2(t)x_1'(t)\right]\mathbf{j}$$
$$+\left[x_1'(t)y_2(t)+x_1(t)y_2'(t)-x_2(t)y_1'(t)-x_2(t)y_1'(t)\right]\mathbf{k}$$

증명

$$\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)] = \mathbf{r}_1'(t)\times\mathbf{r}_2(t) + \mathbf{r}_1(t)\times\mathbf{r}_2'(t)$$

$L.H.S:$

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}_1(t)\times\mathbf{r}_2(t)] = &\left[y_1'(t)z_2(t) + y_1(t)z_2'(t) - y_2'(t)z_1(t) - y_2(t)z_1'(t)\right]\mathbf{i} \\ &+\left[z_1'(t)x_2(t) + z_1(t)x_2'(t) - z_2'(t)x_1(t) - z_2(t)x_1'(t)\right]\mathbf{j} \\ &+\left[x_1'(t)y_2(t) + x_1(t)y_2'(t) - x_2(t)y_1'(t) - x_2(t)y_1'(t)\right]\mathbf{k}\end{aligned}$$

$R.H.S:$

$$\begin{aligned}\mathbf{r}_1'(t)\times\mathbf{r}_2(t) + \mathbf{r}_1(t)\times\mathbf{r}_2'(t) = &\left[y_1'(t)z_2(t) + y_1(t)z_2'(t) - y_2'(t)z_1(t) - y_2(t)z_1'(t)\right]\mathbf{i} \\ &+\left[z_1'(t)x_2(t) + z_1(t)x_2'(t) - z_2'(t)x_1(t) - z_2(t)x_1'(t)\right]\mathbf{j} \\ &+\left[x_1'(t)y_2(t) + x_1(t)y_2'(t) - x_2(t)y_1'(t) - x_2(t)y_1'(t)\right]\mathbf{k}\end{aligned}$$

$\therefore L.H.S. = R.H.S$

Reference slides

Optimization – Steepest Descent Method

비제약 최적화 기법 중

Gradient 방법

- **Steepest Descent 방법**
- 공액 경사도 방법(Conjugate Gradient 방법)
- Newton의 방법
- Davidon-Fletcher-Powell(DFP) 방법
- Broyden-Fletcher-Goldfarb-Shanno(BFGS) 방법

황금 분할에 의한 1차원 탐색 방법

직접 탐사법(Direct Search Method)

- Hooke & Jeeves의 직접 탐사법
- Nelder & Mead의 Simplex 방법

Gradient 방법- Steepest Descent 방법(최속 강하법)

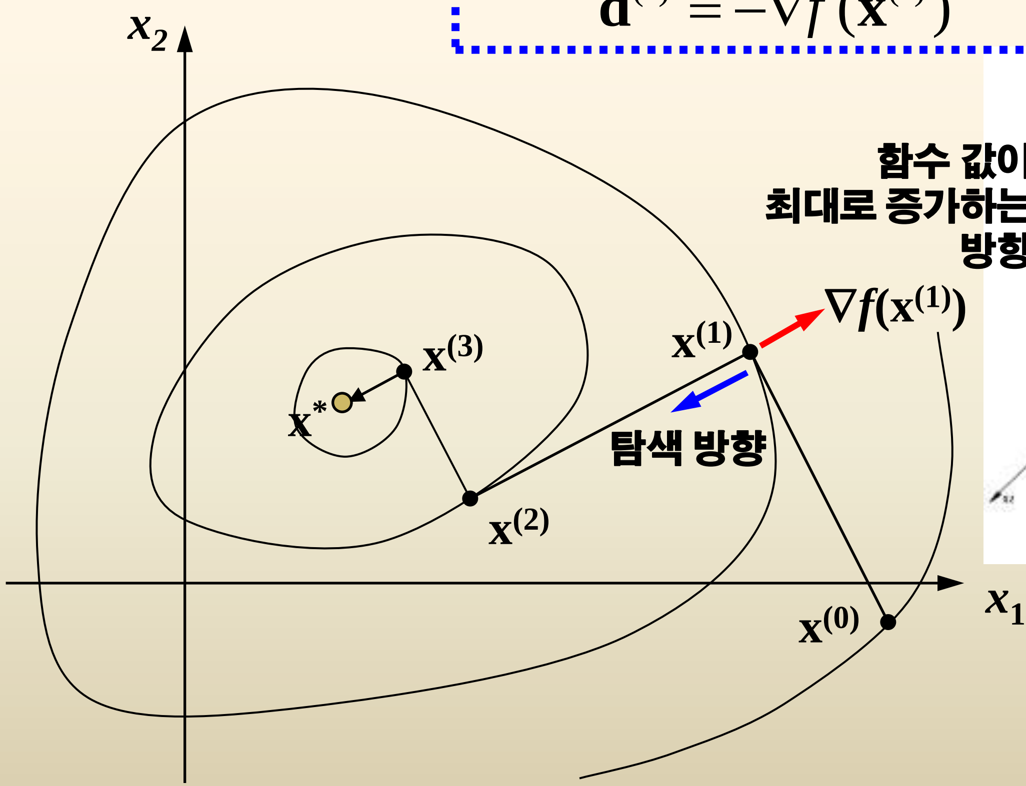
☑ 탐색 방향(Search Direction)을 목적 함수의 Gradient Vector의 반대 방향으로 가정하고 순차적으로 최적해를 찾는 방법

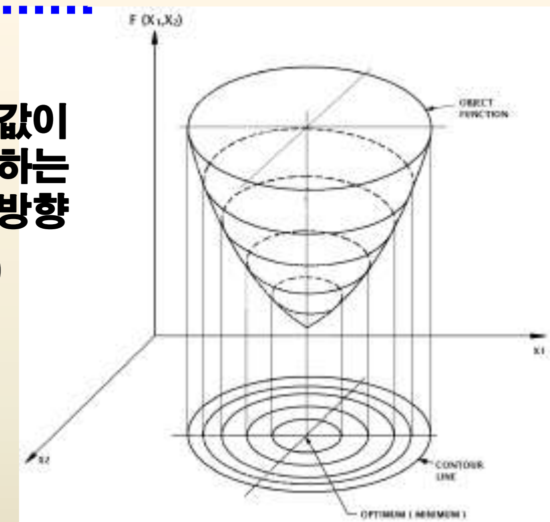
➡ Gradient Vector($\nabla f(\mathrm{x})$): 함수 값이 최대로 증가하는 방향

<u>목적 함수를 최소화 하는 문제일 경우</u>

탐색 방향

$$\mathbf{d}^{(1)} = -\nabla f(\mathbf{x}^{(1)})$$





비제약 최적화 문제- Steepest descent 방법을 이용한 해법(1)

$$\textit{Minimize} \quad f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$$

$$\nabla f(\mathbf{x}) = \nabla f(x_1, x_2) = \begin{pmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{pmatrix}$$

☑ 단계 1 - $\mathbf{x}^{(1)}$구하기

$$\nabla f(\mathbf{x}^{(0)}) = \nabla f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - \alpha^{(0)} \nabla f(\mathbf{x}^{(0)})$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix}$$

