Naval Architecture & Ocean Engineering

[2008][06-2]

Engineering Mathematics 2

October, 2008

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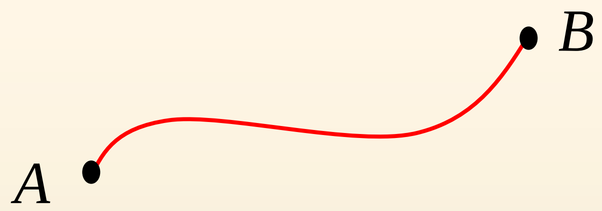
Vector Calculus (3)
: Line, Double and Triple Integrals

- Line Integrals
- Independence Path
- Double Integrals
- Surface Integrals
- Triple Integrals

Line Integrals

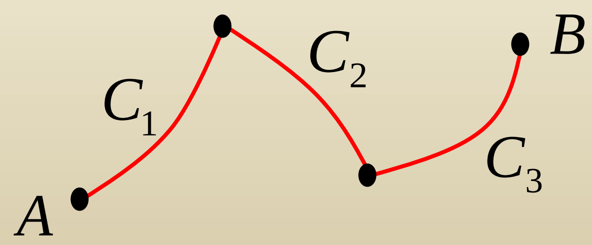
Terminology

(i) C is smooth curve if f' and g' are continuous on the closed interval [a,b] and not simultaneously zero on the open interval (a,b)



(ii) C is piecewise smooth if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end – that is,

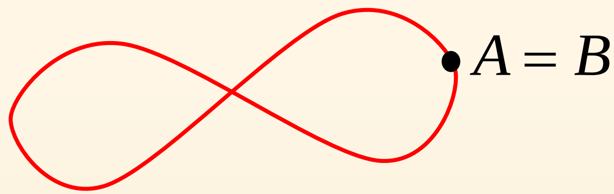
$$C = C_1 \cup C_2 \cup \ldots \cup C_n$$



Line Integrals

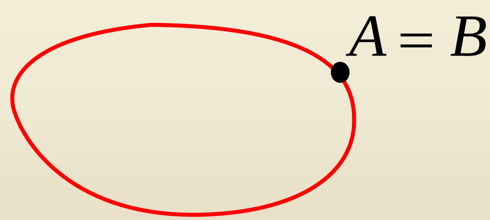
Terminology

(iii) C is closed curve if A=B.



Closed but not simple

(iv) C is simple closed curve if A=B and the curve does not cross itself.

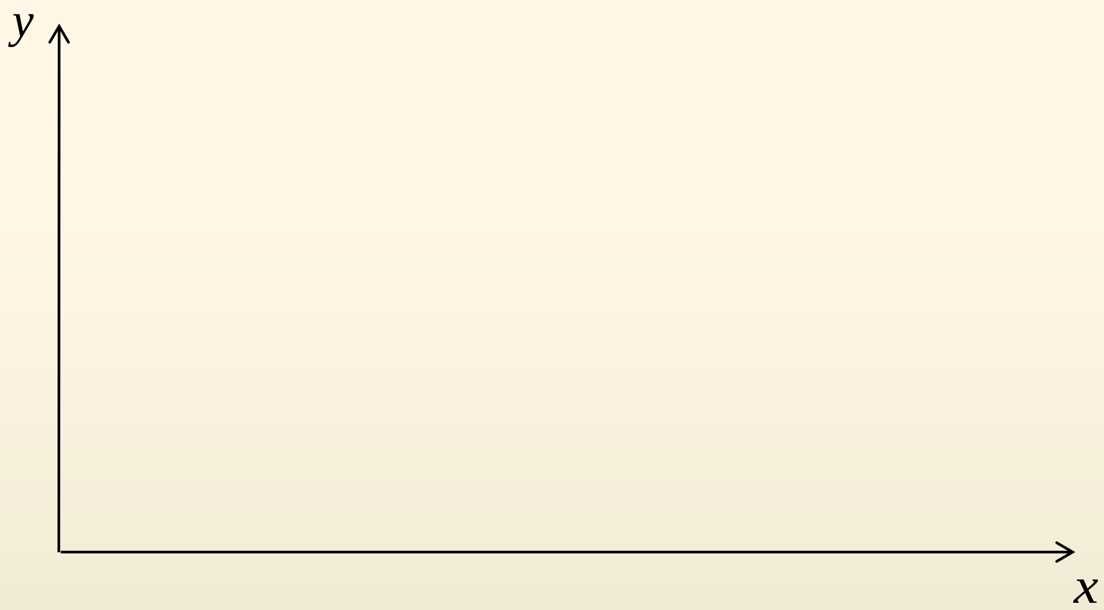


Closed and simple

(v) If C is not a closed curve, then the positive direction on C is the direction corresponding to increasing values of t.

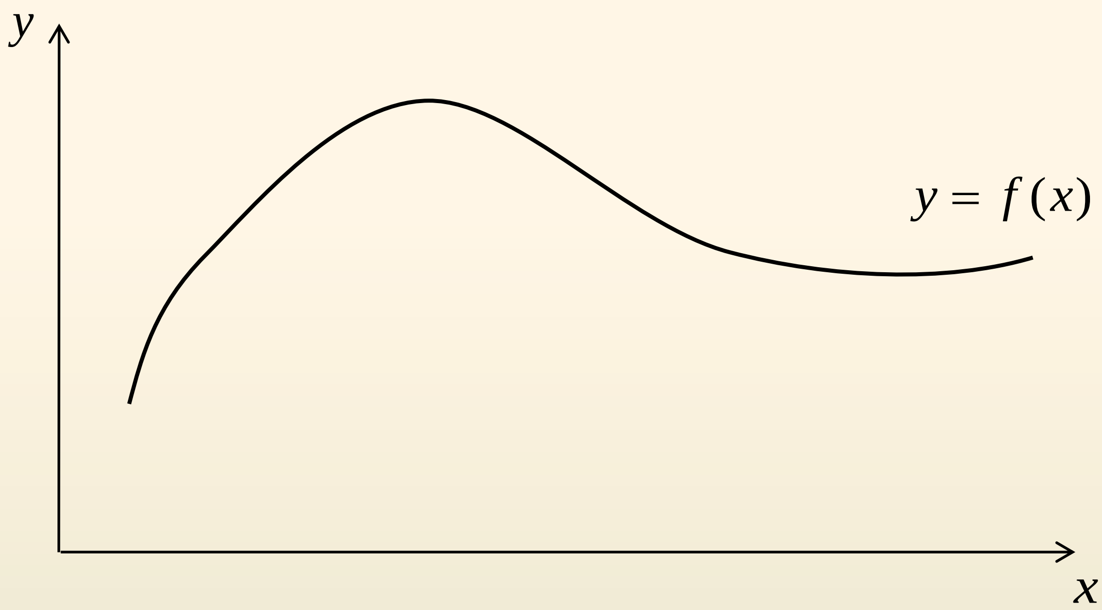
Line Integrals

Definite Integral



Line Integrals

Definite Integral



Line Integrals

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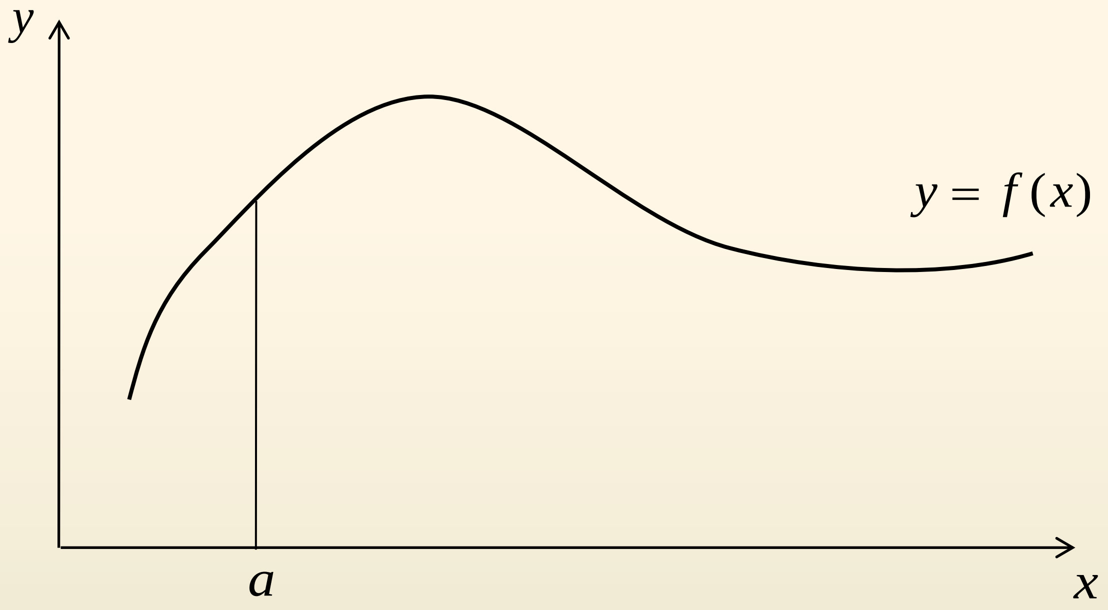

Line Integrals

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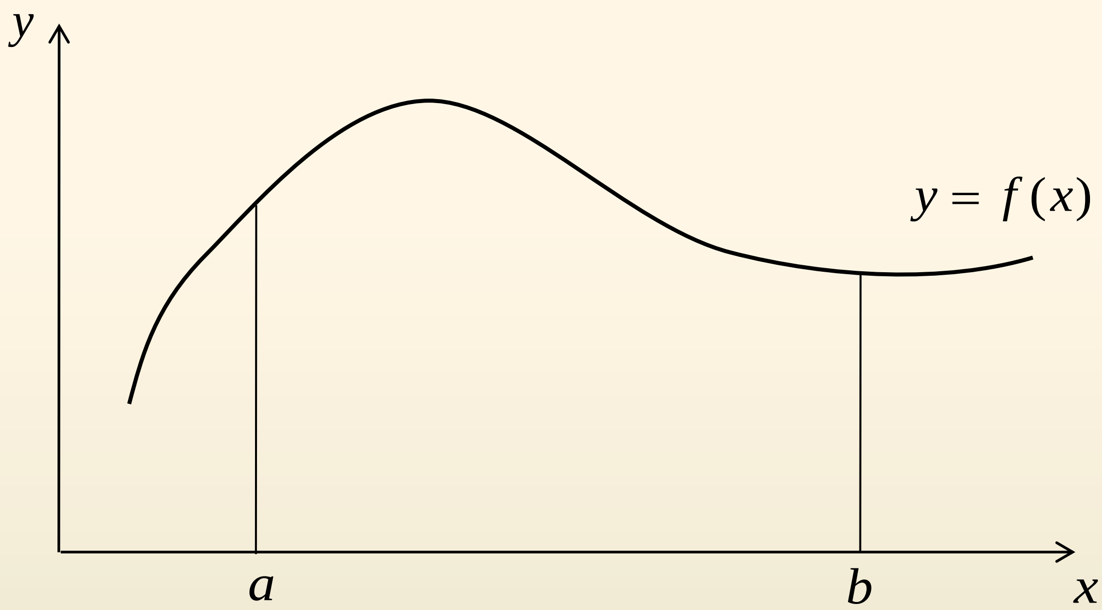

Line Integrals

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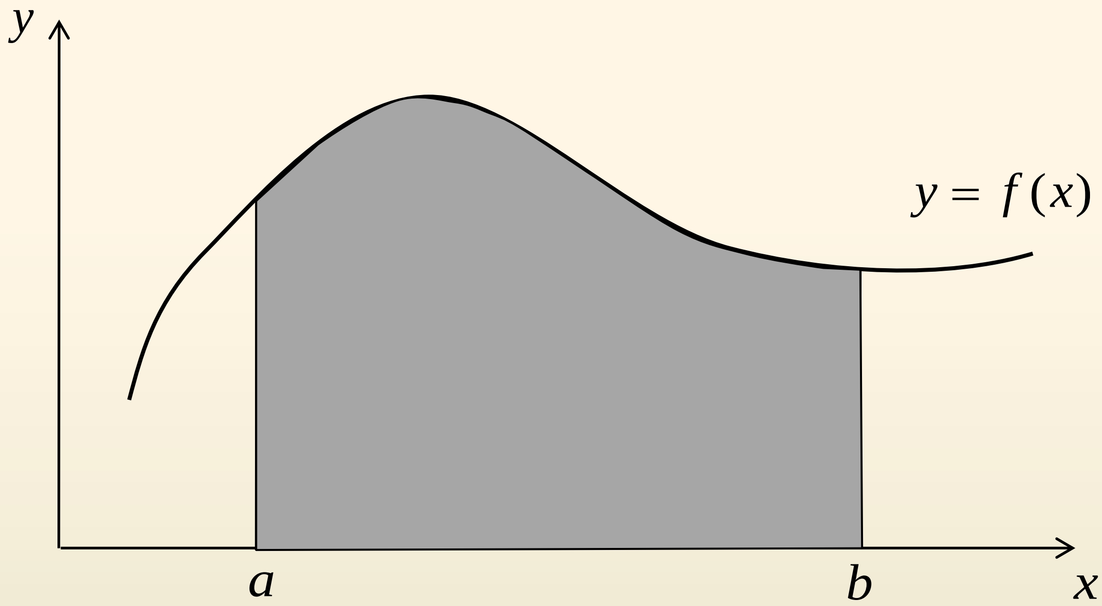

Line Integrals

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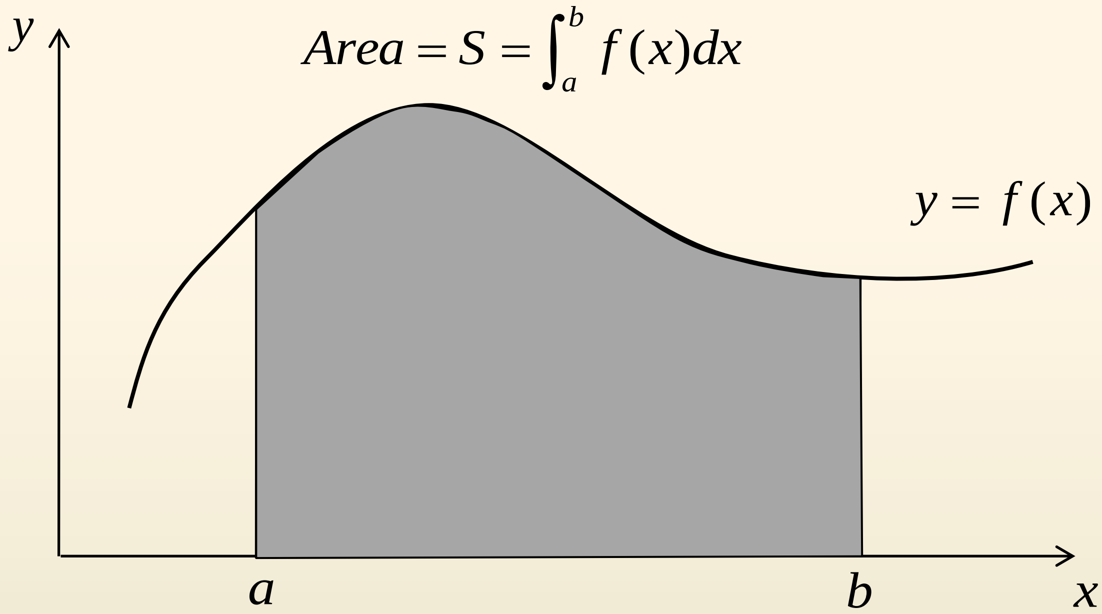
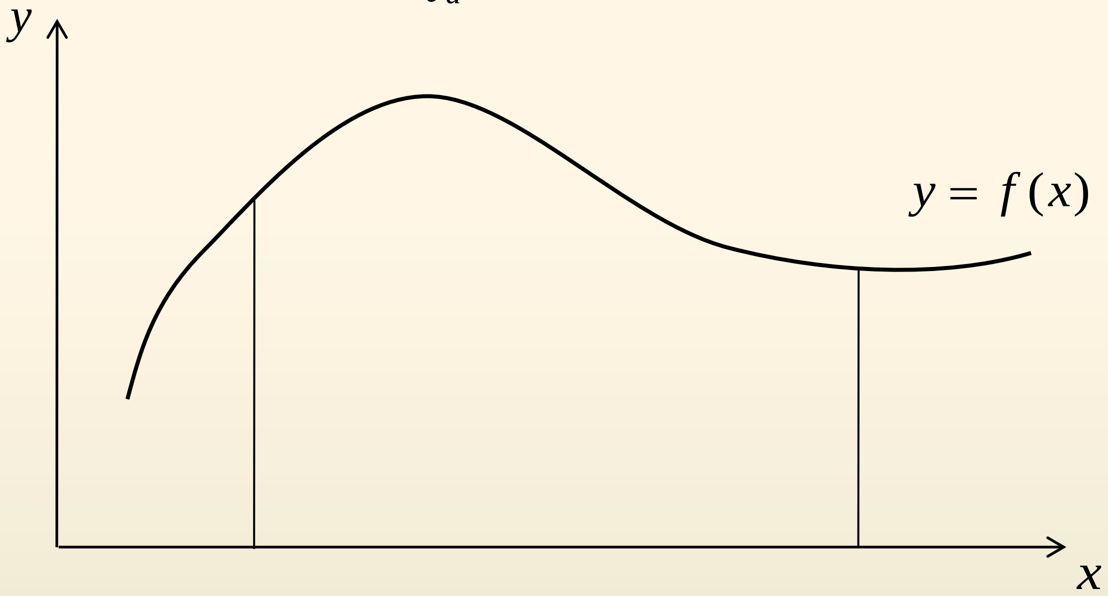


Line Integrals

Definite Integral

$$S = \int_a^b f(x)dx$$



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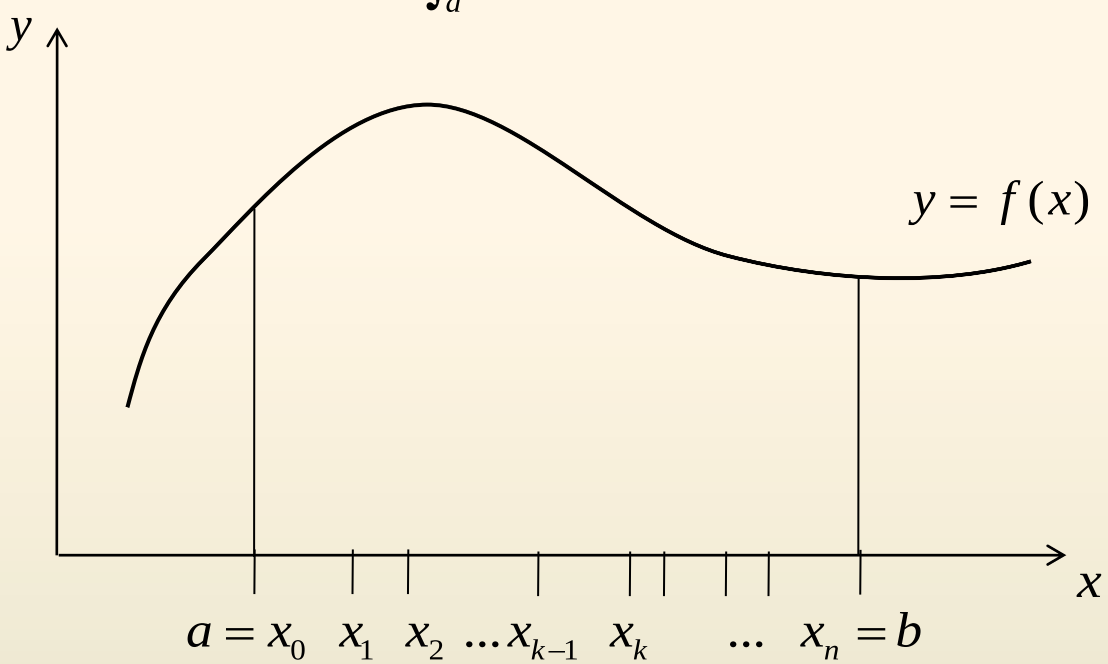

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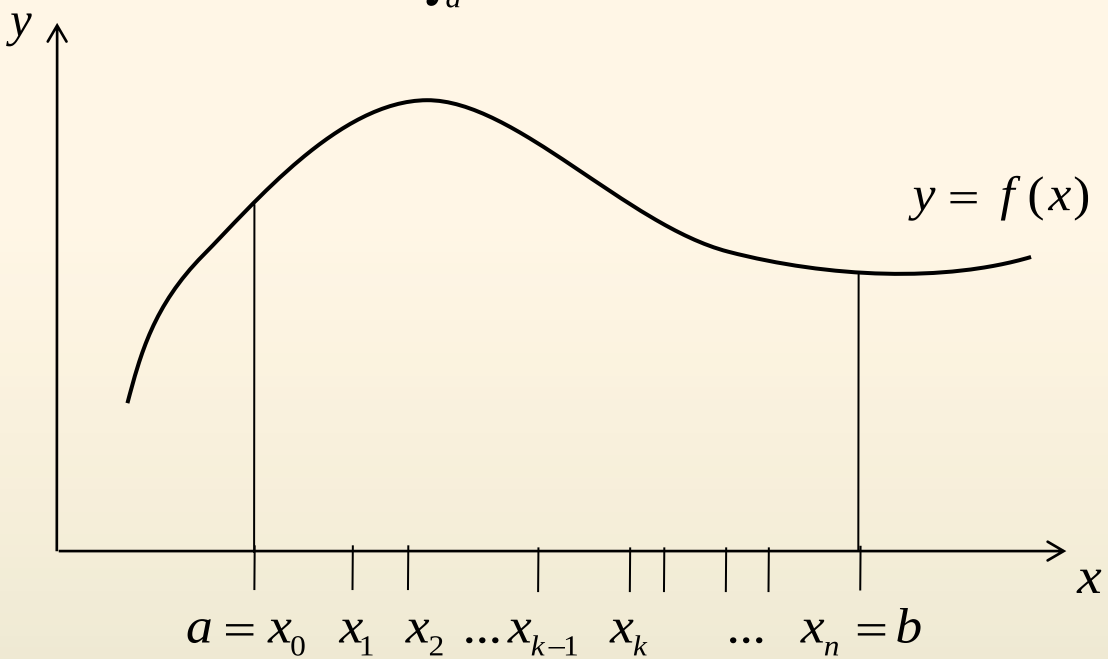

subinterval

$$x_1^* = \frac{x_0 + x_1}{2} \quad , \Delta x_1 = x_1 - x_0$$

Line Integrals

Definite Integral

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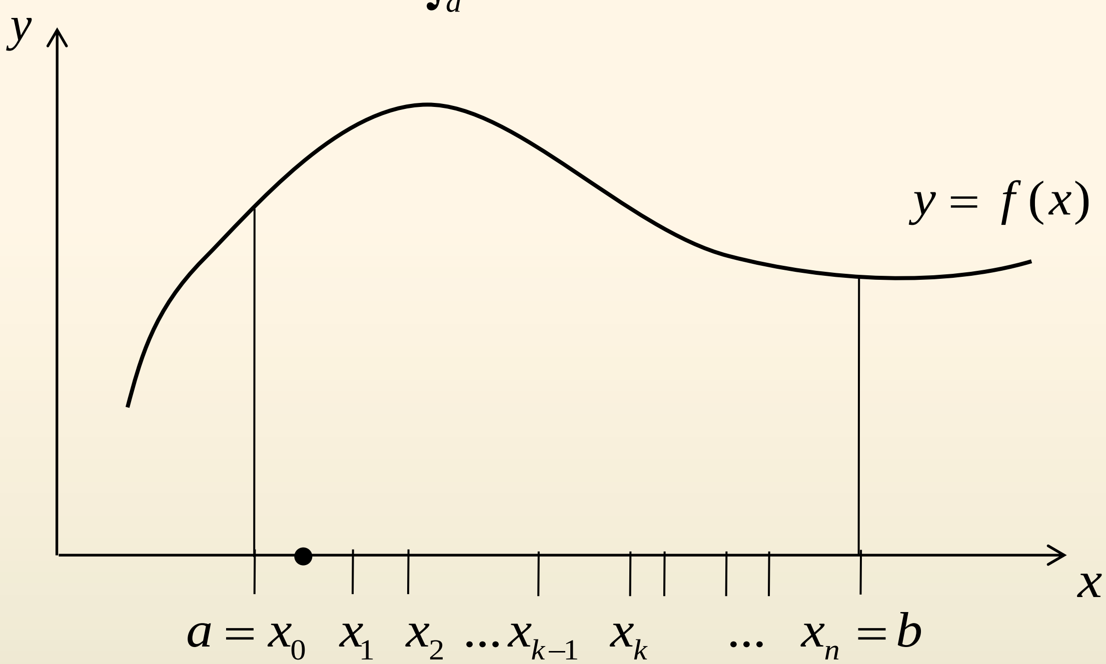

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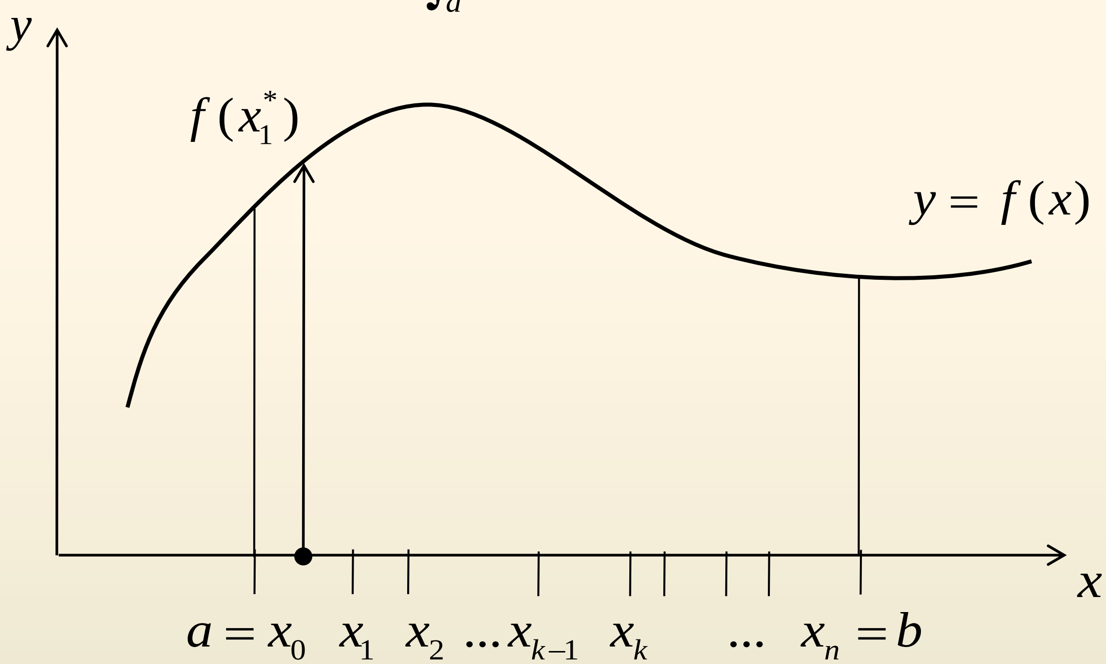

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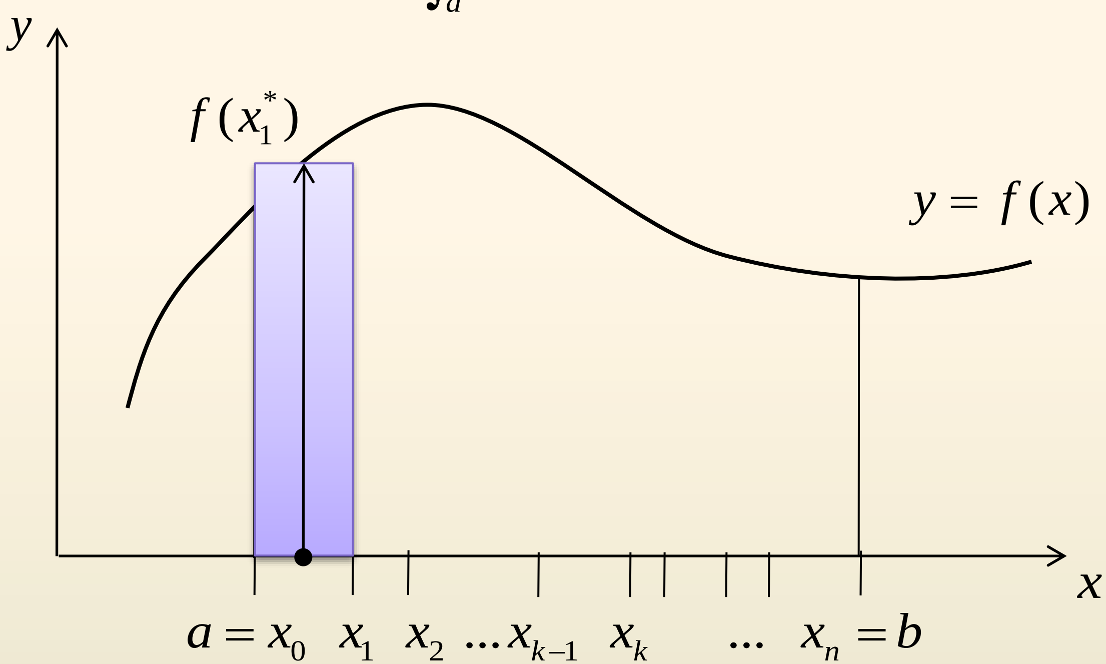

$$f(x_1^*)\Delta x_1$$

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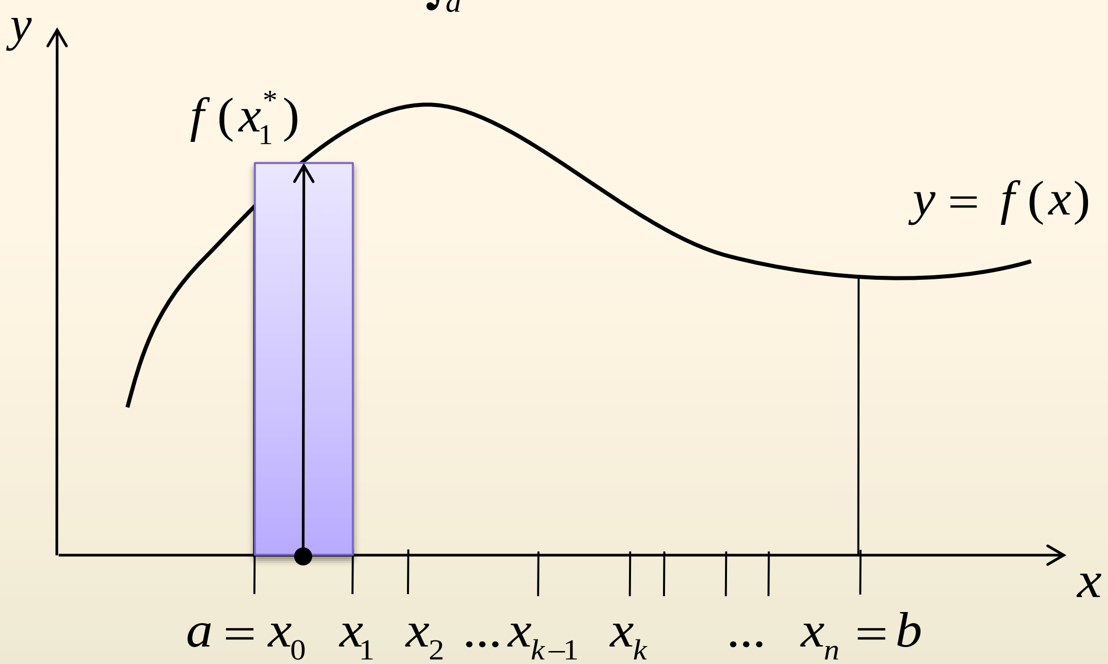

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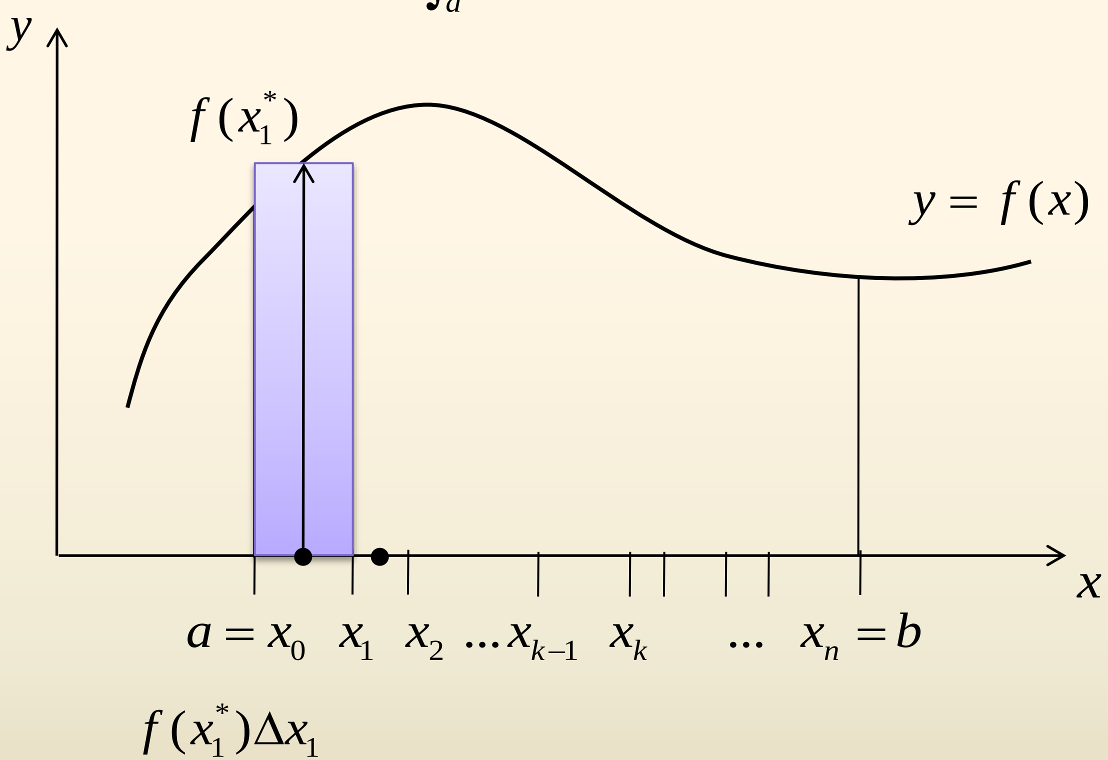

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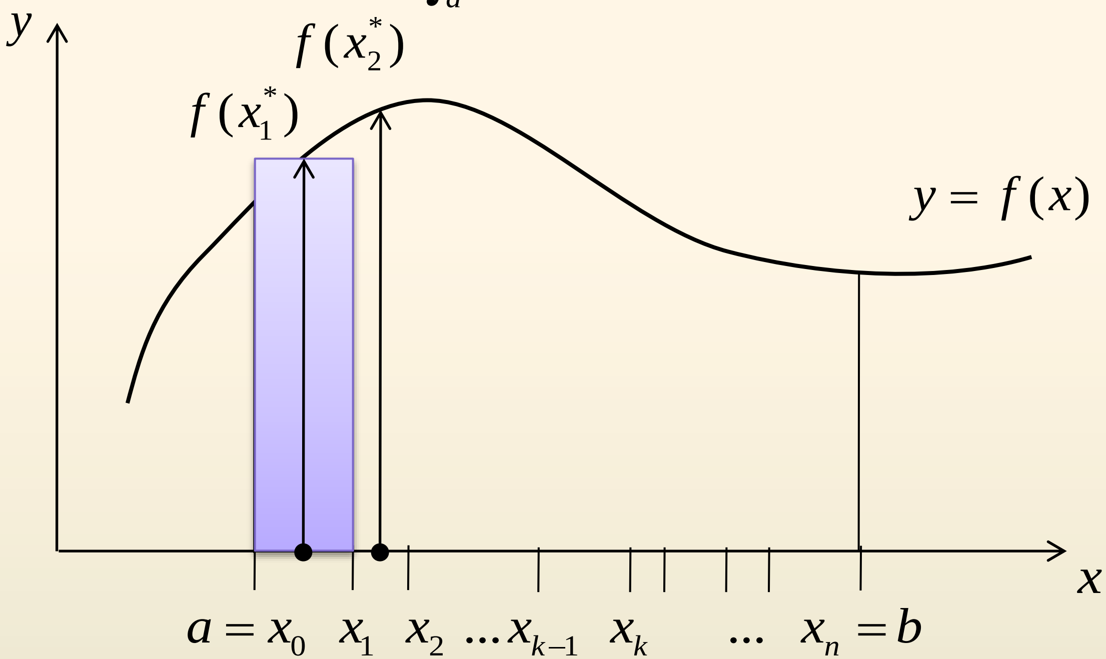

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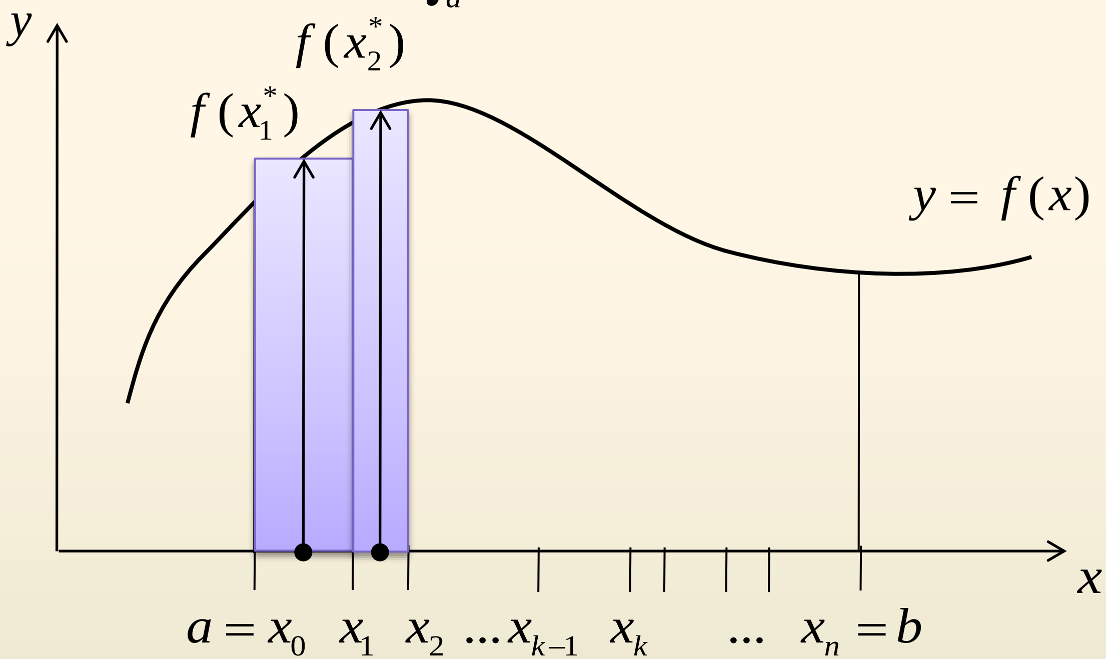

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$$f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2$$

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Definite Integral

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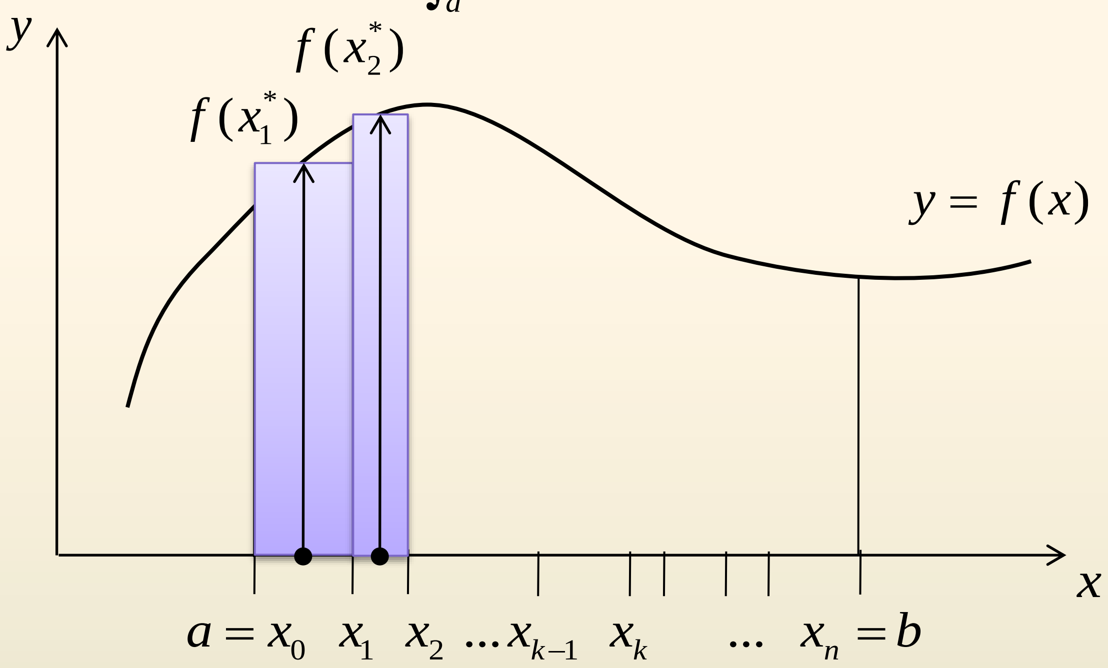

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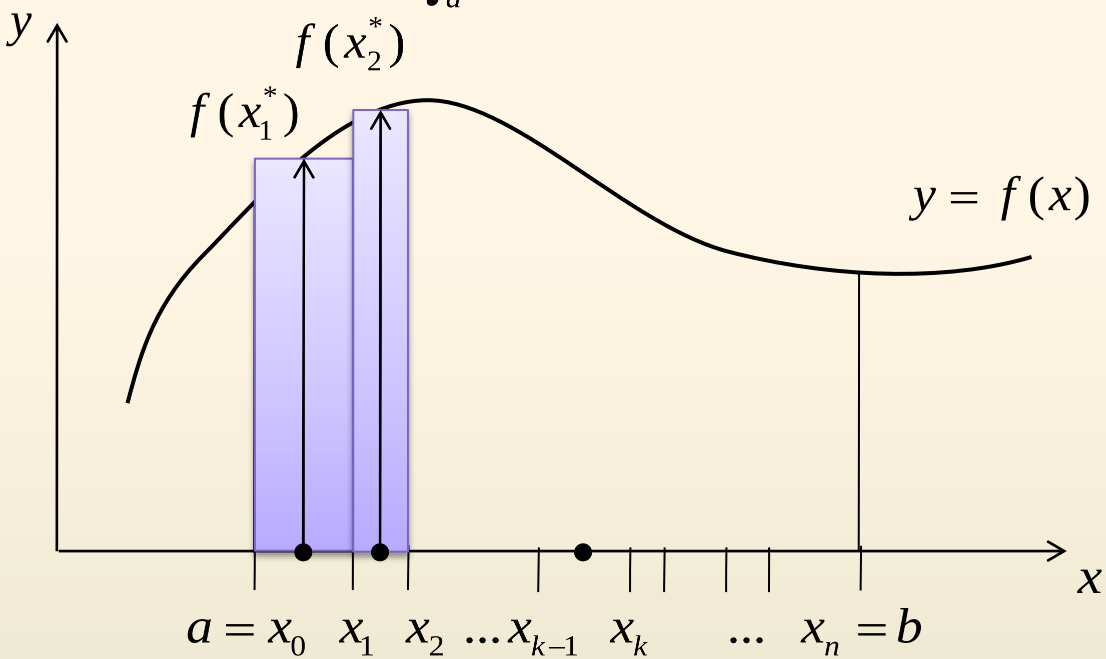

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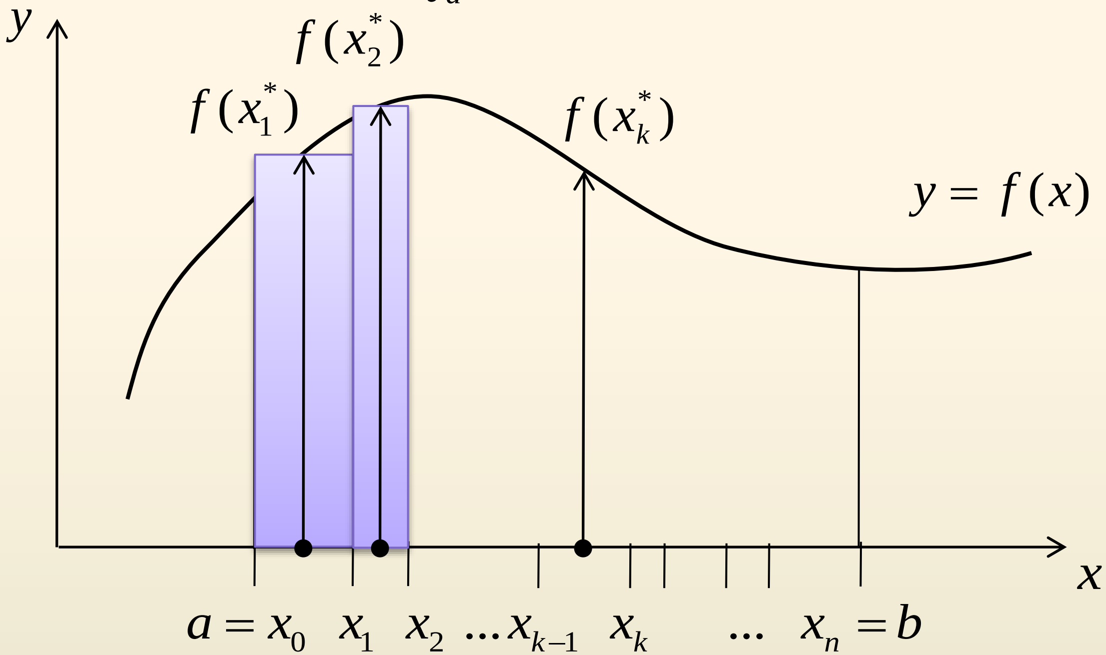