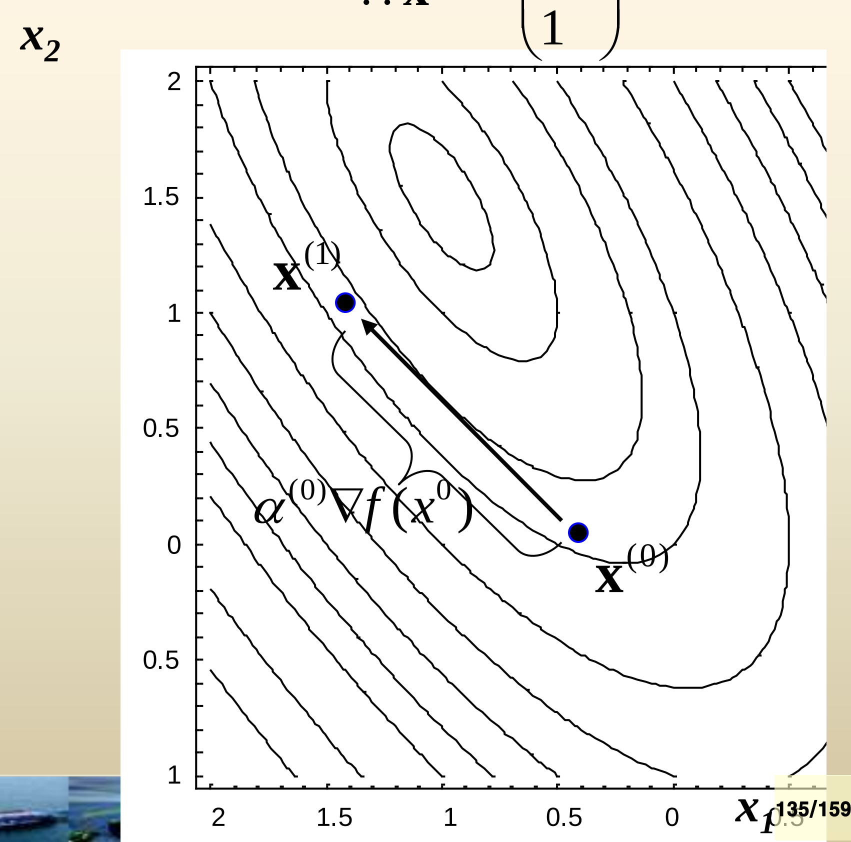
편의상 $\alpha^{(0)}$을 α로 대체함

$$f(\mathbf{x}^{(1)}) = \alpha^2 - 2\alpha$$

함수 $f(\mathbf{x}^{(1)})$가 최소값을 가질 조건

$$\frac{df(\mathbf{x}^{(1)})}{d\alpha} = 0 \text{으로부터 } \alpha = 1.0$$

$$\therefore \mathbf{x}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



비제약 최적화 문제- Steepest descent 방법을 이용한 해법(2)

☑ **단계 2 -**

$\mathbf{x}^{(2)}$구하기

$$\nabla f(\mathbf{x}^{(1)}) = \nabla f\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1+4x_1+2x_2\\-1+2x_1+2x_2\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$$

$$\mathbf{x}^{(2)} = \mathbf{x}^{(1)} - \alpha^{(1)}\nabla f(\mathbf{x}^{(1)})$$

$$= \begin{pmatrix}-1\\1\end{pmatrix} - \alpha\begin{pmatrix}-1\\-1\end{pmatrix} = \begin{pmatrix}-1+\alpha\\1+\alpha\end{pmatrix}$$

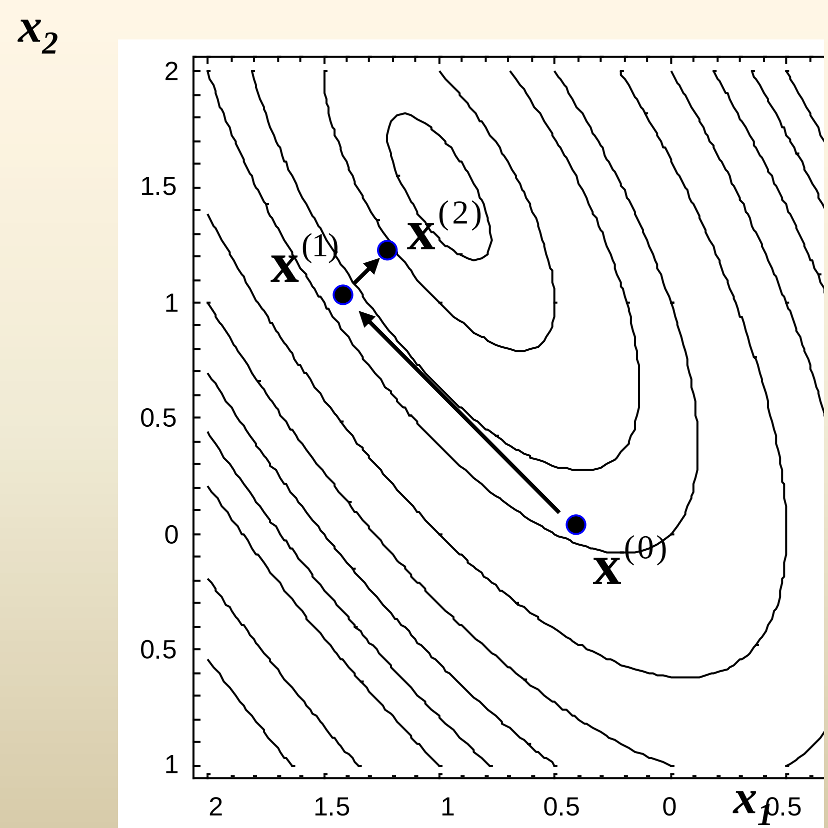
편의상 $\alpha^{(1)}$을 α로 대체함

$$f(\mathbf{x}^{(2)}) = 5\alpha^2 - 2\alpha - 1$$

함수 $f(\mathbf{x}^{(2)})$가 최소값을 가질 조건

$\dfrac{df(\mathbf{x}^{(2)})}{d\alpha} = 0$으로부터 $\alpha = 0.2$

$$\therefore \mathbf{x}^{(2)} = \begin{pmatrix}-0.8\\1.2\end{pmatrix}$$



비제약 최적화 문제- Steepest descent 방법을 이용한 해법(3)

☑ **단계 3 -**

$\mathbf{x}^{(3)}$구하기

$$\nabla f(\mathbf{x}^{(1)}) = \nabla f\begin{pmatrix} -0.8 \\ 1.2 \end{pmatrix} = \begin{pmatrix} 1+4x_1+2x_2 \\ -1+2x_1+2x_2 \end{pmatrix} = \begin{pmatrix} 0.2 \\ -0.2 \end{pmatrix}$$

$$\mathbf{x}^{(3)} = \mathbf{x}^{(2)} - \alpha^{(2)}\nabla f(\mathbf{x}^{(2)})$$

$$= \begin{pmatrix} -0.8 \\ 1.2 \end{pmatrix} - \alpha\begin{pmatrix} 0.2 \\ -0.2 \end{pmatrix} = \begin{pmatrix} -0.8-0.2\alpha \\ 1.2+0.2\alpha \end{pmatrix}$$

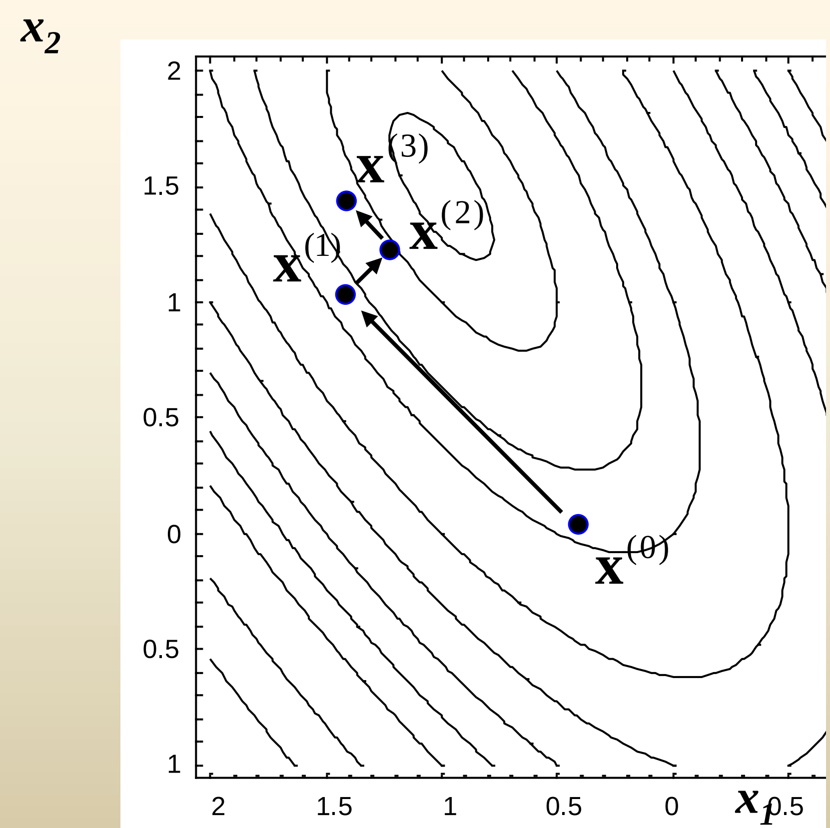
편의상 $\alpha^{(2)}$을 α로 대체함

$$f(\mathbf{x}^{(3)}) = 0.04\alpha^2 - 0.08\alpha - 1.2$$

함수 $f(\mathbf{x}^{(3)})$가 최소값을 가질 조건

$$\frac{df(\mathbf{x}^{(3)})}{d\alpha} = 0 \text{으로부터 } \alpha = 1.0$$

$$\therefore \mathbf{x}^{(3)} = \begin{pmatrix} -1 \\ 1.4 \end{pmatrix}$$



비제약 최적화 문제- Steepest descent 방법을 이용한 해법(4)

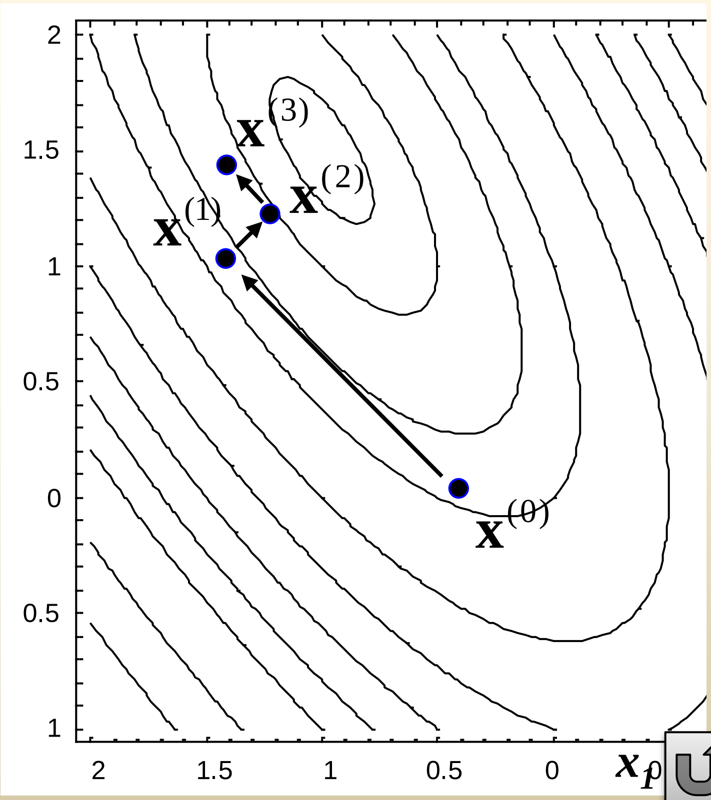
☑ **단계 4 -**

최적해 구하기

이와 같은 과정을 반복하여 $\left|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\right| \le \varepsilon$일 경우

중지하며 그때의 $\mathbf{x}^{(k+1)}$이 최적해가 된다.

x_2



x_1

Reference slides

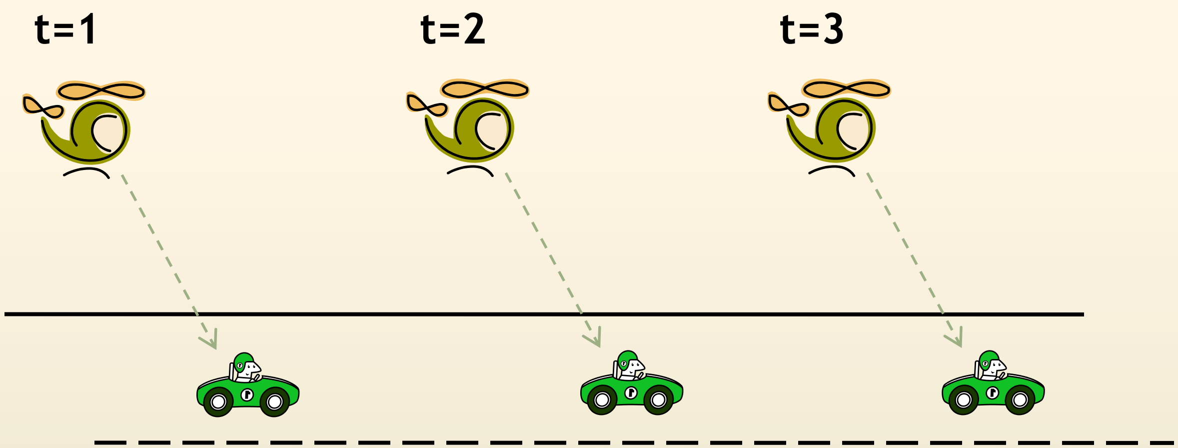
Lagrangian & Eulerian Description[1)]

1) 김찬중, 길잡이 유체공학입문, 문운당, 2002, p76~77, 유체의 운동 기술 방법

Lagrangian & Eulerian Description

☑ **Lagrangian Description**

: 하나의 유체입자의 운동을 표현하는데 어떠한 위치에서의 속도, 가속도를 **시간(t)의 함수로만 표현하는 방법**. 관측자가 계속 이동하며 그 입자를 따라가며 운동을 기술함



시간 t에서 차량의 속도

$$\mathbf{V} = \mathbf{V}(t)$$

시간 t에서 차량의 가속도

$$\mathbf{a} = \frac{d\mathbf{V}}{dt}$$

(Q) 유체의 운동을 Lagrangian Description으로 표현한다면??

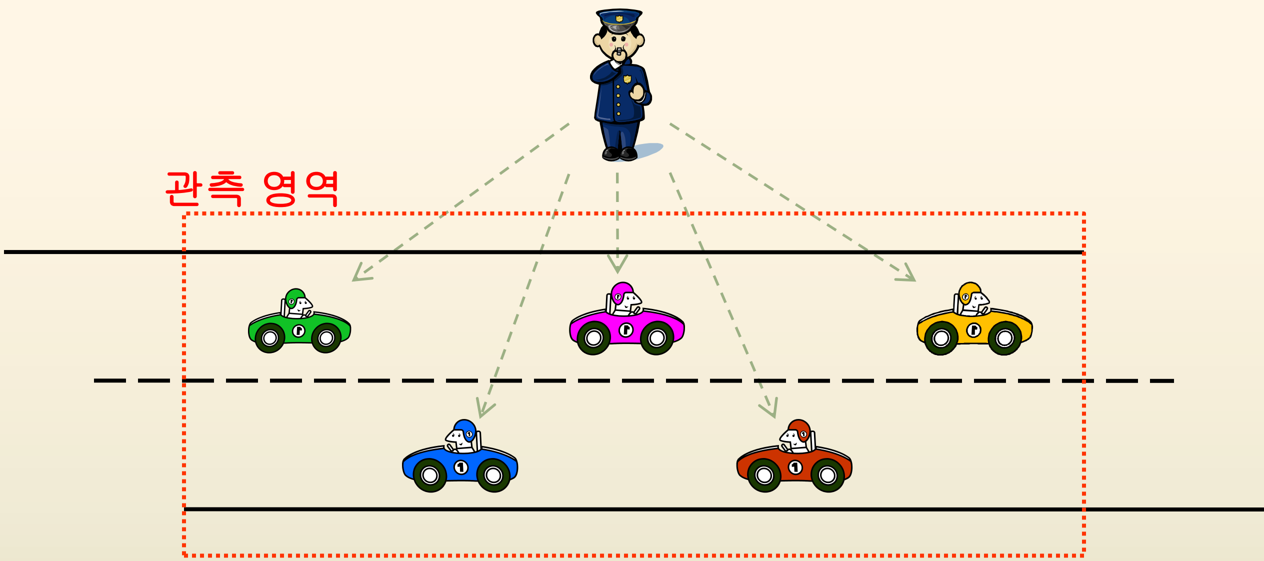
Lagrangian Desciption에 따르면, 수백만개의 유체 입자를 하나하나 관측해야 함