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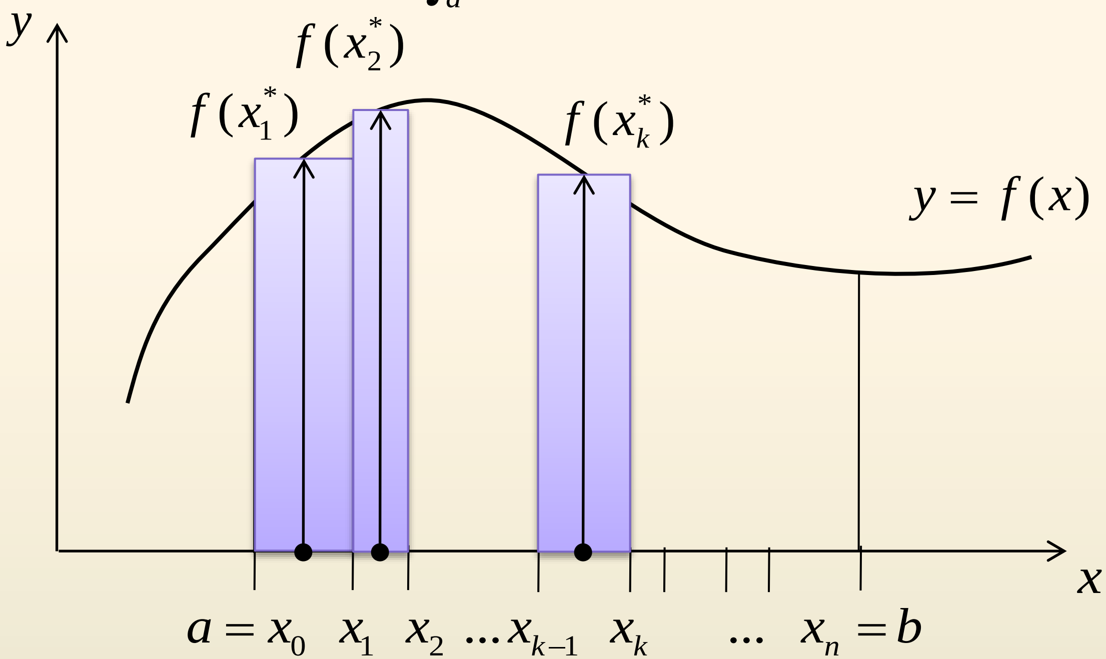

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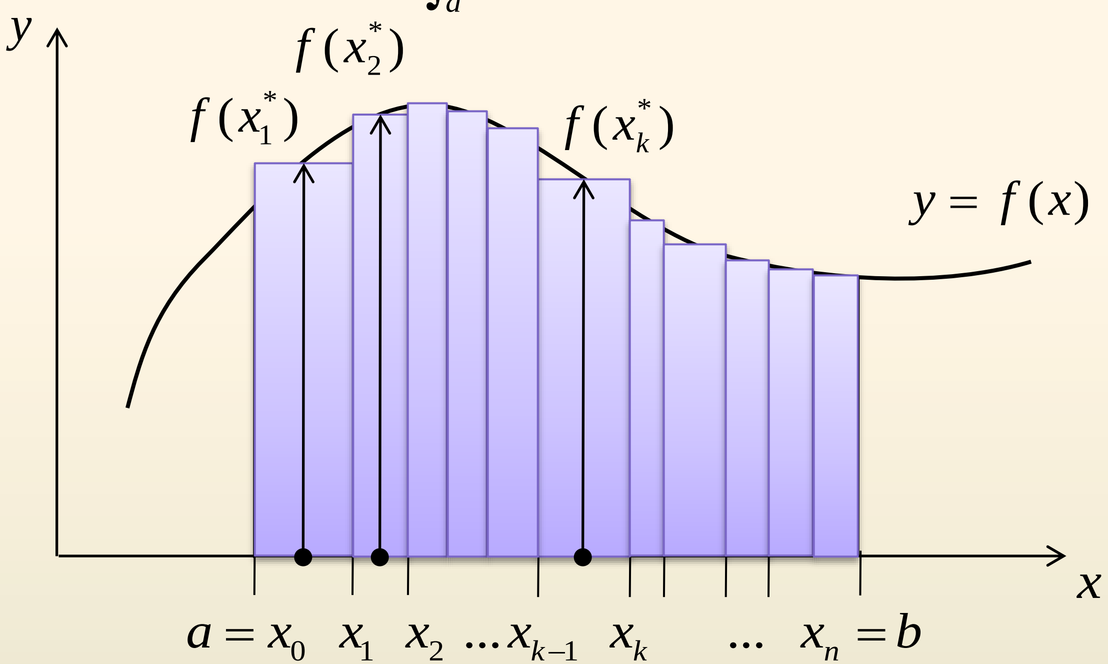

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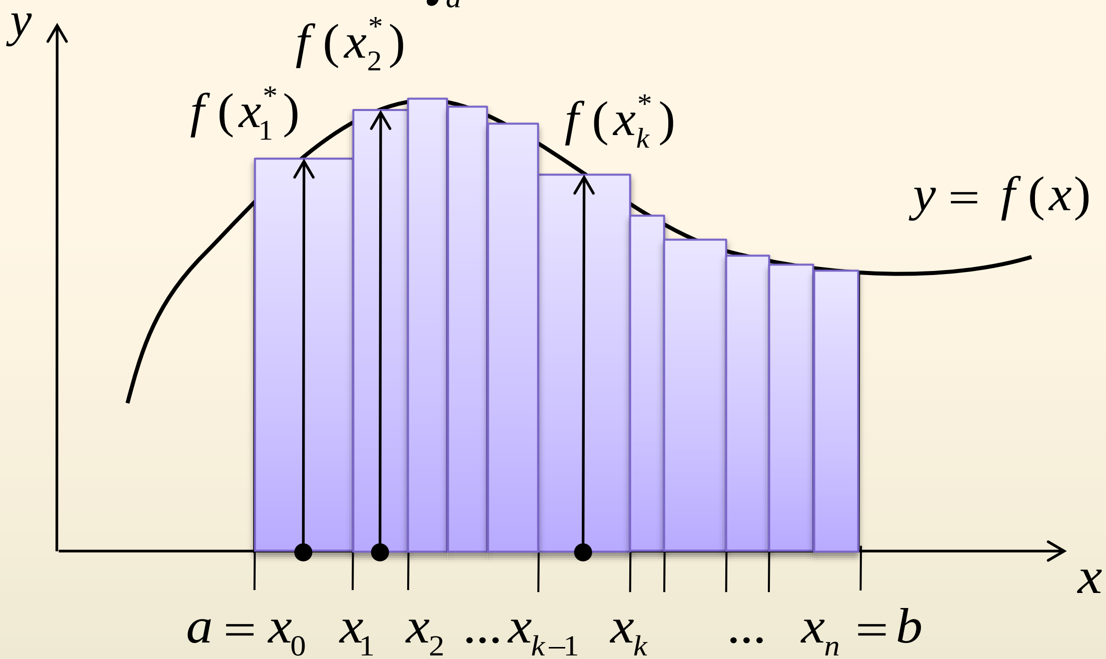

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$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Line Integrals

Definite Integral

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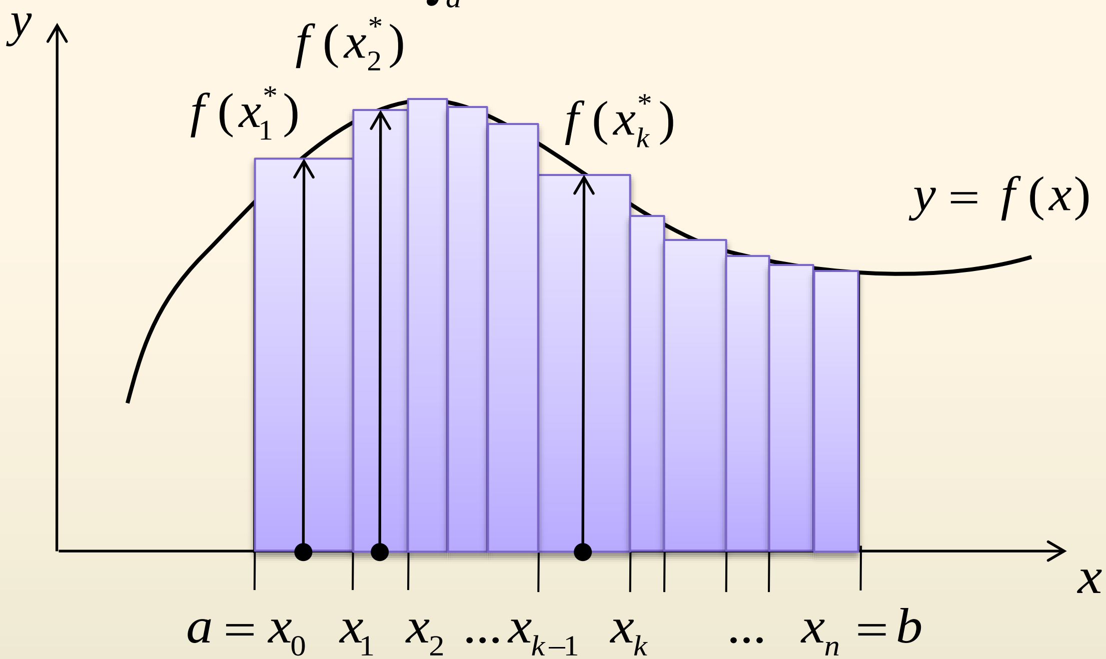

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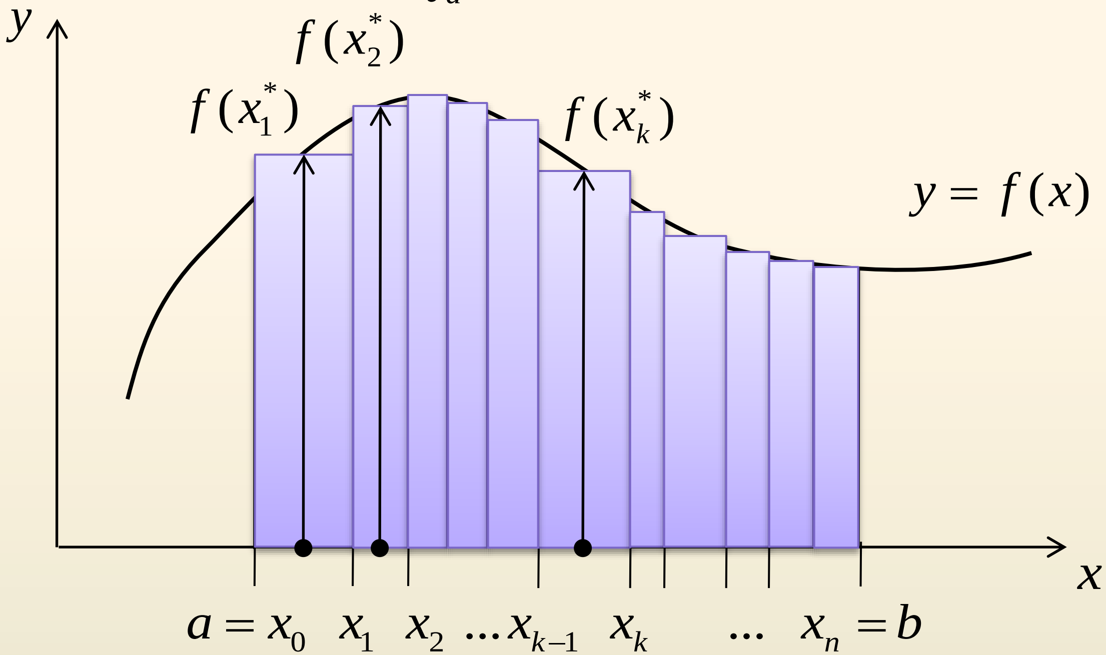

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$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

$\|P\|$: length of the longest subinterval

Line Integrals

Definite Integral

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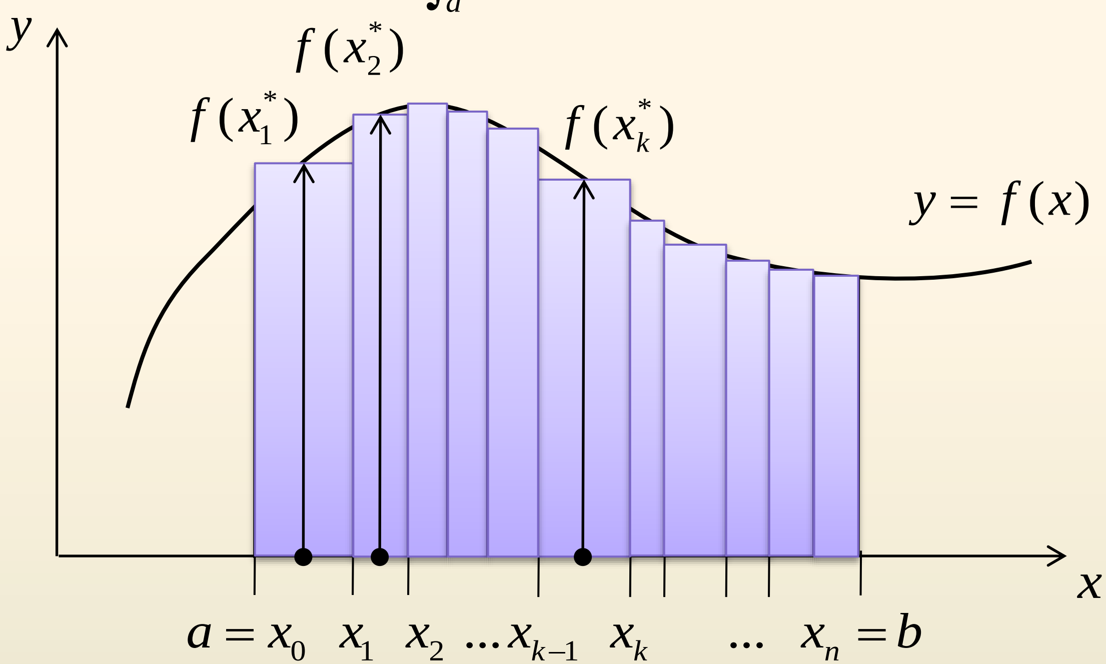

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The definite integral of a function of a single variable is given by the limit of a sum

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

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Line Integrals

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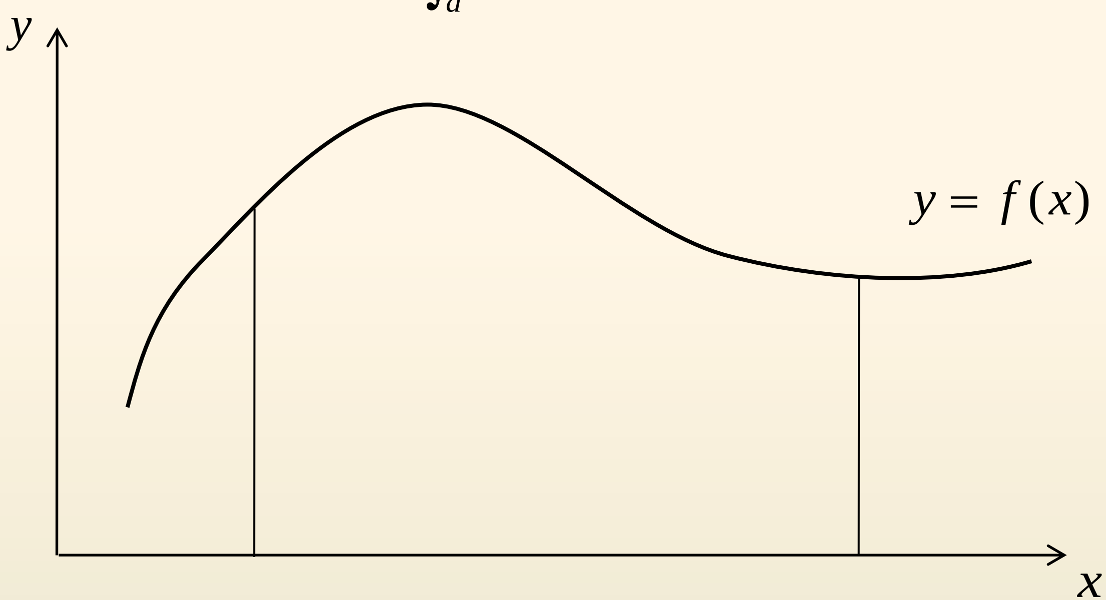

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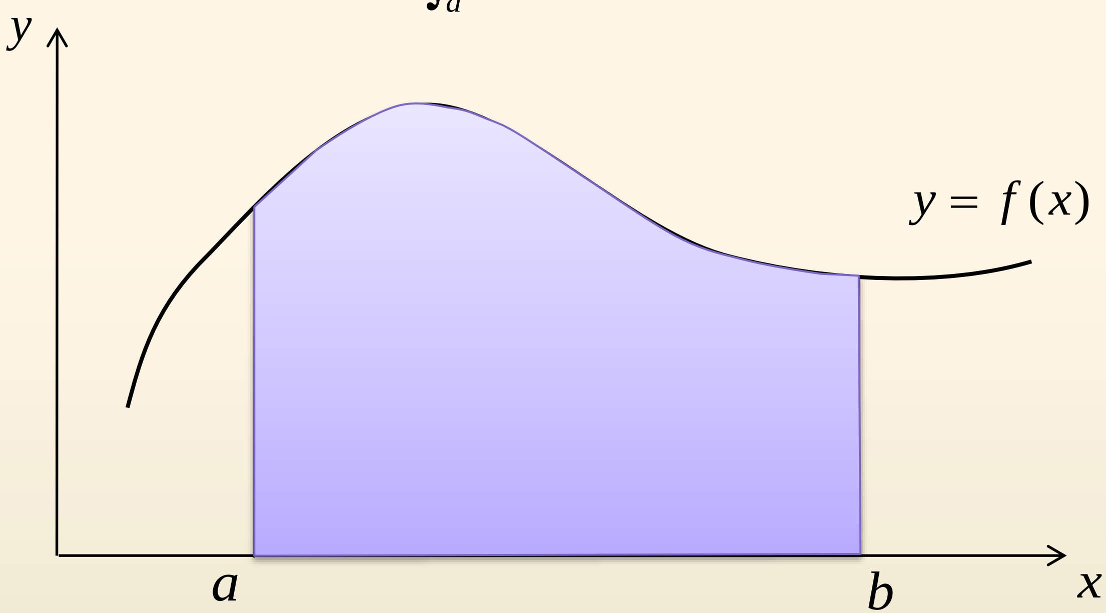

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Line Integrals

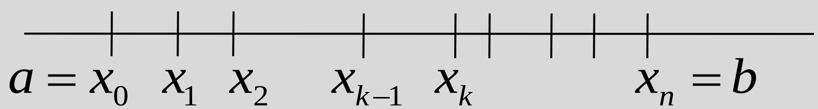
Definite Integral

y=*f* (x)

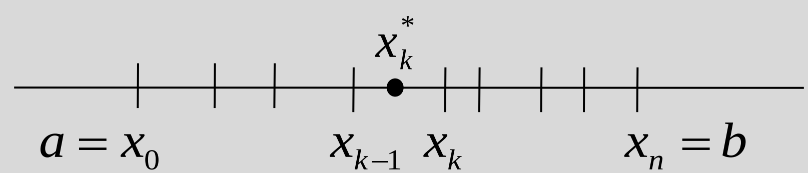
1. Let f be defined on a closed interval [a,b].
2. Partition the interval [a,b] into n subintervals $[x_{k-1}, x_k]$ of length $\Delta x_k = x_k - x_{k-1}$

   Let *P* denote the partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$



3. Let $\|P\|$ be the length of the longest subinterval. The number $\|P\|$ is called the **norm** of the partition *P*
4. Choose a number x_k^* in each subinterval.
5. Form the sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$



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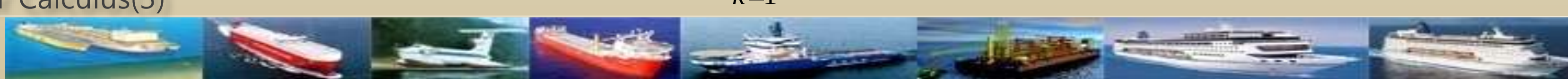
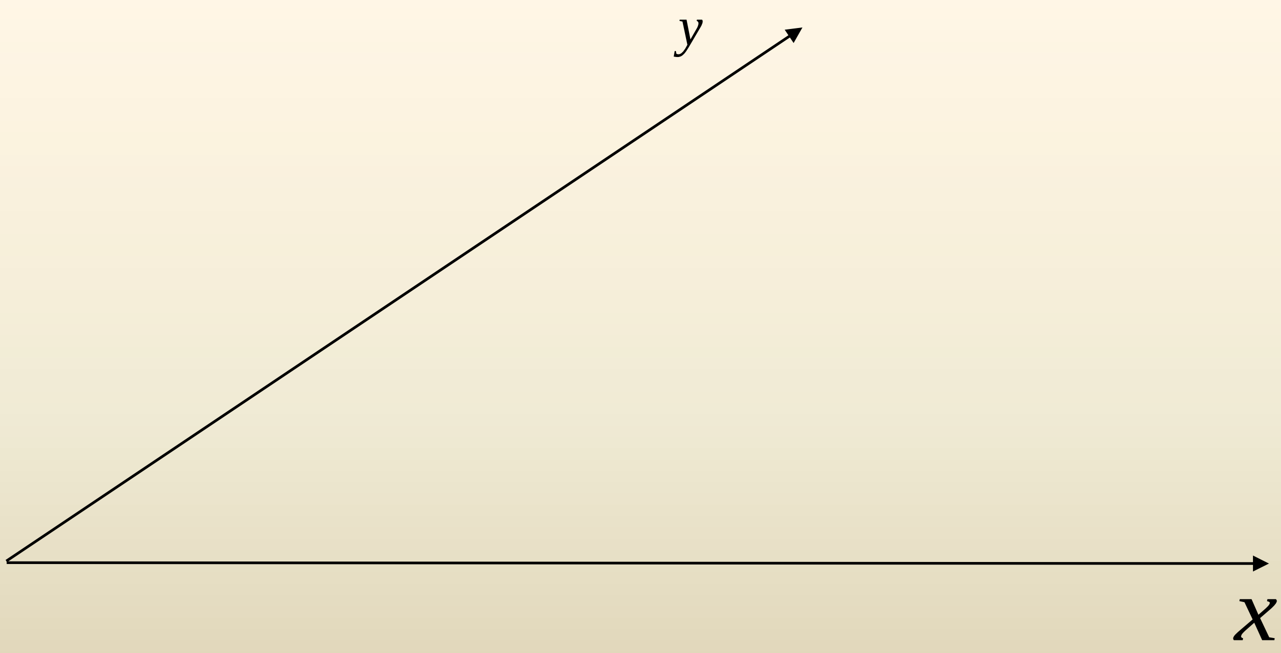


Line Integrals

Line Integral in the Plane



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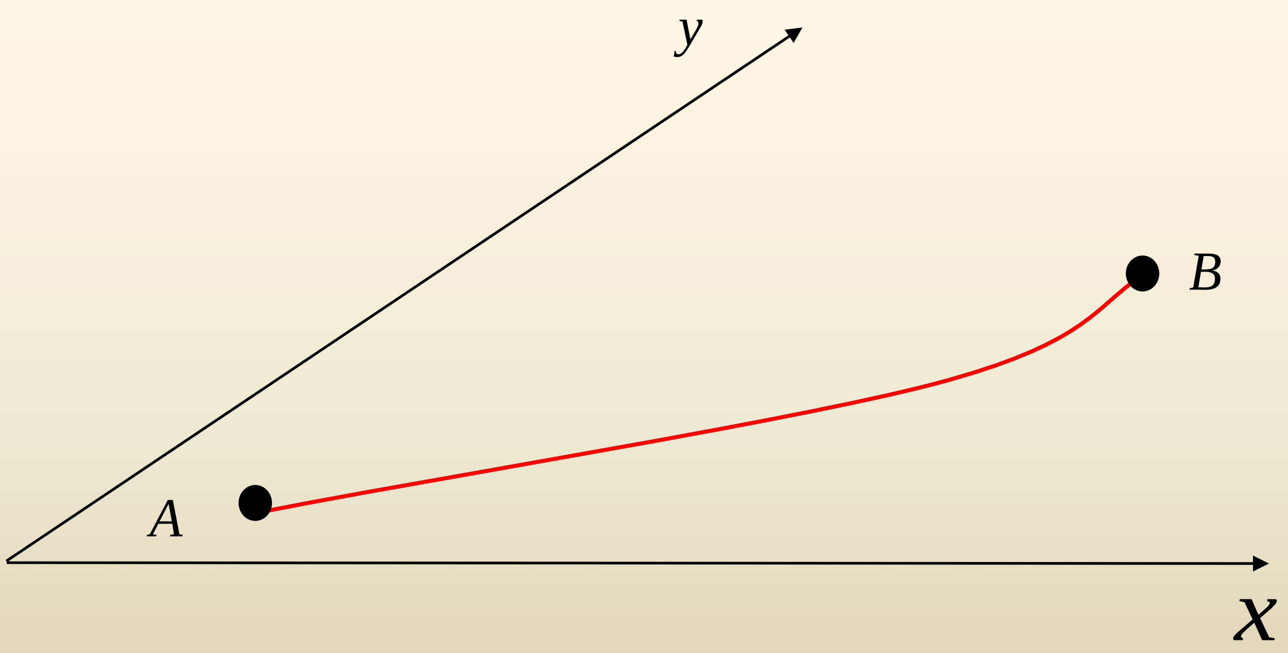

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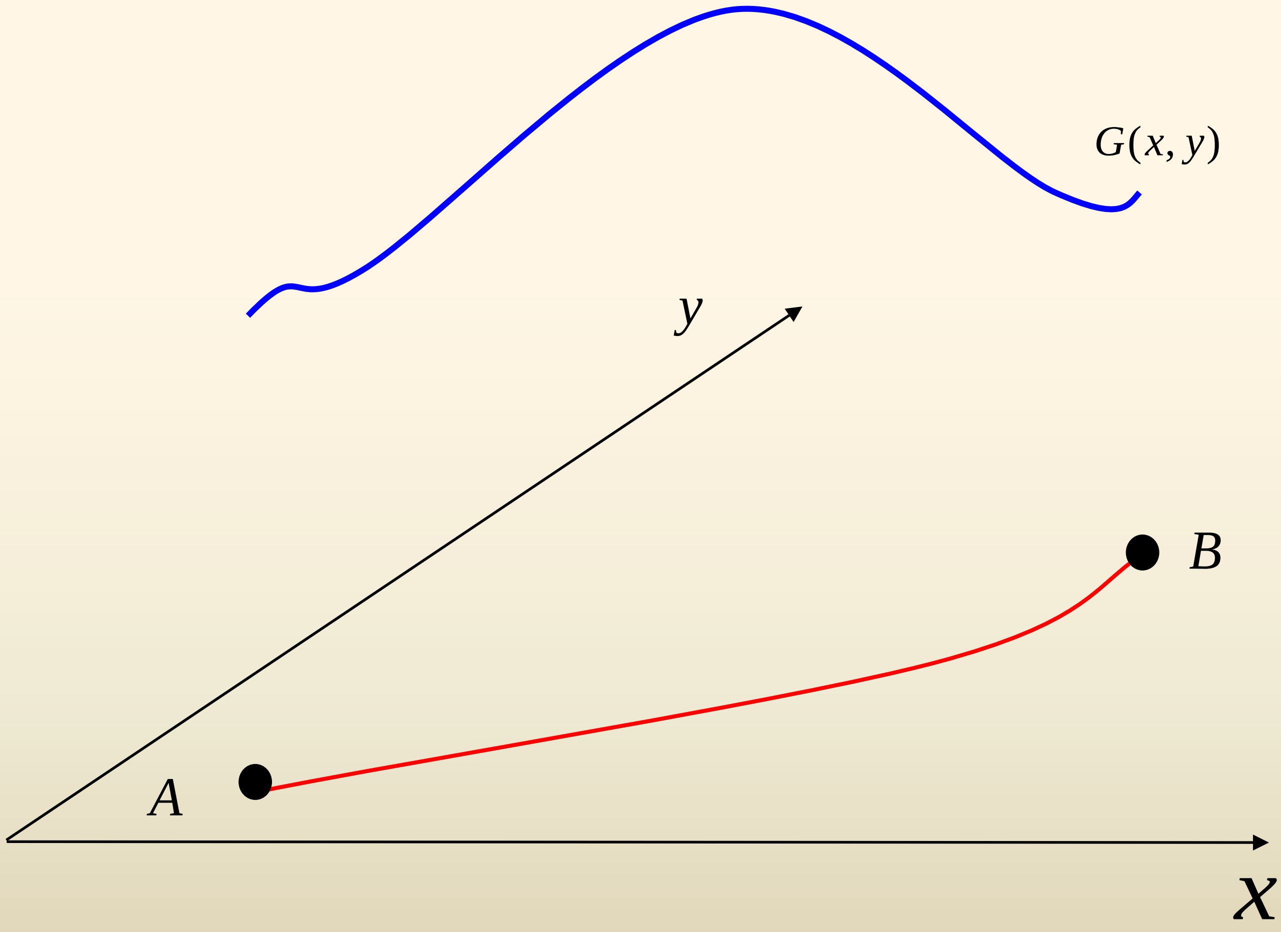

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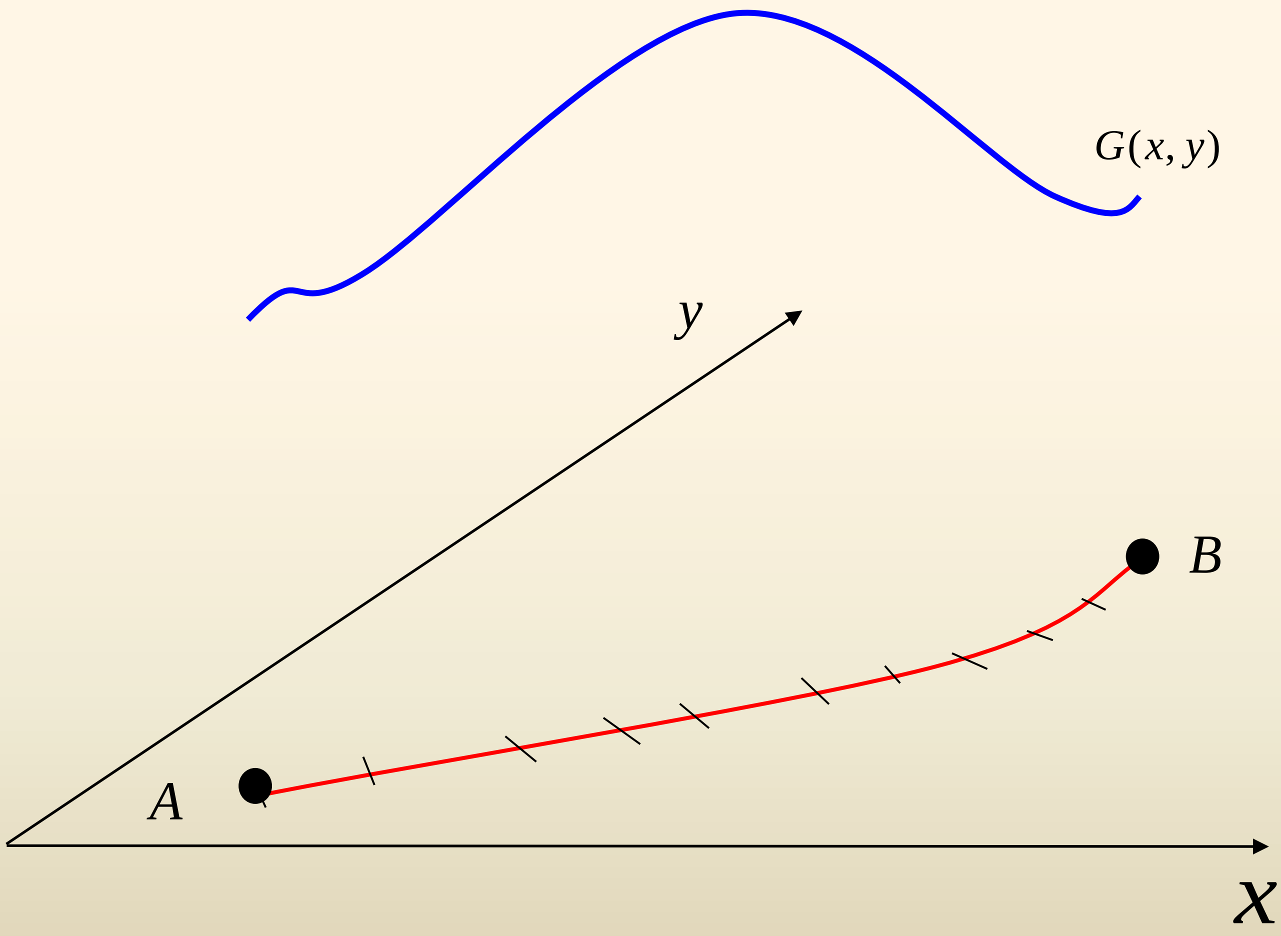

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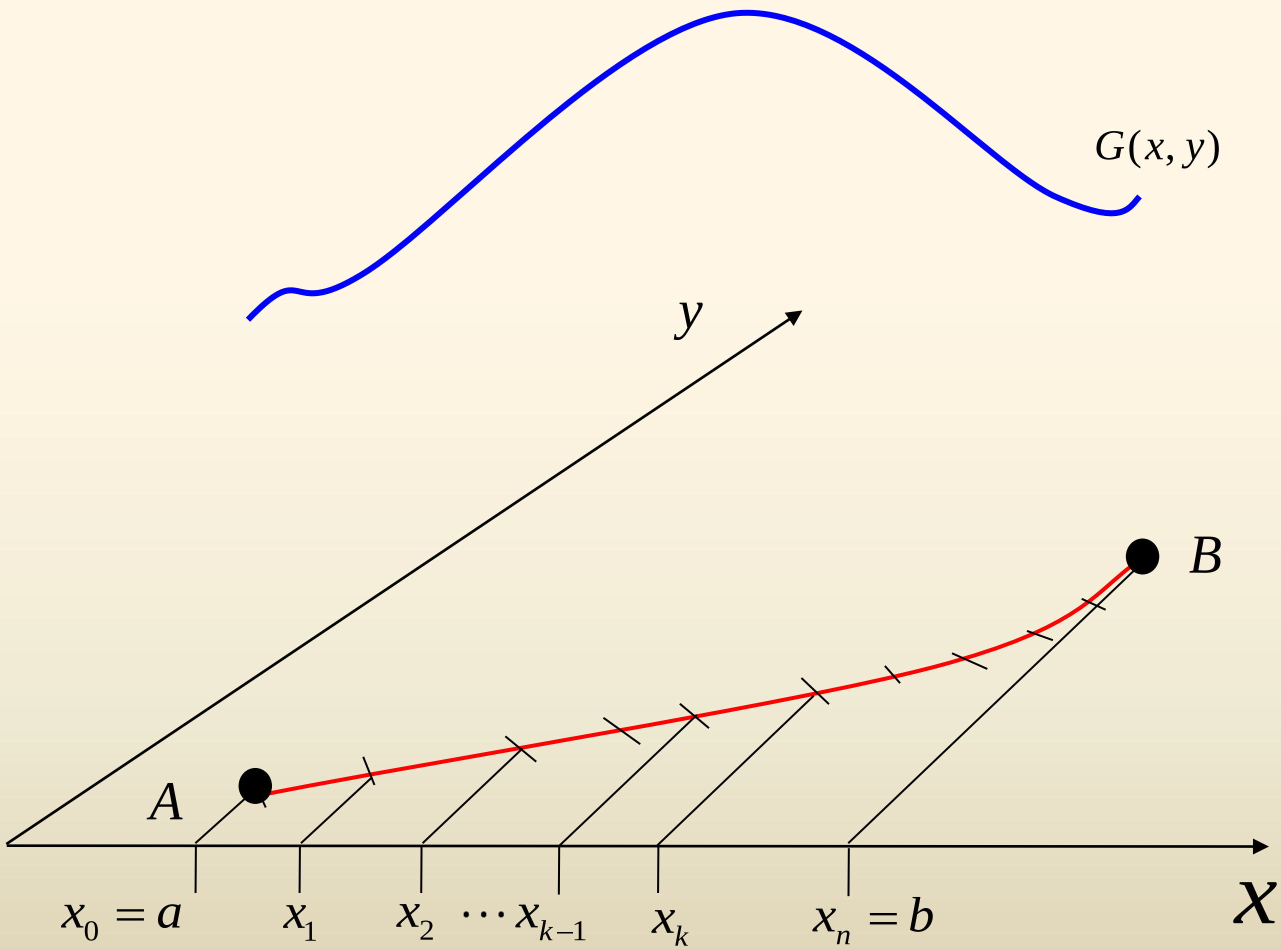
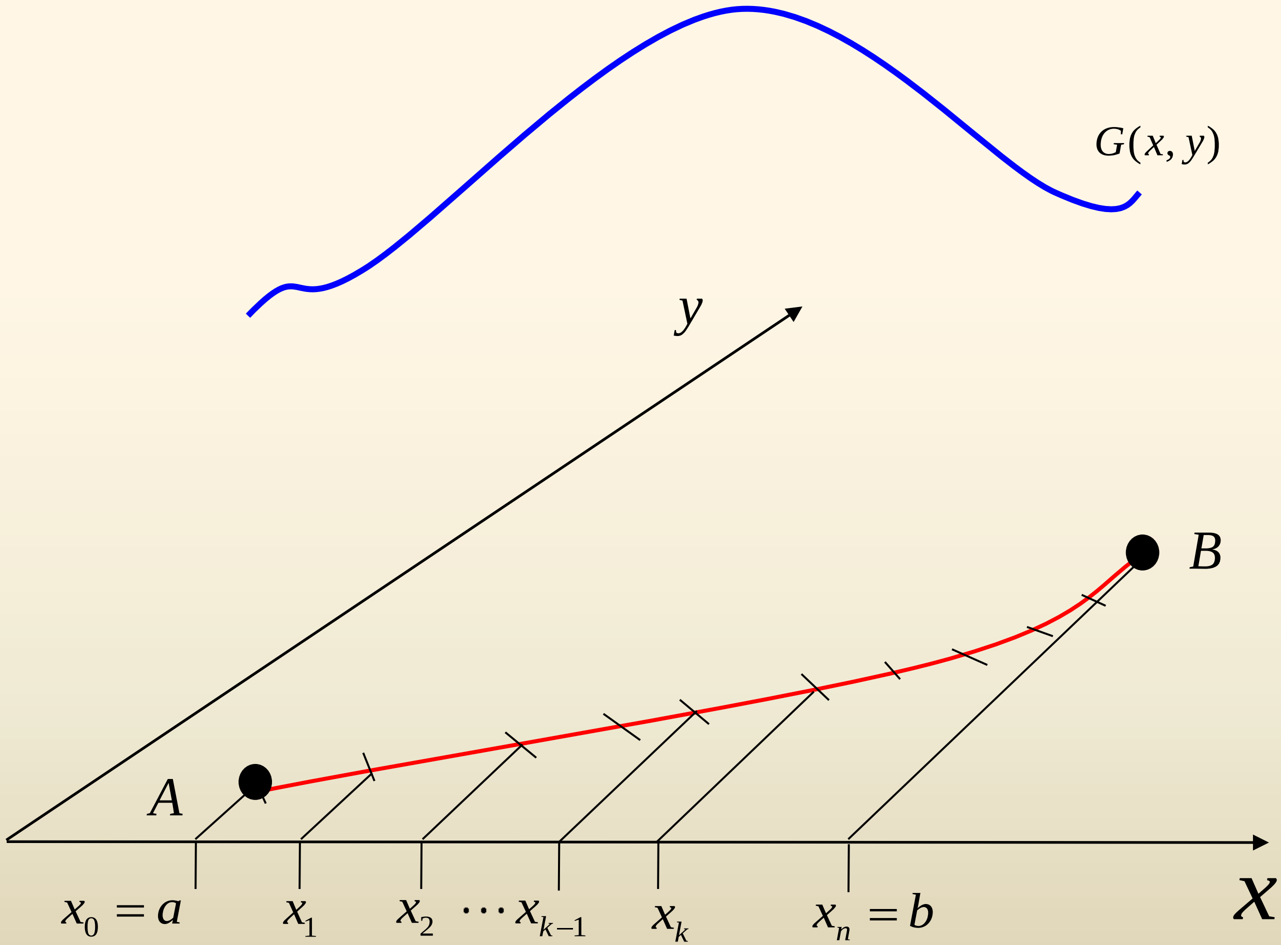