따라서, Lagrangian Desciption을 주로 강체의 운동을 기술할 때 적합함

Lagrangian & Eulerian Description

☑ **Eulerian Description**

: 유체가 출입하는 검사체적 (Control Volume)을 정의하고, 어떠한 위치에서의 속도, 가속도를 위치(x,y,z)와 시간(t)의 함수로 표현하는 방법. 관측자(관측기)가 한자리에서 특정 영역을 통과하는 유체 입자의 운동을 기술함



(Q) 유체의 운동을 **Eulerian Description**으로 표현한다면??

입자 하나하나가 아니라, 관측 영역(검사 체적) 내부의 유체 입자의 운동을 기술하는데 적합하다.

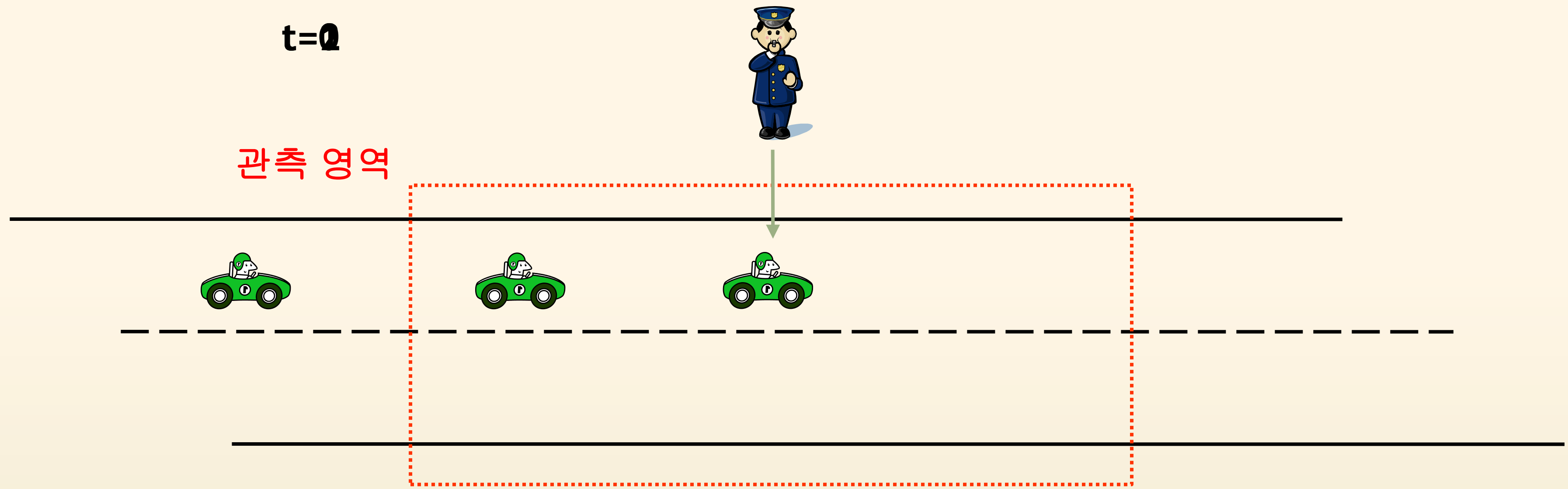
✓ 시간 **t**에서 **(x,y,z)**위치에 있는 차량의 속도

$$\mathbf{V} = \mathbf{V}(x, y, z, t)$$

✓ 시간 **t**에서 **(x,y,z)**위치에 있는 차량의 가속도

$$\mathbf{a} = \frac{d\mathbf{V}(x, y, z, t)}{dt}$$

Lagrangian & Eulerian Description



✓ 가속도 각 성분의 의미

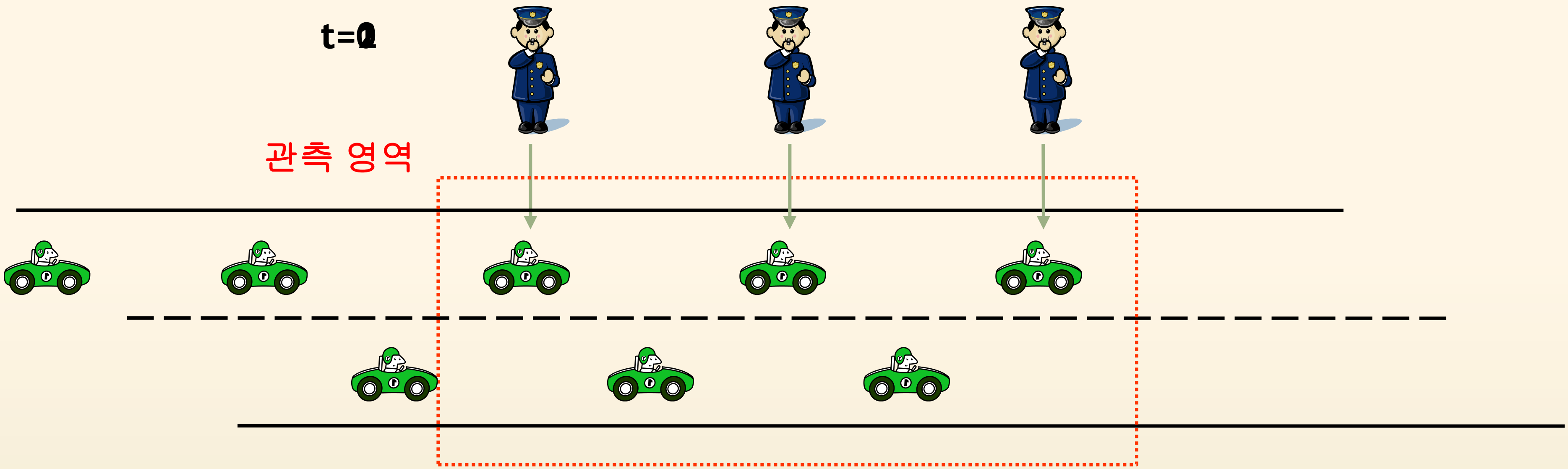
$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

위치 고정 $\left(\frac{\partial \mathbf{V}}{\partial x} = \frac{\partial \mathbf{V}}{\partial y} = \frac{\partial \mathbf{V}}{\partial z} = 0 \right)$

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t}$$

검사 체적 내부의 한 지점에서
시간의 흐름에 따라 변하는 가속도
(국소 가속도 : Local acceleration)