Line Integrals

Line Integral in the Plane



$$x_0 \le x_1^* \le x_1$$
$$y_0 \le y_1^* \le y_1$$

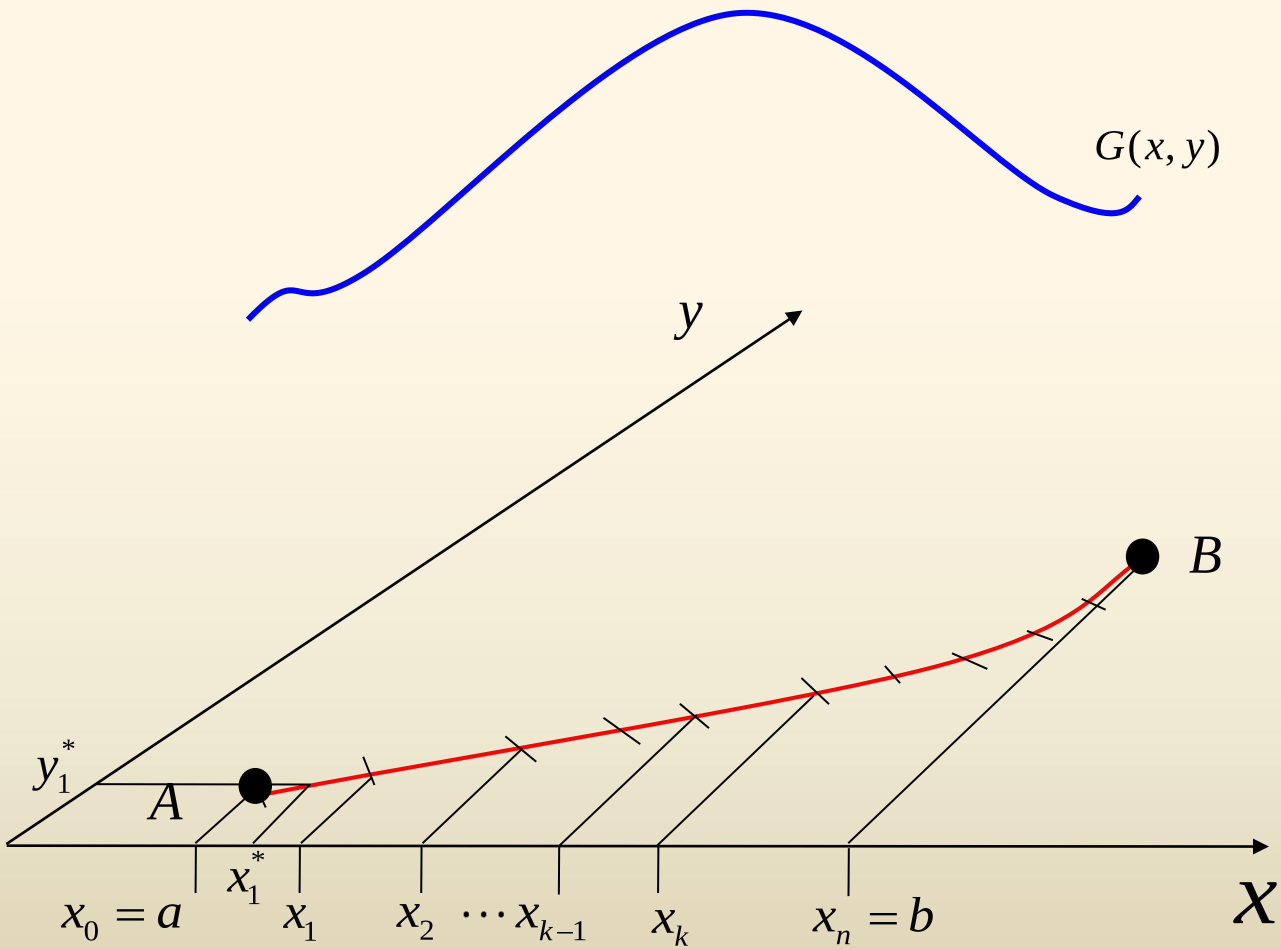
Δs_1

Line Integrals

Line Integral in the Plane

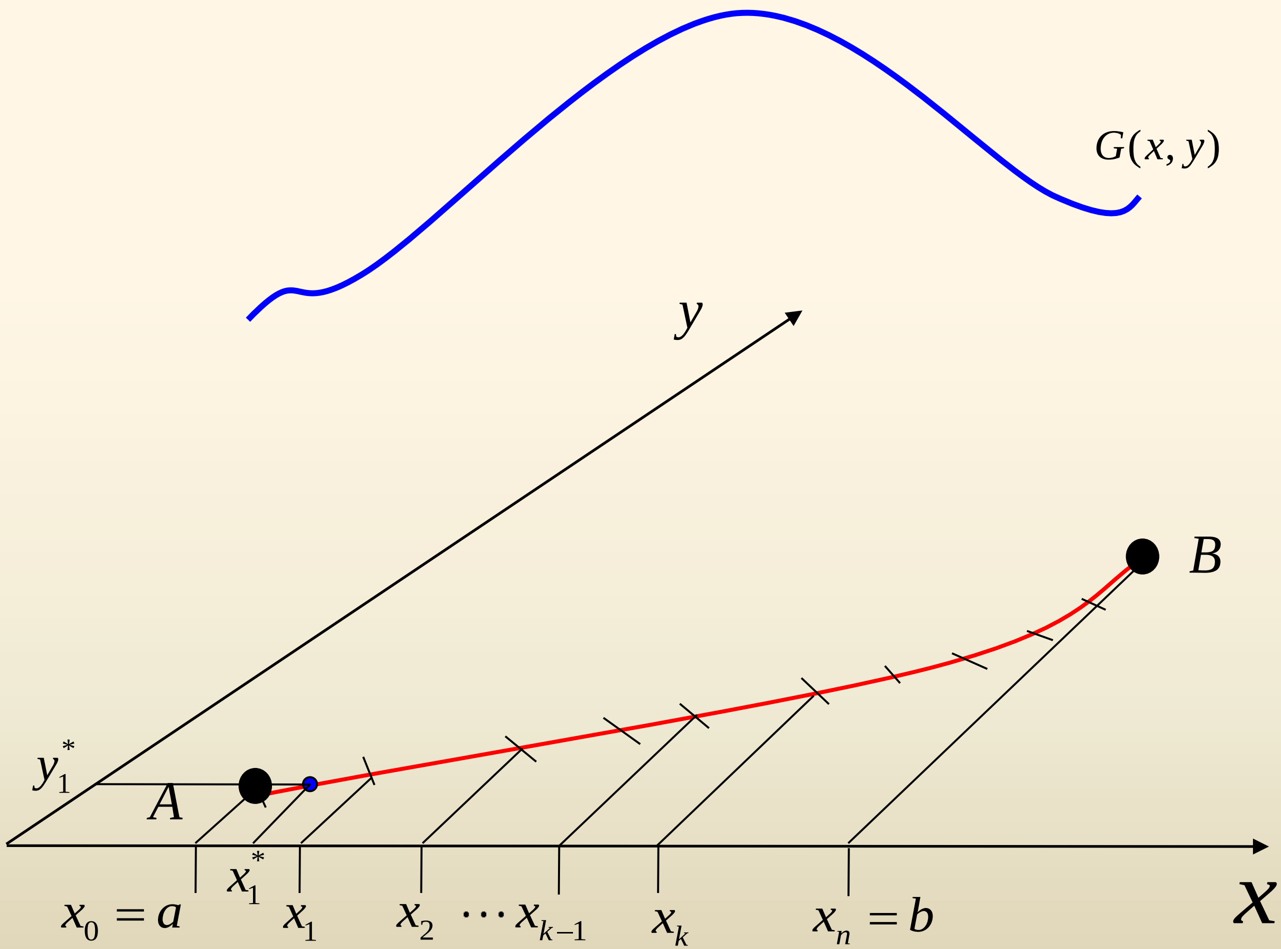
$x_0 \le x_1^* \le x_1$ subinterval

$y_0 \le y_1^* \le y_1$ Δs_1



Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$

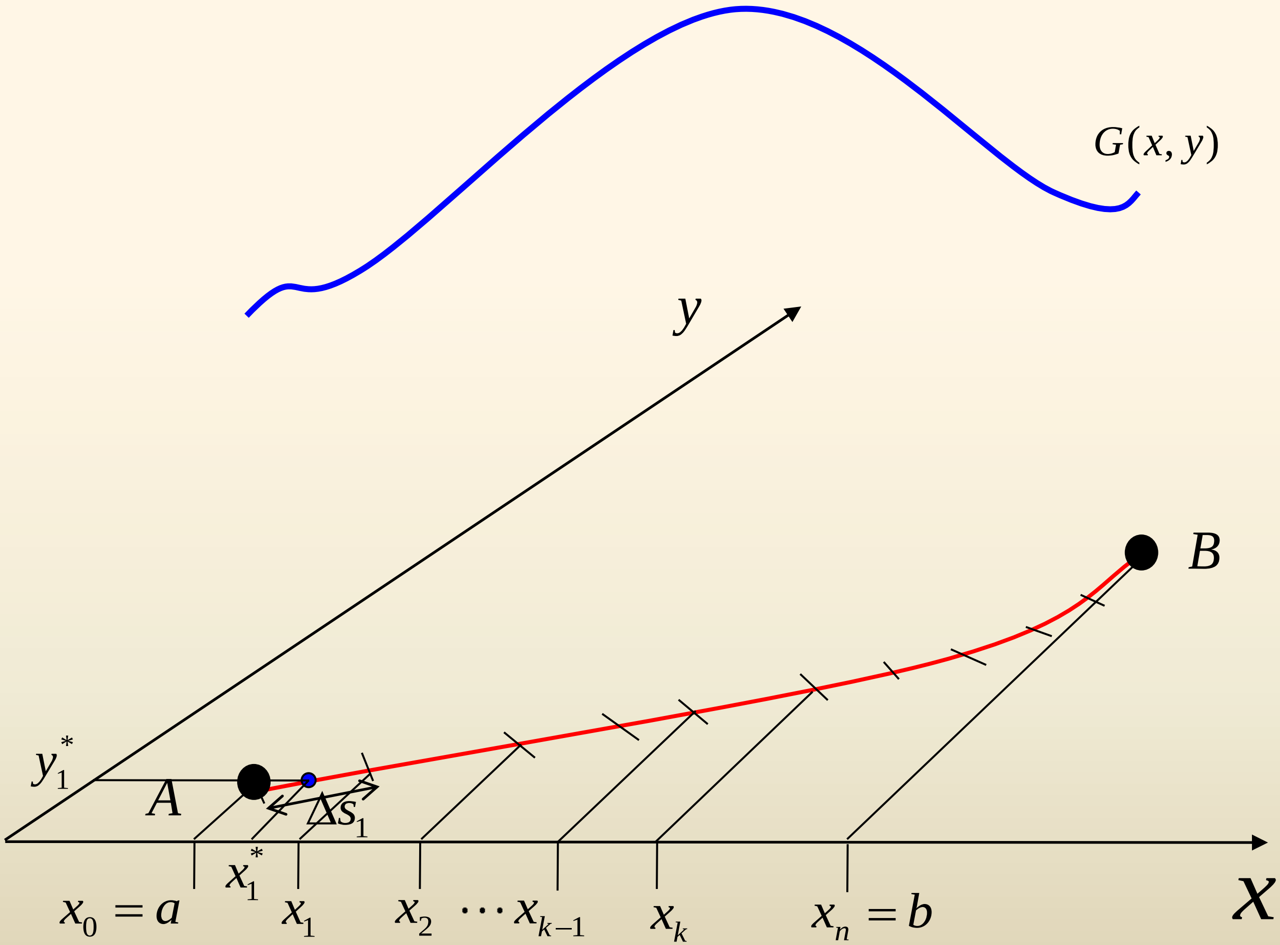
$y_0 \le y_1^* \le y_1$

subinterval

Δs_1

Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$

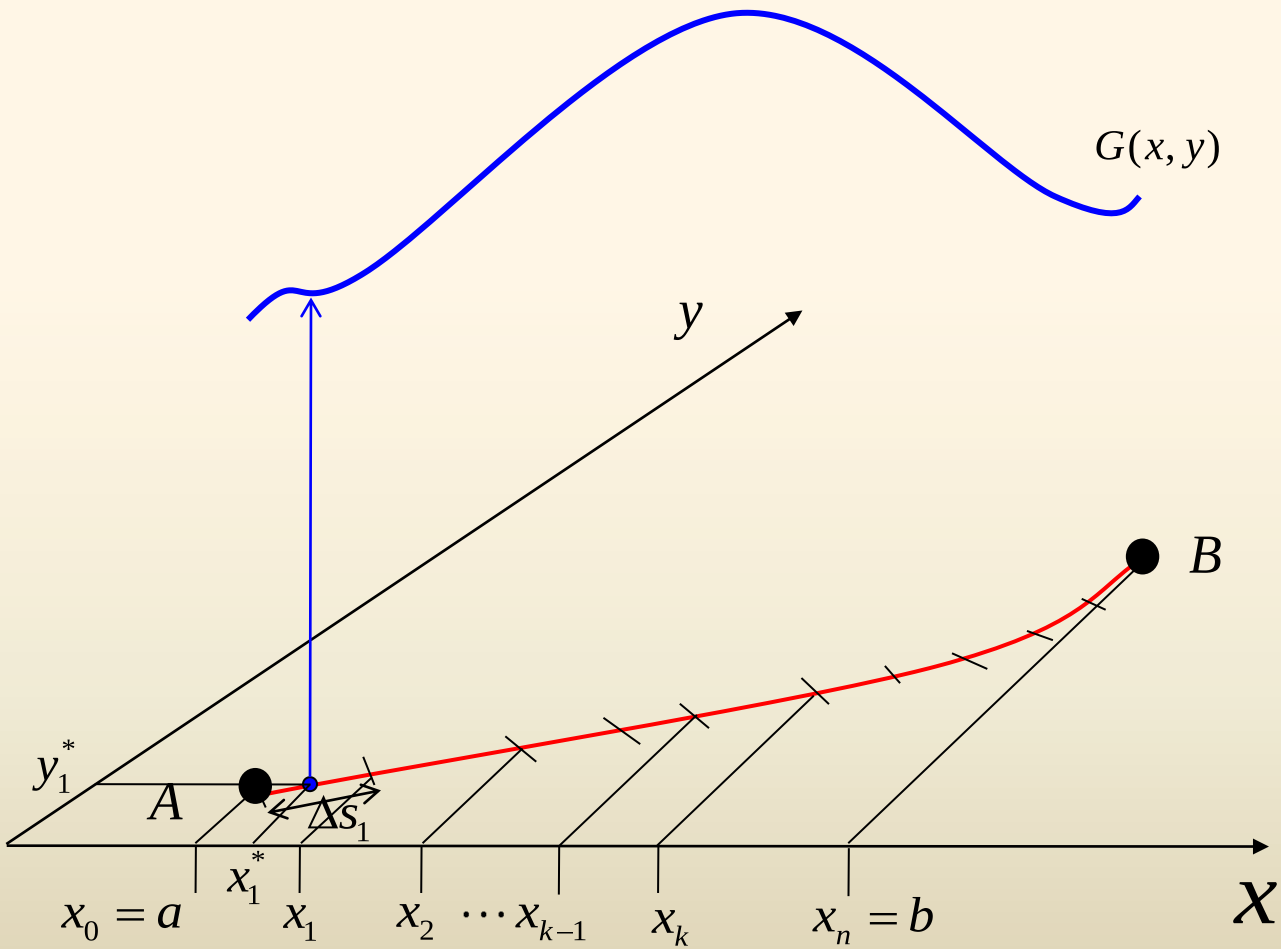
$y_0 \le y_1^* \le y_1$

subinterval

Δs_1

Line Integrals

Line Integral in the Plane



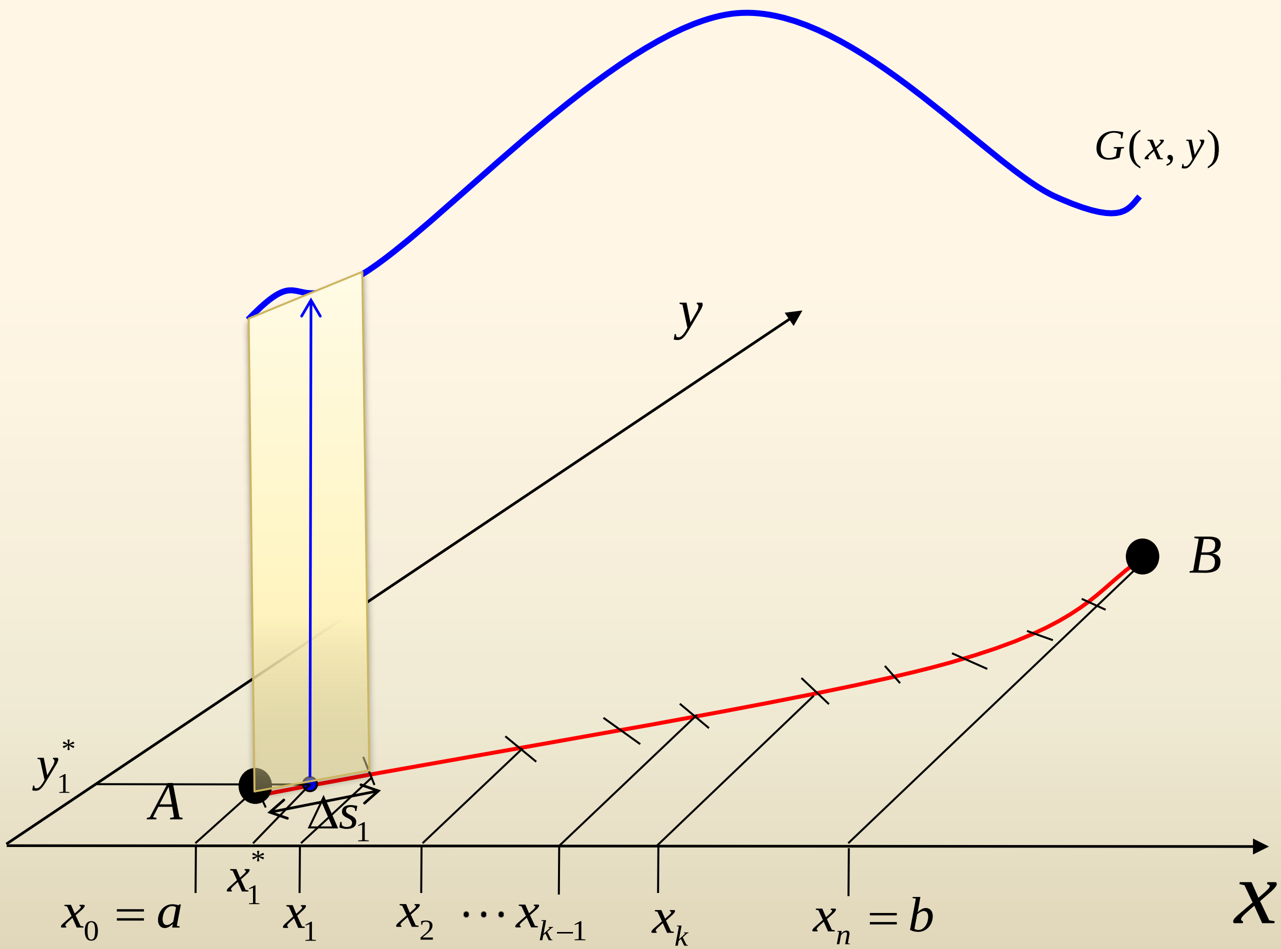
$x_0 \le x_1^* \le x_1$

$y_0 \le y_1^* \le y_1$

subinterval Δs_1

Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$

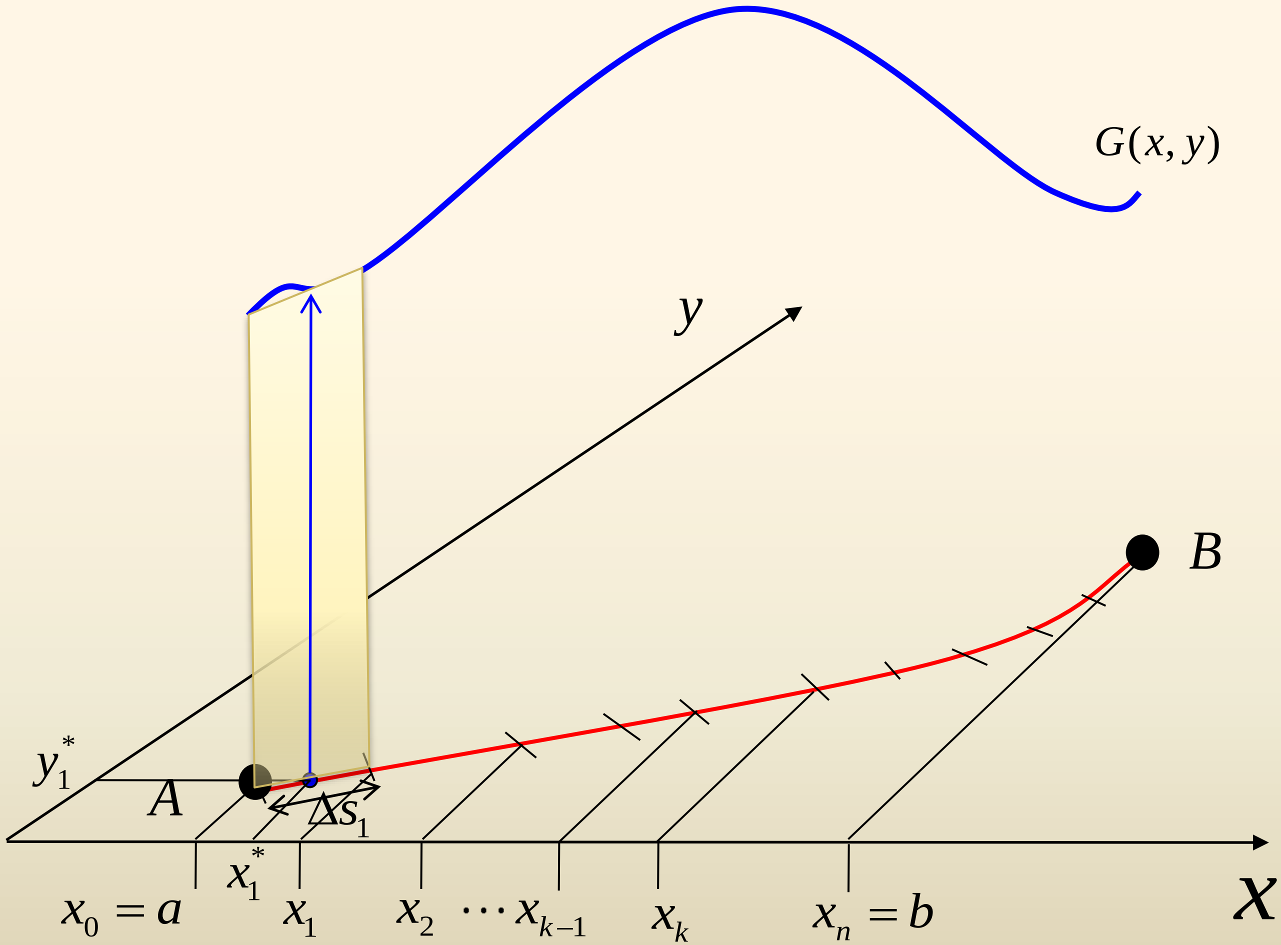
$y_0 \le y_1^* \le y_1$

subinterval

Δs_1

Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$

$y_0 \le y_1^* \le y_1$

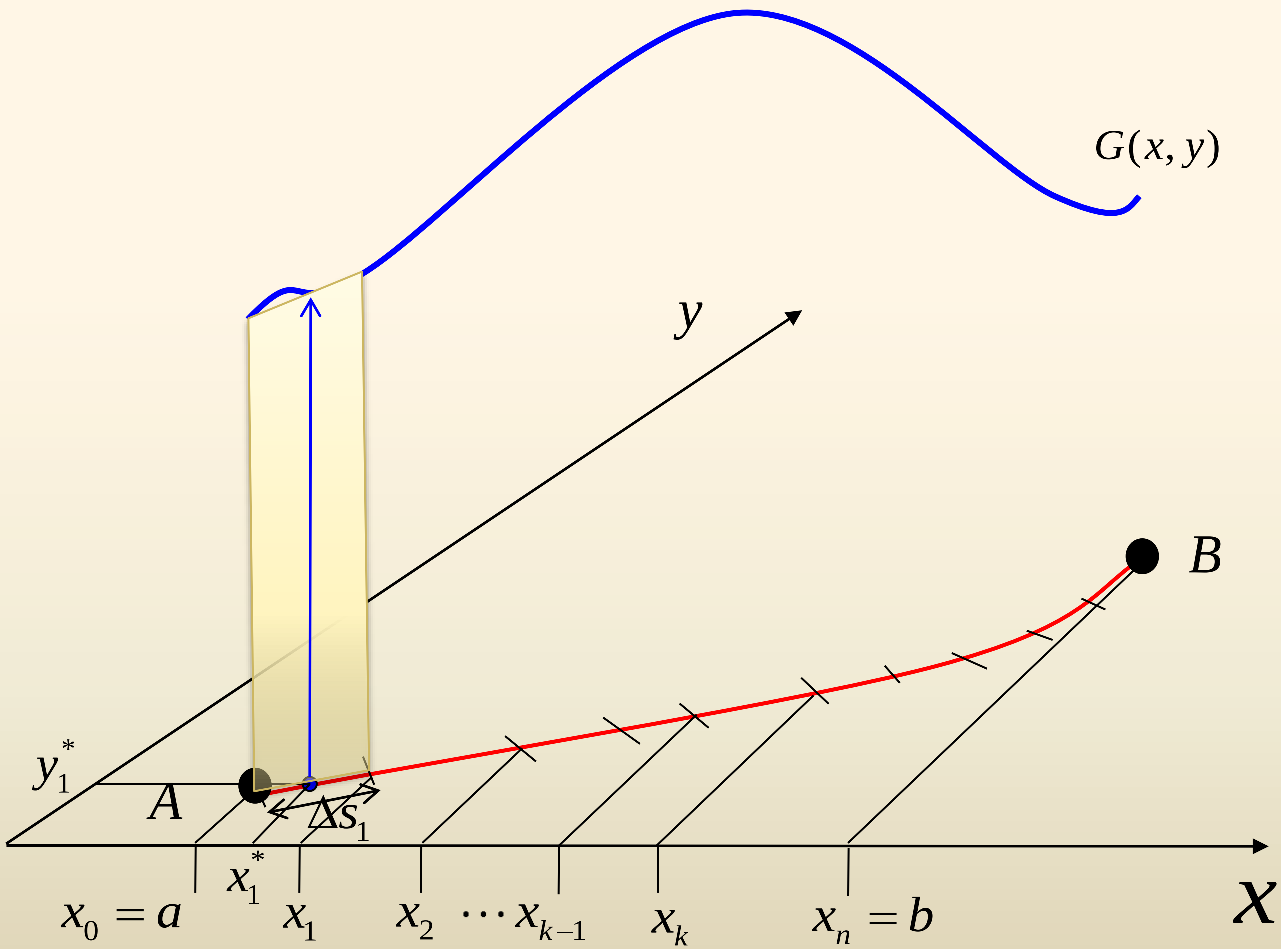
subinterval

Δs_1

$G(x_1^*, y_1^*)\Delta s_1$

Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$ subinterval

$y_0 \le y_1^* \le y_1$ Δs_1

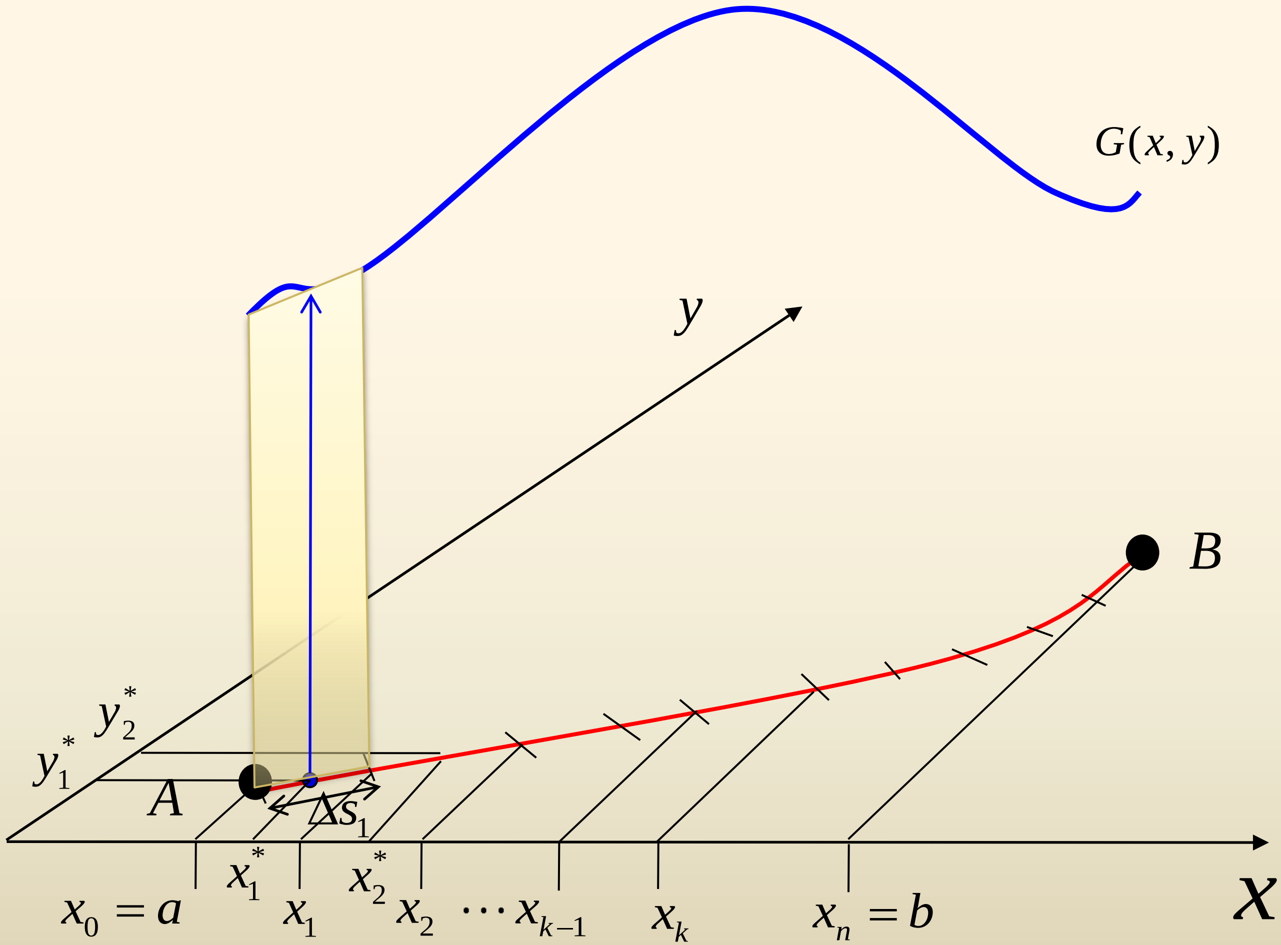
$x_1 \le x_2^* \le x_2$

$y_1 \le y_2^* \le y_2$ Δs_2

$G(x_1^*, y_1^*)\Delta s_1$

Line Integrals

Line Integral in the Plane

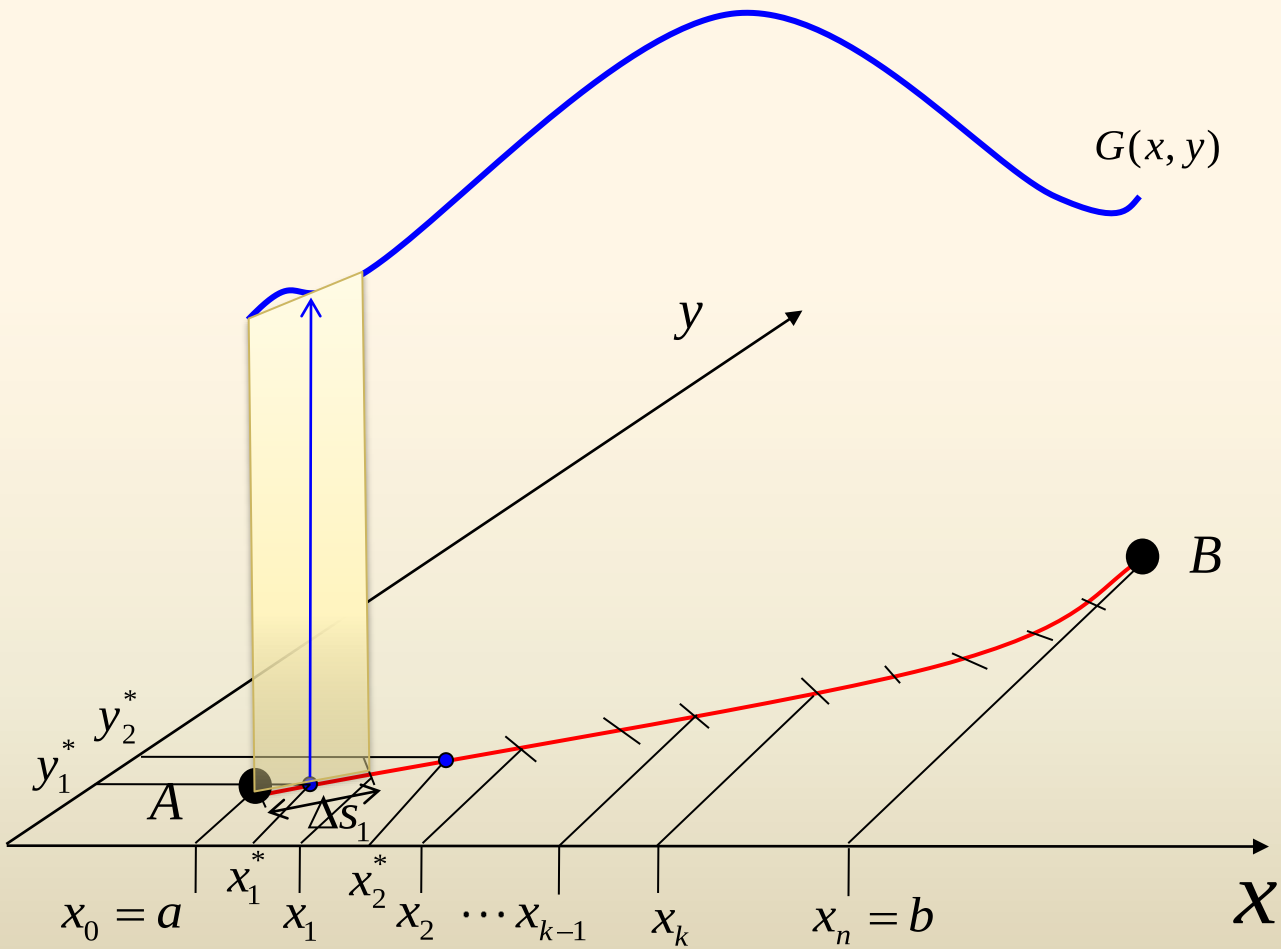


	subinterval
$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2

$G(x_1^*, y_1^*)\Delta s_1$

Line Integrals

Line Integral in the Plane



subinterval

$x_0 \le x_1^* \le x_1$

$y_0 \le y_1^* \le y_1$

Δs_1

$x_1 \le x_2^* \le x_2$

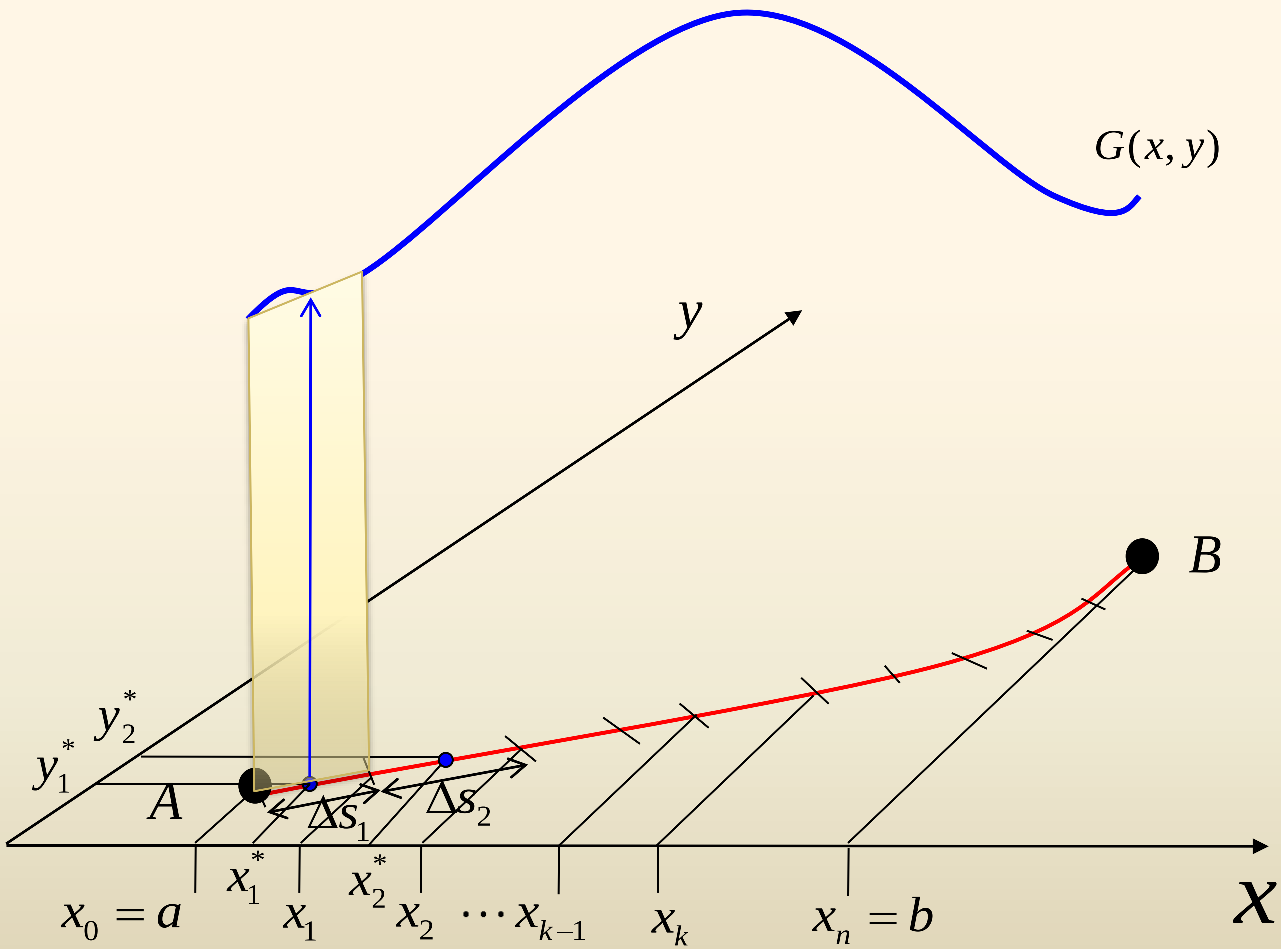
$y_1 \le y_2^* \le y_2$

Δs_2

$G(x_1^*, y_1^*)\Delta s_1$

Line Integrals

Line Integral in the Plane

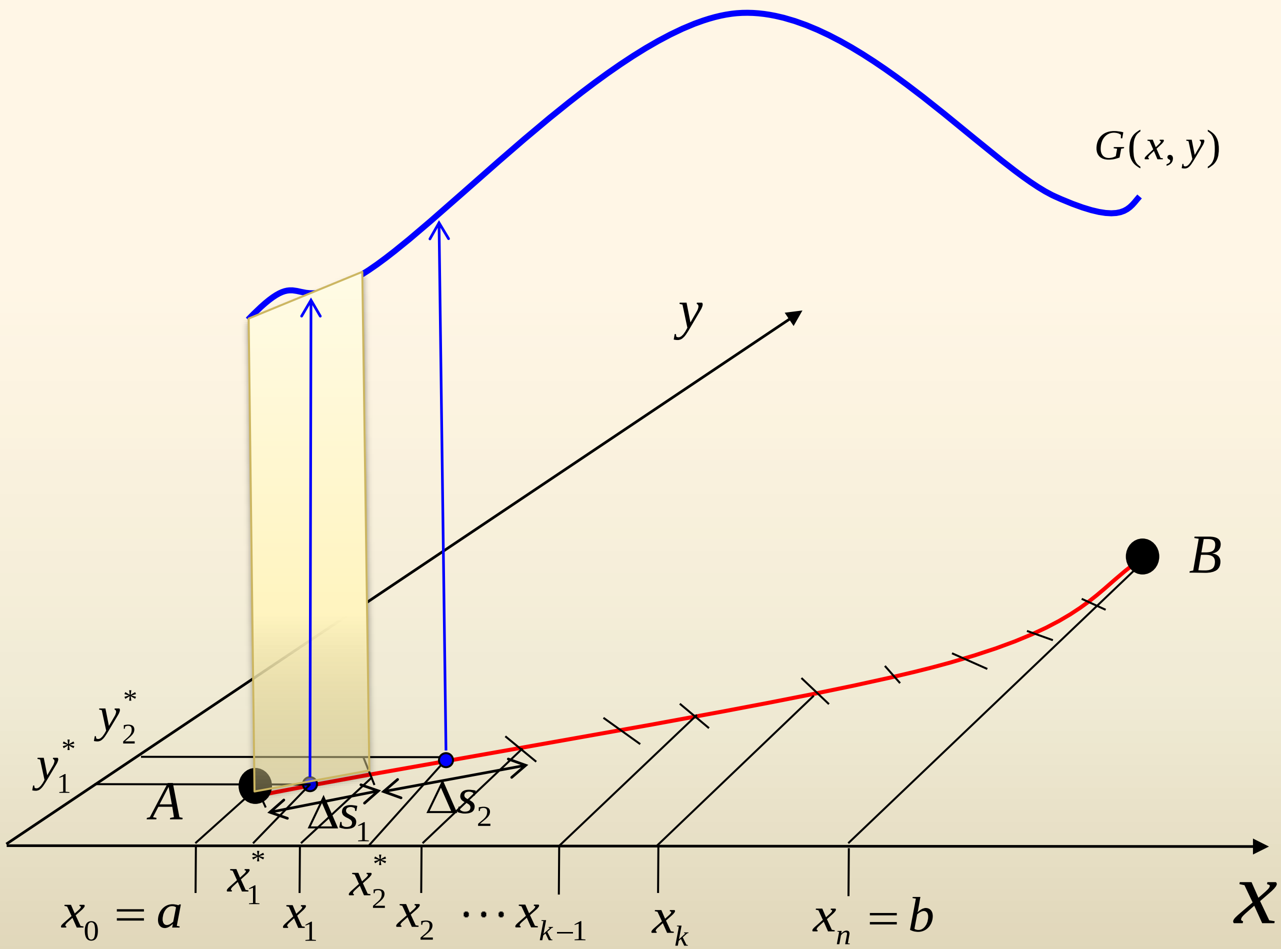


$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	subinterval Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2