Lagrangian & Eulerian Description



✓ 가속도 각 성분의 의미

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

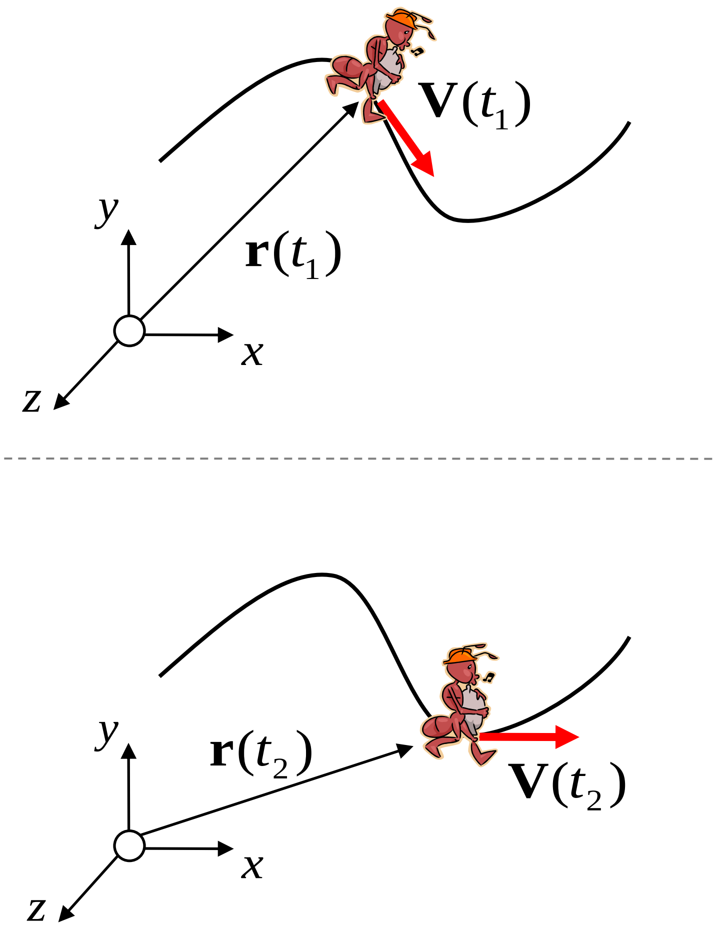
시간 고정 $\left(\frac{\partial \mathbf{V}}{\partial t} = 0\right)$ →

$$\mathbf{a} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

시간이 고정일 때, 공간 상의
위치 변화에 따른 가속도
(대류 가속도 : **Convective acceleration**)