$G(x_1^*, y_1^*)\Delta s_1$

Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$ subinterval

$y_0 \le y_1^* \le y_1$ Δs_1

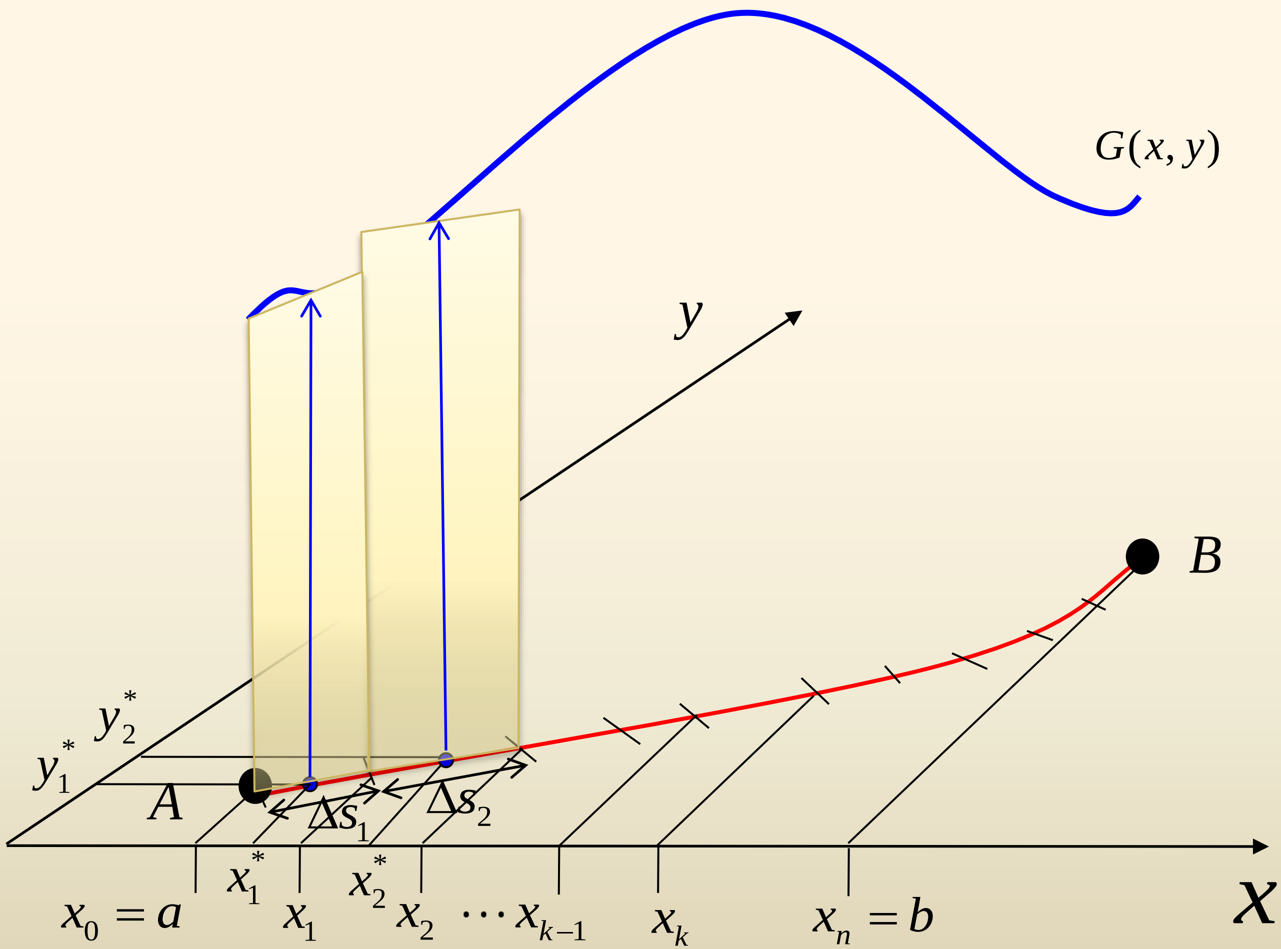
$x_1 \le x_2^* \le x_2$

$y_1 \le y_2^* \le y_2$ Δs_2

$G(x_1^*, y_1^*)\Delta s_1$

Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$

$y_0 \le y_1^* \le y_1$

subinterval Δs_1

$x_1 \le x_2^* \le x_2$

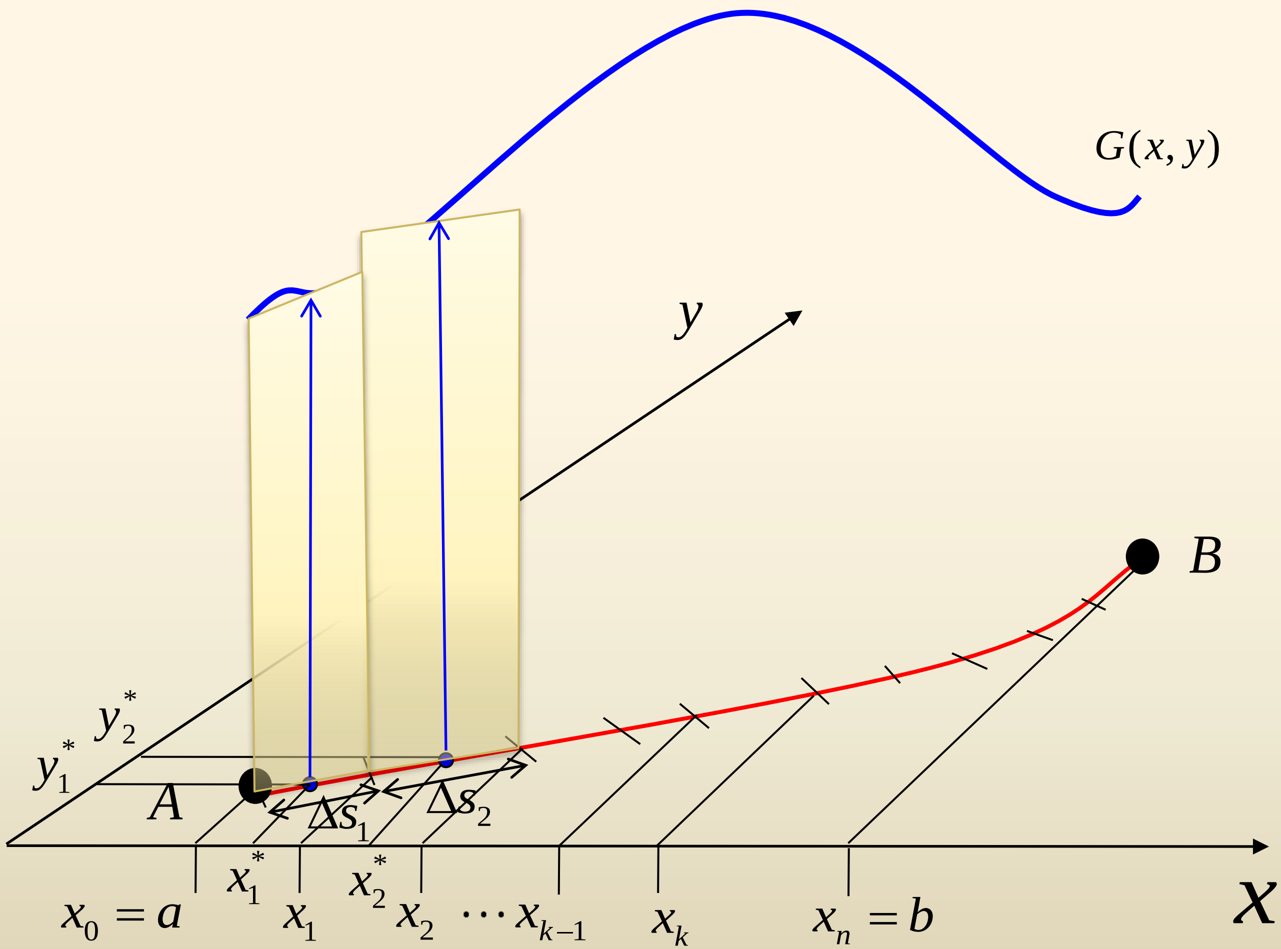
$y_1 \le y_2^* \le y_2$

Δs_2

$G(x_1^*, y_1^*)\Delta s_1$

Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$ subinterval

$y_0 \le y_1^* \le y_1$ Δs_1

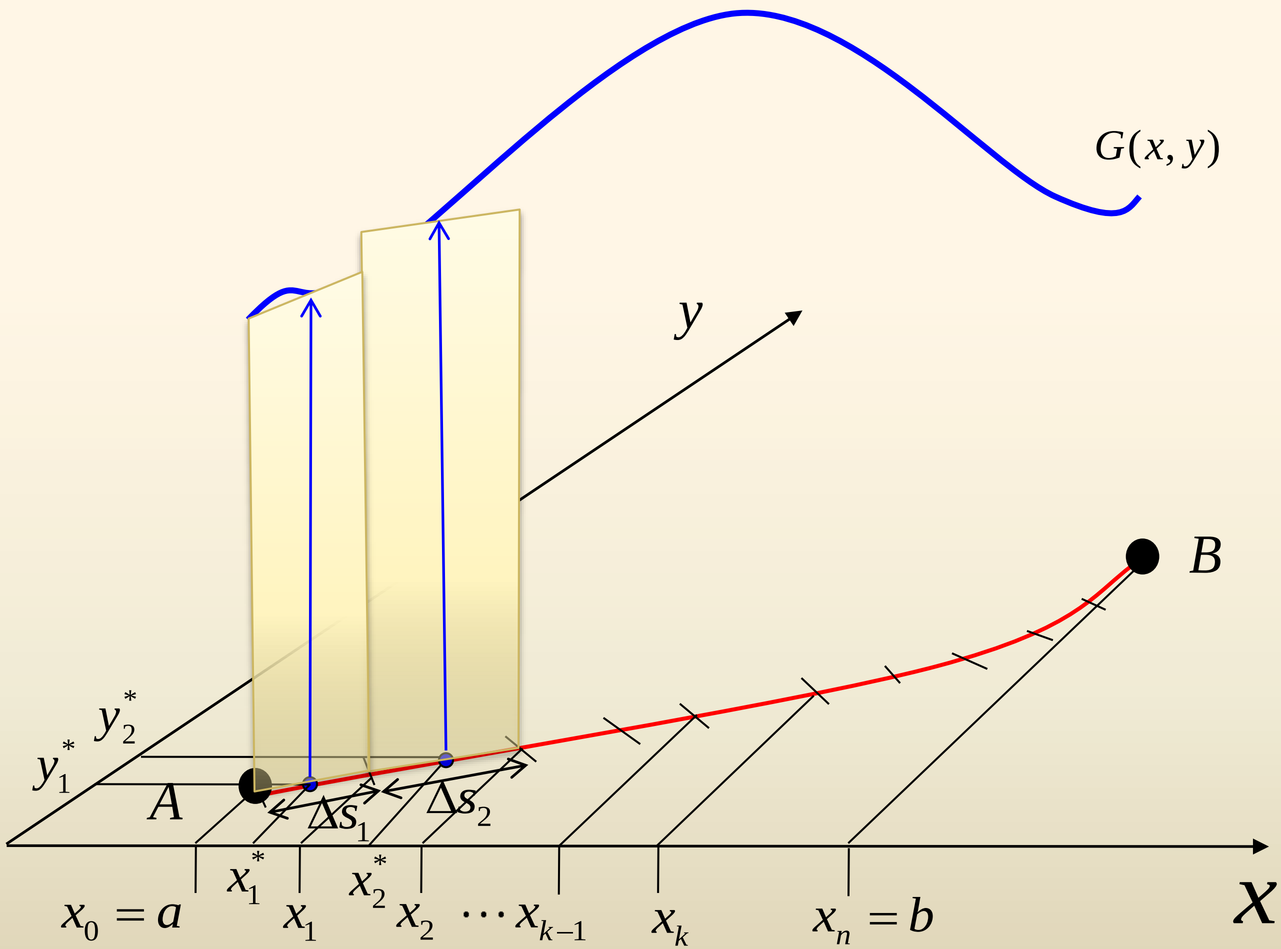
$x_1 \le x_2^* \le x_2$

$y_1 \le y_2^* \le y_2$ Δs_2

$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2$

Line Integrals

Line Integral in the Plane

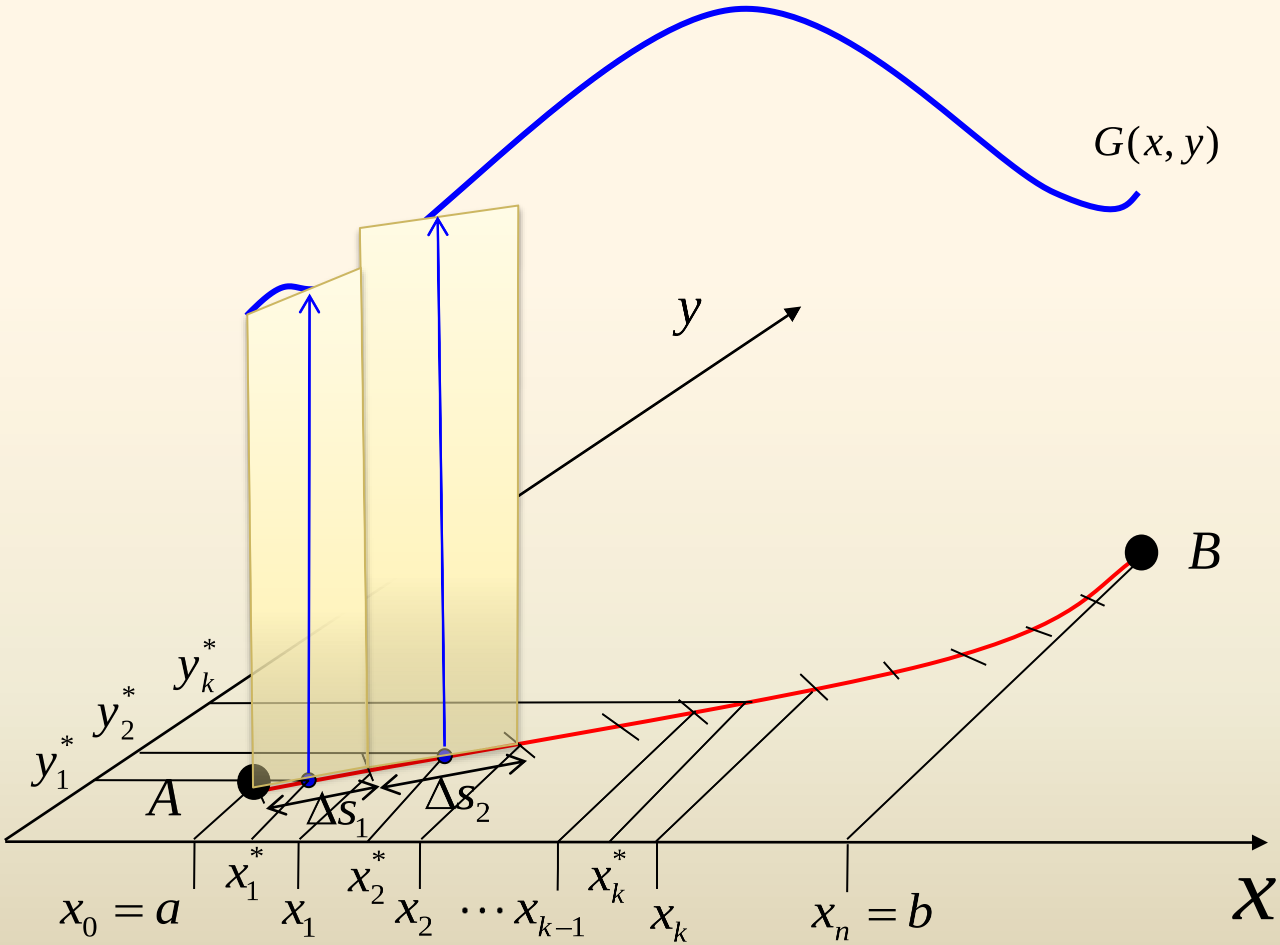


	subinterval
$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2
⋮	
$x_{k-1} \le x_k^* \le x_k$ $y_{k-1} \le y_k^* \le y_k$	Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2$$

Line Integrals

Line Integral in the Plane

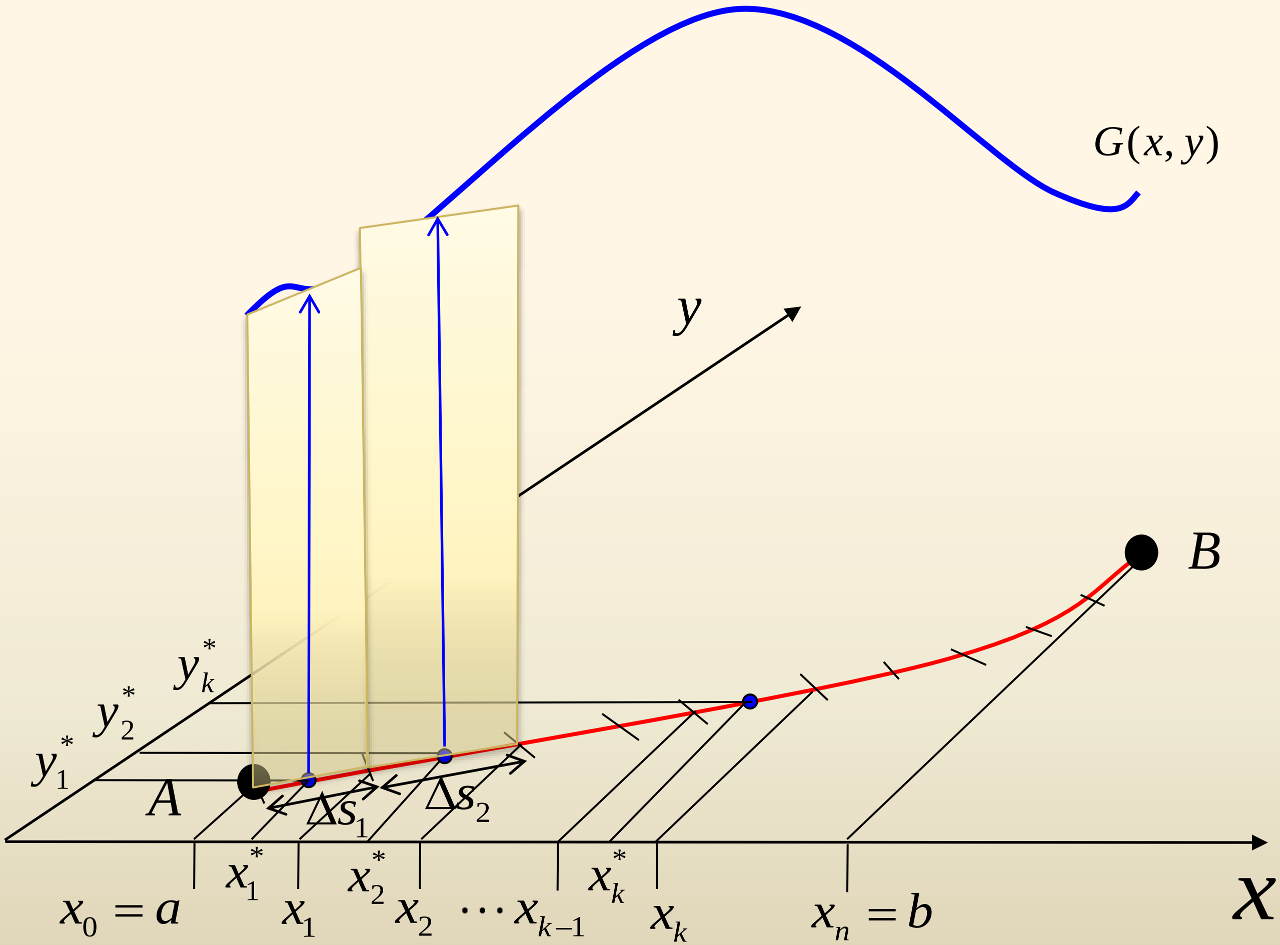


	subinterval
$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2
$\vdots$	
$x_{k-1} \le x_k^* \le x_k$ $y_{k-1} \le y_k^* \le y_k$	Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2$$

Line Integrals

Line Integral in the Plane

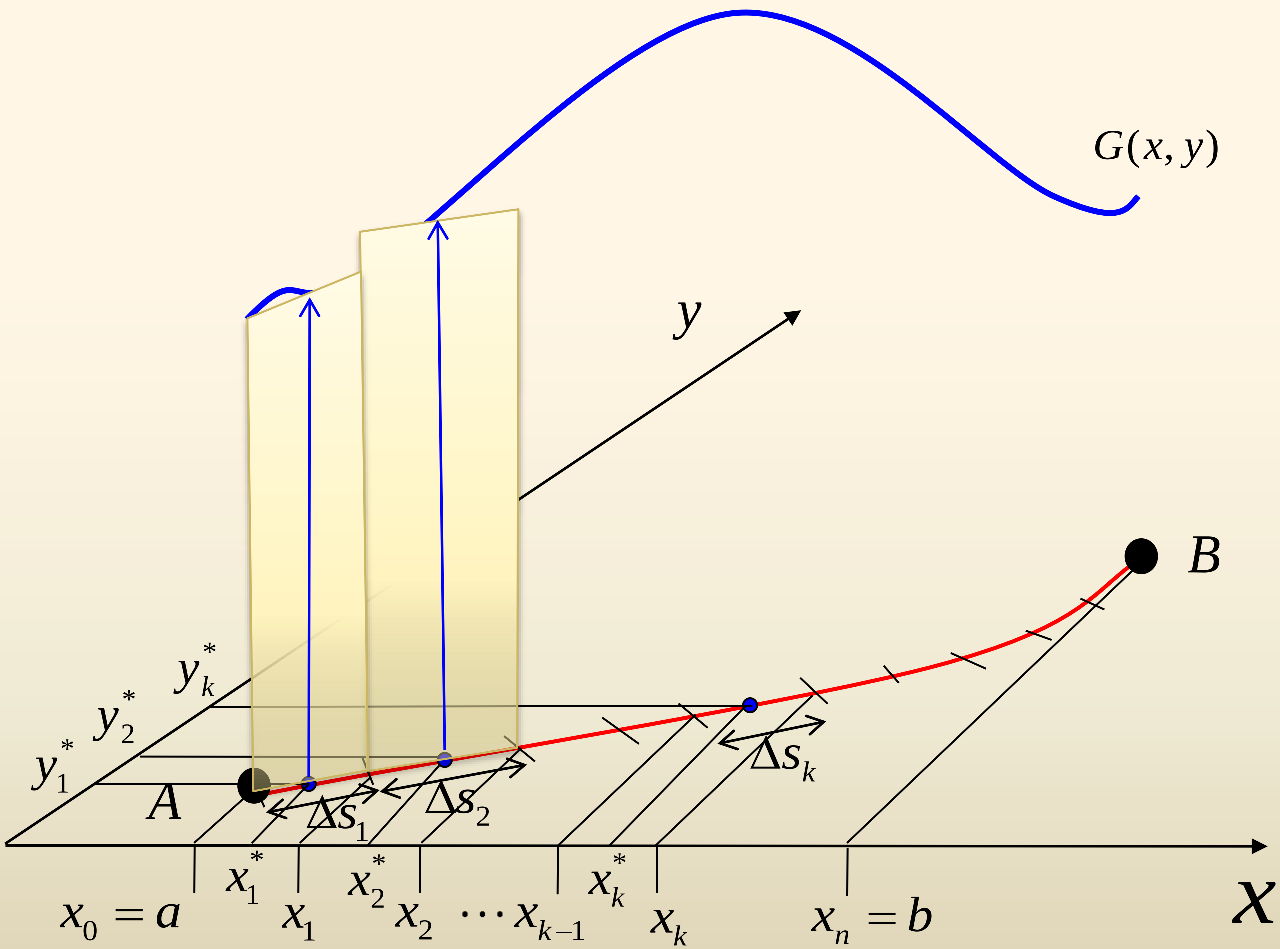


$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	subinterval Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2
$\vdots$	
$x_{k-1} \le x_k^* \le x_k$ $y_{k-1} \le y_k^* \le y_k$	Δs_k

$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2$

Line Integrals

Line Integral in the Plane

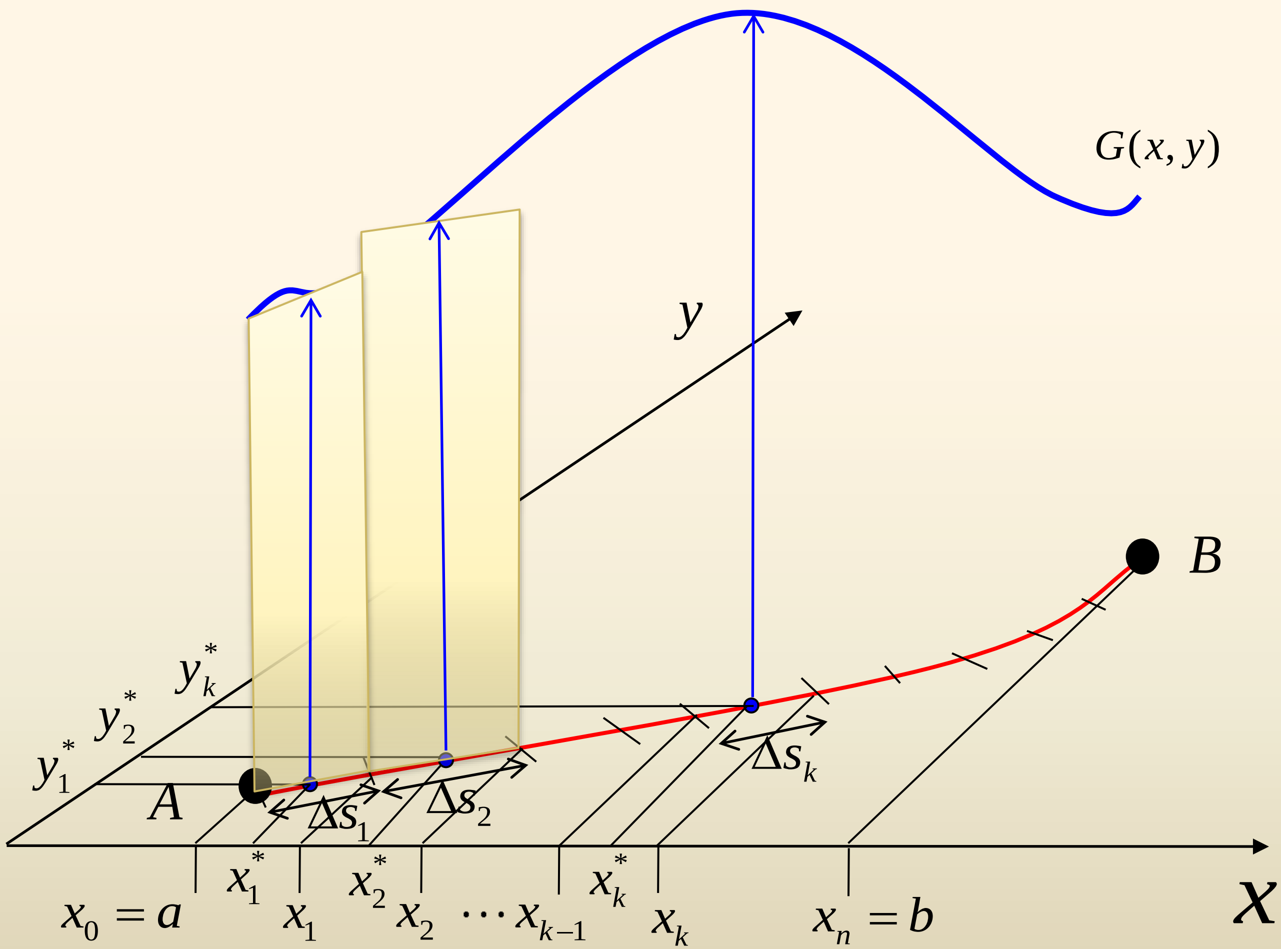


$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	subinterval Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2
$\vdots$	
$x_{k-1} \le x_k^* \le x_k$ $y_{k-1} \le y_k^* \le y_k$	Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2$$

Line Integrals

Line Integral in the Plane



subinterval

$x_0 \le x_1^* \le x_1$, $y_0 \le y_1^* \le y_1$ — Δs_1

$x_1 \le x_2^* \le x_2$, $y_1 \le y_2^* \le y_2$ — Δs_2

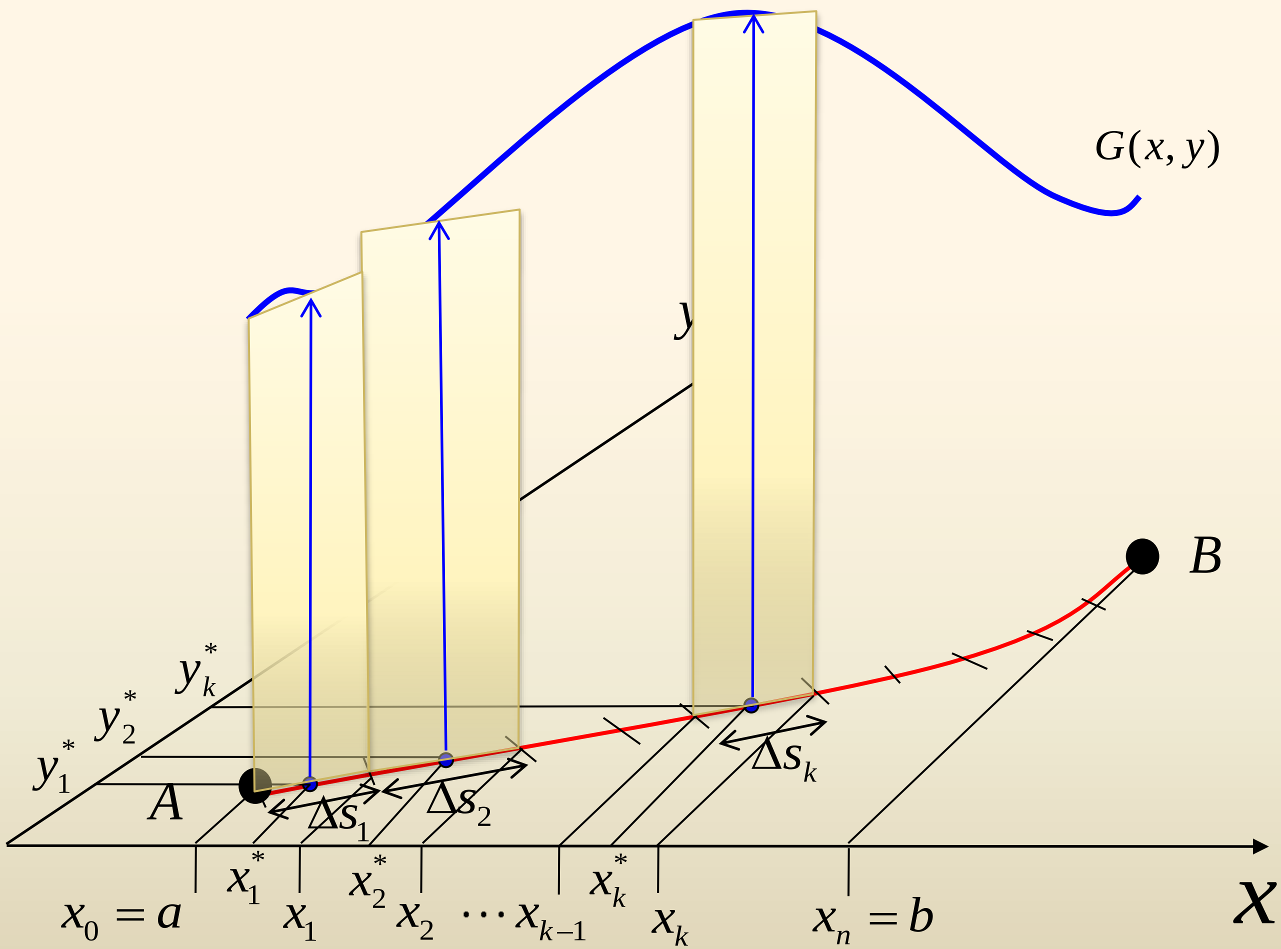
$\vdots$

$x_{k-1} \le x_k^* \le x_k$, $y_{k-1} \le y_k^* \le y_k$ — Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2$$

Line Integrals

Line Integral in the Plane

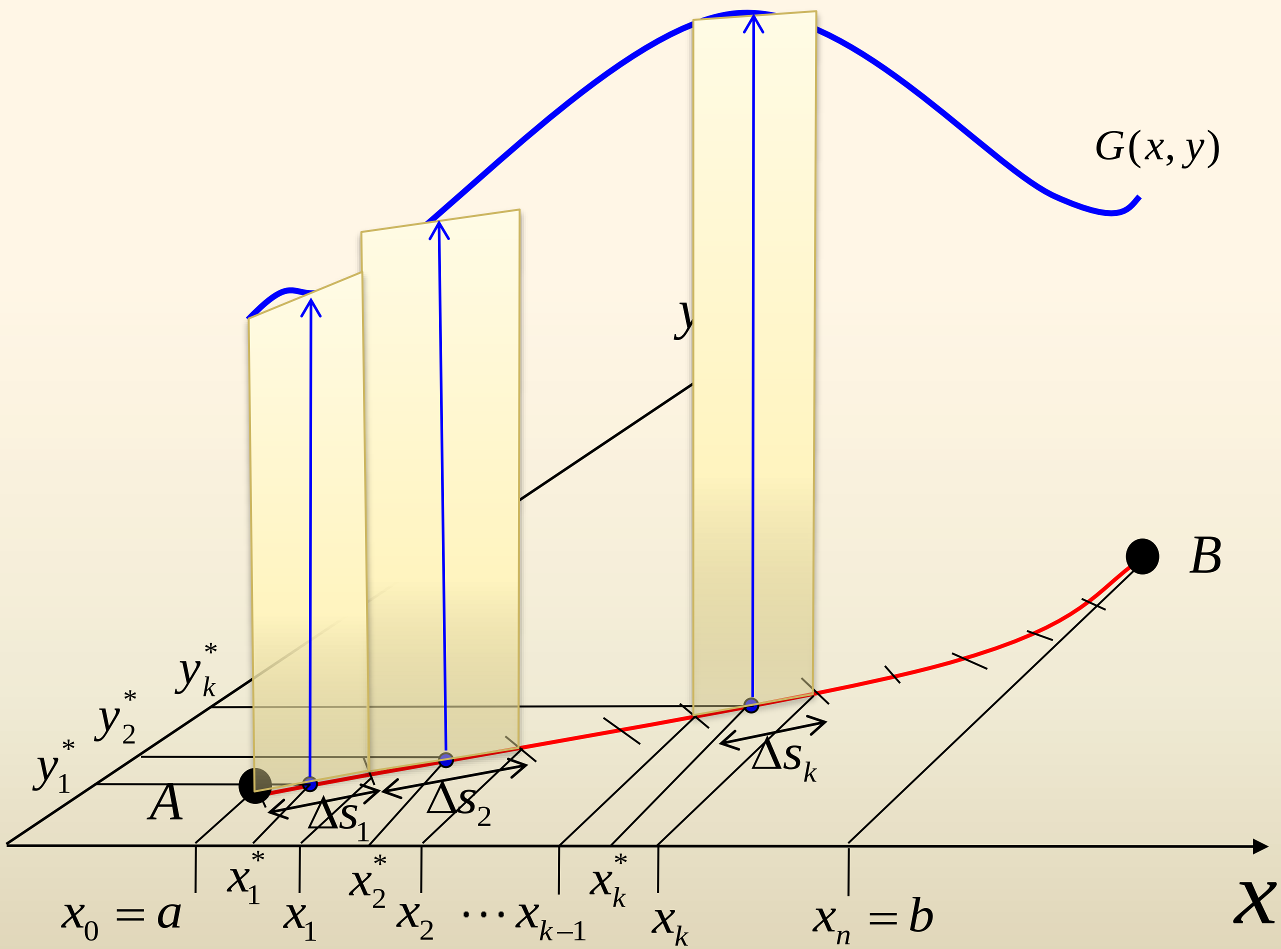


	subinterval
$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2
$\vdots$	
$x_{k-1} \le x_k^* \le x_k$ $y_{k-1} \le y_k^* \le y_k$	Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2$$

Line Integrals

Line Integral in the Plane

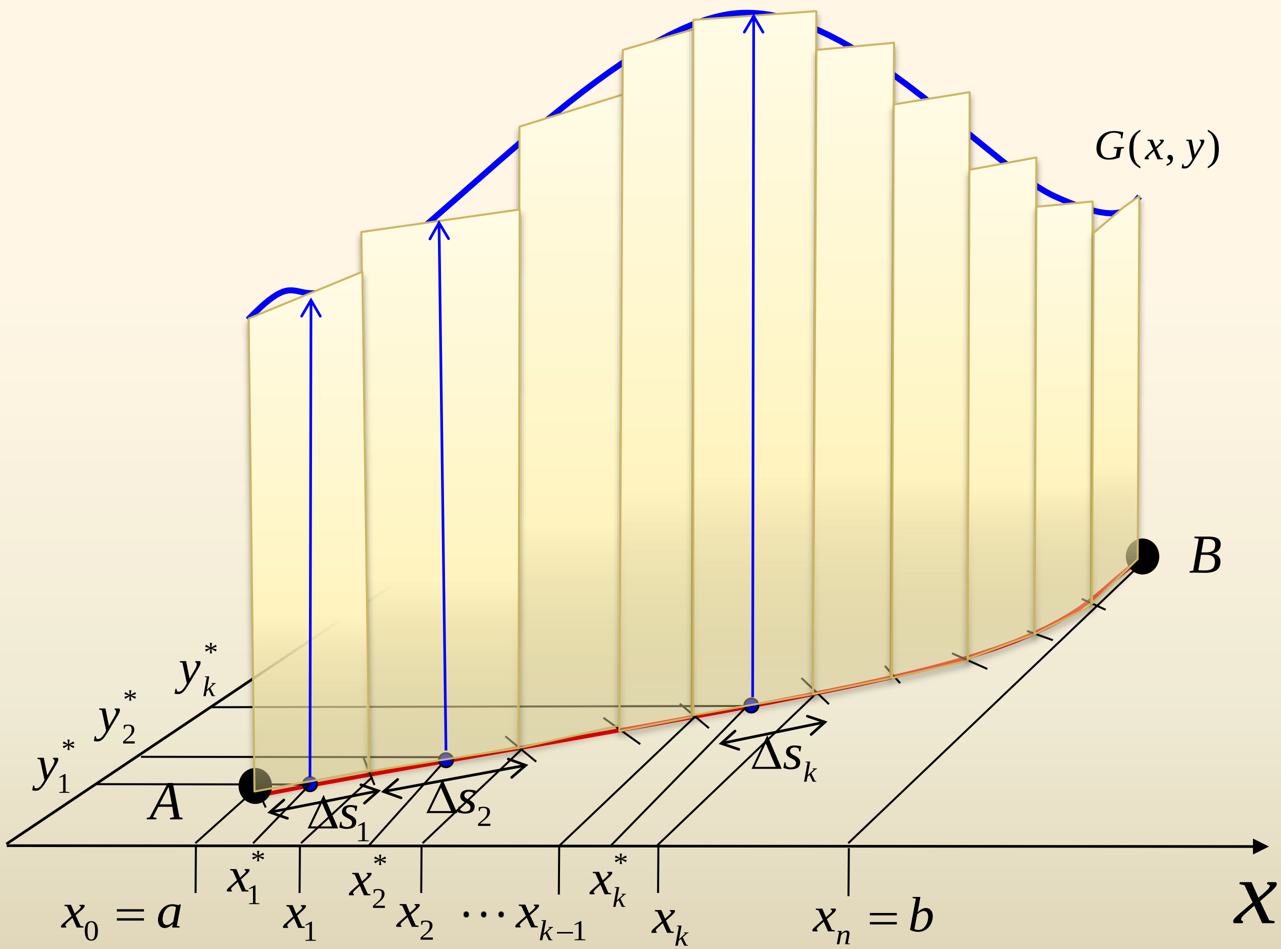


	subinterval
$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2
$\vdots$	
$x_{k-1} \le x_k^* \le x_k$ $y_{k-1} \le y_k^* \le y_k$	Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2 + \cdots + G(x_k^*, y_k^*)\Delta s_k$$

Line Integrals

Line Integral in the Plane



subinterval

$x_0 \le x_1^* \le x_1$, $y_0 \le y_1^* \le y_1$ — Δs_1

$x_1 \le x_2^* \le x_2$, $y_1 \le y_2^* \le y_2$ — Δs_2

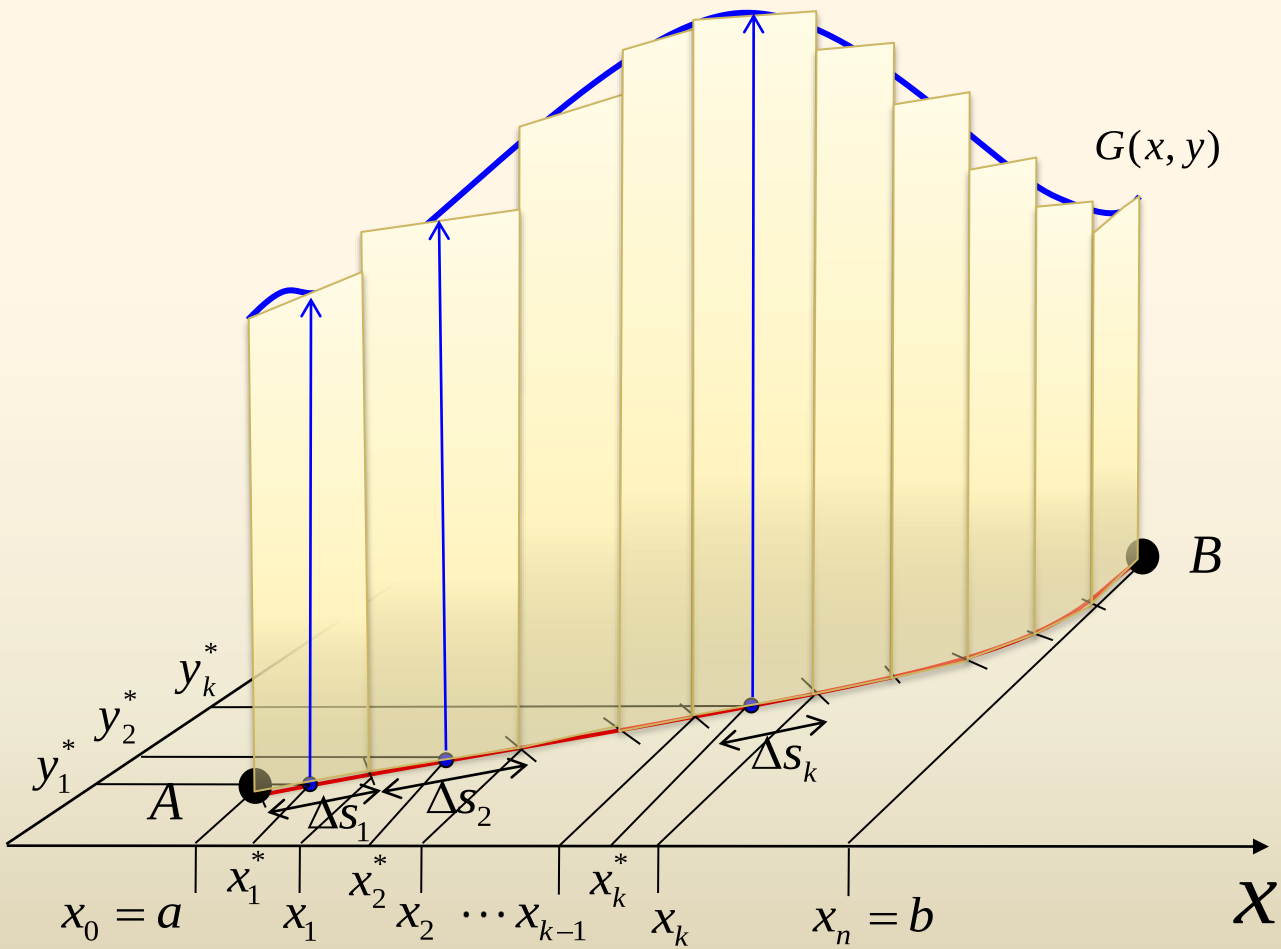
$\vdots$

$x_{k-1} \le x_k^* \le x_k$, $y_{k-1} \le y_k^* \le y_k$ — Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2 + \cdots + G(x_k^*, y_k^*)\Delta s_k$$

Line Integrals

Line Integral in the Plane

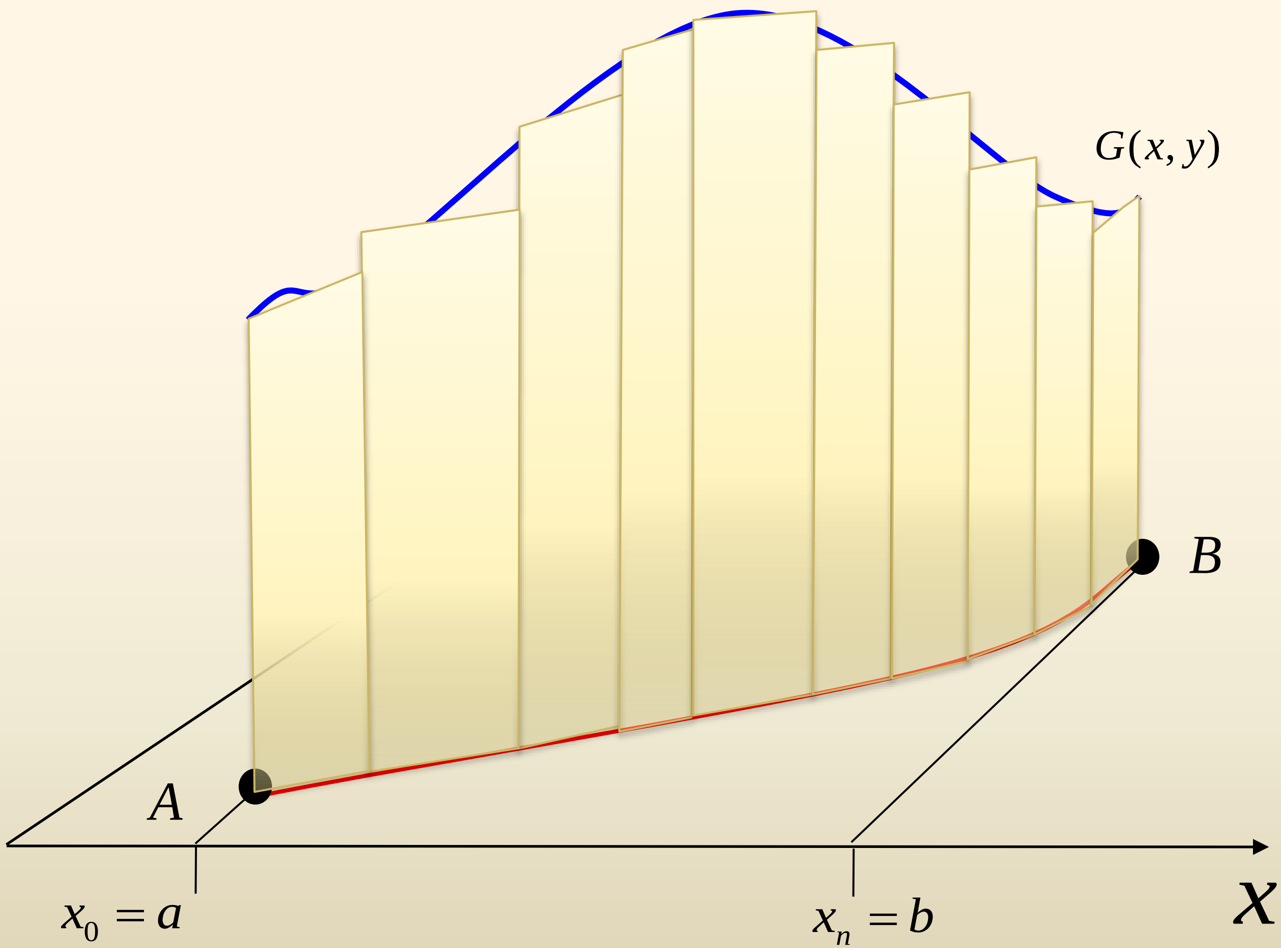


	subinterval
$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2
$\vdots$	
$x_{k-1} \le x_k^* \le x_k$ $y_{k-1} \le y_k^* \le y_k$	Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2 + \cdots + G(x_k^*, y_k^*)\Delta s_k + \cdots + G(x_b^*, y_b^*)\Delta s_b$$

Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$ subinterval

$y_0 \le y_1^* \le y_1$ Δs_1

$x_1 \le x_2^* \le x_2$

$y_1 \le y_2^* \le y_2$ Δs_2

$\vdots$

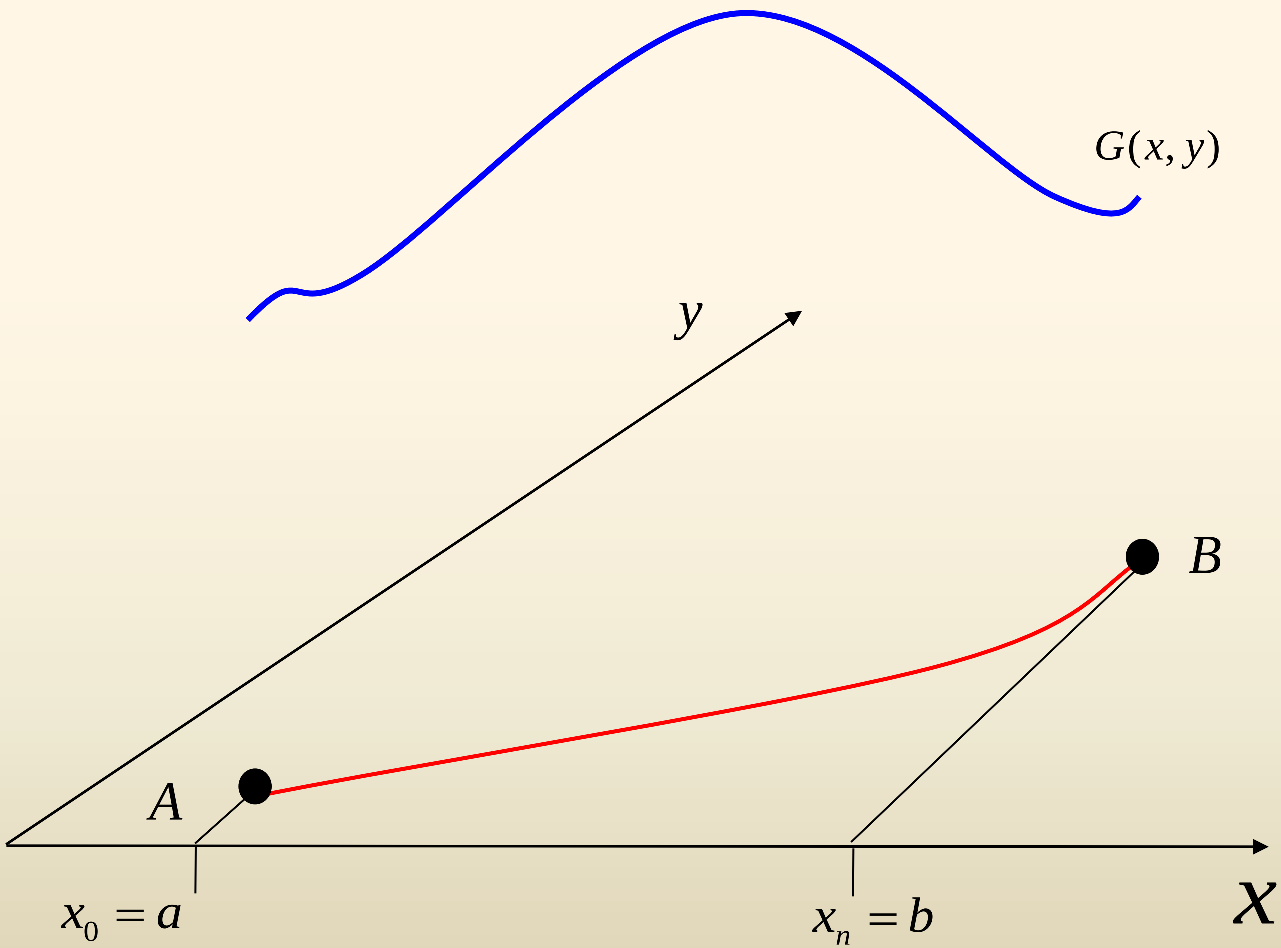
$x_{k-1} \le x_k^* \le x_k$

$y_{k-1} \le y_k^* \le y_k$ Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2 + \cdots + G(x_k^*, y_k^*)\Delta s_k + \cdots + G(x_b^*, y_b^*)\Delta s_b$$