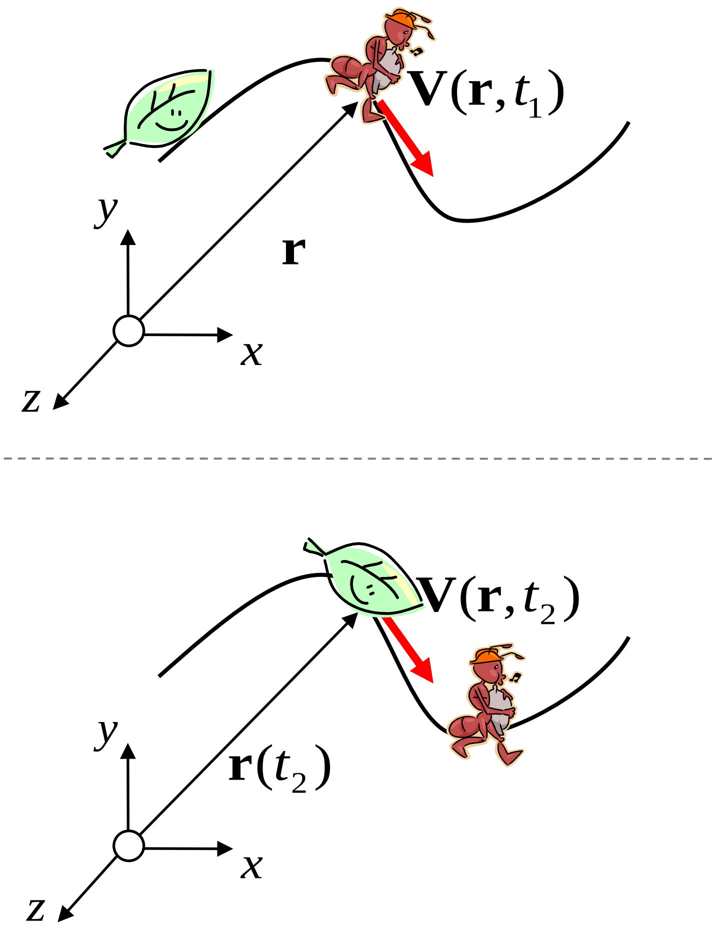
Lagrangian & Eulerian Description

✓ Lagrangian Description



“동일한 입자가 서로 다른 위치”로 시간에 따라 운동하는 것을 관찰

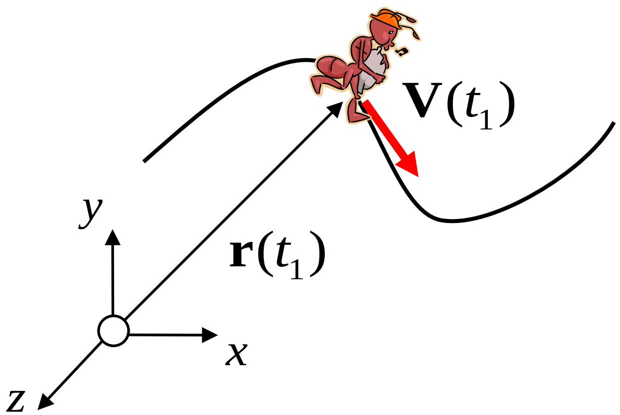
✓ Eulerian Description

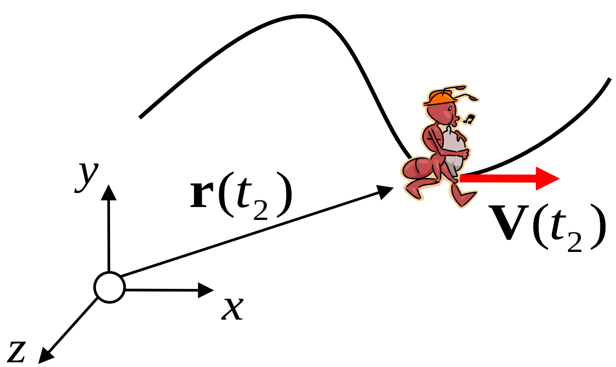


“동일한 위치에서 서로 다른 입자”가 시간에 따라 운동하는 것을 관찰

Lagrangian Description

✓ Lagrangian Description





"동일한 입자가 서로 다른 위치"로
시간에 따라 운동하는 것을 관찰

✓ 시간 t에서 개미의 속도

$$\mathbf{V} = \mathbf{V}(t)$$

✓ 시간 t에서 개미의 가속도

$$\mathbf{a} = \frac{d\mathbf{V}(t)}{dt}$$

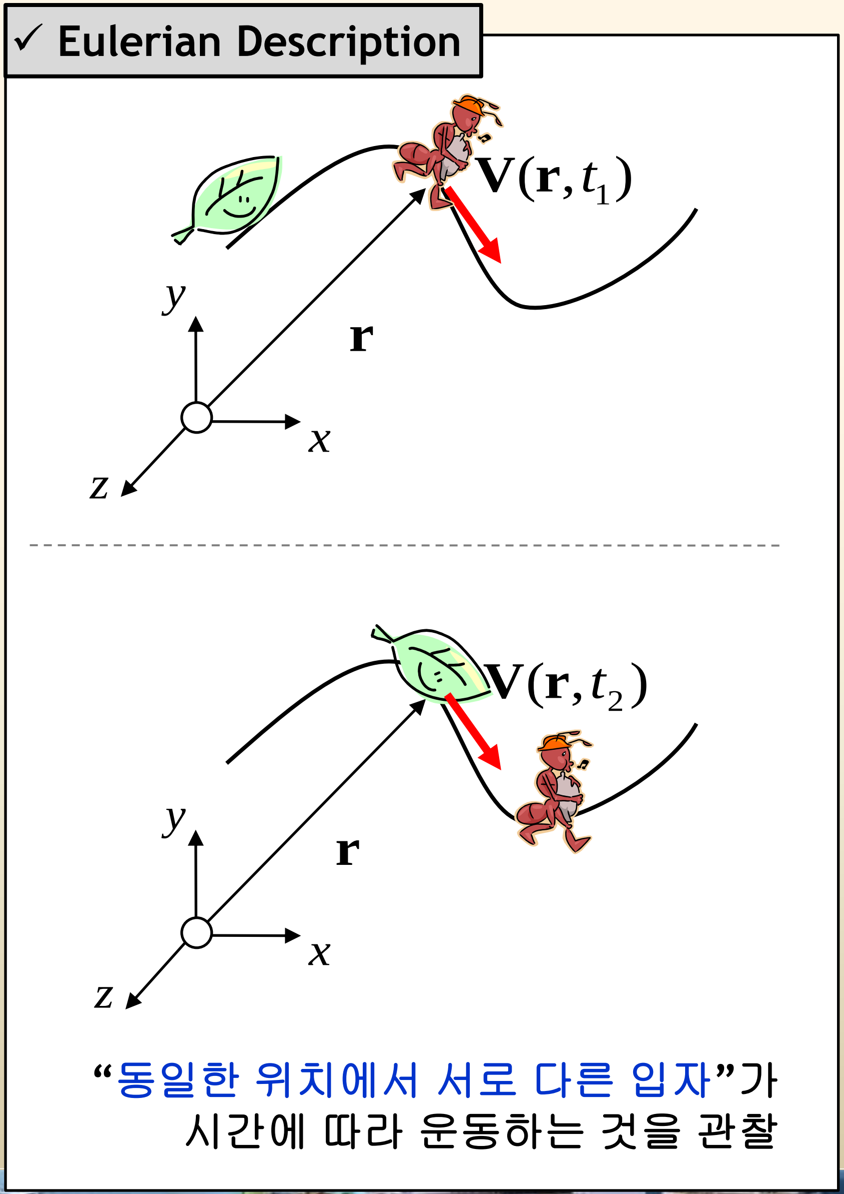
Eulerian Description

✓ 시간 t에서 (*x,y,z*)위치에 있는 물체의 속도

$$\mathbf{V} = \mathbf{V}(x, y, z, t)$$

✓ 시간 t에서 (x,y,z)위치에 있는 물체의 가속도

$$\mathbf{a} = \frac{d\mathbf{V}(x, y, z, t)}{dt}$$



Eulerian Description

✓ 시간 t에서 **(*x,y,z*)**위치에 있는 물체의 가속도

$$d\mathbf{V}(x,y,z,t)=\frac{\partial \mathbf{V}}{\partial t}dt+\frac{\partial \mathbf{V}}{\partial x}dx+\frac{\partial \mathbf{V}}{\partial y}dy+\frac{\partial \mathbf{V}}{\partial z}dz$$

$$\mathbf{a}=\frac{d\mathbf{V}(x,y,z,t)}{dt}=\frac{\partial \mathbf{V}}{\partial t}\frac{dt}{dt}+\frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt}+\frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt}+\frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt}\quad\left(\frac{dt}{dt}=1,\frac{dx}{dt}=u,\frac{dy}{dt}=v,\frac{dz}{dt}=w\right)$$

$$=\frac{\partial \mathbf{V}}{\partial t}+u\frac{\partial \mathbf{V}}{\partial x}+v\frac{\partial \mathbf{V}}{\partial y}+w\frac{\partial \mathbf{V}}{\partial z}$$

Eulerian Description

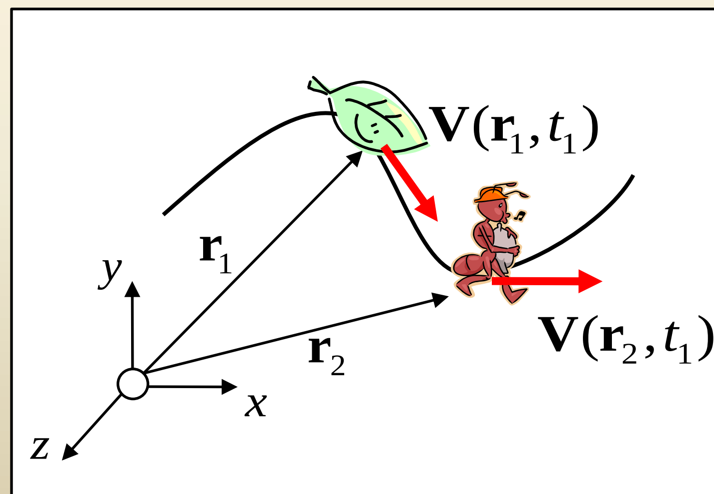
✓ 가속도 각 성분의 의미

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

시간 고정 $\left(\frac{\partial \mathbf{V}}{\partial t} = 0\right)$ →

$$\mathbf{a} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

시간이 고정일 때, 공간 상의
위치 변화에 따른 가속도
(대류 가속도 : Convective acceleration)



Eulerian Description

✓ 시간 t에서 **(*x,y,z*)**위치에 있는 물체의 가속도 **(Eulerian Description으로 기술할 때의 가속도)**

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

Nabla(▽)를 사용하여 정리하면,

$$= \frac{\partial \mathbf{V}}{\partial t} + \left(u\hat{i} + v\hat{j} + w\hat{k}\right) \bullet \left(\frac{\partial \mathbf{V}}{\partial x}\hat{i} + \frac{\partial \mathbf{V}}{\partial y}\hat{j} + \frac{\partial \mathbf{V}}{\partial z}\hat{k}\right)$$

$$= \frac{\partial \mathbf{V}}{\partial t} + \left(u\hat{i} + v\hat{j} + w\hat{k}\right) \bullet \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\mathbf{V}$$

$$= \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}$$

성분으로 분해하여 표현

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Q) 선형 (Linear)? 비선형(Nonlinear)?

A) 비선형(Nonlinear)
미지수**(*u,v,w*)**의 곱으로 표현된 항이 존재.

Reference slides

Froude-Krylov Force & Diffraction Force

Froude-Krylov Force & Diffraction Force (2)

Froude-Krylov Force & Diffraction Force

(Continue)

$$F_{FK,k}+F_{D,k}=-\rho\iint_{S_B}(\phi_I+\phi_D)e^{i\omega t}i\omega n_k dS$$

$$\frac{\partial\phi_k}{\partial n}=i\omega n_k$$

$$=-\rho\iint_{S_B}(\phi_I+\phi_D)e^{i\omega t}\frac{\partial\phi_k}{\partial n}dS$$

$$=-\rho e^{i\omega t}\iint_{S_B}\left(\phi_I\frac{\partial\phi_k}{\partial n}+\phi_D\frac{\partial\phi_k}{\partial n}\right)dS$$

Green's 2nd formular 사용

$$\left(\iint_{S_B}\phi_D\frac{\partial\phi_k}{\partial n}dA=\iint_{S_B}\phi_k\frac{\partial\phi_D}{\partial n}dA\right)$$

$$=-\rho e^{i\omega t}\iint_{S_B}\left(\phi_I\frac{\partial\phi_k}{\partial n}+\phi_k\frac{\partial\phi_D}{\partial n}\right)dS$$

Body B.C. 적용

$$\frac{\partial\phi_D}{\partial n}=-\frac{\partial\phi_I}{\partial n}\quad(on\ \ S_B)$$

$$=-\rho e^{i\omega t}\iint_{S_B}\left(\phi_I\frac{\partial\phi_k}{\partial n}-\phi_k\frac{\partial\phi_I}{\partial n}\right)dS$$