Line Integrals

Line Integral in the Plane



$x_0 \le x_1^* \le x_1$ subinterval

$y_0 \le y_1^* \le y_1$ Δs_1

$x_1 \le x_2^* \le x_2$

$y_1 \le y_2^* \le y_2$ Δs_2

$\vdots$

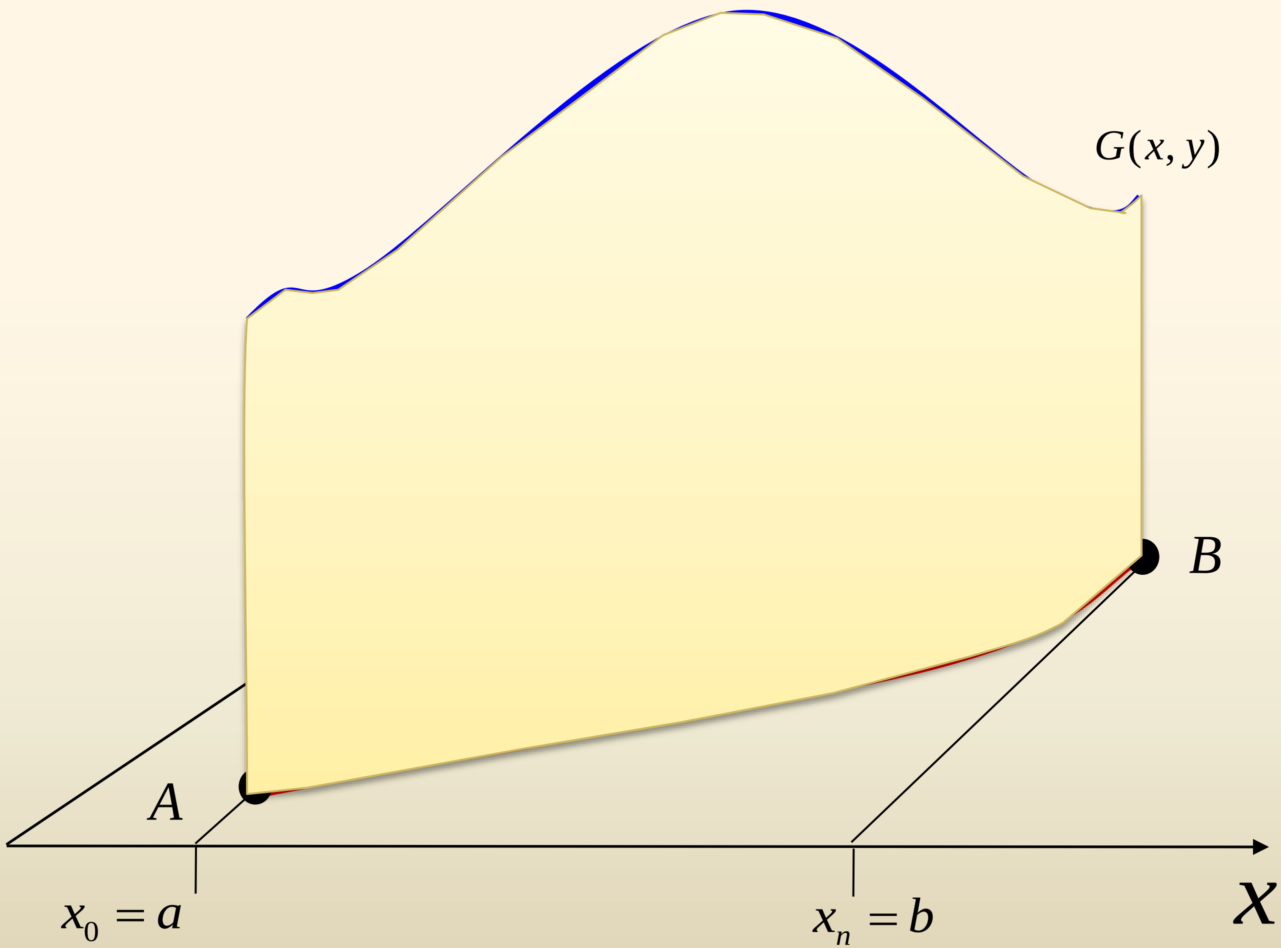
$x_{k-1} \le x_k^* \le x_k$

$y_{k-1} \le y_k^* \le y_k$ Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2 + \cdots + G(x_k^*, y_k^*)\Delta s_k + \cdots + G(x_b^*, y_b^*)\Delta s_b$$

Line Integrals

Line Integral in the Plane

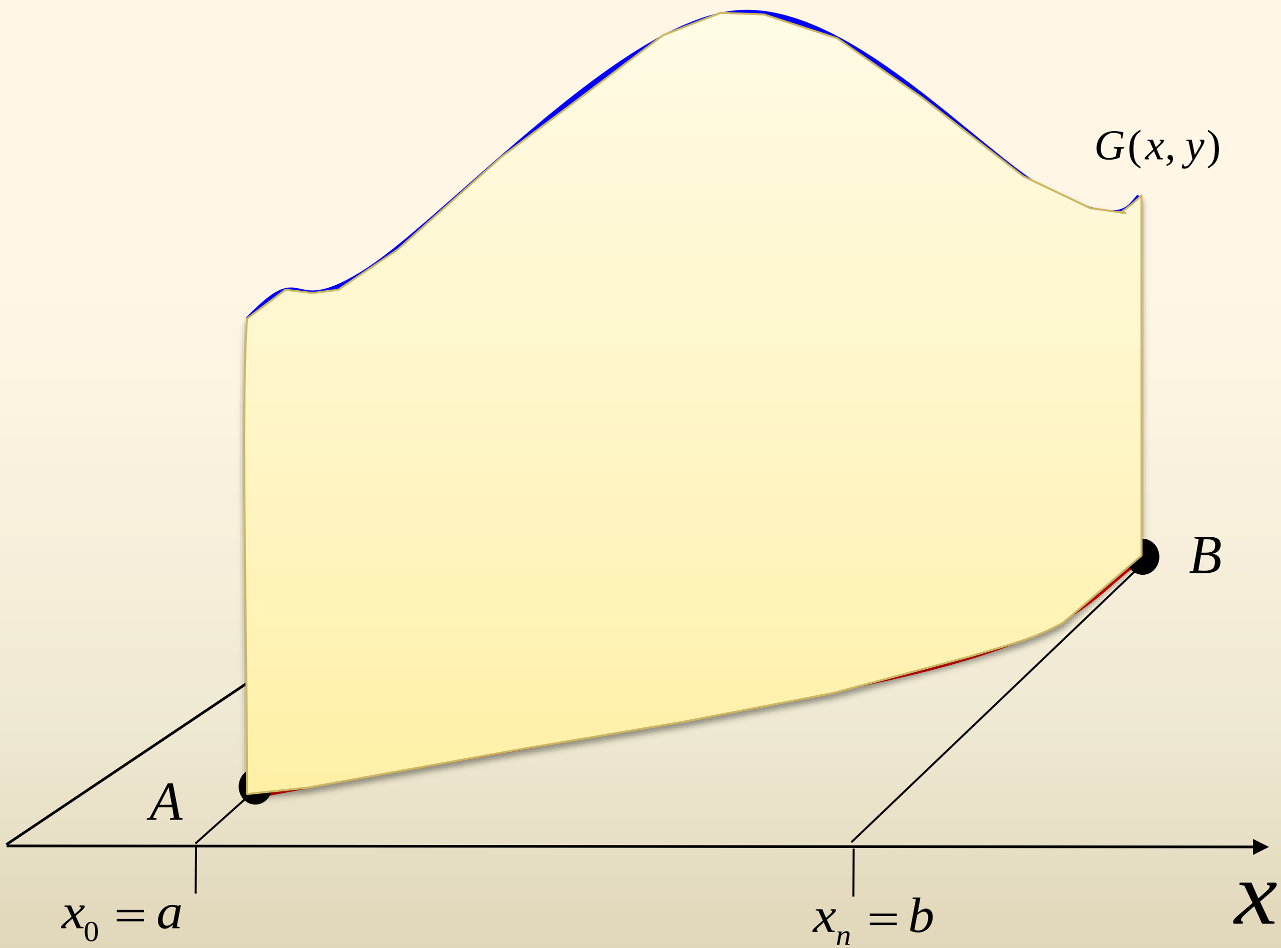


$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	subinterval Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2
$\vdots$	
$x_{k-1} \le x_k^* \le x_k$ $y_{k-1} \le y_k^* \le y_k$	Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2 + \cdots + G(x_k^*, y_k^*)\Delta s_k + \cdots + G(x_b^*, y_b^*)\Delta s_b$$

Line Integrals

Line Integral in the Plane



	subinterval
$x_0 \le x_1^* \le x_1$ $y_0 \le y_1^* \le y_1$	Δs_1
$x_1 \le x_2^* \le x_2$ $y_1 \le y_2^* \le y_2$	Δs_2
$\vdots$	
$x_{k-1} \le x_k^* \le x_k$ $y_{k-1} \le y_k^* \le y_k$	Δs_k

$$G(x_1^*, y_1^*)\Delta s_1 + G(x_2^*, y_2^*)\Delta s_2 + \cdots + G(x_k^*, y_k^*)\Delta s_k + \cdots + G(x_b^*, y_b^*)\Delta s_b \Rightarrow \int_C G(x, y)ds = \lim_{\|P\| \to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k$$

$\|P\|$: length of the longest subinterval

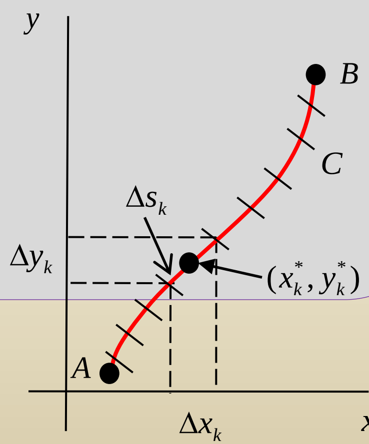
Line Integrals

Line Integral in the Plane

z=*G* (x,y)

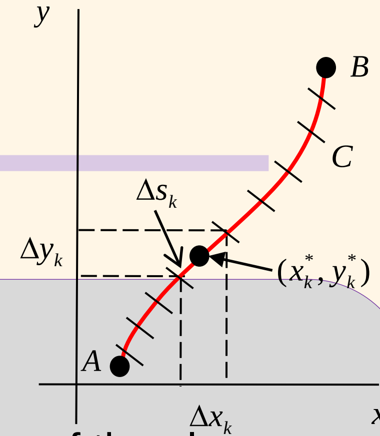
1. Let G be defined in some region that contains the smooth curve *C* defined by
   $x = f(t), y = g(t),\ a \le t \le b$
2. Divide *C* into n subarcs of length Δs_k according to the partition
   $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$ of $[a,b]$. Let the projection of each subarc onto the x- and y=axes have length Δx_k and Δy_k,respectively.
3. Let $\|P\|$ be the **norm** of the partition or the length of the longest subarc.
4. Choose a point (x_k^*, y_k^*) in each subarc.
5. Form the sum

$$\sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k, \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k, \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k$$



Line Integrals

Line Integral in the Plane



Definition 9.9

Line Integrals in the Plane

Let G be a function of two variables x and y defined on a region of the plane containing a smooth curve *C.*

(i) The line integral of *G* along *C* from *A* to *B* with respect to *x* is

$$\int_C G(x,y)\,dx = \lim_{||P||\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta x_k$$

(ii) The line integral of *G* along *C* from *A* to *B* with respect to y is

$$\int_C G(x,y)\,dy = \lim_{||P||\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta y_k$$

(iii) The line integral of *G* along *C* from *A* to *B* with respect arc length is

$$\int_C G(x,y)\,ds = \lim_{||P||\to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*)\Delta s_k$$

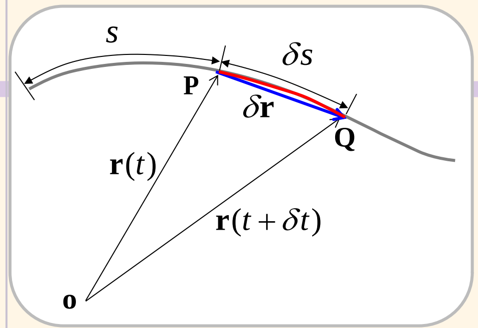
Line Integrals

Method of Evaluation
- Curve Defined Parametrically

Line Integrals

Method of Evaluation
- Curve Defined Parametrically

Tangent Vector and Unit Tangent Vector



The chord PQ

$$\delta \mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

The arc length s

unit tangent vector : direction

magnitude

tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{T}\left|\frac{d\mathbf{r}}{dt}\right| = \dot{\mathbf{r}}$$

unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta \mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

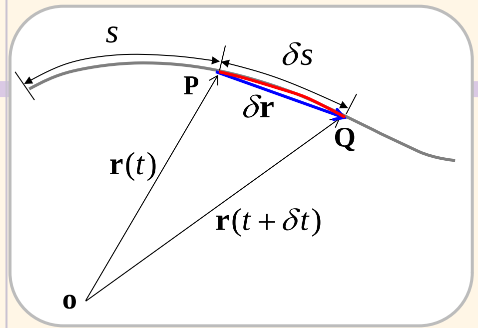
$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}$$

$$\therefore \frac{ds}{dt} = \dot{s} = \left|\frac{d\mathbf{r}}{dt}\right| = \left|\dot{\mathbf{r}}\right|$$

Line Integrals

Method of Evaluation
- Curve Defined Parametrically

Tangent Vector and Unit Tangent Vector



The chord PQ

$$\delta\mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

The arc length s

(circle) unit tangent vector : direction
(box) magnitude

tangent vector at P

$$\lim_{\delta t \to 0} \frac{\delta\mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt} = \mathbf{T}\left|\frac{d\mathbf{r}}{dt}\right| = \dot{\mathbf{r}}$$

unit tangent vector at P

$$\lim_{\delta s \to 0} \frac{\delta\mathbf{r}}{\delta s} = \frac{d\mathbf{r}}{ds} = \mathbf{T}$$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}$$

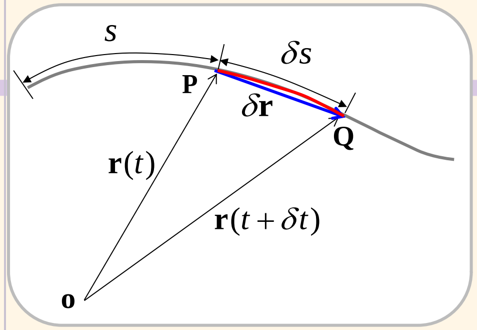
$$\therefore \frac{ds}{dt} = \dot{s} = \left|\frac{d\mathbf{r}}{dt}\right| = |\dot{\mathbf{r}}|$$

Line Integrals

Method of Evaluation
- Curve Defined Parametrically

If C is a smooth curve parameterized by

Tangent Vector and Unit Tangent Vector



The chord PQ

$$\delta \mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t)$$

The arc length s

unit tangent vector : direction

magnitude

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Line Integrals

Method of Evaluation
- Curve Defined Parametrically

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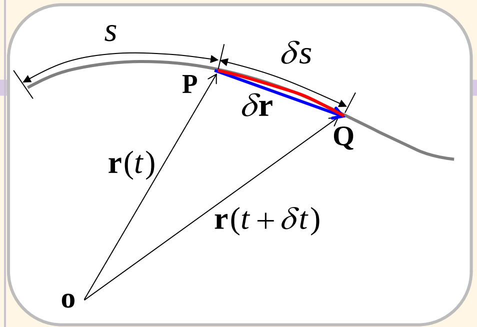
$$x = f(t),\ y = g(t), a \le t \le b$$

Tangent Vector and Unit Tangent Vector



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The arc length s

unit tangent vector : direction (circled)
magnitude (boxed)

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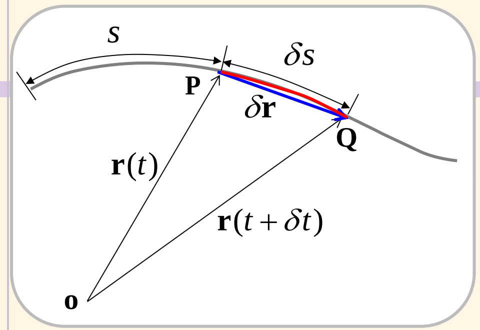

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○ unit tangent vector : direction

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Line Integrals

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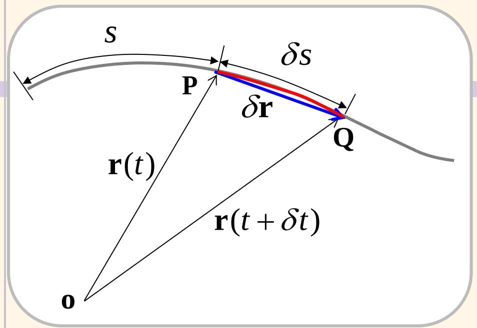

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Line Integrals

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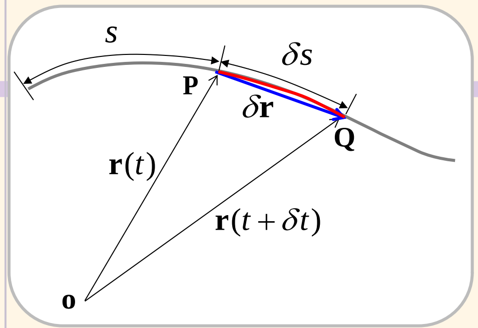

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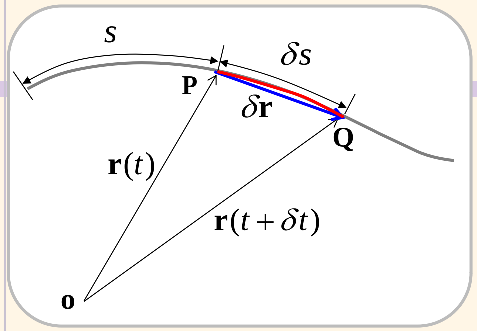

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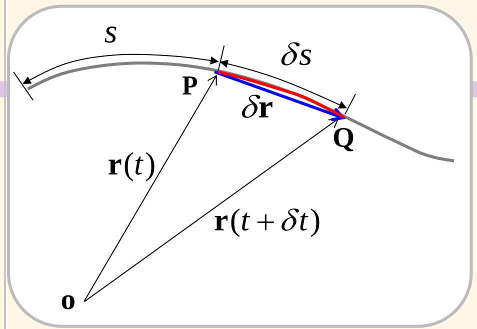

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Convert the line integral to a definite integral in a single variable

Line Integrals

Notation

Line Integrals

Notation

$$\int_C P(x,y)dx + \int_C Q(x,y)dy$$

In practice, It can be written as

$$\int_C P(x,y)dx + Q(x,y)dy$$ **or simply** $$\int_C Pdx + Qdy$$

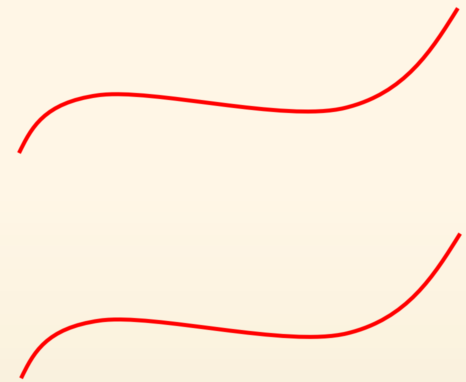
A line integral along a closed curve C is very often denoted by

$$\oint_C Pdx + Qdy$$

$||P||$: the **norm** of the partition or the length of the longest subarc

Line Integrals

Curves with opposite orientation



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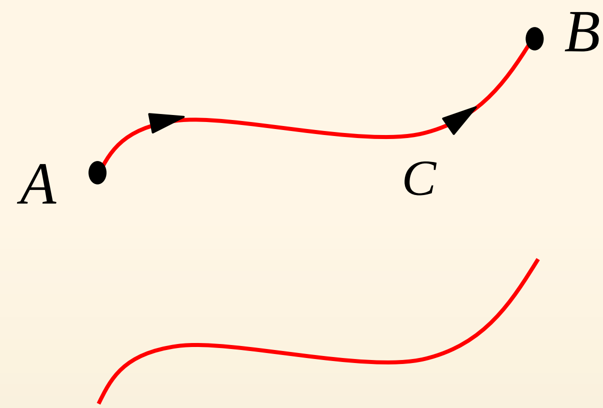
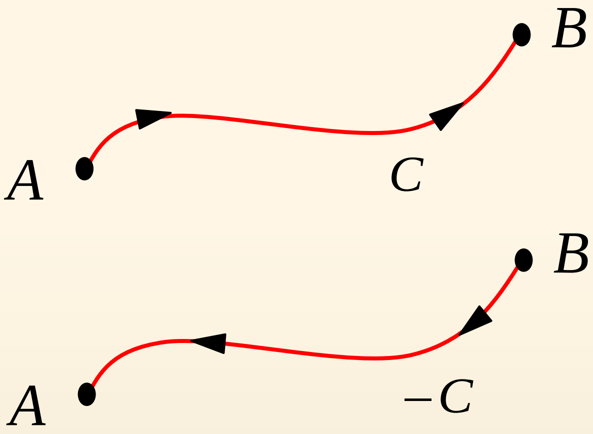
Line Integrals

Curves with opposite orientation



Line Integrals

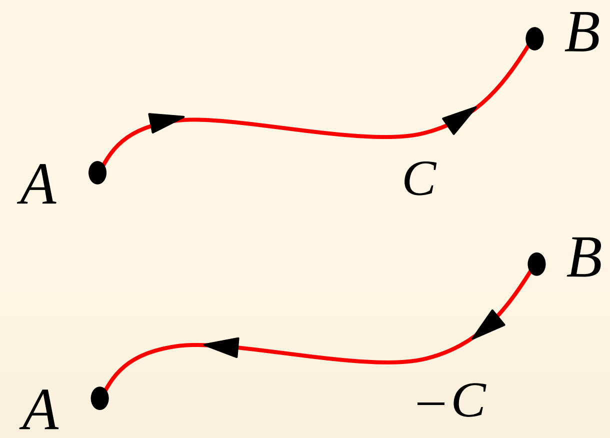
Curves with opposite orientation



Line Integrals

Curves with opposite orientation

$$\int_{-C} P\,dx + Q\,dy = -\int_{C} P\,dx + Q\,dy$$



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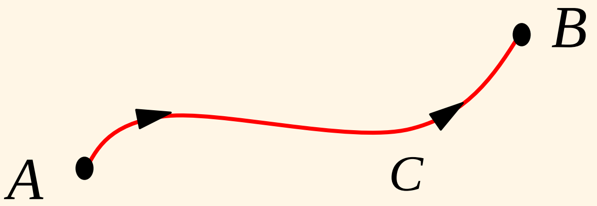
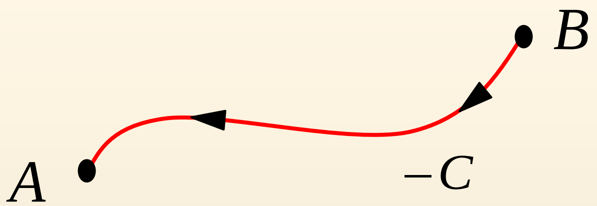


$$\int_{-C} P\,dx + Q\,dy + \int_{C} P\,dx + Q\,dy = 0$$



||*P*|| : the **norm** of the partition or the length of the longest subarc

Line Integrals

Curves with opposite orientation

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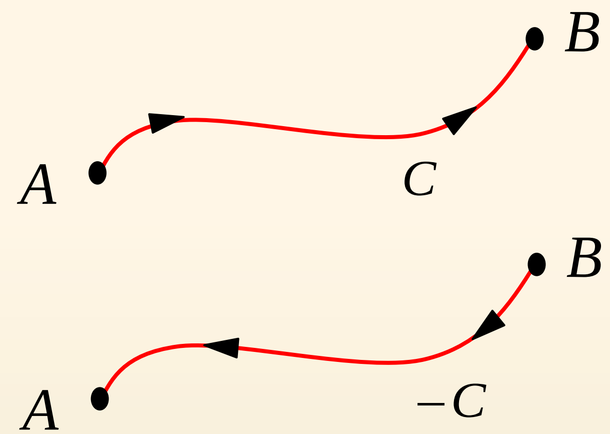

Line integrals in space

Line Integrals

Curves with opposite orientation

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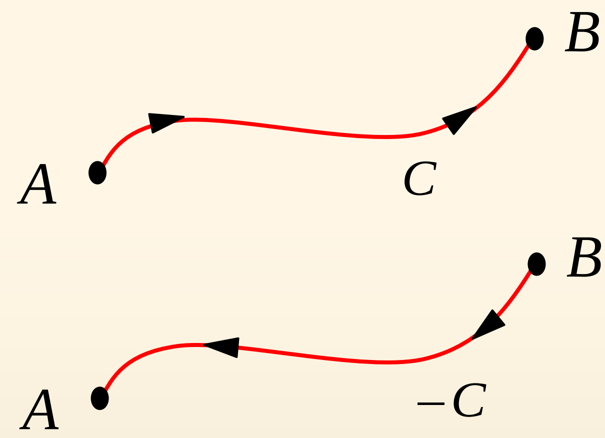


Line integrals in space

$$\int_{C} G(x, y, z) dz = \lim_{||P|| \to 0} \sum_{k=1}^{n} G(x_k^*, y_k^*, z_k^*) \Delta z_k$$

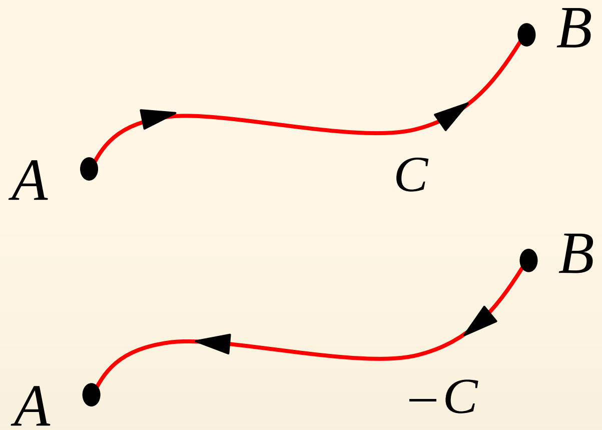
$\|P\|$: length of the longest subinterval

Line Integrals

Curves with opposite orientation

$$\int_{-C} Pdx + Qdy = -\int_C Pdx + Qdy$$

$$\int_{-C} Pdx + Qdy + \int_C Pdx + Qdy = 0$$



Line integrals in space

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Line integral along a space curve C with respect to z

Line Integrals

Method of Evaluation

Line Integrals

Method of Evaluation

If C is a smooth curve in 3-space defined parametric equation

Line Integrals

Method of Evaluation

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Line Integrals

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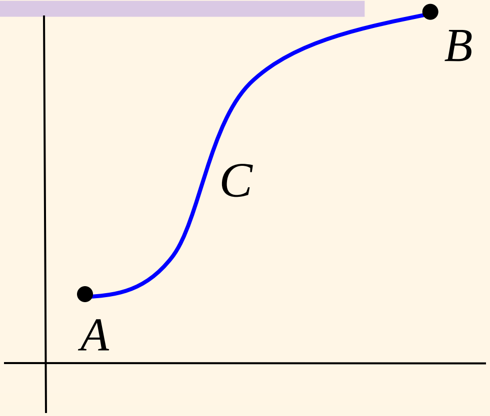
Line Integrals

Line Integrals

$$C : x = f(t),\ y = g(t),\ a \le t \le b$$

$\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j}$: position vector of points on *C*

Line Integrals



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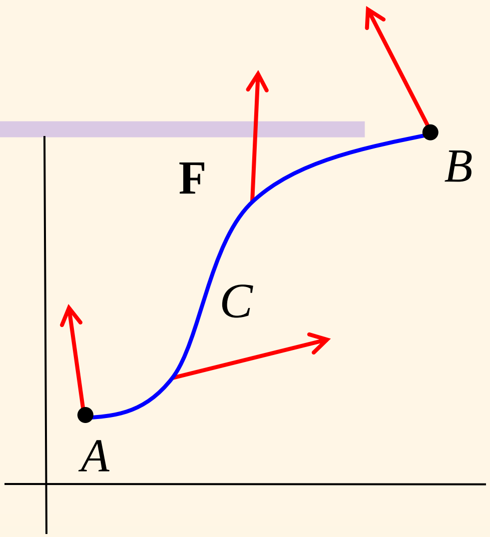

$F(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is defined along a curve

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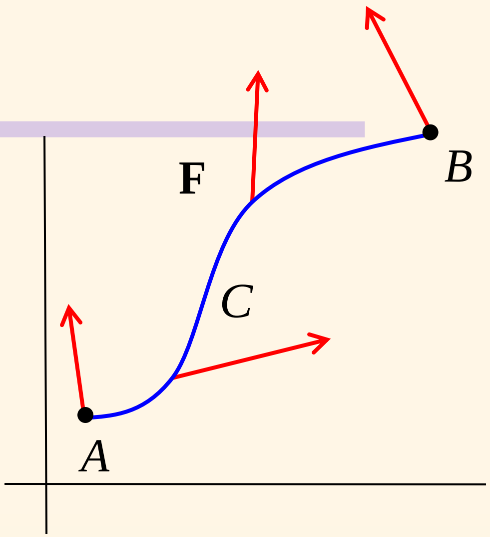

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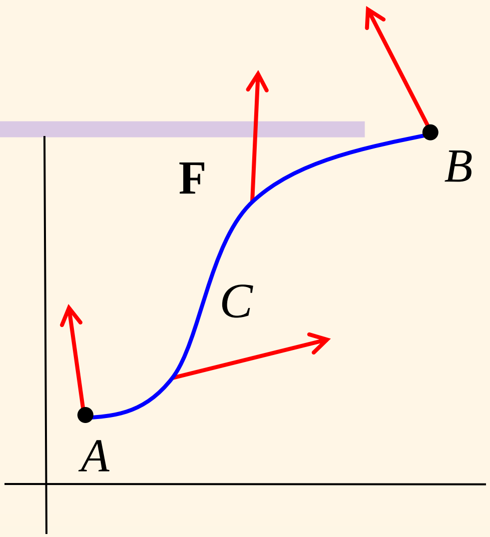

$$\frac{d\mathbf{r}}{dt} = f'(t)\mathbf{i} + g'(t)\mathbf{j} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \qquad dr = \frac{d\mathbf{r}}{dt}dt = dx\mathbf{i} + dy\mathbf{j}$$

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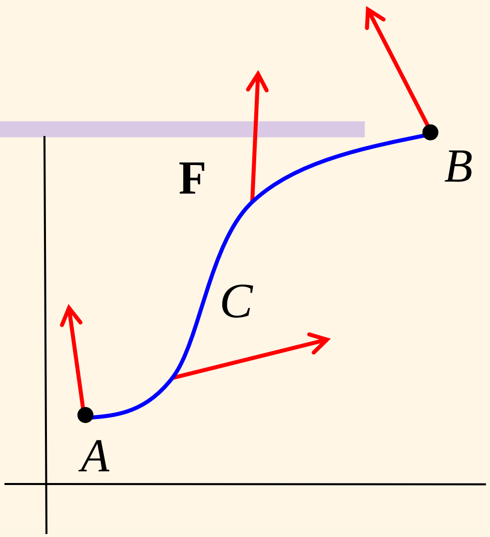

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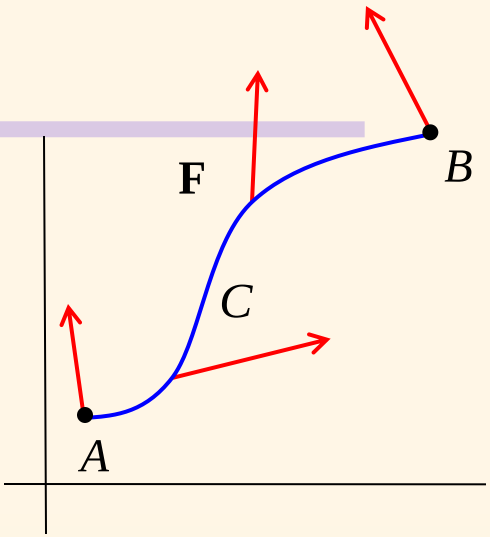
Line Integrals



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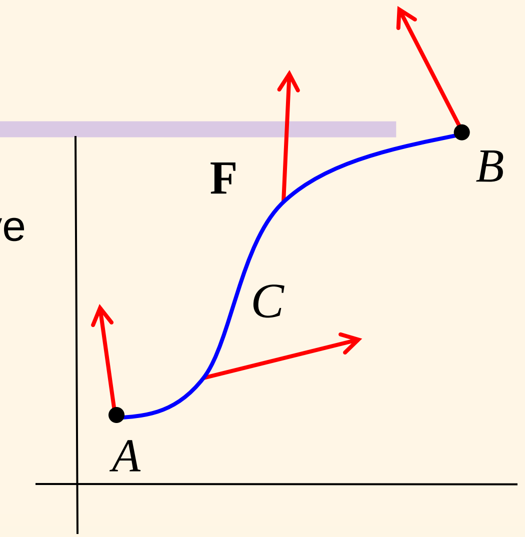
Ref. Line Integrals

Line Integrals



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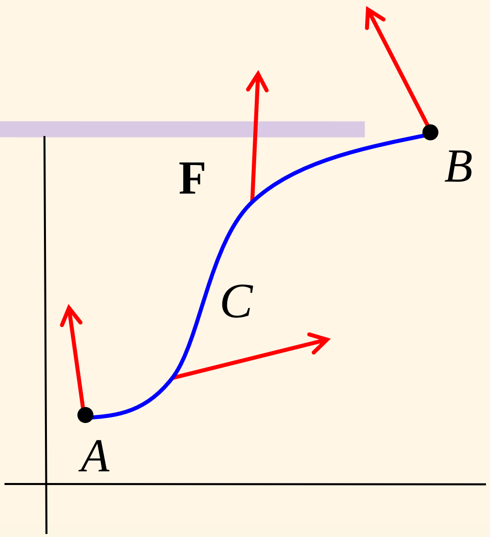
Ref. Line Integrals

$$\int_C P(x,y)dx + Q(x,y)dy = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Line Integrals



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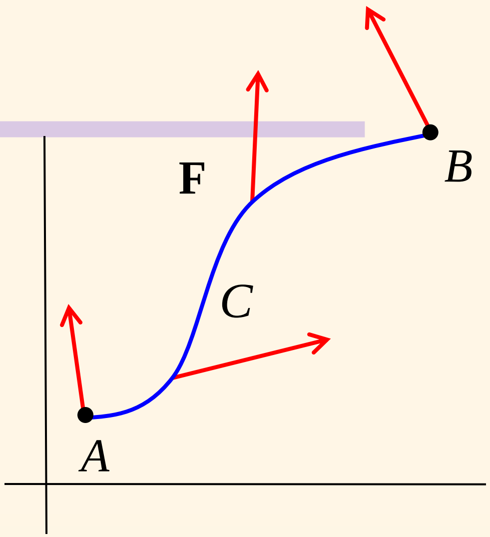
Ref. Line Integrals

$$\int_C P(x,y)dx + Q(x,y)dy = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Similarly, for a line integral on a space curve,

Line Integrals



$F(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is defined along a curve

$C : x = f(t),\ y = g(t),\ a \le t \le b$

$\mathbf{r} = f(t)\mathbf{i} + g(t)\mathbf{j}$: position vector of points on C

$$\frac{d\mathbf{r}}{dt} = f'(t)\mathbf{i} + g'(t)\mathbf{j} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \qquad dr = \frac{d\mathbf{r}}{dt}dt = dx\mathbf{i} + dy\mathbf{j}$$

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Ref. Line Integrals

$$\int_C P(x,y)dx + Q(x,y)dy = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Similarly, for a line integral on a space curve,

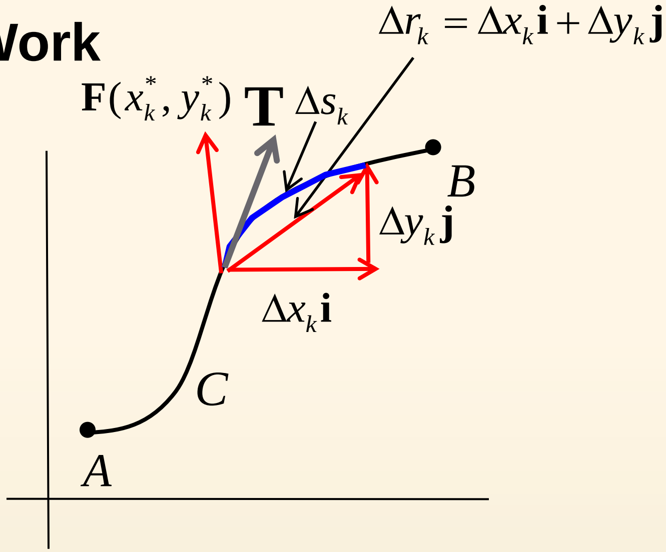
$$\int_C P(x,y)dx + Q(x,y)dy + R(x,y)dz = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Line Integrals

Work

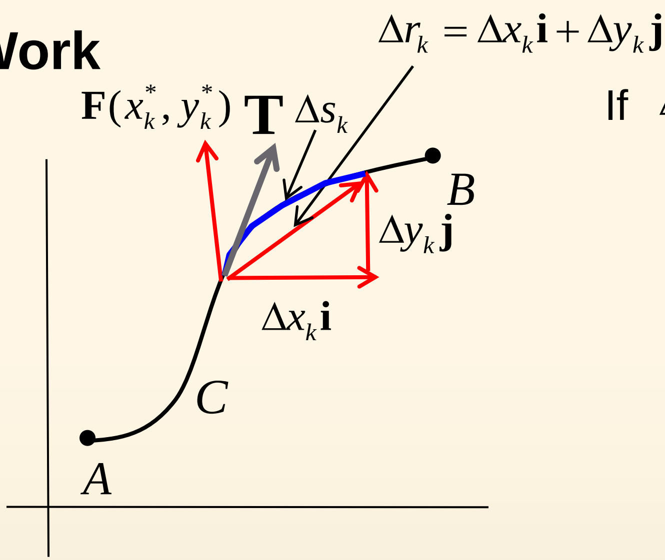
Line Integrals

Work



Line Integrals

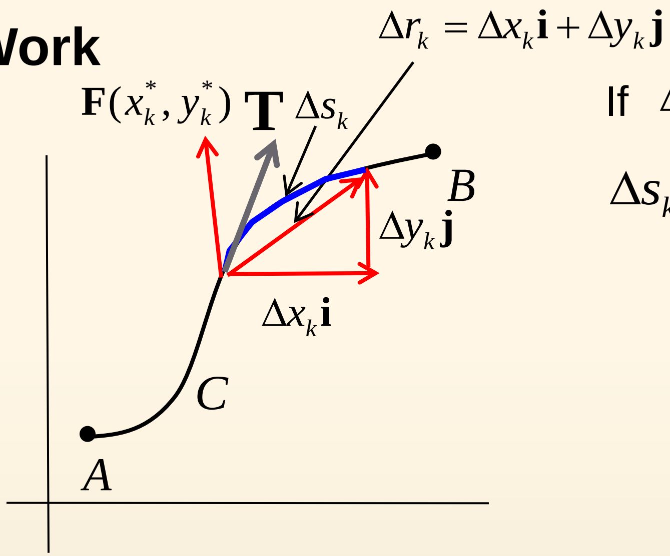
Work



If Δs_k is small, $\mathbf{F}(x_k^*, y_k^*)$ is constant force, and

Line Integrals

Work

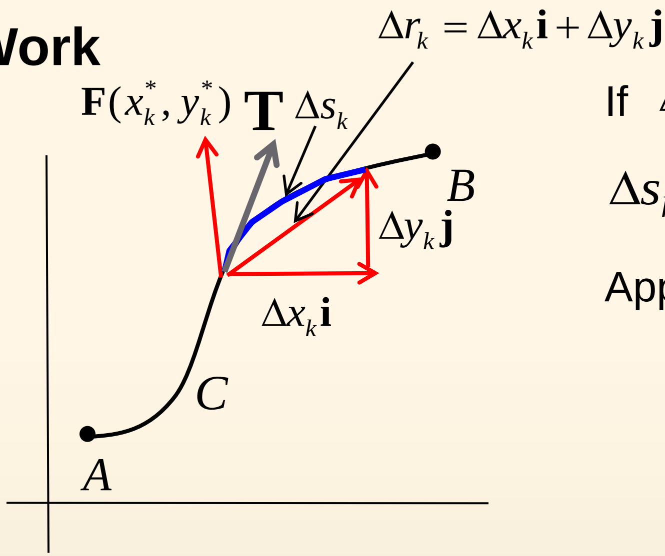


If Δs_k is small, $\mathbf{F}(x_k^*, y_k^*)$ is constant force, and

$$\Delta s_k \approx \Delta r_k$$

Line Integrals

Work



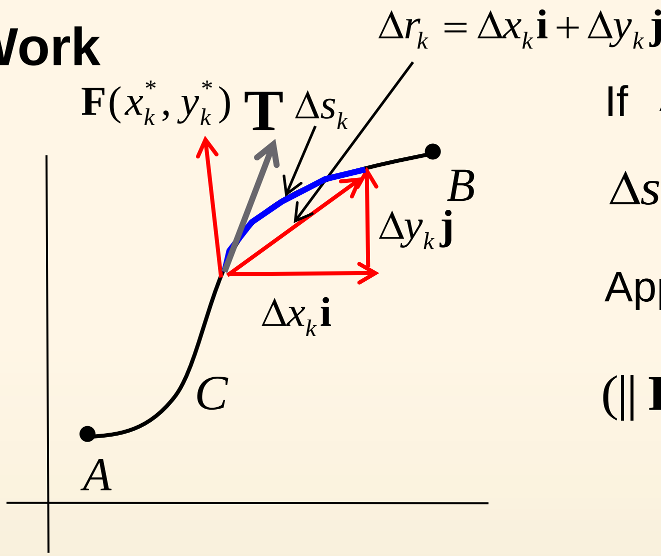
If Δs_k is small, $\mathbf{F}(x_k^*, y_k^*)$ is constant force, and

$\Delta s_k \approx \Delta r_k$

Approximate work done by F over the subarc is

Line Integrals

Work



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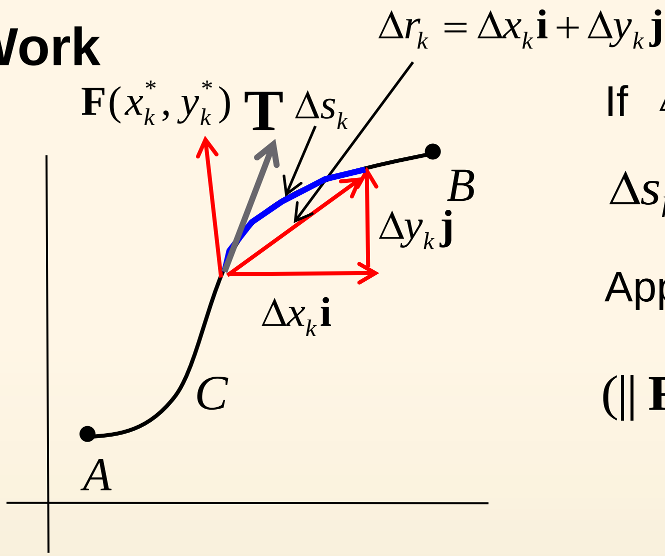
$$(\| \mathbf{F}(x_k^*, y_k^*) \| \cos\theta) \| \Delta r_k \| = \mathbf{F}(x_k^*, y_k^*) \cdot \Delta r_k$$
$$= P(x_k^*, y_k^*)\Delta x_k + Q(x_k^*, y_k^*)\Delta y_k$$

Line Integrals

Work



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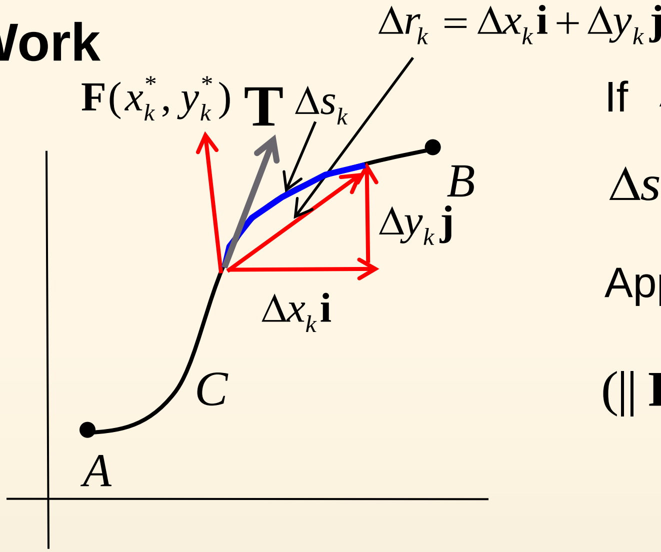
By summing the elements of work and passing to limit,

Line Integrals

Work



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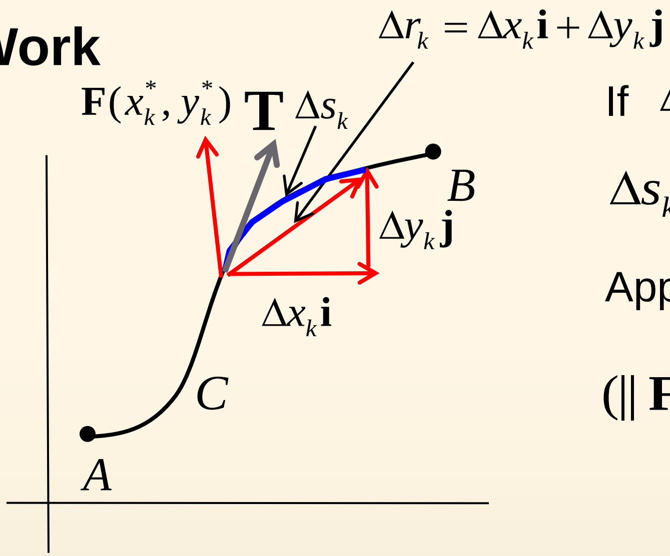
Line Integrals

Work



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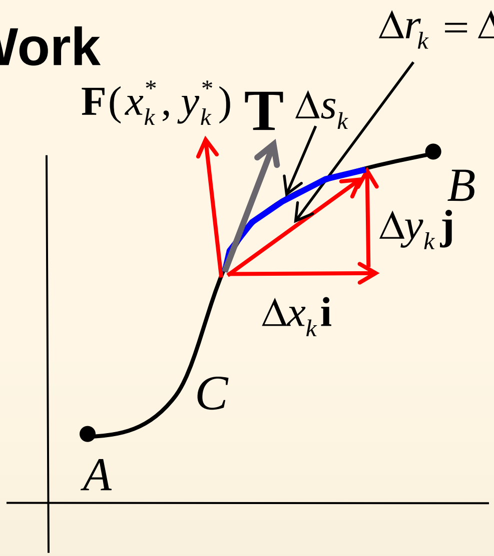
$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} \qquad d\mathbf{r} = \mathbf{T}ds \; (\mathbf{T} = d\mathbf{r}/ds)$$

Line Integrals

Work



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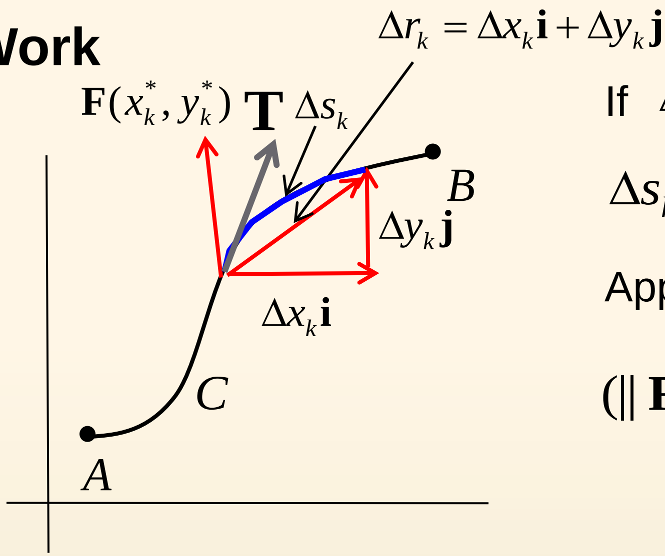
Line Integrals

Work



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$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}ds$$

The work done by a force *F* along a curve *C* is due entirely to the tangential component of *F*

Line Integrals

Circulation

$$\text{Circulation} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Circulation is a measure of the amount by which the fluid tends to turn the curve *C* by rotating around it.