Green's 2nd formular와 Body B.C을 이용하여 Diffraction Potential을 직접 구하지 않음, Incidence Potential로 치환

Reference slides

Radiation Velocity Potential

Radiation Velocity Potential (10)

✓ Singularity Distribution Method[2] (2-D)

Laplace Equation을 만족함

물체 표면에 특이점 (source, doublet, vortex)을 분포시켜 수학적으로 **물체 경계면**을 생성시키고, 이들 특이점들의 **강도(Strength)**를 구하여 **전체 유장의 velocity potential**을 구하는 방법

※ Laplace equation on polar coordinate

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)}_{=0}+\underbrace{\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}}_{=0}=0$$

Let 2-D source $\phi = \ln r$

$$\frac{\partial \phi}{\partial r}=\frac{1}{r},\quad r\frac{\partial \phi}{\partial r}=1,\quad \frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)=0,\quad \frac{\partial^2 \phi}{\partial \theta^2}=0$$

Given : 특이점 (source, doublet, vortex)

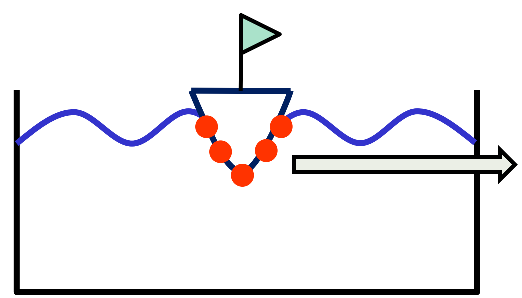
Find : 특이점의 강도(Strength)

Find (→ q_m) Given (→ ϕ_m)

$$\phi = \sum_{m=1}^{N} q_m \phi_m$$

N개의 미지수가 존재함

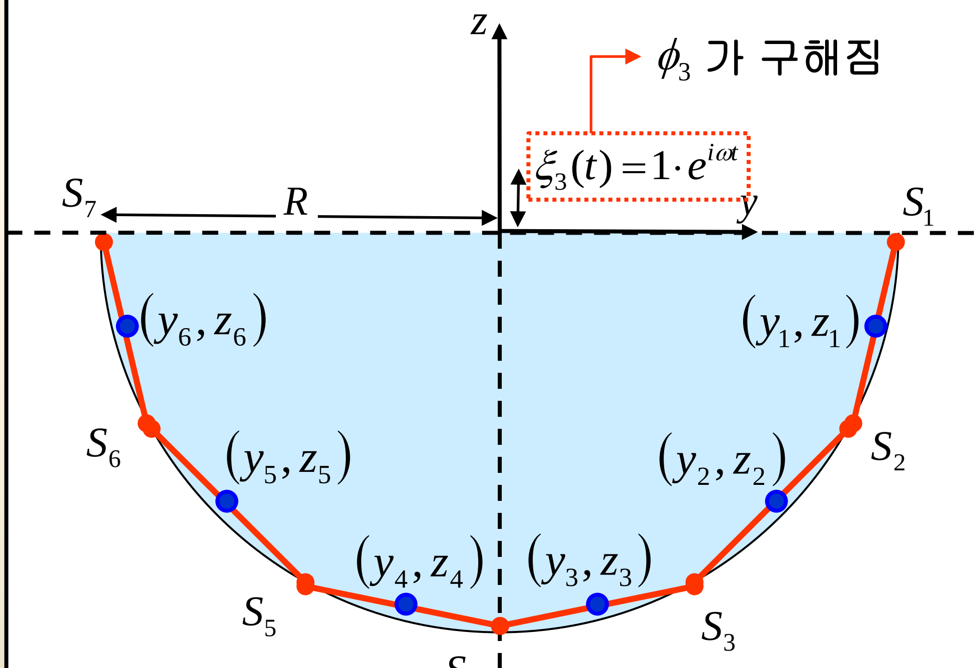
➔ N개의 경계조건으로 부터 방정식 구함



$$\frac{\partial \phi_m}{\partial n} = V_n$$

Radiation Velocity Potential (11)

ex) 반원이 $\xi_3(t) = 1 \cdot e^{i\omega t}$ 로 운동 중일 때, Velocity potential을 구하시오.



(y_m, z_m): $\overline{S_m S_{m+1}}$ 의 중점

Step 1. 반원을 6등분 한다.$(S_1, \cdots, S_6)$

Step 2. 등분된 점과 점 사이를 선으로 연결하고, 각 Line segment에 line source를 분포시킨다. 한 Line에 분포된 source는 같은 강도(strength)를 가짐

- 점 $(\eta(s), \zeta(s))$ 에 위치한 크기 q_j 인 source

➔ $q_m \ln\sqrt{(y-\eta(s))^2 + (z-\zeta(s))^2}$

- Line($\overline{S_m S_{m+1}}$)에 분포된 Line source

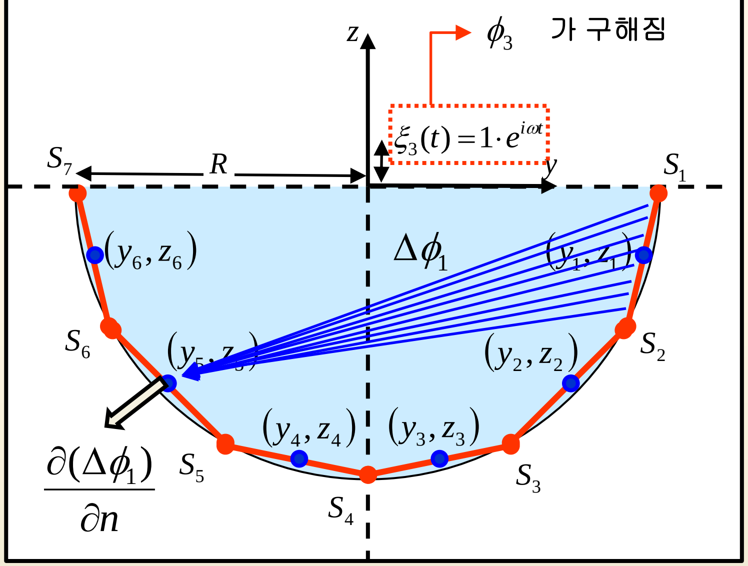
➔ $\Delta\phi_m = q_m \int_{\overline{S_m S_{m+1}}} \ln\sqrt{(y-\eta(s))^2 + (z-\zeta(s))^2}\, ds$

Step 3. Source를 다음과 같이 각 Line Source들의 합으로 가정함

$$\phi_3(y,z) = \sum_{m=1}^{6} \Delta\phi_m = \sum_{m=1}^{6} q_m \int_{\overline{S_m S_{m+1}}} \ln\sqrt{(y-\eta(s))^2 + (z-\zeta(s))^2}\, ds$$

Radiation Velocity Potential (11)

ex) 반원이 $\xi_3(t)=1\cdot e^{i\omega t}$ 로 운동 중일 때,
Velocity potential을 구하시오.



Step 4. 물체 경계 조건(Body boundary condition)

$$\frac{\partial \phi_3}{\partial n} = i\omega n_3$$

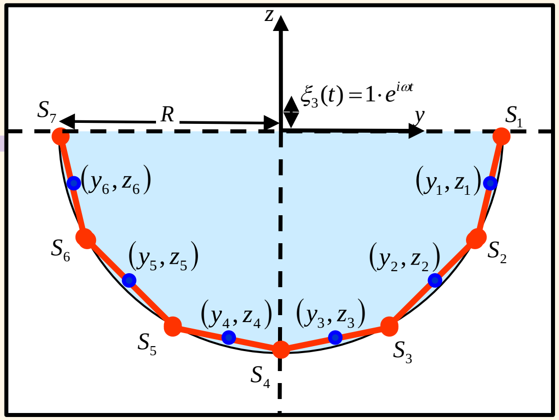
➔ **LHS:**

$$\left.\frac{\partial \phi_3}{\partial n}\right|_{(y_m, z_m)} = q_1 \frac{\partial}{\partial n}\left[\int_{S_1S_2} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_m,z_m)}$$
$$+ q_2 \frac{\partial}{\partial n}\left[\int_{S_2S_3} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_m,z_m)}$$
$$+\cdots$$
$$+ q_6 \frac{\partial}{\partial n}\left[\int_{S_6S_7} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_m,z_m)}$$

➔ **RHS :** $i\omega n_3 = -i\omega \cos\theta \big|_{(y_m, z_m)}$

$$\phi_3(y,z) = \sum_{m=1}^{6} \Delta\phi_m = \sum_{m=1}^{6} q_m \int_{S_mS_{m+1}} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds$$

Radiation Velocity Potential (13)



↓

물체 경계 조건(Body boundary condition)

$$q_1 \frac{\partial}{\partial n}\left[\int_{\overline{S_1S_2}} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_1,z_1)} + \cdots + q_6 \frac{\partial}{\partial n}\left[\int_{\overline{S_6S_7}} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_1,z_1)} = -i\omega\cos\theta_{(y_1,z_1)}$$

$$q_1 \frac{\partial}{\partial n}\left[\int_{\overline{S_1S_2}} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_2,z_2)} + \cdots + q_6 \frac{\partial}{\partial n}\left[\int_{\overline{S_6S_7}} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_2,z_2)} = -i\omega\cos\theta_{(y_2,z_2)}$$

$$\vdots$$

$$q_1 \frac{\partial}{\partial n}\left[\int_{\overline{S_1S_2}} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_6,z_6)} + \cdots + q_6 \frac{\partial}{\partial n}\left[\int_{\overline{S_6S_7}} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_6,z_6)} = -i\omega\cos\theta_{(y_6,z_6)}$$

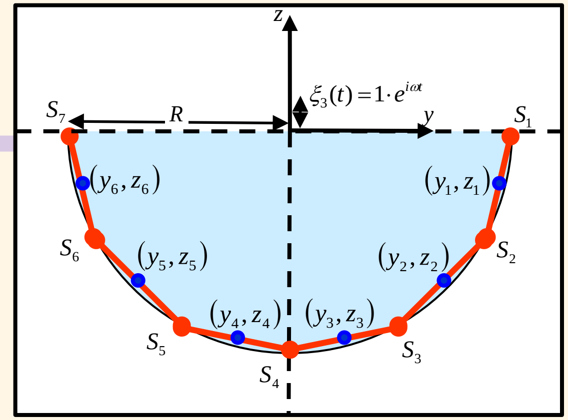
방정식 : 6개

⇕

미지수 : 6개 $q_1, \cdots, q_6$ ⇨

Now we can find the solution !!!

Radiation Velocity Potential (14)



⇩

Step 5. 방정식을 Matrix 형태로 나타내면,

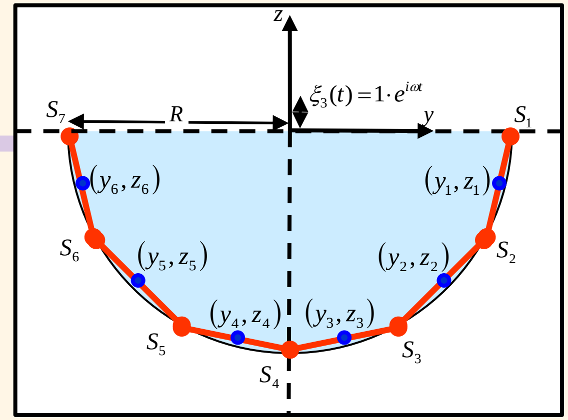
$$\underbrace{q_1 \frac{\partial}{\partial n}\left[\int_{\overline{S_1S_2}} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_1,z_1)}}_{I_{11}} + \cdots + \underbrace{q_6 \frac{\partial}{\partial n}\left[\int_{\overline{S_6S_7}} \ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_1,z_1)}}_{I_{16}} = -i\omega\cos\theta_{(y_1,z_1)}$$

$$\Rightarrow \left.\begin{aligned} q_1 I_{11} + \cdots + q_6 I_{16} &= b_1 \\ q_1 I_{21} + \cdots + q_6 I_{26} &= b_2 \\ &\vdots \\ q_1 I_{61} + \cdots + q_6 I_{66} &= b_6 \end{aligned}\right) \Rightarrow \mathbf{Aq} = \mathbf{b} \quad \left(\mathbf{A} = \begin{bmatrix} I_{11} & \cdots & I_{16} \\ \vdots & \ddots & \vdots \\ I_{61} & \cdots & I_{66} \end{bmatrix}, \mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_6 \end{bmatrix} \right)$$

⇩

$$\mathbf{q} = \mathbf{A}^{-1}\mathbf{b}$$

Radiation Velocity Potential (15)



⇩

(참고) I_{jk} **의 계산**

$f(y,z)=\ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}$ 라 하면,

$$\frac{\partial f(y,z)}{\partial n}=\nabla f \bullet \mathbf{n}=\left(\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)\bullet(n_2,n_3) \qquad \left(\begin{aligned}\frac{\partial f}{\partial y}&=\frac{y-\eta(s)}{(y-\eta(s))^2+(z-\zeta(s))^2}\\ \frac{\partial f}{\partial z}&=\frac{z-\zeta(s)}{(y-\eta(s))^2+(z-\zeta(s))^2}\end{aligned}\right)$$

⇩

$$I_{jk}=\frac{\partial}{\partial n}\left[\int_{\overline{S_kS_{k+1}}}\ln\sqrt{(y-\eta(s))^2+(z-\zeta(s))^2}\,ds\right]_{(y_j,z_j)}$$

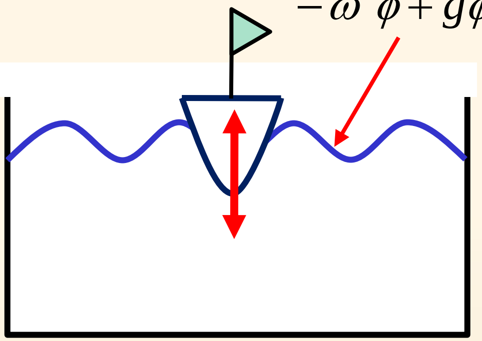
$$=\int_{\overline{S_kS_{k+1}}}\left\{n_2\frac{y_j-\eta(s)}{(y_j-\eta(s))^2+(z_j-\zeta(s))^2}+n_3\frac{z_j-\zeta(s)}{(y_j-\eta(s))^2+(z_j-\zeta(s))^2}\right\}ds$$

(발췌) 유체동역학 총정리.ppt, 2008.7.31. 함승호, p154

Radiation Wave Velocity Potential (16)

Linearized Free Surface B.C.

$$-\omega^2\phi + g\phi_z = 0 \quad (\text{on } z=0)$$



따라서, 단순한 형태의 2차원 source $(q\ln r)$ 대신
Free surface condition을 만족하는 Green function을 사용함

ex) Green function introduced by Wehausen and Laitone(1960)

$$G(z,\zeta,t) = \frac{1}{2\pi}\left\{\ln(z-\zeta) - \ln(z-\bar{\zeta}) + 2\cdot PV\int_0^{\infty}\frac{e^{-ik(z-\bar{\zeta})}}{\nu-k}dk\right\}\cos\omega t$$

$$-e^{-i\nu(z-\bar{\zeta})}\sin\omega t$$

complex notation : $z = x+iy, \zeta = \xi + i\eta$

Wave number : $\nu(=\omega^2/g)$