$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$: Fluid does not Circulate in curve C

$\int_C \mathbf{F} \cdot \mathbf{T}\, ds > 0$: Fluid tend to rotate C in counterclockwise

$\int_C \mathbf{F} \cdot \mathbf{T}\, ds < 0$: Fluid tend to rotate C in clockwise

Line Integrals

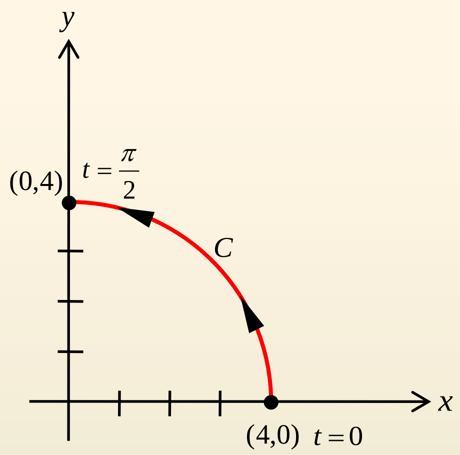
☑ Example 1
Evaluation of Line Integrals

Evaluate

(a) $\int_C xy^2 dx$,

(b) $\int_C xy^2 dy$,

(c) $\int_C xy^2 ds$



on the quarter-circle C defined by x=4cos$t$, y=4sin$t$, $0 \le t \le \pi/2$. See 9.47.

Solution)

(a) $\int_C xy^2 dx$

$$= \int_0^{\pi/2} \overbrace{(4\cos t)}^{x}\overbrace{(16\sin^2 t)}^{y^2}\overbrace{(-4\sin t\,dt)}^{dx}$$

$$= -256\int_0^{\pi/2} \sin^3 t \cos t \, dt$$

$$= -256\left[\frac{1}{4}\sin^4 t\right]_0^{\pi/2}$$

$$= -64$$

Line Integrals

☑ Example 1
Evaluation of Line Integrals

Evaluate

(a) $\int_C xy^2 dx$,

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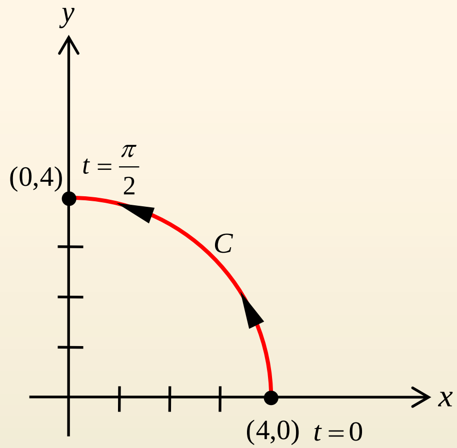

on the quarter-circle C defined by $x=4\cos t$, $y=4\sin t$, $0\le t\le \pi/2$. See Figure 9.47.

Solution)

(b) $\int_C xy^2 dy$

$$= \int_0^{\pi/2} \overbrace{(4\cos t)}^{x}\overbrace{(16\sin^2 t)}^{y^2}\overbrace{(4\cos t dt)}^{dx}$$

$$= 256\int_0^{\pi/2} \sin^2 t\cos^2 t\, dt$$

$$= 64\int_0^{\pi/2} (2\sin t\cos t)^2 dt$$

$$= 64\int_0^{\pi/2} \sin^2 2t\, dt$$

$$= 32\int_0^{\pi/2} 2\sin^2 2t\, dt$$

$$= 32\int_0^{\pi/2} (\cos^2 2t+\sin^2 2t)+(-\cos^2 2t+\sin^2 2t)\, dt$$

$$= 32\int_0^{\pi/2} 1-(\cos^2 2t-\sin^2 2t)\, dt$$

$$= 32\int_0^{\pi/2} 1-\cos 4t\, dt$$

$$= 32\left[t-\frac{1}{4}\sin 4t\right]_0^{\pi/2} = 16\pi$$

Line Integrals

☑ Example 1
Evaluation of Line Integrals

Evaluate

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(b) $\int_C xy^2 dy$,

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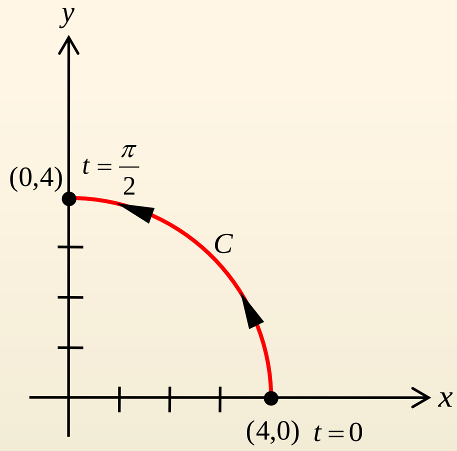


on the quarter-circle C defined by $x=4\cos t$, $y=4\sin t$, $0 \le t \le \pi/2$. See 9.47.

Solution)

(c) $\int_C xy^2 ds$

$$= \int_0^{\pi/2} \overbrace{(4\cos t)}^{x} \overbrace{(16\sin^2 t)}^{y^2} \overbrace{\sqrt{(4\cos t)^2 + (4\sin t)^2}\, dt}^{ds}$$

$$= 256 \int_0^{\pi/2} \sin^2 t \cos t\, dt$$

$$= 256 \left[\frac{1}{3} \sin^3 t \right]_0^{\pi/2} = \frac{256}{3}$$

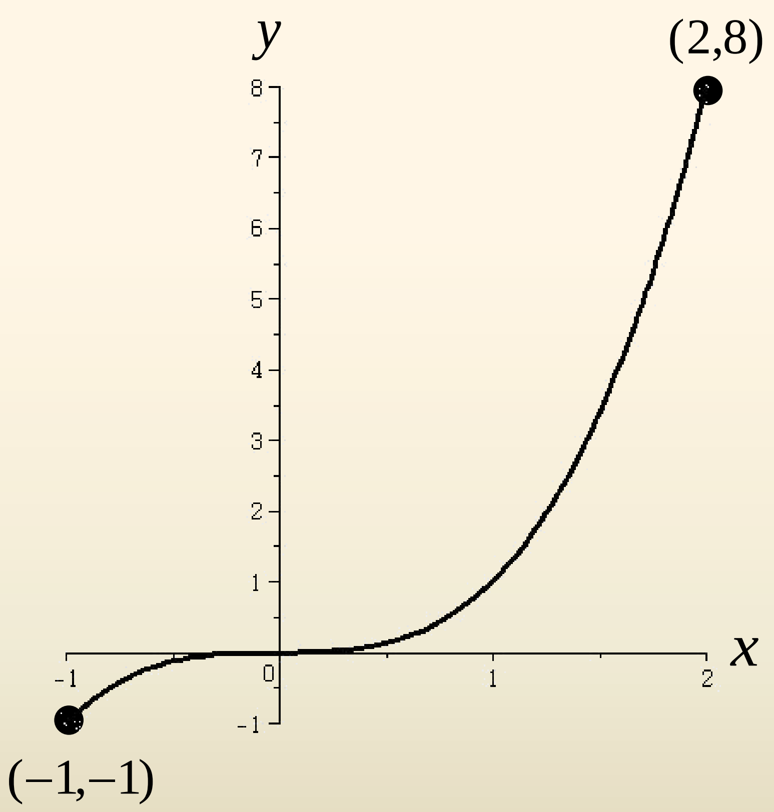
Line Integrals

☑ Example 2
Curve Defined by an Explicit Function

Evaluate $\int_C xydx + x^2dy$, where C is given by $y=x^3$, $-1 \le x \le 2$.

Solution)

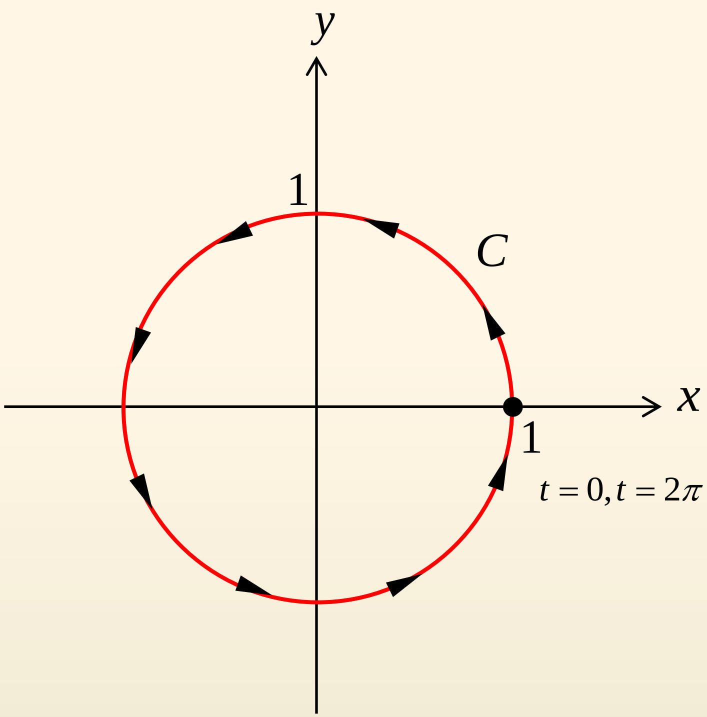
$$\int_C xydx + x^2dy$$
$$= \int_C xydx + \int_C x^2dy$$
$$= \int_{-1}^{2} x(x^3)dx + \int_{-1}^{2} x^2(3x^2dx)$$
$$= \int_{-1}^{2} x^4 + 3x^4dx = \int_{-1}^{2} 4x^4dx$$
$$= \left[\frac{4}{5}x^5\right]_{-1}^{2} = \frac{132}{5}$$



Line Integrals

☑ Example 3
Curve Defined Parametrically

Evaluate $\oint_C x\,dx$, where C is the circle
x=cost, y=sint, 0≤t≤2π.



Solution)

$$\oint_C x\,dx = \int_0^{2\pi} \cos t(-\sin t\,dt)$$

$$= \int_0^{2\pi} -\cos t \sin t\,dt$$

$$= \left[\frac{1}{2}\cos^2 t\right]_0^{2\pi} = 0$$

Line Integrals

☑ Example 4
Closed Curve

Evaluate $\oint_C y^2dx - x^2dy$ on the closed curve C that is shown in Figure 9.49(a).

Solution)

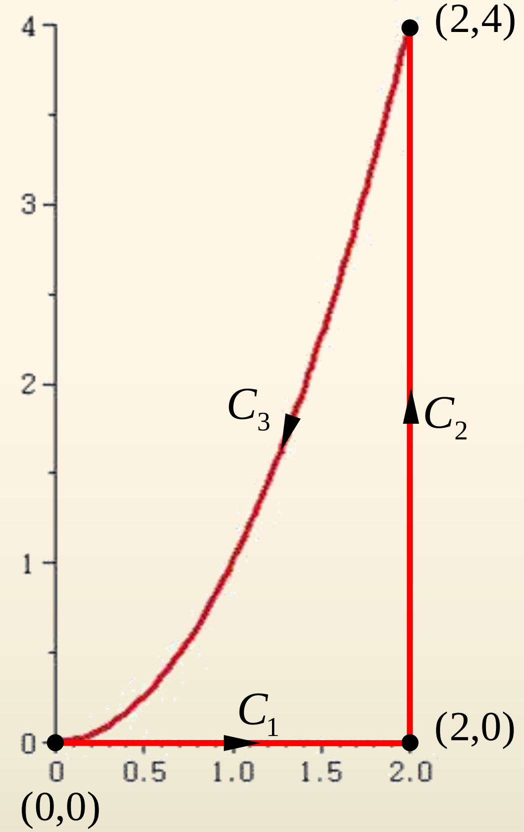
$$\oint_C y^2dx - x^2dy$$

$$= \int_{C_1} y^2dx - x^2dy + \int_{C_2} y^2dx - x^2dy + \int_{C_3} y^2dx - x^2dy$$

$$= \left[\int_0^2 0dx - x^2(0)\right] + \left[\int_0^4 y^2(0) - 4dy\right] + \left[\int_2^0 x^4dx - x^2(2xdx)\right]$$

$$= [0] - [4y]_{y=0}^{y=4} + \left[\frac{1}{5}x^5 - \frac{1}{2}x^4\right]_{x=2}^{x=0}$$

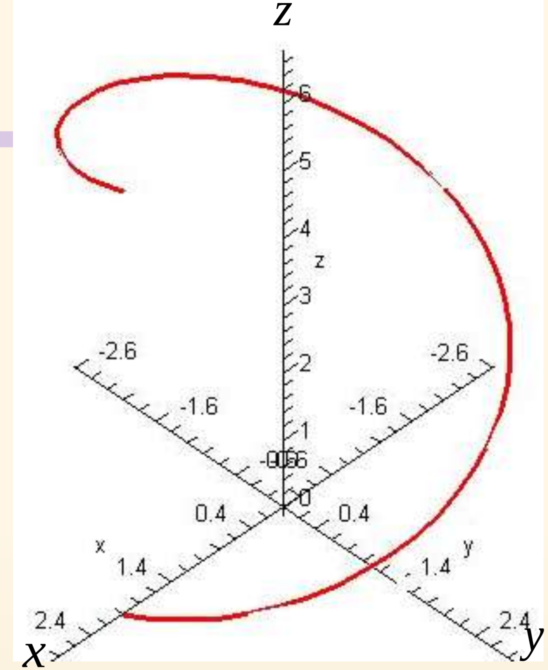
$$= 0 - 16 + \frac{8}{5} = -\frac{72}{5}$$



Line Integrals

☑ Example 5
Line Integral on a Curve in 3-Space

Evaluate $\int_C ydx + xdy + zdz$, where C is the helix $x=2\cos t$, $y=2\sin t$, $z=t$, $0 \le t \le 2\pi$.



Solution)

$$x = 2\cos t \quad ➡ \quad dx = -2\sin t \cdot dt$$

$$y = 2\sin t \quad ➡ \quad dy = 2\cos t \cdot dt$$

$$z = t \quad ➡ \quad dz = 1 \cdot dt = dt$$

$$\int_C ydx + xdy + zdz$$

$$= \int_0^{2\pi} 2\sin t \cdot (-2\sin t \cdot dt) + 2\cos t \cdot (2\cos t \cdot dt) + t \cdot (dt)$$

$$= \int_0^{2\pi} (-4\sin^2 t + 4\cos^2 t + t)dt$$

$$= \int_0^{2\pi} (4\cos 2t + t)dt$$

$$= \left[2\sin 2t + \frac{t^2}{2} \right]_0^{2\pi} = 2\pi^2$$

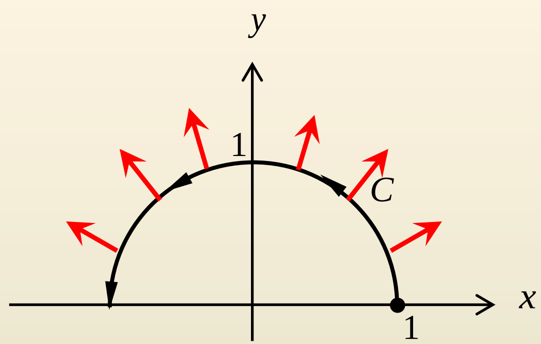
Line Integrals

☑ Example 6
Work Done by a Force

Find the work done by (a) $\mathbf{F}=x\mathbf{i}+y\mathbf{j}$ and (b) $\mathbf{F}=\frac{3}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}$ along the curve C traced by $\mathbf{r}(t)=\cos t\mathbf{i}+\sin t\mathbf{j}$ from $t=0$ to $t=\pi$.

Solution)

$(a)\ \mathbf{F}=x\mathbf{i}+y\mathbf{j}$

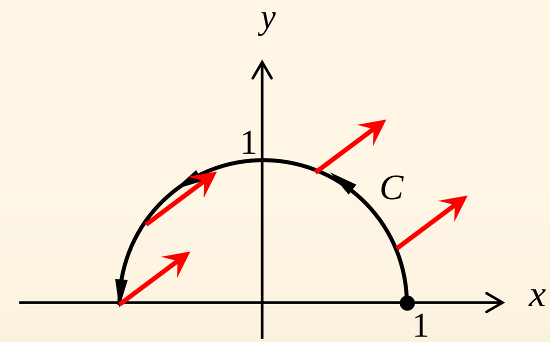


$$W=\int_C \mathbf{F}\bullet d\mathbf{r}=\int_C (x\mathbf{i}+y\mathbf{j})\bullet d\mathbf{r}$$

$$=\int_0^{\pi}(\cos t\mathbf{i}+\sin t\mathbf{j})\bullet(-\sin t\mathbf{i}+\cos t\mathbf{j})dt$$

$$=\int_0^{\pi}(-\cos t\sin t+\sin t\cos t)dt=0$$

$(b)\ \mathbf{F}=\frac{3}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}$



$$W=\int_C \mathbf{F}\bullet d\mathbf{r}=\int_C\left(\frac{3}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)\bullet d\mathbf{r}$$

$$=\int_0^{\pi}\left(\frac{3}{4}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)\bullet(-\sin t\mathbf{i}+\cos t\mathbf{j})dt$$

$$=\int_0^{\pi}\left(-\frac{3}{4}\sin t+\frac{1}{2}\cos t\right)dt$$

$$=\left[\frac{3}{4}\cos t+\frac{1}{2}\sin t\right]_0^{\pi}=-\frac{3}{2}$$

Independence of path

Differential – Functions of Two Variables

Independence of path

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The differential of a function two variables $\phi(x, y)$ is

Independence of path

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Independence of path

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It is said to be exact differential equation if there exists a function $\phi(x, y)$ such that

Independence of path

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Independence of path

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$d\phi = x^2 y^3 dx + x^3 y^2 dy$ Is differential of $\phi(x, y) = \frac{1}{3} x^3 y^3$ => Exact differential

Independence of path

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Independence of path

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=> Not an exact differential

Independence of path

Path Independence

Independence of path

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A line integral whose value is the same for ***every*** curve or path connection A and B is said to be **independent of the path**.

Independence of path

Path Independence

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Theorem 9.8 — **Fundamental Theorem for Line Integrals**

Suppose there exists a function $\phi(x, y)$ **such that** $d\phi = Pdx + Qdy;$ **that is,** $Pdx + Qdy$ **is an exact differential. Then** $\int_C Pdx + Qdy$ **depends on only the endpoints A and B of the path C and**

$$\int_C Pdx + Qdy = \phi(B) - \phi(A).$$

Independence of path

Path Independence

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$$\int_C Pdx + Qdy = \phi(B) - \phi(A).$$

=> If $\phi(x, y)$ **is exact, line integral of** $\phi(x, y)$ **is said to be path independent.**

Independence of path

Theorem 9.8 **Fundamental Theorem for Line Integrals**

Suppose there exists a function $\phi(x, y)$ such that $d\phi = Pdx + Qdy;$ that is, $Pdx + Qdy$ is an exact differential. Then $\int_C Pdx + Qdy$ depends on only the endpoints A and B of the path C and

$$\int_C Pdx + Qdy = \phi(B) - \phi(A).$$

Proof)

By chain rule, $$\int_C Pdx + Qdy = \int_a^b \left(\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} \right) dt$$

$$= \int_a^b \frac{d\phi}{dt} dt = \phi(f(t), g(t))\Big]_a^b$$

$$= \phi(f(b), g(b)) - \phi(f(a), g(a))$$

$$= \phi(B) - \phi(A).$$

Ref. Simply Connected

Independence of path

Test for Path Independence in plane

Theorem 9.9 **Test for Path Independence**

Let P **and** Q **have continuous first partial derivatives in an open simply connected region. Then** $\int_C Pdx + Qdy$ **is independent of path** C **if and only if**

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

For all (x,y) in the region

Independence of path

Conservative Vector Fields

Ref. Conservative Force and Mechanical Energy Conservation

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = Pdx + Qdy$$

$$= (P\mathbf{i} + Q\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = \mathbf{F} \cdot d\mathbf{r}$$

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} \quad (P = \frac{\partial \phi}{\partial x}, \quad Q = \frac{\partial \phi}{\partial y})$$

$\Rightarrow$ Vector field **F** is a gradient of the function ϕ

$\mathbf{F}$ is said to be **gradient field** and ϕ is said to be a **potential function**.

Gradient force field **F** is path independent and the work done by the force along a closed path is *zero*. For this reason, such a force field is also said to be **conservative**. In a conservative field **F** the *law of conservation* of mechanical energy holds.

Independence of path

Test for Path Independence in space

Theorem 9.10 **Test for Path Independence**

Let P **,** Q **and** R **have continuous first partial derivatives in an open simply connected region of space. Then** $\int_C Pdx + Qdy + Rdz$ **is independent of path** C **if and only if**

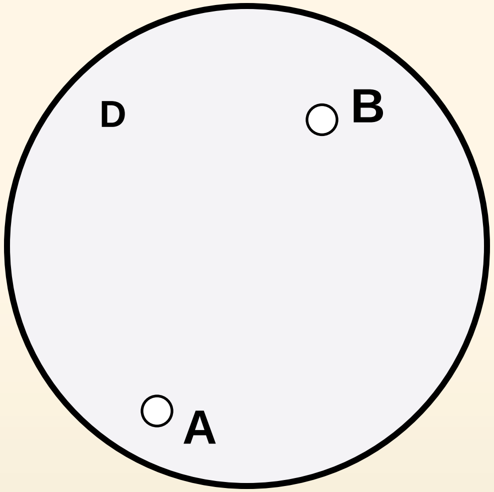
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Independence of path

$$\mathbf{F} = \text{grad } f$$

$$\text{curl } \mathbf{F} = 0$$

Independence of path



path independent in a domain D in space

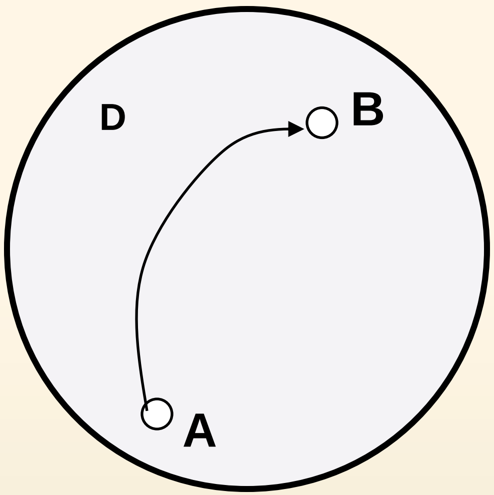
$$\textbf{(1)} \quad \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz\right)$$

The integral (1) has the same value for all paths in D that begin at A and end at B.

$$\mathbf{F} = \text{grad } f$$

$$\text{curl } \mathbf{F} = 0$$

Independence of path



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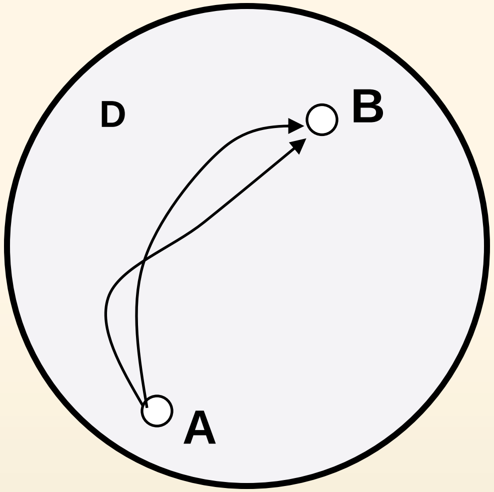
Independence of path



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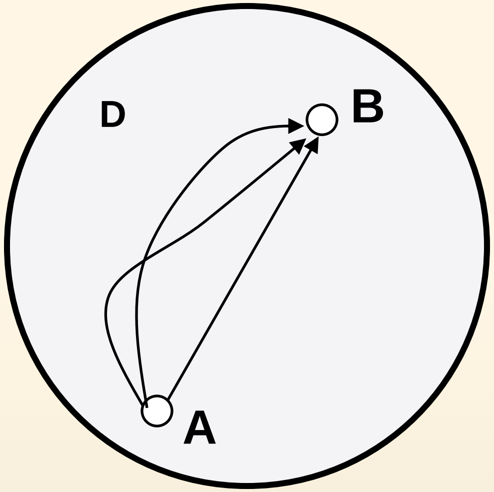
Independence of path



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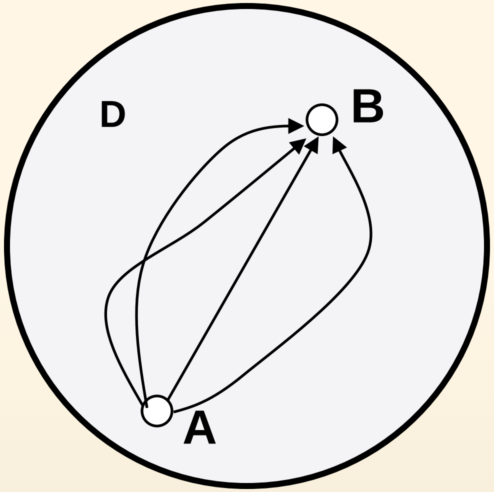
Independence of path



path independent in a domain D in space

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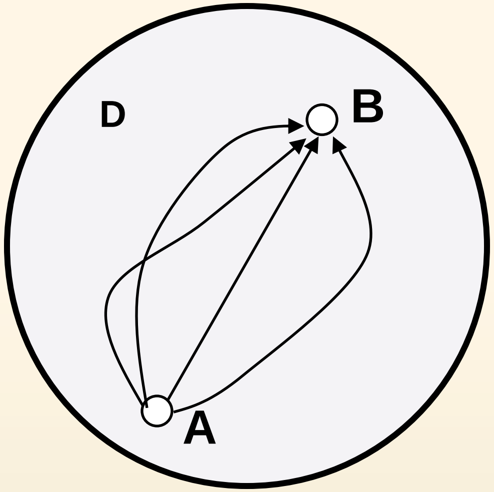
Independence of path



path independent in a domain D in space

$$\text{(1)} \quad \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz \right)$$

The integral (1) has the same value for all paths in D that begin at A and end at B.

Next 3 ideas give path independence of (1) in a domain D if and only if :

$\mathbf{F} = \text{grad } f$

Integration around closed curves C in D always gives 0.

$\text{curl } \mathbf{F} = 0$

Independence of path

Independence of path

$$\text{(1)} \quad \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz \right)$$

(Theorem 1)

A line integral (1) with continuous F_1, F_2, F_3 **in a domain D in space is path independent in D if and only if** $\mathbf{F} = [F_1, F_2, F_3]$ **is the gradient of some function f in D**

$$\text{(2)} \quad \mathbf{F} = \text{grad } f \quad \text{, thus,} \quad F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y}, F_3 = \frac{\partial f}{\partial z}$$

Independence of path

Independence of path

(proof) $\mathbf{F} = \text{grad } f \quad \left([F_1, F_2, F_3] = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \right)$

Independence of path

(proof) $\mathbf{F} = \text{grad } f \quad \left([F_1, F_2, F_3] = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \right)$

$$\int_C (F_1 dx + F_2 dy + F_3 dz) = \int_C \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{df}{dt} dt = f[x(t), y(t), z(t)] \Big|_{t=a}^{t=b}$$

$$= f\big(x(b), y(b), z(b)\big) - f\big(x(a), y(a), z(a)\big)$$

$$= f(B) - f(A)$$

Independence of path

(proof) $\mathbf{F} = \operatorname{grad} f \quad \left([F_1, F_2, F_3] = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \right)$

$$\int_C \left(F_1 dx + F_2 dy + F_3 dz \right) = \int_C \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right)$$

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$$= \int_a^b \frac{df}{dt} dt = f\left[x(t), y(t), z(t) \right]_{t=a}^{t=b}$$

$$= f\left(x(b), y(b), z(b) \right) - f\left(x(a), y(a), z(a) \right)$$

$$= f(B) - f(A)$$

$$\therefore \int_C \left(F_1 dx + F_2 dy + F_3 dz \right) = f(B) - f(A)$$

Independence of path

(proof) $\mathbf{F} = \operatorname{grad} f \quad \left([F_1, F_2, F_3] = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \right)$

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$$= f\left(x(b), y(b), z(b)\right) - f\left(x(a), y(a), z(a)\right)$$

$$= f(B) - f(A)$$

$$\therefore \int_C (F_1 dx + F_2 dy + F_3 dz) = f(B) - f(A)$$

a line integral is independent of path

Independence of path

(Theorem 2)

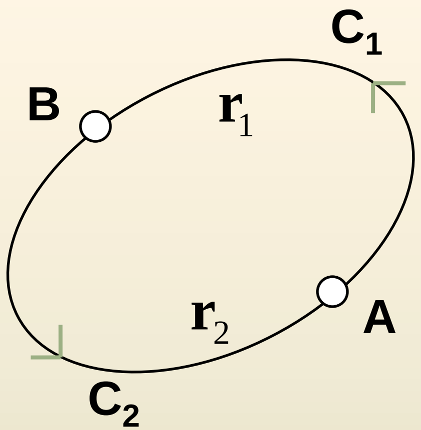
Integration around closed curves C in D always gives 0.

Independence of path

(Theorem 2)

Integration around closed curves C in D always gives 0.

(proof)



Independence of path

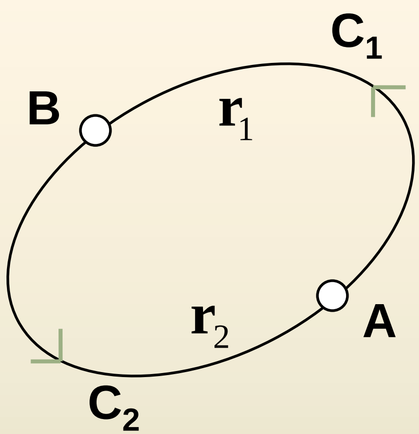
(Theorem 2)

Integration around closed curves C in D always gives 0.

(proof)

$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_{C_2} \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

$$C_1 : A \rightarrow B, C_2 : B \rightarrow A$$



Independence of path

(Theorem 2)

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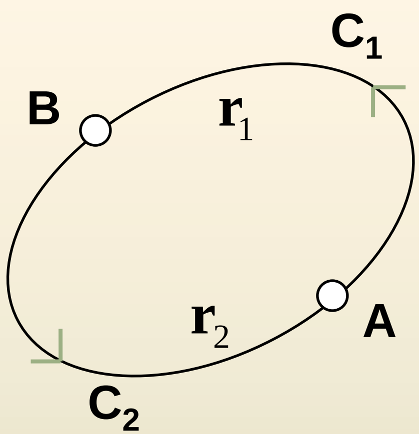


$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_{C_2} \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

$$C_1 : A \rightarrow B, C_2 : B \rightarrow A$$

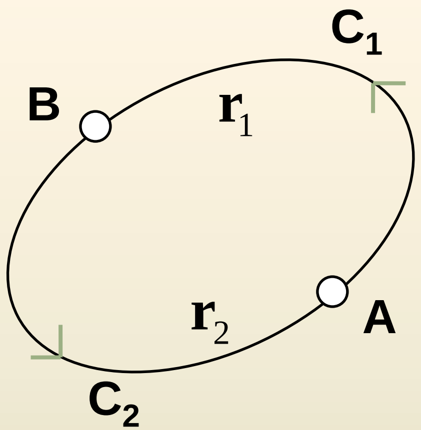
$$\int_A^B \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_B^A \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

Independence of path

(Theorem 2)

Integration around closed curves C in D always gives 0.

(proof)



$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_{C_2} \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

$$C_1 : A \to B, C_2 : B \to A$$

$$\int_A^B \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_B^A \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

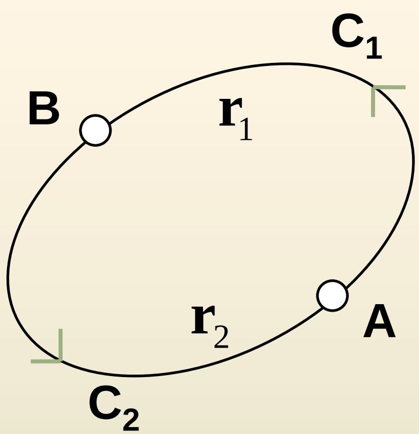
Move 2nd term to the right.

Independence of path

(Theorem 2)

Integration around closed curves C in D always gives 0.

(proof)



$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_{C_2} \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

$$C_1 : A \to B, C_2 : B \to A$$

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Move 2nd term to the right.

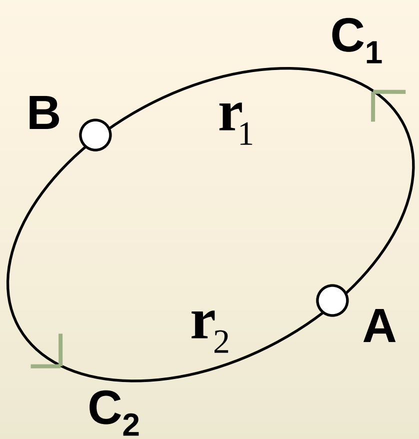
$$\int_A^B \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 = -\int_B^A \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = \int_A^B \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2$$

Independence of path

(Theorem 2)

Integration around closed curves C in D always gives 0.

(proof)



$$\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_{C_1} \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_{C_2} \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

$$C_1 : A \to B, C_2 : B \to A$$

$$\int_A^B \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 + \int_B^A \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = 0$$

Move 2nd term to the right.

$$\int_A^B \mathbf{F}(\mathbf{r}_1) \bullet d\mathbf{r}_1 = -\int_B^A \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2 = \int_A^B \mathbf{F}(\mathbf{r}_2) \bullet d\mathbf{r}_2$$

In conclusion, a line integral is path independent

Independence of path

(4) $\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$

(Theorem 3)

Let F_1, F_2, F_3 in the line integral (1) $\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz \right)$

be continuous and have continuous first partial derivatives in a domain D in space. Then :

(a) If the differential form (4) is exact in D - and thus (1) is path independent by theorem (3*)- then in D,

(6) $\operatorname{curl} \mathbf{F} = 0$

$$\left(\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \right)$$

in components

(6’) $$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Independence of path

(4) $\mathbf{F} \bullet d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz$

(Theorem 3)

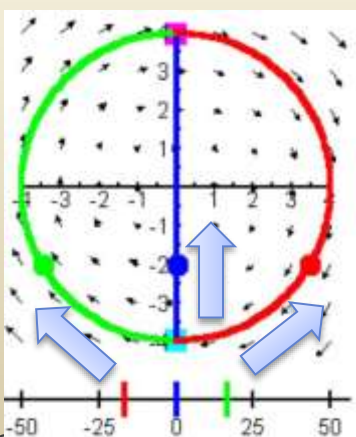
Let F_1, F_2, F_3 in the line integral (1) $\int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} = \int_C \left(F_1 dx + F_2 dy + F_3 dz \right)$

(6) $\operatorname{curl} \mathbf{F} = 0$

$$\left(\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right] \right)$$

in components

(6') $$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$



If it is rotational field,

$\int \mathbf{F} \bullet d\mathbf{r} > 0$

$\int \mathbf{F} \bullet d\mathbf{r} = 0$ **Path dependent → not a conservative field**

$\int \mathbf{F} \bullet d\mathbf{r} < 0$

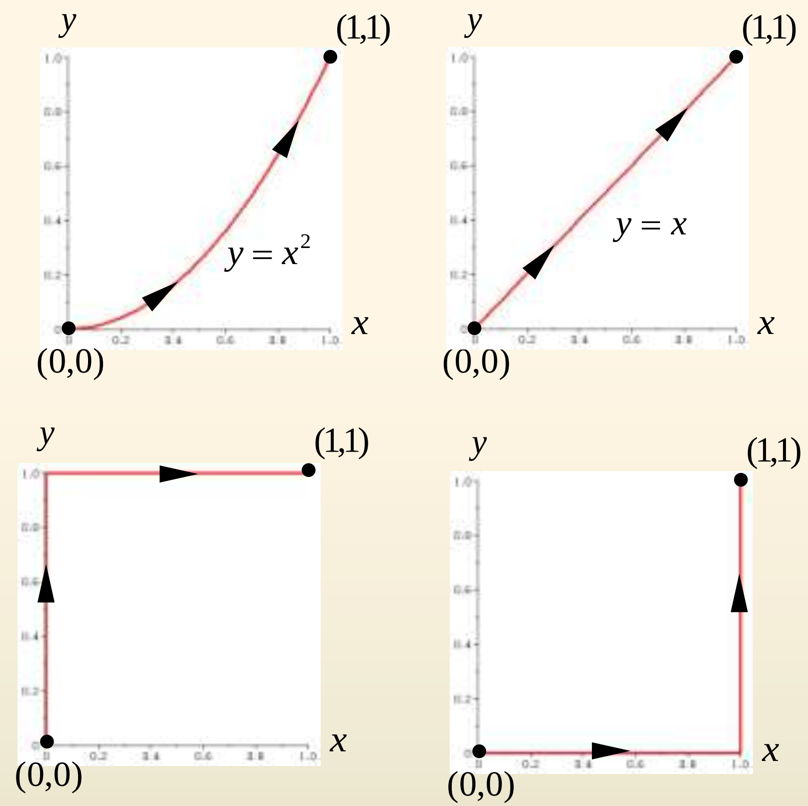
Independence of path

☑ Example 1
An Integral Independent of the Path

The integral $\int_C y\,dx + x\,dy$ has the same value on each path C between (0,0) and (1,1) shown in Figure 9.65. You may recall from Problems 11-14 of Exercises 9.8 that on these paths

$$\int_C y\,dx + x\,dy = 1$$

In example 2 we shall prove that the given integral is independent of the path.



Independence of path

☑ Example 2
Using Theorem 9.8

In Example 1 note that $d(xy) = y\,dx + x\,dy$; that is, $y\,dx + x\,dy$ is an exact differential. Hence, $\int_C y\,dx + x\,dy$ is independent of the path between any two points A and B. Specifically, if A and B are, respectively, $(0,0)$, and $(1,1)$, we then have, from theorem 9.8,

$$\int_{(0,0)}^{(1,1)} ydx + xdy = \int_{(0,0)}^{(1,1)} d(xy) = xy\Big]_{(0,0)}^{(1,1)} = 1.$$

9.9 Example 1.

$$\int_C ydx + xdy = 1$$

Independence of path

☑ Example 3

A Path-Dependent Integral

Show that the integral $\int_C (x^2 - 2y^3)dx + (x + 5y)dy$ is not independent of the path C.

Solution)

$$P = x^2 - 2y^3,\ \ Q = x + 5y$$

$$\frac{\partial P}{\partial y} = -6y^2,\ \ \frac{\partial Q}{\partial x} = 1$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}.$$

∴Integral is not independent of the path.

Independence of path

☑ Example 4
An Integral Independent of the Path

Show that $\int_C (y^2-6xy+6)dx+(2xy-3x^2)dy$ is independent of any path C between (-1,0) and (3,4). Evaluate.

Solution)

$$\int_C (y^2-6xy+6)dx+(2xy-3x^2)dy$$
$$=\int_C P(x,y)dx+Q(x,y)dy$$

$$P=y^2-6xy+6 \quad Q=2xy-3x^2 \Rightarrow \frac{\partial P}{\partial y}=2y-6x \quad \frac{\partial Q}{\partial x}=2y-6x$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}.$$

∴Integral is independent of the path.

$$P=\frac{\partial \phi}{\partial x}=y^2-6xy+6$$
$$Q=\frac{\partial \phi}{\partial y}=2xy-3x^2$$
$$\phi=\int y^2-6xy+6dx=xy^2-3x^2y+6x+g(y)$$
$$\frac{\partial \phi}{\partial y}=2xy-3x^2+g'(y)$$
$$\therefore g'(y)=0,\ g(y)=C$$

$$\int_{(-1,0)}^{(3,4)}(y^2-6xy+6)dx+(2xy-3x^2)dy$$
$$=\int_{(-1,0)}^{(3,4)}d(xy^2-3x^2y+6x)$$
$$=\left[xy^2-3x^2y+6x\right]_{(-1,0)}^{(3,4)}$$
$$=(48-108+18)-(-6)=-36$$

Independence of path

☑ Example 5
Gradient Field

Show that the vector field
$\mathbf{F}=(y^2+5)\mathbf{i}+(2xy-8)\mathbf{j}$ is a gradient field.
Find a potential function for $\mathbf{F}$.

Solution)

$$\mathbf{F}=(y^2+5)\mathbf{i}+(2xy-8)\mathbf{j}$$
$$=P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}$$

$$\frac{\partial P}{\partial y}=2y,\ \frac{\partial Q}{\partial x}=2y$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

$\therefore$Vector field is a gradient field.

$$P=\frac{\partial \phi}{\partial x}=y^2+5$$

$$Q=\frac{\partial \phi}{\partial y}=2xy-8$$

$$\phi=\int y^2+5dx=y^2x+5x+g(y)$$

$$\frac{\partial \phi}{\partial y}=2xy+g'(y)$$

$$\therefore g'(y)=-8,\ g(y)=-8y+C$$

$$\phi=y^2x+5x-8y+C$$

$$\nabla\phi=\frac{\partial \phi}{\partial x}\mathbf{i}+\frac{\partial \phi}{\partial y}\mathbf{i}=(y^2+5)\mathbf{i}+(2xy-8)\mathbf{j}$$

Independence of path

☑ Example 6

An Integral Independent of the Path

Show that $\int_C (y+yz)dx+(x+3z^3+xz)dy+(9yz^2+xy-1)dz$ is independent of any path C between (1,1,1) and (2,1,4). Evaluate

Solution)

$$\int_C (y+yz)dx+(x+3z^3+xz)dy+(9yz^2+xy-1)dz$$

$$=\int_C P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$$

$$\frac{\partial P}{\partial y}=1+z=\frac{\partial Q}{\partial x},\ \frac{\partial P}{\partial z}=y=\frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial z}=9z^2+x=\frac{\partial R}{\partial y}$$

∴Integral is independent of the path.

$$P=\frac{\partial Q}{\partial x}=y+yz$$

$$\phi=\int y+yzdx=yx+xyz+g(y,z)$$

$$\frac{\partial \phi}{\partial y}=x+xz+\frac{\partial g(y,z)}{\partial y}=x+3z^3+xz$$

$$\phi=xy+xyz+3yz^3+h(z)$$

$$\frac{\partial \phi}{\partial z}=xy+9yz^2+\frac{dh(z)}{dz}=9yz^2+xy-1$$

$$\therefore \phi=xy+xyz+3yz^3-z$$

$$\int_C (y+yz)dx+(x+3z^3+xz)dy+(9yz^2+xy-1)dz$$

$$=\int_{(1,1,1)}^{(2,1,4)} (y+yz)dx+(x+3z^3+xz)dy+(9yz^2+xy-1)dz$$

$$=\int_{(1,1,1)}^{(2,1,4)} d(xy+xyz+3yz^3)$$

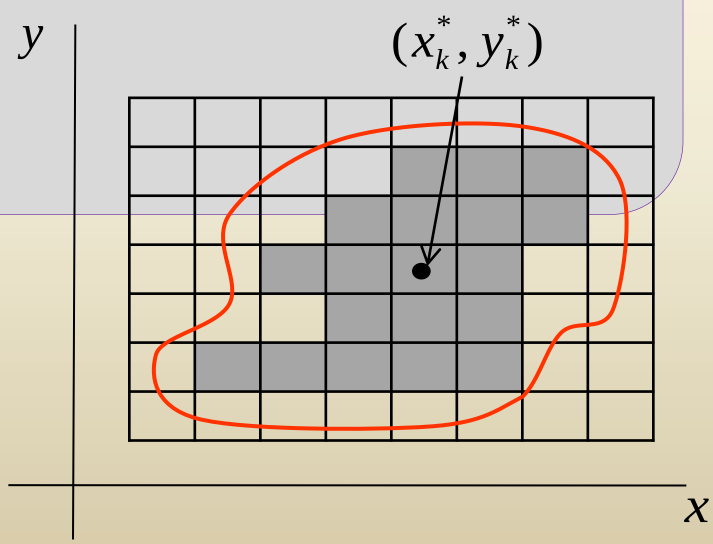
$$=(xy+xyz+3yz^3-z)\Big|_{(1,1,1)}^{(2,1,4)}$$

$$=198-4-194$$

Double Integrals

z=*f* (x,y)

1. Let f be defined in closed and bounded region *R*
2. By means of a grid of vertical and horizontal lines parallel to the coordinate axes, form a partition *P* of *R* into n rectangular subregions of areas ΔA_k that lie entirely in *R*.
3. Let $\|P\|$ be the **norm** of the partition or the length of the longest diagonal of the R_k.
4. Choose a point (x_k^*, y_k^*) in each subregion R_k.
5. Form the sum $\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$



Double Integrals

Definition 9.10 **The Double Integral**

Let f be a function of two variables defined on a closed region R. Then the **double integral of f over *R*** is given by

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \cdots (1)$$

Integrability

If the limit in integral exists, we say that f is **integrable** over *R* and that R is the region of integration. When f is continuous on *R*, then f is necessarily integrable over *R*

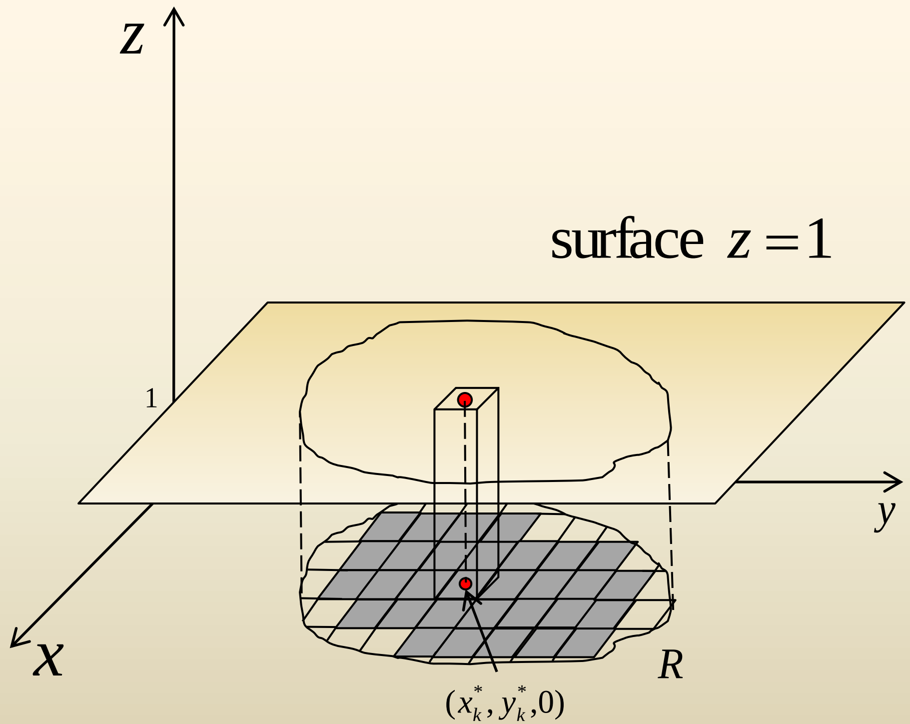
Double Integrals

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \cdots (1)$$

Area

when $f(x,y) = 1$ on R, then $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta A$ simply give the **area** A of the region

$$A = \iint_R dA$$



Double Integrals

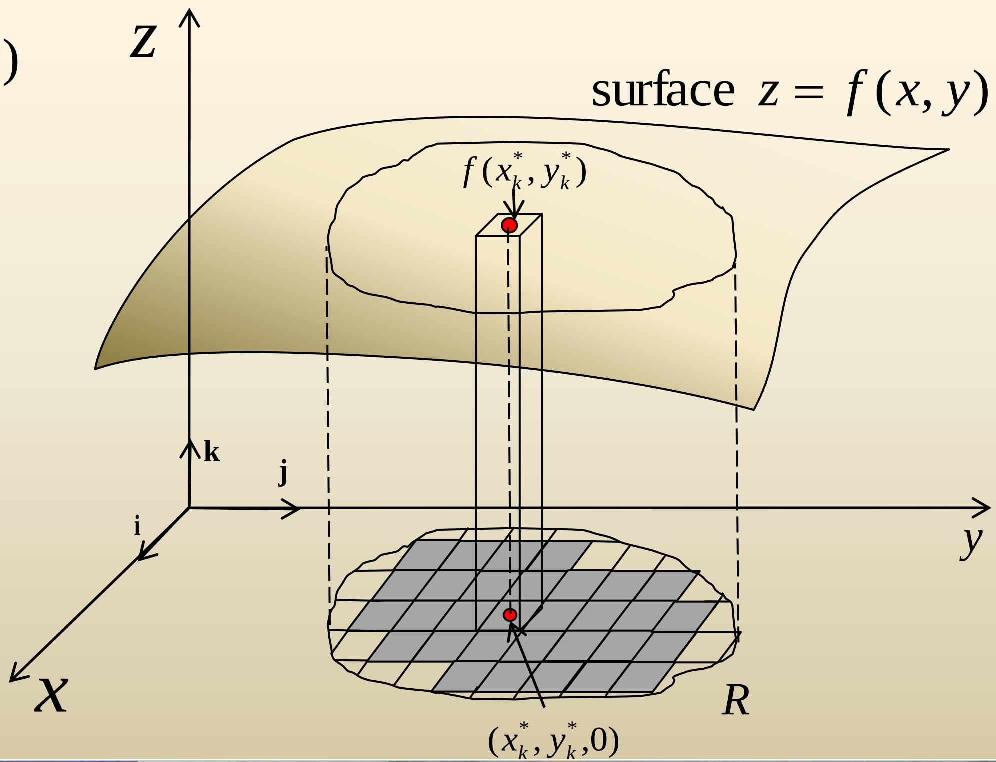
$$\iint_R f(x,y)dA = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \cdots (1)$$

Volume

If $f(x,y) > 0$ on *R*, then the product $f(x_k^*, y_k^*)\Delta A_k$ give the volume of rectangular prism. The summation of volume $\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ is approximation to the **volume** *V*, of the solid *above* the region R and *below* the surface $z = f(x,y)$

The limit of this sum as $\|P\| \to 0$

$$V = \iint_R f(x,y)dA$$



Double Integrals

Theorem 9.11 **Properties of Double Integrals**

Let f and g be functions of two variables that are integrable over a region *R*, Then

$$(i)\ \iint_R kf(x,y)\,dA = k\iint_R f(x,y)\,dA$$

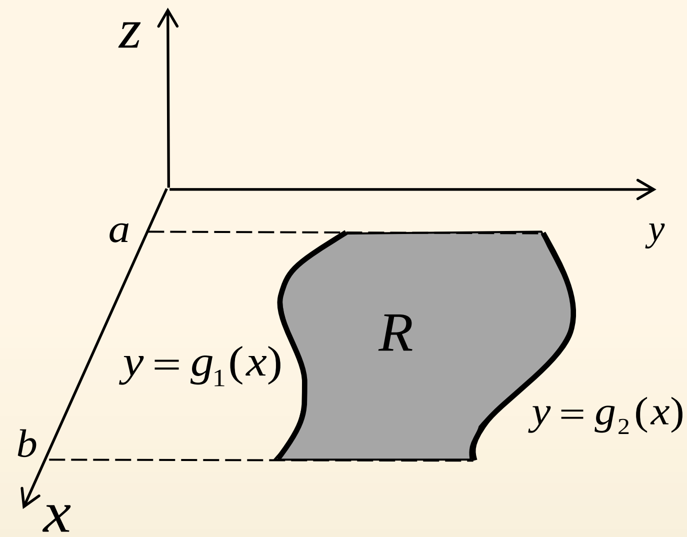
$$(ii)\ \iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

$$(iii)\ \iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} g(x,y)\,dA, \text{ where } R_1 \text{ and } R_2$$

are subregions of R that do not overlap and $R = R_1 \cup R_2$

Double Integrals

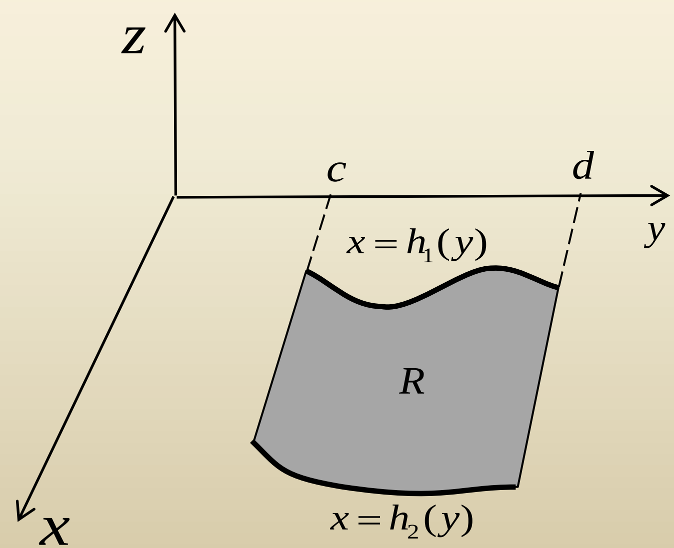
Regions of Type I and Type II



Regions of Type I

$$R : a \le x \le b, \quad g_1(x) \le y \le g_2(x)$$

Where g_1 and g_2 are continuous.



Regions of Type II

$$R : c \le y \le d, \quad h_1(y) \le x \le h_2(y)$$

Where h_1 and h_2 are continuous.

Double Integrals

Type I:

$R: a \le x \le b, \quad g_1(x) \le y \le g_2(x)$

Type II:

$R: c \le y \le d, \quad h_1(y) \le x \le h_2(y)$

Iterated Integrals

For type I

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

For type II

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right] dy$$

Double Integrals

Type I:

$R: a \le x \le b, \quad g_1(x) \le y \le g_2(x)$

Type II:

$R: c \le y \le d, \quad h_1(y) \le x \le h_2(y)$

Theorem 9.12 **Evaluation of Double Integrals**

Let f **be continuous on region** ***R***

(i) If R is Type I, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(ii) If R is Type II, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Double Integrals

Laminas with Variable Density – Center of Mass

If density is constant,

$$m = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \rho f(x_k^*) \Delta x_k = \int_a^b \rho f(x)\,dx$$

If density is not constant,

$$m = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \rho f(x_k^*) \Delta x_k = \iint_R \rho(x, y)\,dA$$

Center of mass $\quad \bar{x} = \dfrac{M_y}{m}, \; \bar{y} = \dfrac{M_x}{m},$

Where,

$$M_y = \iint_R x\rho(x, y)\,dA \qquad M_x = \iint_R y\rho(x, y)\,dA$$

Moment of lamina about y- and x- axis

Double Integrals

Moments of Inertia

Moments of inertia about the x-axis

$$I_x = \iint_R y^2 \rho(x, y) dA$$

Moments of inertia about the y-axis

$$I_y = \iint_R x^2 \rho(x, y) dA$$

Double Integrals in Polar Coordinates

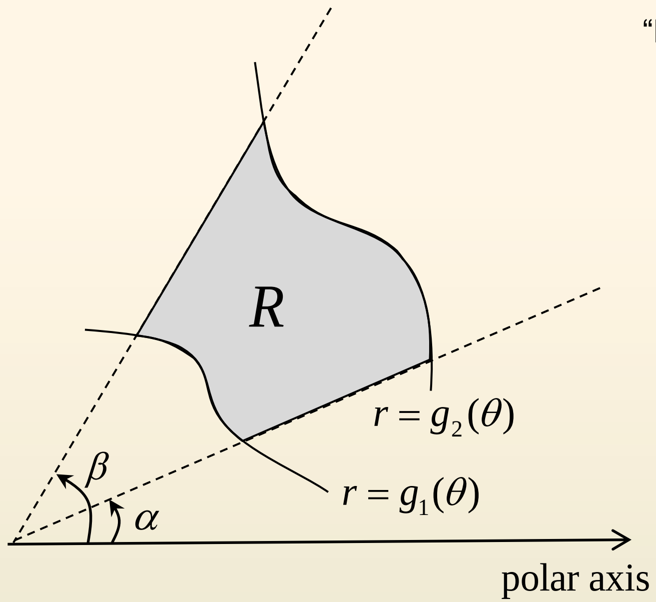
Polar Rectangles

"Polar Rectangles"

ΔA_k

Double Integrals in Polar Coordinates

Polar Rectangles

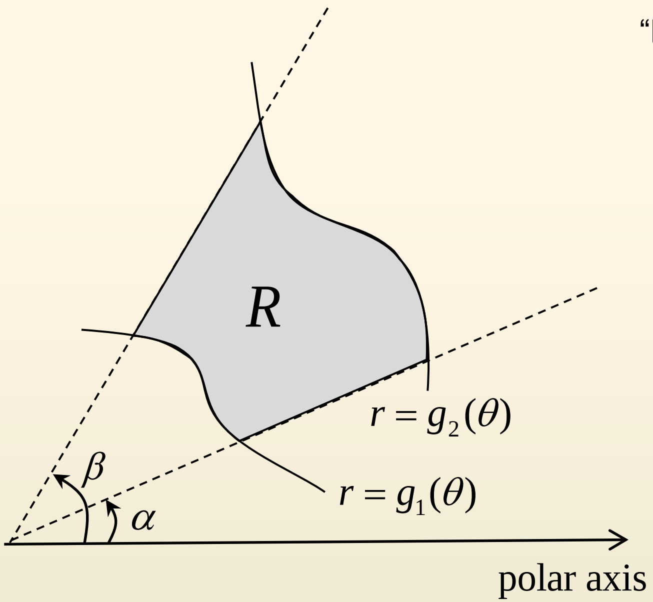


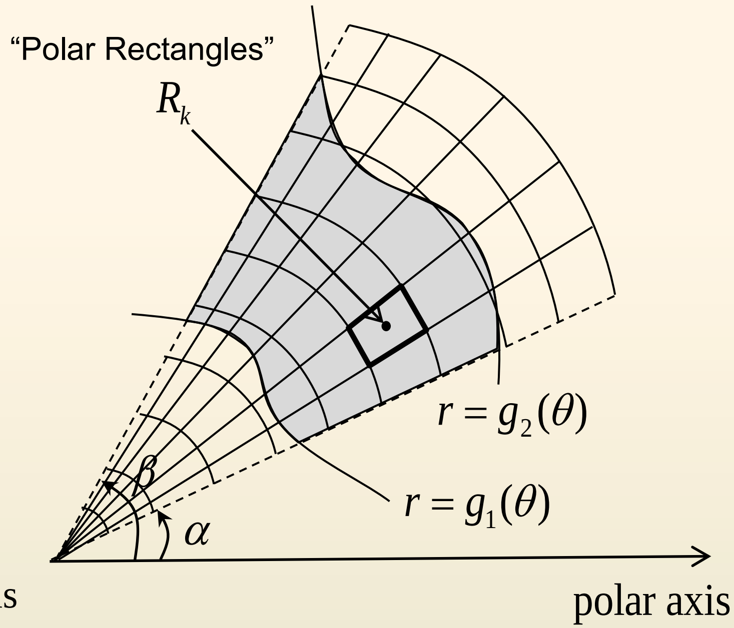
"Polar Rectangles"

ΔA_k

Double Integrals in Polar Coordinates

Polar Rectangles

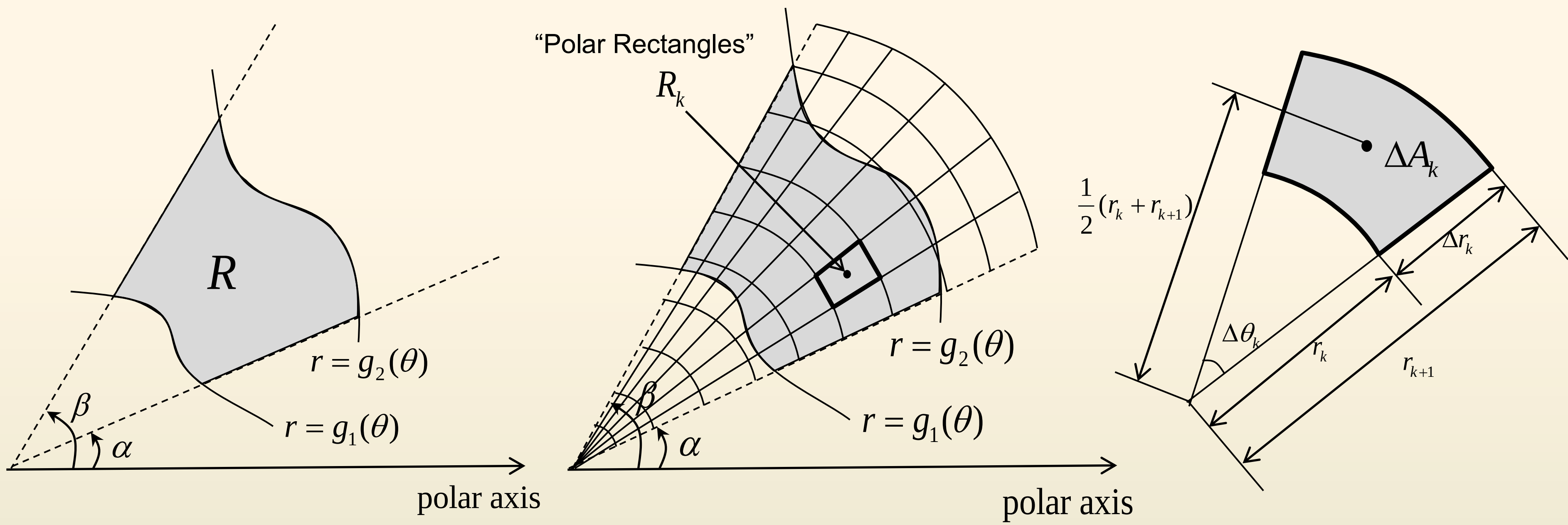




ΔA_k

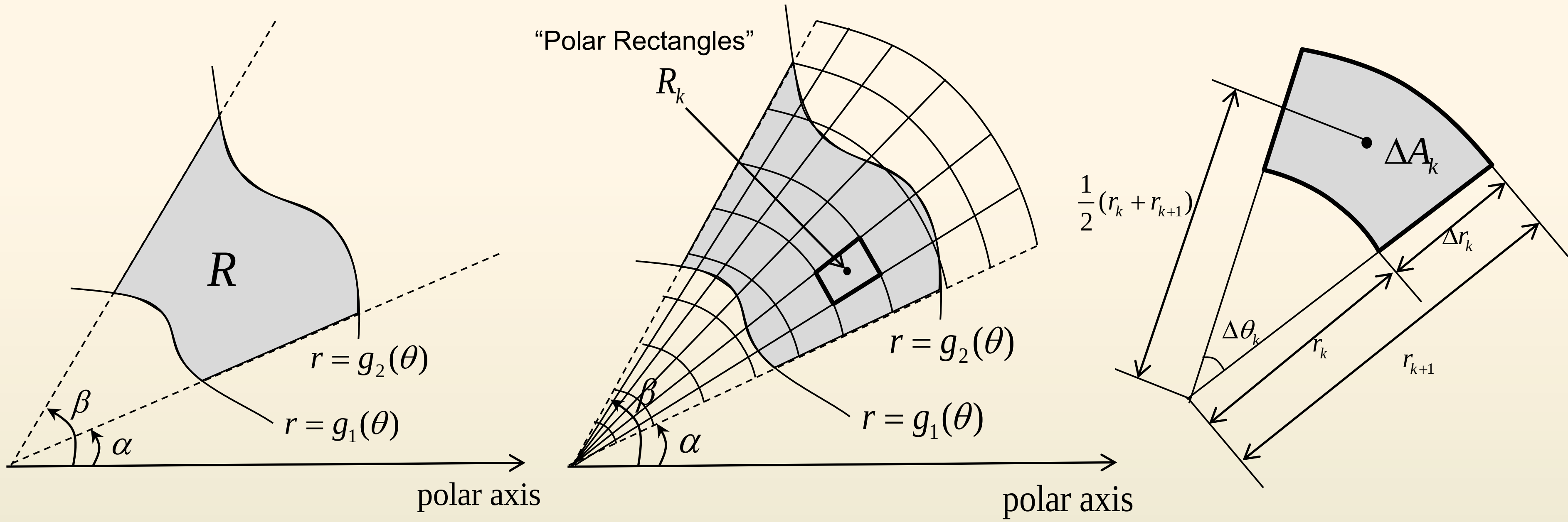
Double Integrals in Polar Coordinates

Polar Rectangles



Double Integrals in Polar Coordinates

Polar Rectangles

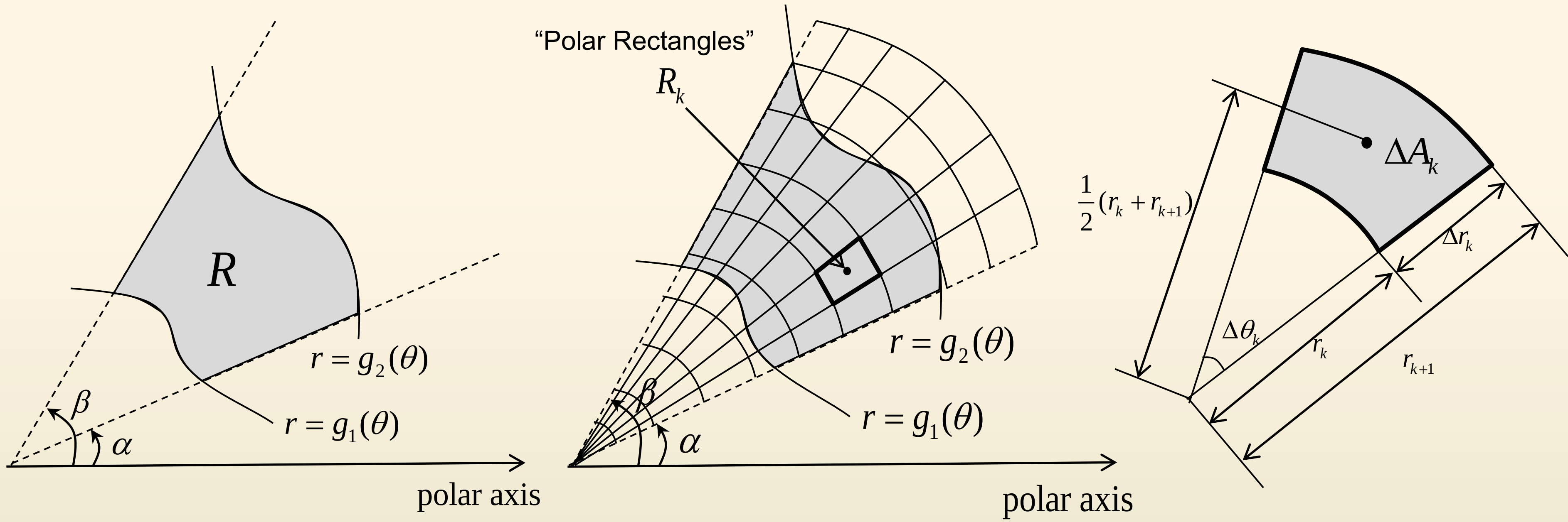


$$\Delta r_k = (r_{k+1} - r_k)$$

$$r_k^* = \frac{1}{2}(r_{k+1} + r_k)$$

Double Integrals in Polar Coordinates

Polar Rectangles



$$\Delta A_k = \frac{1}{2} r_{k+1}^2 \Delta\theta_k - \frac{1}{2} r_k^2 \Delta\theta_k$$

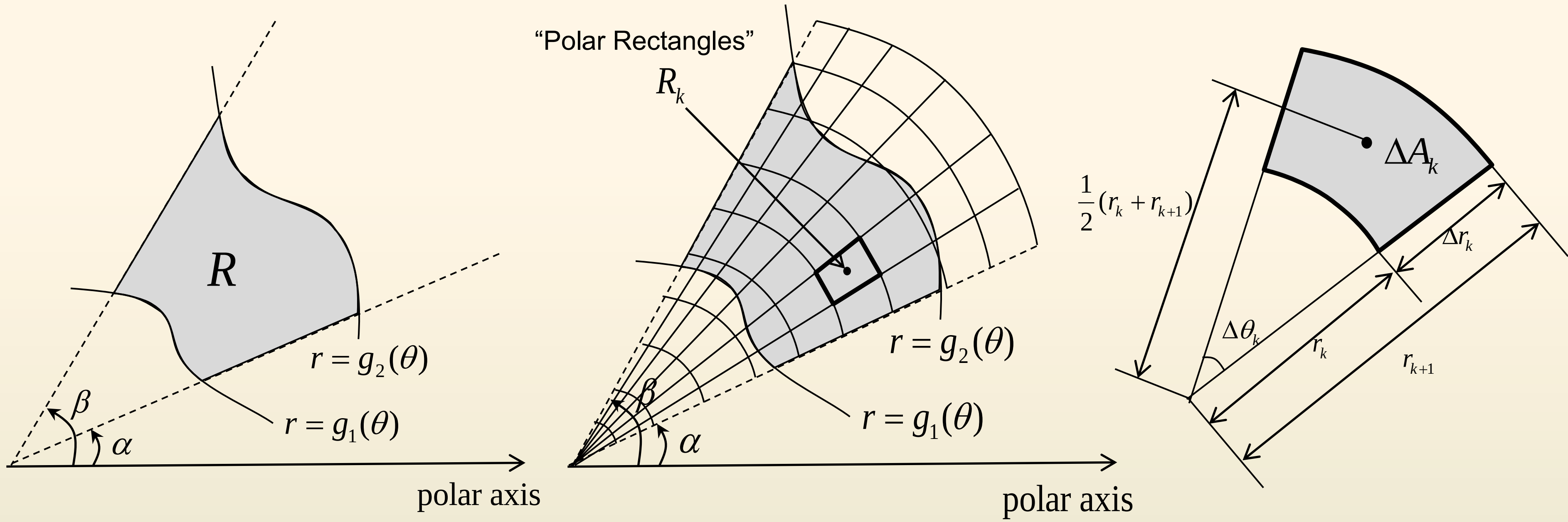
$$= \frac{1}{2}\left(r_{k+1}^2 - r_k^2\right)\Delta\theta_k = \frac{1}{2}(r_{k+1} + r_k)(r_{k+1} - r_k)\Delta\theta_k = r_k^* \Delta r_k \Delta\theta_k$$

$$\Delta r_k = (r_{k+1} - r_k)$$

$$r_k^* = \frac{1}{2}(r_{k+1} + r_k)$$

Double Integrals in Polar Coordinates

Polar Rectangles



$$\Delta A_k = \frac{1}{2} r_{k+1}^2 \Delta\theta_k - \frac{1}{2} r_k^2 \Delta\theta_k$$

$$= \frac{1}{2}\left(r_{k+1}^2 - r_k^2\right)\Delta\theta_k = \frac{1}{2}(r_{k+1} + r_k)(r_{k+1} - r_k)\Delta\theta_k = r_k^* \Delta r_k \Delta\theta_k$$

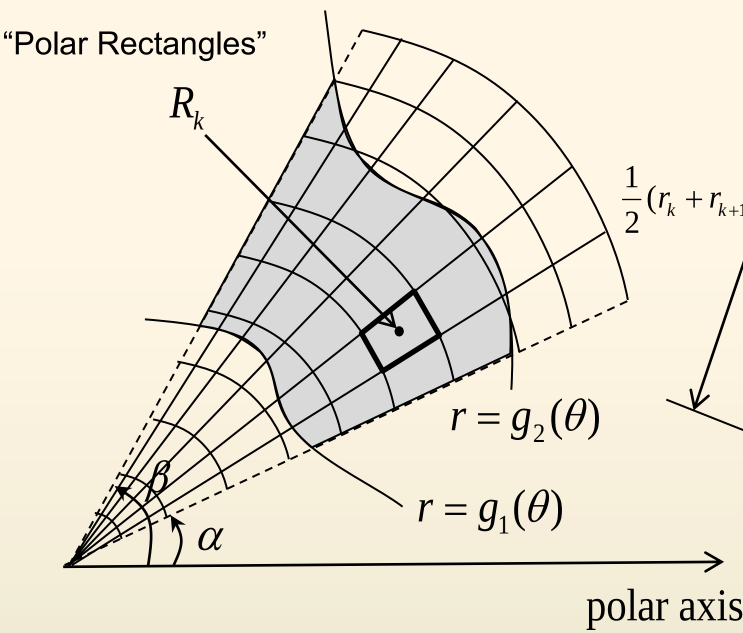
Ref. Area of Polar Rectangles

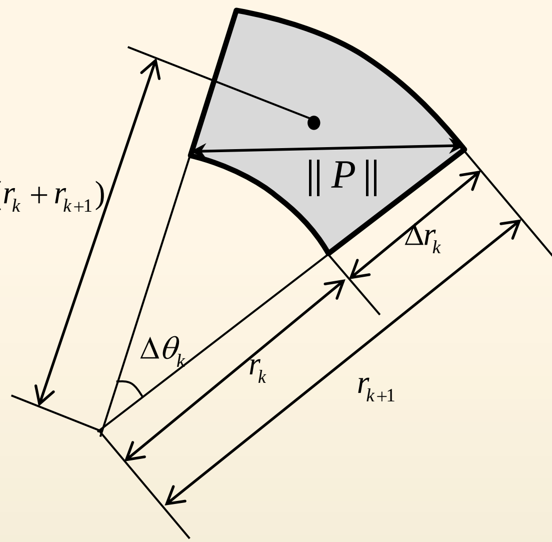
$$\Delta r_k = (r_{k+1} - r_k)$$

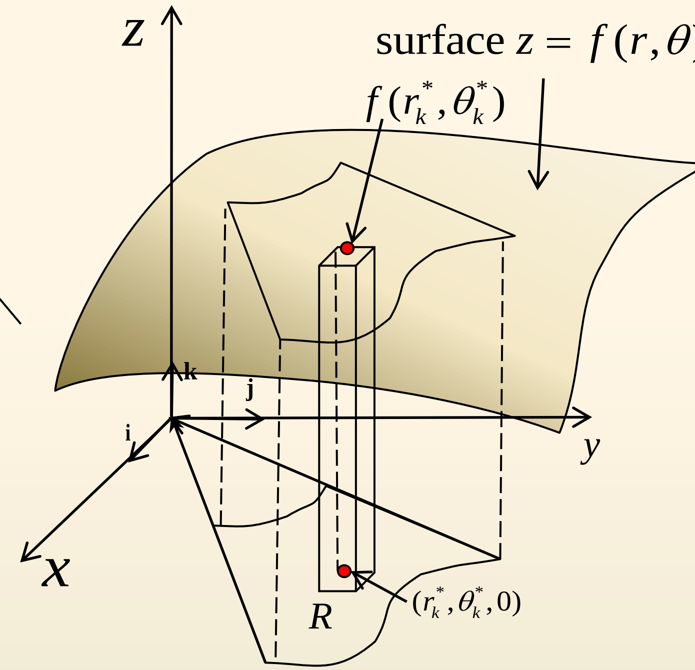
$$r_k^* = \frac{1}{2}(r_{k+1} + r_k)$$

Double Integrals in Polar Coordinates

Polar Rectangles







$$\Delta A_k = r_k^* \Delta r_k \Delta\theta_k$$

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta\theta_k = \iint_R f(r,\theta)\,dA$$

$$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Double Integrals in Polar Coordinates

Change of Variables : Rectangular to Polar Coordinates

If *R* is describes in polar coordinates as

$$0 \le g_1(\theta) \le r \le g_2(\theta)$$

$$\alpha \le \theta \le \beta$$

$$0 \le \beta - \alpha \le 2\pi$$

Then, $$\iint_R f(x,y)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

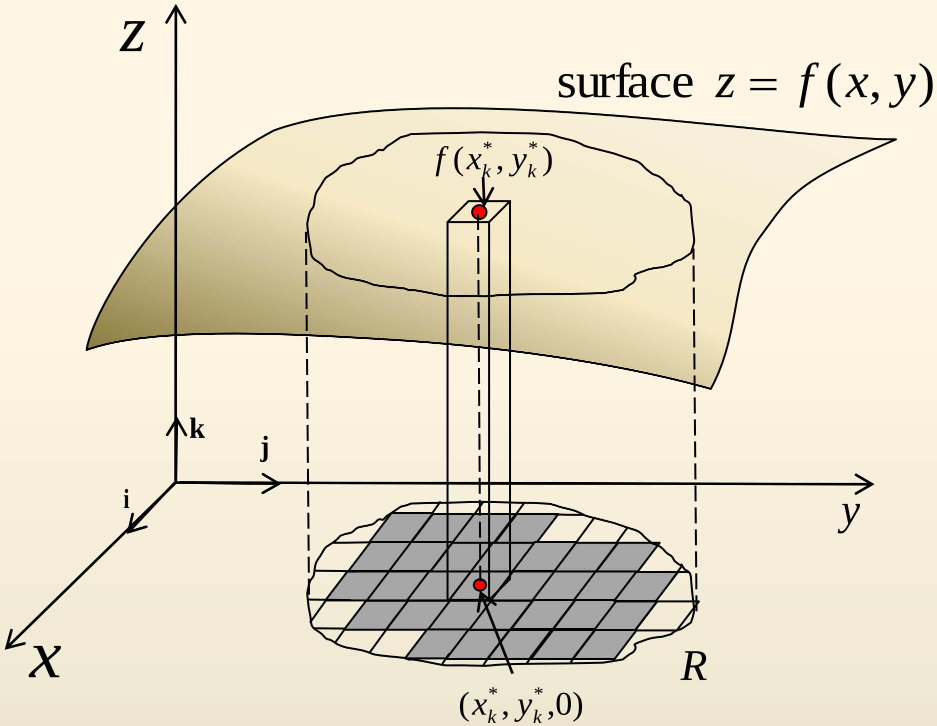
Above equation is particularly useful when *f* contains the expression

$x^2 + y^2$ Since, in polar coordinates, we can write

$$x^2 + y^2 = r^2 \quad and \quad \sqrt{x^2 + y^2} = r$$

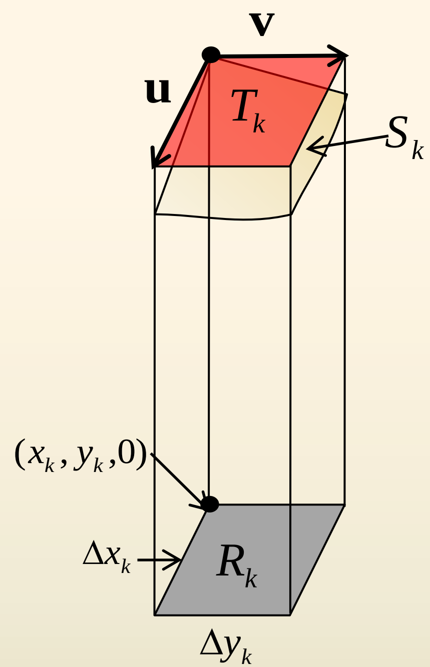
Surface Integrals

Surface Area



$$\mathbf{u} = \Delta x_k \mathbf{i} + f_x(x_k, y_k)\Delta x_k \mathbf{k}$$

$$\mathbf{v} = \Delta y_k \mathbf{j} + f_y(x_k, y_k)\Delta y_k \mathbf{k}$$

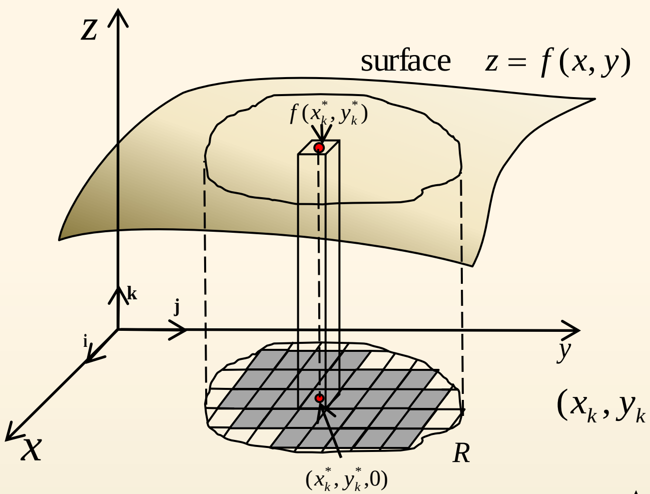


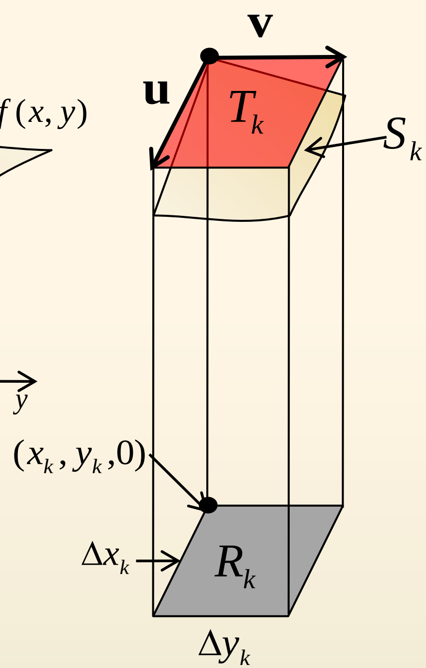
If R_k is small, $\Delta T_k \approx \Delta S_k$ $\quad (\Delta T_k, \Delta S_k$: Area of $T_k, S_k)$

$$\Delta T_k = \|\mathbf{u}\times\mathbf{v}\| \quad , \mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_k & 0 & f_x(x_k, y_k)\Delta x_k \\ 0 & \Delta y_k & f_y(x_k, y_k)\Delta y_k \end{vmatrix} = [-f_x(x_k, y_k)\mathbf{i} - f_y(x_k, y_k)\mathbf{j} + \mathbf{k}]\Delta x_k \Delta y_k$$

Surface Integrals

Surface Area





$$\mathbf{u} = \Delta x_k \mathbf{i} + f_x(x_k, y_k)\Delta x_k \mathbf{j}$$

$$\mathbf{v} = \Delta y_k \mathbf{i} + f_y(x_k, y_k)\Delta y_k \mathbf{j}$$

If R_k is small, $\Delta T_k \approx \Delta S_k$

($\Delta T_k, \Delta S_k$: Area of T_k, S_k)

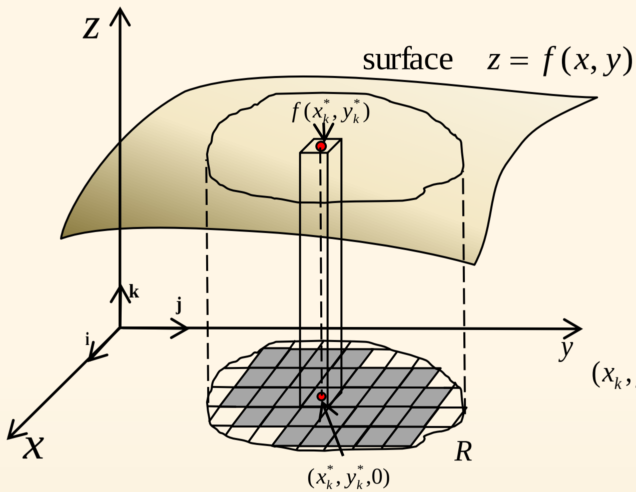
$$\Delta T_k = \| \mathbf{u} \times \mathbf{v} \|$$

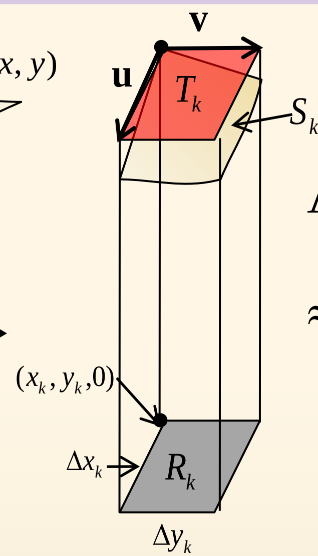
$$\mathbf{u} \times \mathbf{v} = [-f_x(x_k, y_k)\mathbf{i} - f_y(x_k, y_k)\mathbf{j} + \mathbf{k}]\Delta x_k \Delta y_k$$

$$\Delta T_k = \sqrt{[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2 + 1}\ \Delta x_k \Delta y_k$$

$$\text{Area of surface} \approx \sum_{k=1}^{n} \sqrt{[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2 + 1}\ \Delta x_k \Delta y_k$$

Surface Integrals





Area of surface

$$\approx \sum_{k=1}^{n} \sqrt{[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2 + 1}\ \Delta x_k \Delta y_k$$

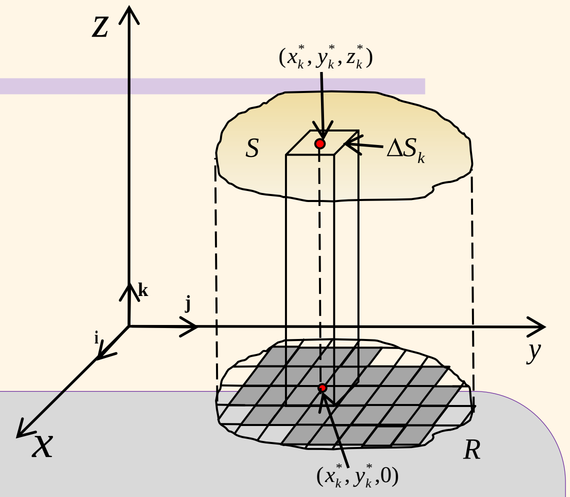
Definition 9.11

Surface Area

Let f be a function for the first partial derivatives f_x and f_y are continuous on a closed region *R*. Then the area of the surface over ***R*** is given by

$$A(s) = \iint_R \sqrt{[f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2 + 1}\ dA \cdots (2)$$

Surface Integrals



Differential of Surface Area

$$dS = \sqrt{1 + [f_x(x_k, y_k)]^2 + [f_y(x_k, y_k)]^2}\; dA$$

$w = G(x, y, z)$

1. Let G be defined in a region of 3-space that contains a surface *S*, which is the graph of a function $z = f(x, y)$. Let the projection *R* of the surface onto the xy-plane be either a Type I or Type II region
2. Divide the surface into n pieces of areas ΔS_k corresponding to a partition *P* of *R* into n rectangles R_k of area ΔA_k
3. Let $\|P\|$ be the **norm** of the partition or the length of the longest diagonal of the R_k
4. Choose a point (x_k^*, y_k^*, z_k^*) in each element of surface area
5. Form the sum $\displaystyle\sum_{k=1}^{n} G(x_k^*, y_k^*, z_k^*)\Delta S_k$

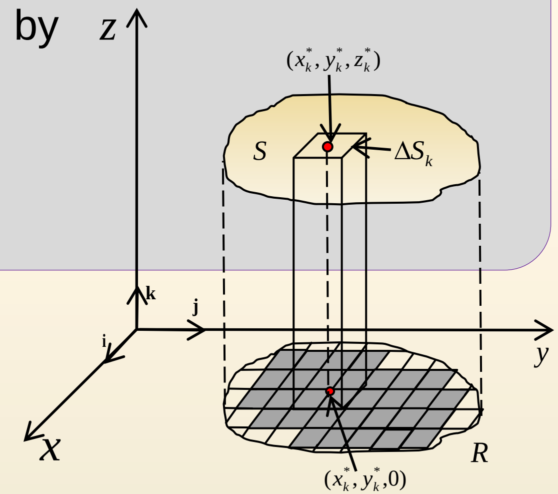
Surface Integrals

Definition 9.12

Surface Integral

Let G be a function of three variables defined over a region of space containing the surface *S*. then the **surface integral of *G* over *S*** is given by

$$\iint_S G(x,y,z)\,dS = \lim_{\|P\|\to 0}\sum_{k=1}^{n} G(x_k^*, y_k^*, z_k^*)\Delta S_k \cdots (4)$$



Method of Evaluation

If $G, f, f_x \text{ and } f_y$ are continuous throughout a region containing *S*, we can evaluate (4) by means of a double integral

$$\iint_S G(x,y,z)\,dS = \iint_R G(x,y,f(x,y))\sqrt{1+[f_x(x,y)]^2+[f_y(x,y)]^2}\,dA \cdots (5)$$

when $G=1$, (5) reduces to furmular (2) for surface area

Surface Integrals

Projection of S into Other Planes

If $y = g(x, z)$ is the equation of a surface *S* that projects onto a region *R* of the xz-plane, then

$$\iint_S G(x, y, z)\,dS = \iint_R G(x, g(x, z), z)\sqrt{1 + [g_x(x, z)]^2 + [g_z(x, z)]^2}\,dA$$

If $x = h(y, z)$ is the equation of a surface *S* that projects onto a region *R* of the yz-plane, then

$$\iint_S G(x, y, z)\,dS = \iint_R G(h(y, z), y, z)\sqrt{1 + [h_y(y, z)]^2 + [h_z(y, z)]^2}\,dA$$

Mass of a Surface

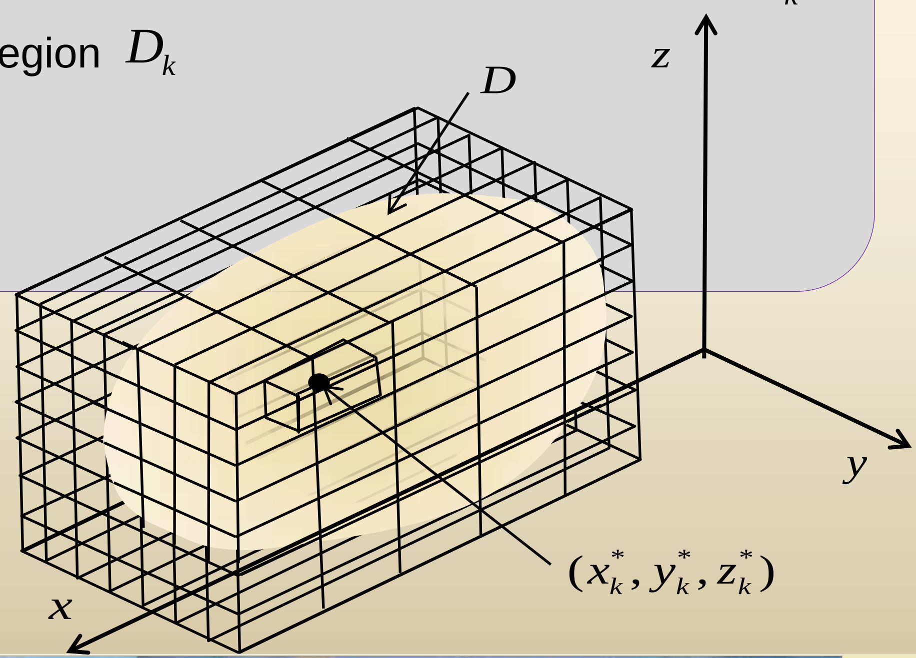
$$m = \iint_S \rho(x, y, z)\,dS$$

Triple Integrals

$w = F(x, y, z)$

1. Let F be defined a closed and bounded region *D* of space.
2. By means of a three- dimensional grid of vertical and horizontal planes parallel to the coordinate planes, form a partition *P* of *D* into n subregions (boxes) D_k of volumes ΔV_k that lie entirely in *D*.
3. Let $\|P\|$ be the **norm** of the partition or the length of the longest diagonal of the D_k
4. Choose a point (x_k^*, y_k^*, z_k^*) in each subregion D_k
5. Form the sum

$$\sum_{k=1}^{n} F(x_k^*, y_k^*, z_k^*)\Delta V_k$$

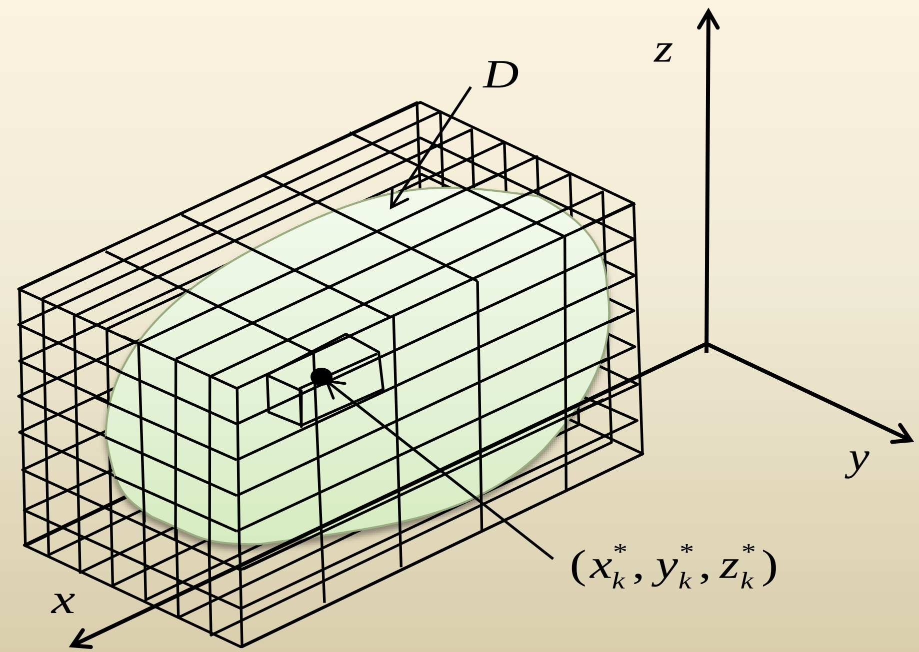


Triple Integrals

Definition 9.13 **The Triple Integral**

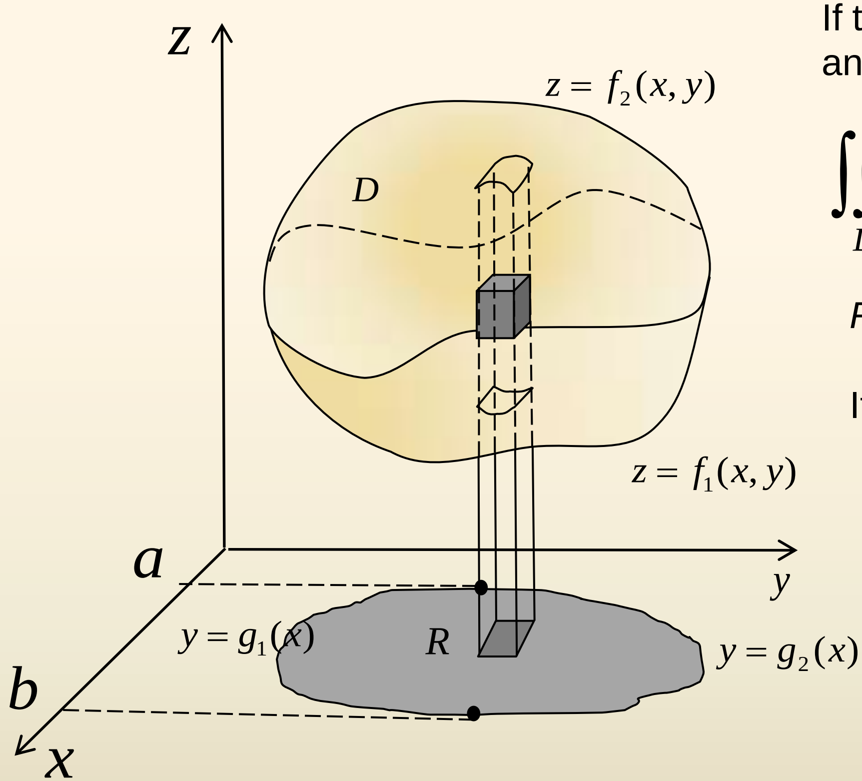
Let F be a function of three variables defined over a closed region *D* of space. Then the **triple integral of *F* over *D*** is given by

$$\iiint_D F(x,y,z)\,dV = \lim_{\|P\|\to 0} \sum_{k=1}^{n} F(x_k^*, y_k^*, z_k^*)\Delta V_k$$



Triple Integrals

Evaluation by Iterated Integrals



If the region D is bounded by above by the graph of $z = f_1(x, y)$ and bounded below by the graph of $z = f_2(x, y)$ then

$$\iiint_D F(x,y,z)\,dV = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz \right] dA$$

R is the orthogonal projection of D onto the xy-plane.

If R is a Type I region,

$$\iiint_D F(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dzdydx$$

Type I:

$R: a \le x \le b, \quad g_1(x) \le y \le g_2(x)$

Type II:

$R: c \le y \le d, \quad h_1(y) \le x \le h_2(y)$

In triple integral, there are three possible orders of integration.

$$dzdydx, \quad dzdxdy, \quad dydxdz,$$
$$dxdydz, \quad dxdzdy, \quad dydzdx,$$

The last two differentials tell the coordinate plane in which the region R is situated.

Triple Integrals

Volume $$V = \iiint_D dV$$

Mass $$m = \iiint_D \rho(x, y, z)\,dV$$

First Moments

The **first moment** of the solid about the coordinate planes indicated by the subscripts are given by

$$M_{xy} = \iiint_D z\rho(x, y, z)\,dV \quad M_{xz} = \iiint_D y\rho(x, y, z)\,dV \quad M_{yz} = \iiint_D x\rho(x, y, z)\,dV$$

Center of Mass $$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m}$$

Centroid

If $\rho(x, y, z) =$ a constant, the center of mass is called the **centroid** of the solid

Triple Integrals

Second Moments

The **Second moment** or **moments of inertia** of D about the coordinate axes indicated by the subscripts, are given by

$$I_x = \iiint_D (y^2 + z^2)\rho(x,y,z)\,dV$$

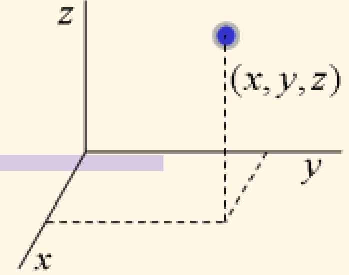
$$I_y = \iiint_D (x^2 + z^2)\rho(x,y,z)\,dV$$

$$I_z = \iiint_D (x^2 + y^2)\rho(x,y,z)\,dV$$

Radius of Gyration

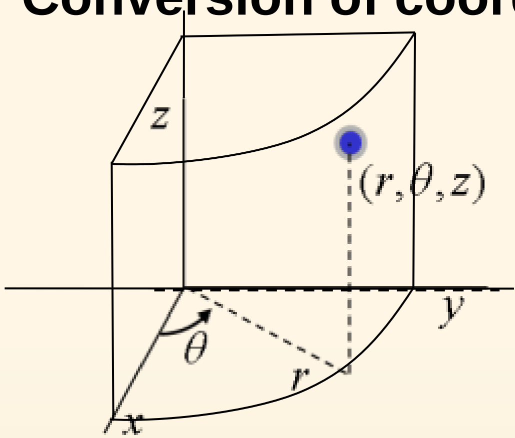
$$R_g = \sqrt{\frac{I}{m}}$$

Triple Integrals



Cartesian coordinates

Conversion of coordinates



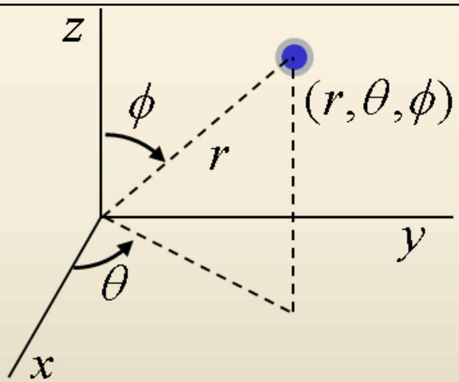
Cylindrical coordinates

Cylindrical → Cartesian

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

Cartesian → Cylindrical

$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad z = z$$



Spherical coordinates

Spherical→ Cylindrical

$$r = \rho\sin\phi \quad \theta = \theta \quad z = \rho\cos\phi$$

Spherical→ Cartesian

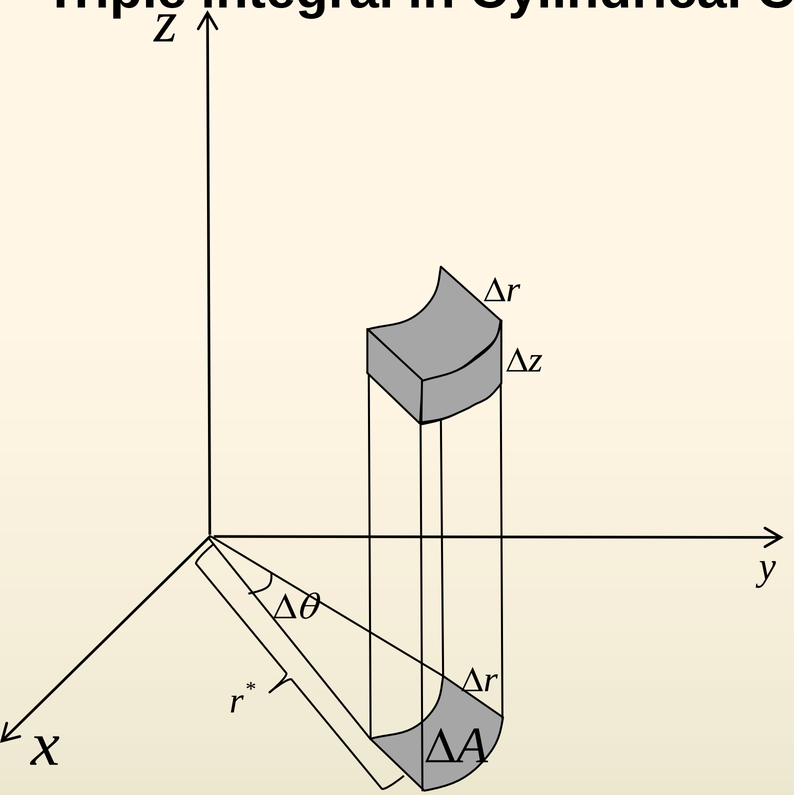
$$x = r\sin\phi\cos\theta \quad y = r\sin\phi\sin\theta \quad z = r\cos\phi$$

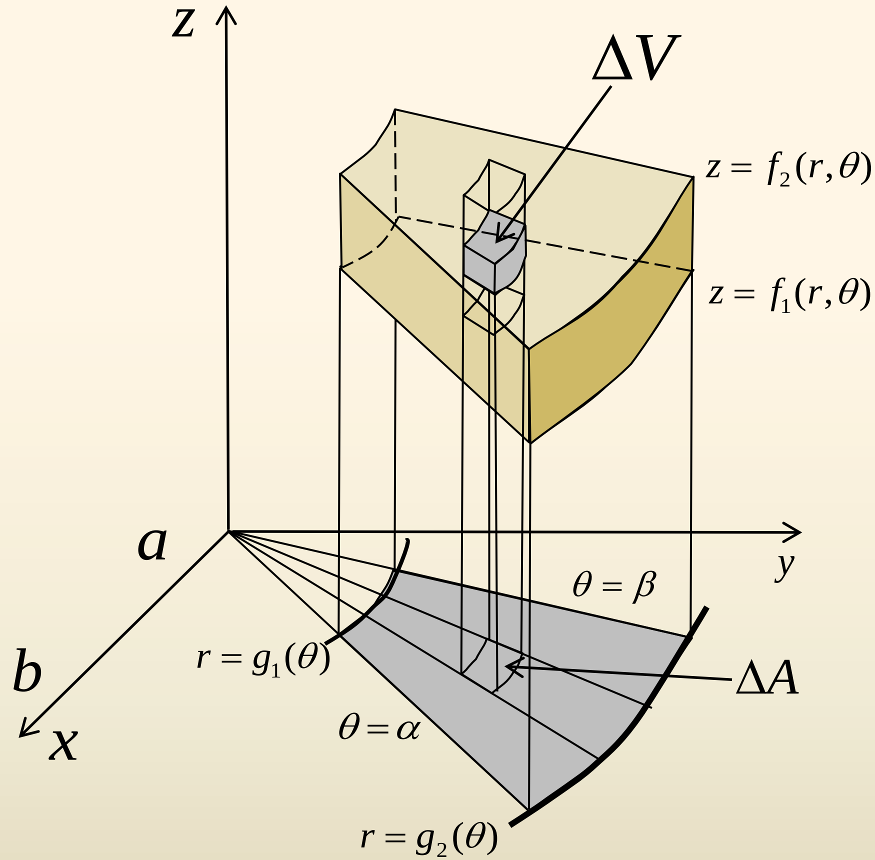
Cartesian → Spherical

$$r = \sqrt{x^2 + y^2 + z^2} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Triple Integrals

Triple integral in Cylindrical Coordinates



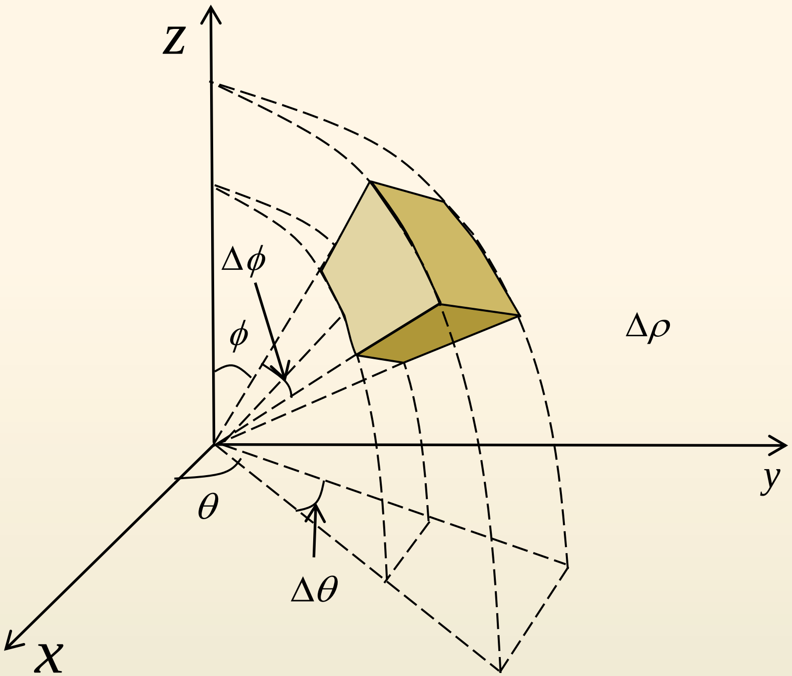


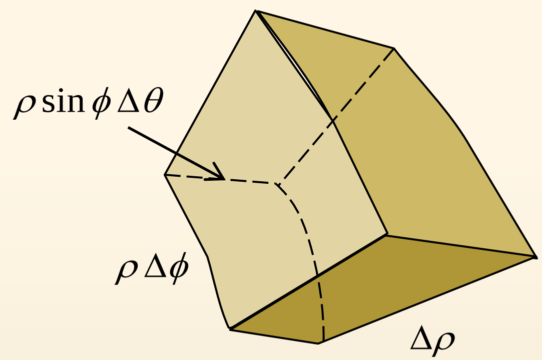
$$\Delta A = (r^{*}\Delta\theta)\cdot\Delta r \qquad \Delta V = \Delta A\,\Delta z = r^{*}\Delta r\Delta\theta\,\Delta z$$

$$\iiint_D F(r,\theta,z)dV = \iint_R \left[\int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r,\theta,z)dz\right]dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} F(r,\theta,z)r\,dz\,dr\,d\theta$$

Triple Integrals

Triple integral in Spherical Coordinates





$$\Delta V \approx \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$$

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$\iiint_D F(r,\theta,z)dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(\phi,\theta)}^{f_2(\phi,\theta)} F(r,\phi,\theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Reference slides

Line Integrals

Proof

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C Pdx + Qdy + Rdz$$

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} = [P, Q, R]$$

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dx\mathbf{k} = [dx, dy, dz]$$

$$\mathbf{F}\bullet d\mathbf{r} = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k})\bullet(dx\mathbf{i} + dy\mathbf{j} + dx\mathbf{k}) = Pdx + Qdy + Rdz$$

$$\therefore \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C Pdx + Qdy + Rdz$$

Reference slides

Simply Connected

Simply connected

Simply connected

- **Simple connect**

A domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

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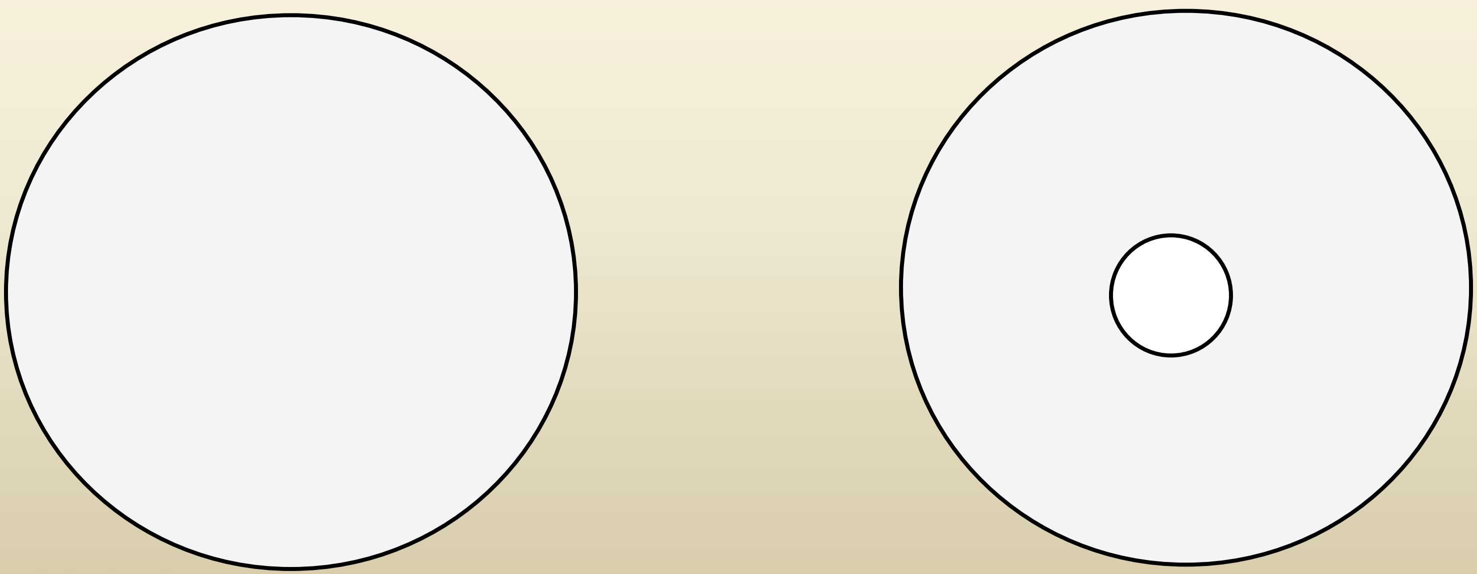

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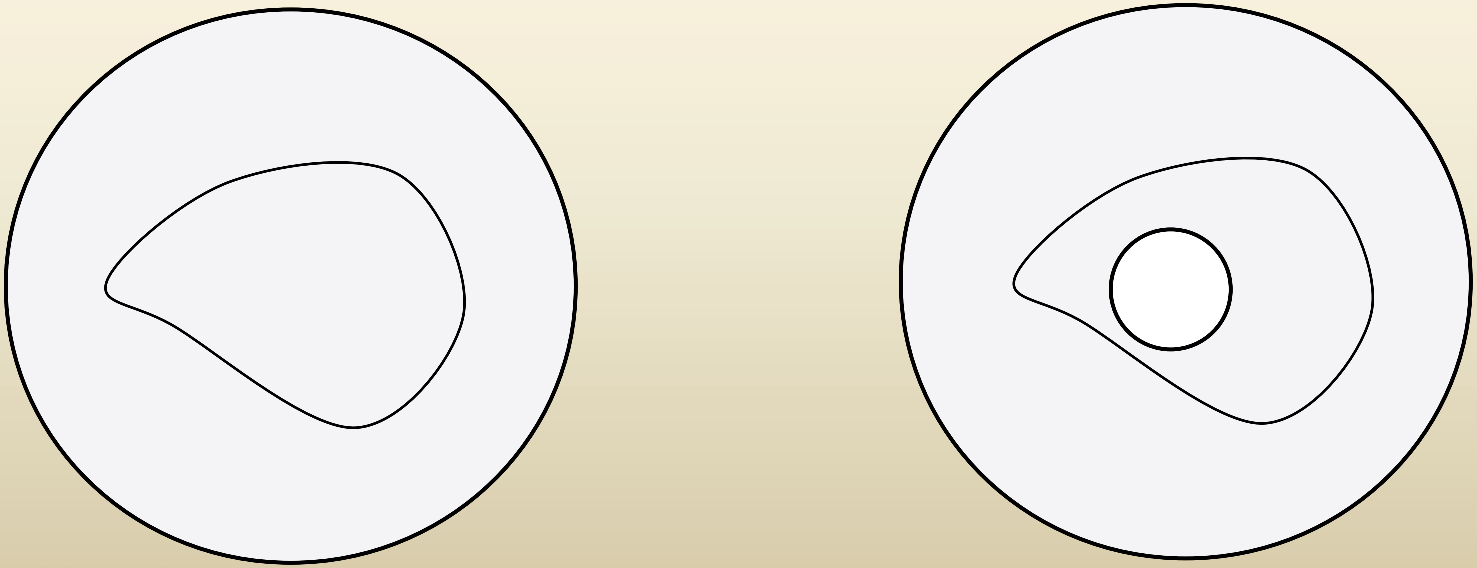

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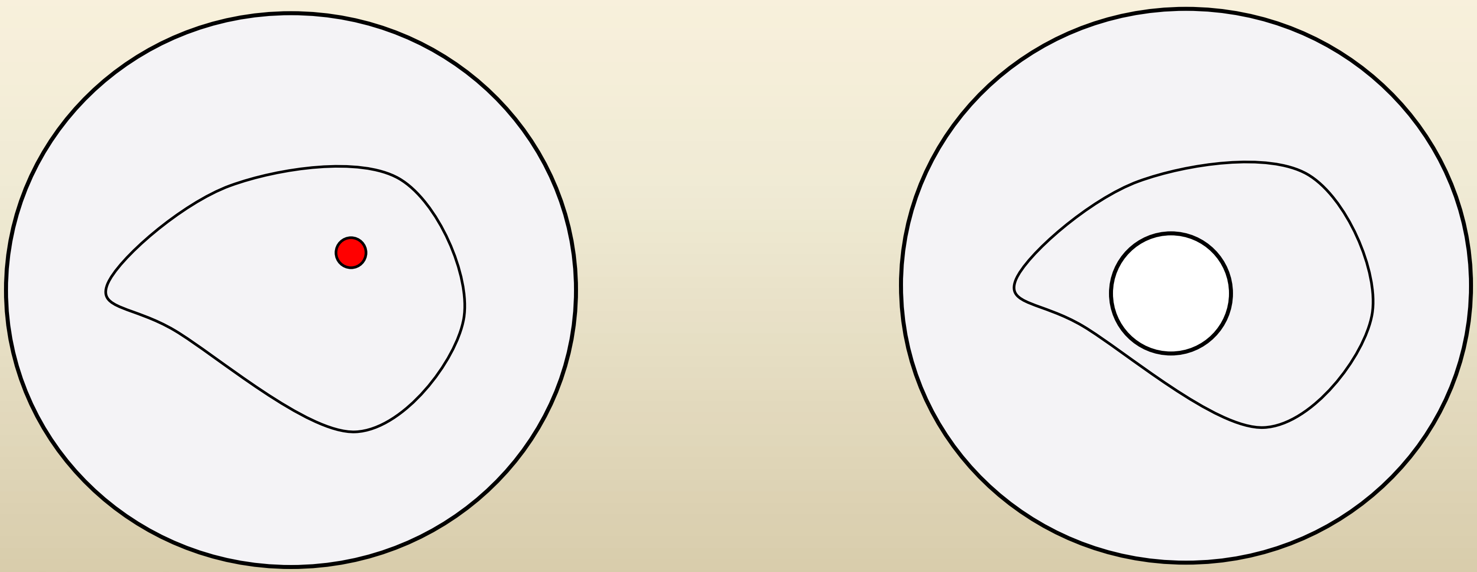

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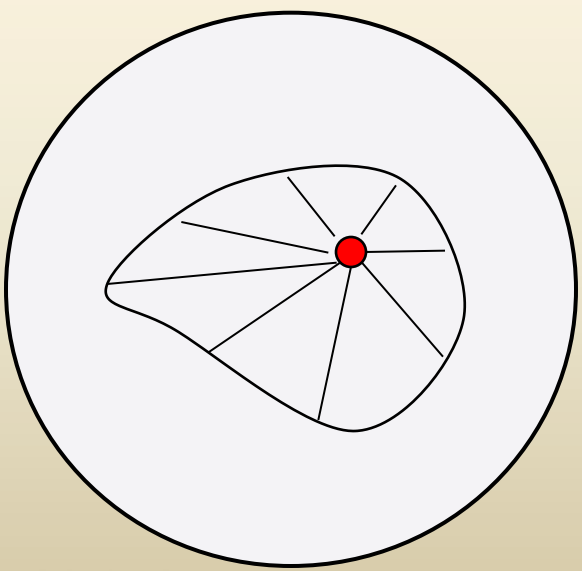
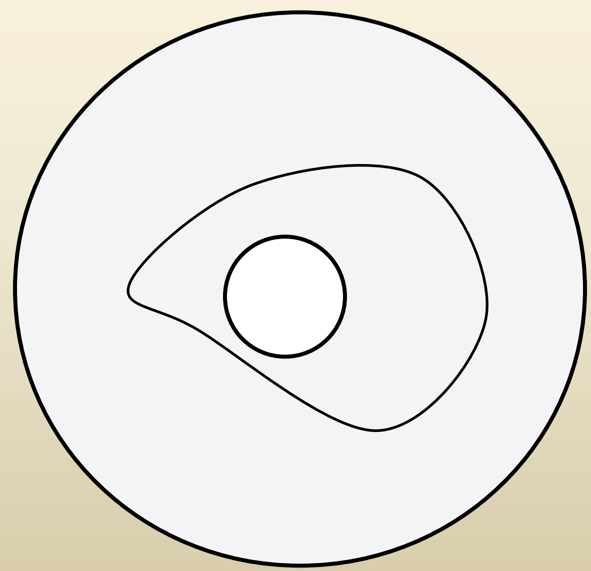


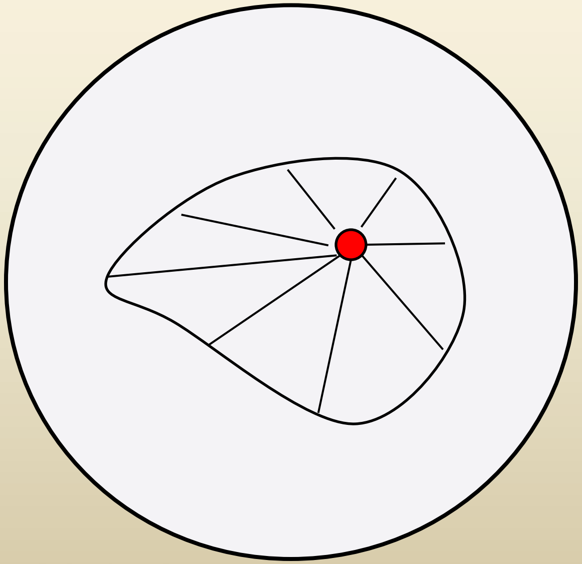


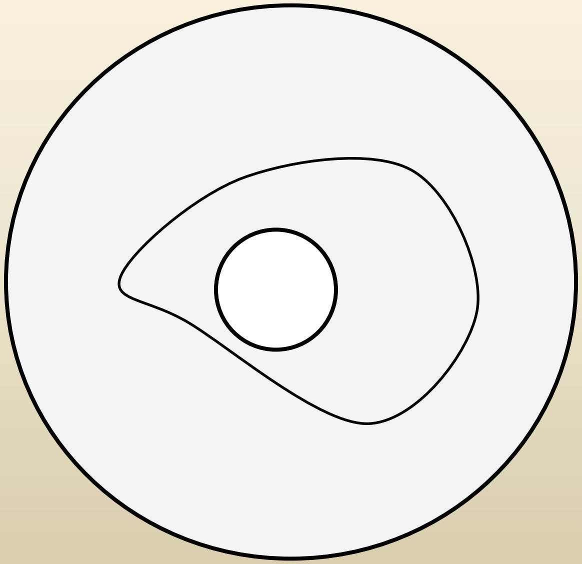
Simply connected

- Simple connect

A domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

it can shrink to a point in D continuously



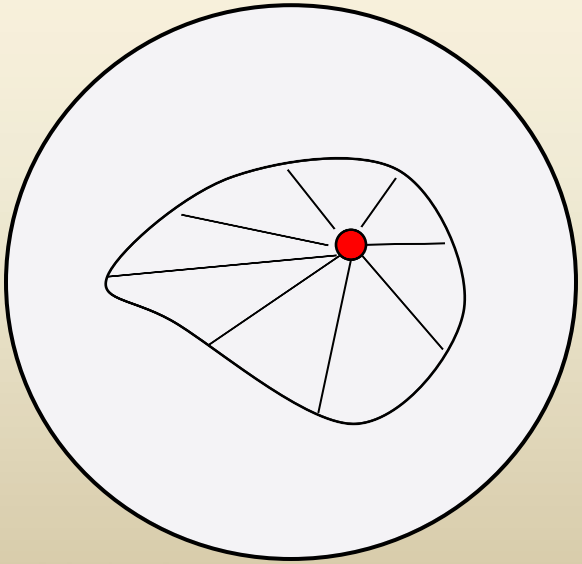


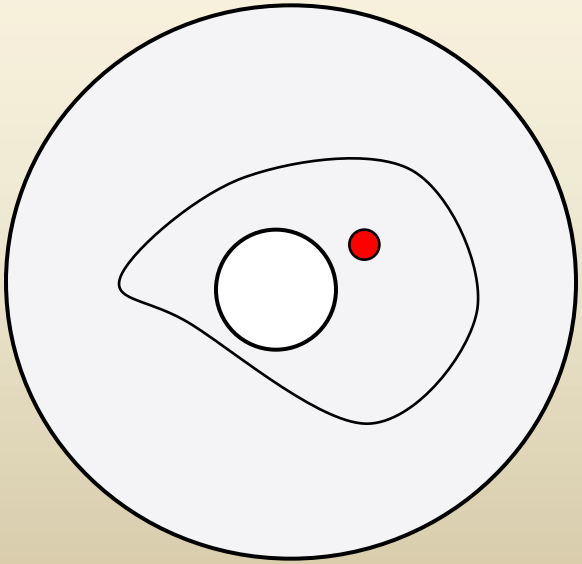
Simply connected

- Simple connect

A domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

it can shrink to a point in D continuously





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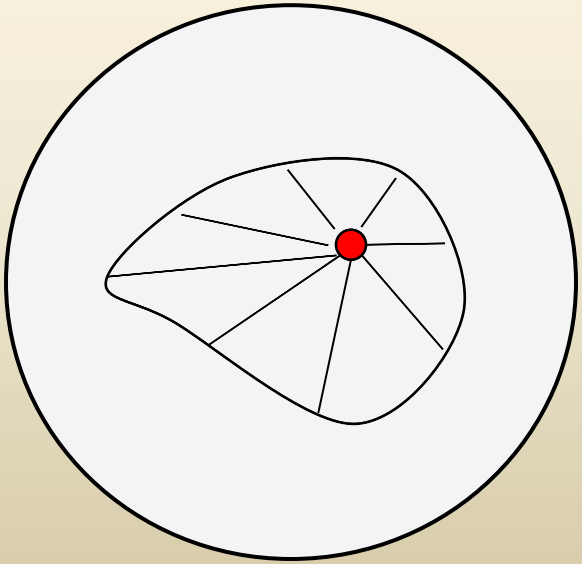
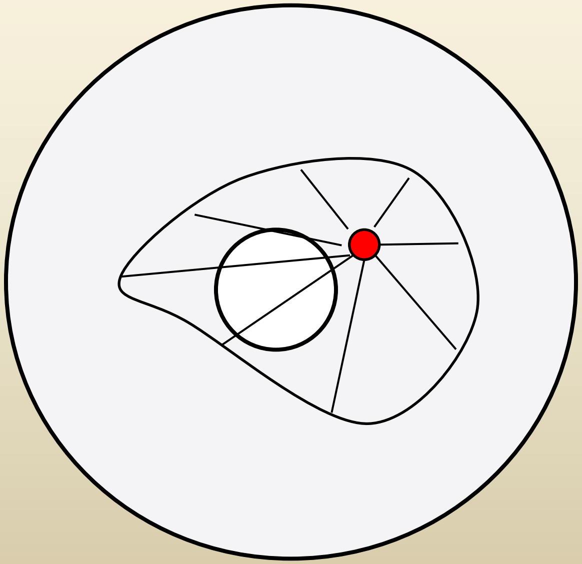
A domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

it can shrink to a point in D continuously



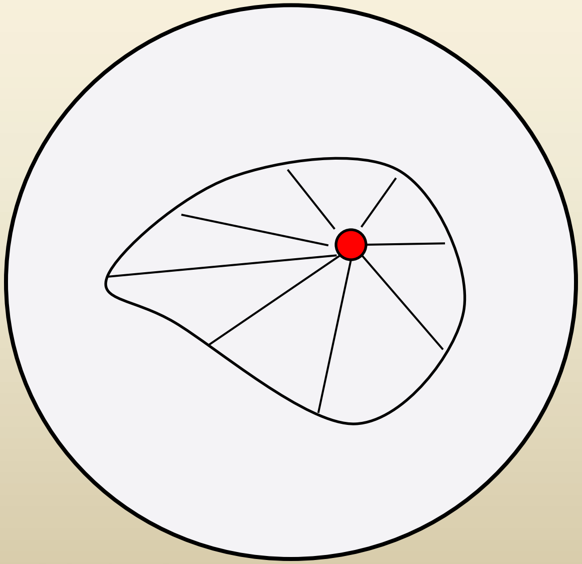


Simply connected

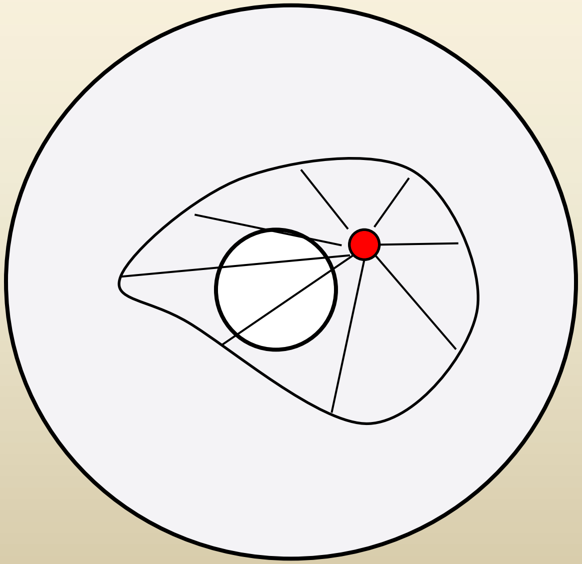
- **Simple connect**

A domain D is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

it can shrink to a point in D continuously



it can't shrink to a point in D continuously



Reference slides

Conservative Force and Mechanical Energy Conservation

(참고) 보존력과 역학적 에너지 보존

▪ 보존력 : 모든 닫힌 경로를 따라 운동하는 입자에 어떠한 힘이 한 알짜 일이 '0'일 때 이 힘을 보존력이라 함

(*닫힌 경로: 어떤 위치를 출발하여 임의의 경로를 거쳐 다시 처음의 위치로 되돌아 올 때)

$$W = \oint \mathbf{F}_{\text{보존력}} \cdot d\mathbf{r} = 0$$

▪보존력의 예 :만유인력,중력, 탄성력, 전기력

단위 입자에 작용하는 보존력을 보존장이라 할 수 있다.

▪보존장의 예 :만유인력장,중력장, 탄성장, 전기장

보존력이 물체에 한일의 음의 값을 보존력에 의한 퍼텐셜에너지의 변화로 정의한다.

$$\Delta U = -W$$

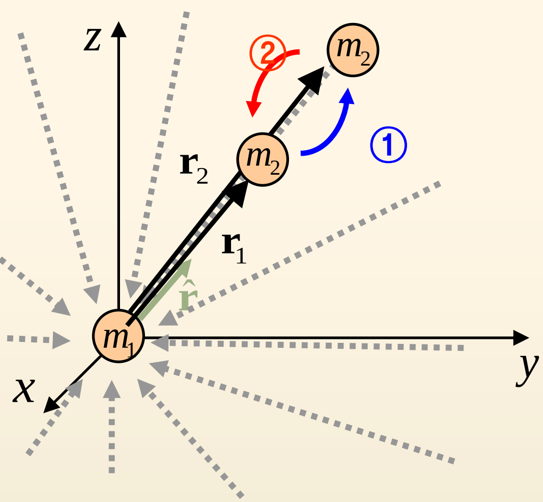
(만유인력과 탄성력 등이 보존력이 아니라면 중력 퍼텐셜에너지나 탄성 퍼텐셜에너지라는 말을 쓰지 않았을 것이다.)

만약, 물체가 보존장 속에서 운동하고, 보존장 외부에서 힘이 작용하지 않는다면 보존장 내에서의 Mechanical Energy는 보존된다.
(Mechanical Energy= Kinetic Energy+ Potential Energy)

(참고) 보존력과 역학적 에너지 보존

$$\mathbf{F}_g = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

힘의 방향은일정하고 거리가 부호가 +,-로 바뀌니까 일은 0이다.



$$|\mathbf{r}_2| > |\mathbf{r}_1|$$

$$\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

$\hat{\mathbf{r}}$:unit position vector

$r = |\mathbf{r}|$

① 물체 m_2가 $\mathbf{r}_1$에서 $\mathbf{r}_2$까지 이동할때 만유인력 $\mathbf{F}_g$가 물체에 한 일 W_1

$$W_1 = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{r_1}^{r_2} -G\frac{m_1 m_2}{r^2}dr = \left[G\frac{m_1 m_2}{r}\right]_{r_1}^{r_2} = G\frac{m_1 m_2}{r_2} - G\frac{m_1 m_2}{r_1}$$

< 0 ,(힘의방향$(-)$, 이동방향$(+)$ 따라서 일은 $(-)$)

② 물체 m_2가 $\mathbf{r}_2$에서 $\mathbf{r}_1$까지 이동할때 만유인력 $\mathbf{F}_g$가 물체에 한 일 W_2

$$W_2 = \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{F}\cdot d\mathbf{r} = \int_{r_2}^{r_1} -G\frac{m_1 m_2}{r^2}dr = \left[G\frac{m_1 m_2}{r}\right]_{r_2}^{r_1} = G\frac{m_1 m_2}{r_1} - G\frac{m_1 m_2}{r_2}$$

> 0 ,(힘의방향$(-)$, 이동방향$(-)$ 따라서 일은 $(+)$)

③ 물체 m_2가 $\mathbf{r}_1$에서 $\mathbf{r}_2$까지 이동한후 다시 $\mathbf{r}_1$으로 돌아왔을때(닫힌 경로) 만유인력 $\mathbf{F}_g$가 물체에 한 일 W

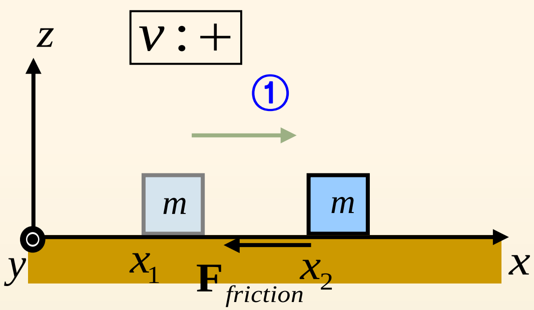
$$W = W_1 + W_2 = G\frac{m_1 m_2}{r_2} - G\frac{m_1 m_2}{r_1} + G\frac{m_1 m_2}{r_1} - G\frac{m_1 m_2}{r_2} = 0$$

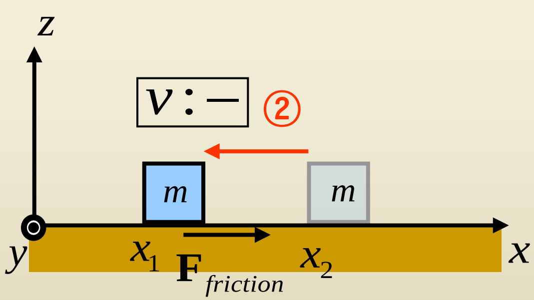
닫힌 경로를 따라 만유인력이 한 일이 0이므로 만유인력은 보존력이다.

(참고) 보존력과 역학적 에너지 보존

마찰력이 작용할때는 포텐셜에너지의 개념이 맞지않는다.

$$\mathbf{F}_{friction} = -\mu mg(\operatorname{sgn} v)\,\mathbf{i}$$





$x_1 < x_2$

①

물체 m이 x_1에서 x_2까지 이동할때
마찰력 $\mathbf{F}_{friction}$이 물체에 한 일 W_1

$$W_1 = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{friction} \cdot d\mathbf{r}$$
$$= \int_{x_1}^{x_2} -\mu mg(\sin v)dx$$
$$= -\mu mg(\sin v)\int_{x_1}^{x_2} dx$$
$$= -\mu mg\int_{x_1}^{x_2} dx$$
$$= -\mu mg(x_2 - x_1)$$

< 0 ,(**힘의방향**$(-)$, **이동방향**$(+)$ **따라서일은** $(-)$)

②

물체 m이 x_2에서 x_1까지 이동할때
마찰력 $\mathbf{F}_{friction}$이 물체에 한 일 W_1

$$W_2 = \int_{\mathbf{r}_2}^{\mathbf{r}_1} \mathbf{F}_{friction} \cdot d\mathbf{r}$$
$$= \int_{x_2}^{x_1} -\mu mg(\sin v)dx$$
$$= -\mu mg(\sin v)\int_{x_2}^{x_1} dx$$
$$= \mu mg\int_{x_2}^{x_1} dx$$
$$= \mu mg(x_1 - x_2)$$

< 0 ,(**힘의방향**$(+)$, **이동방향**$(-)$ **따라서일은** $(-)$)

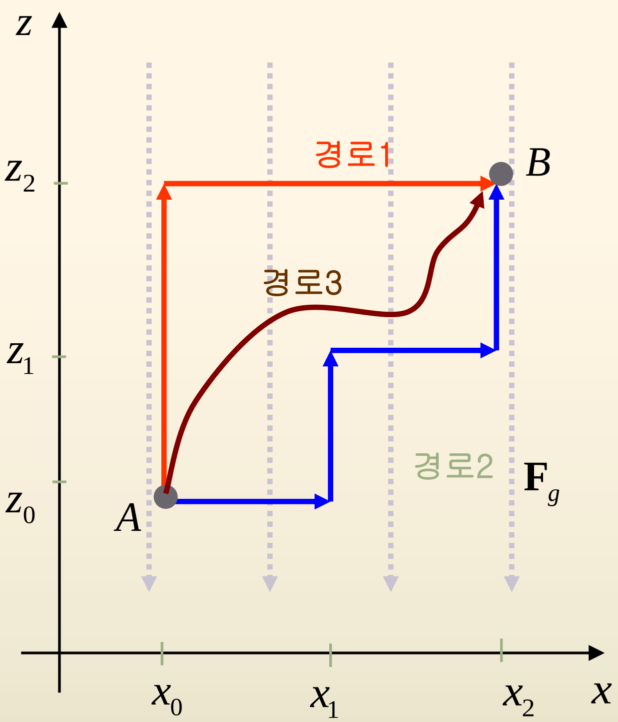
③

물체 m이 x_1에서 x_2까지 이동한후 다시 x_1으로 돌아왔을때(닫힌 경로)
마찰력 $\mathbf{F}_{frition}$이 물체에 한 일 W

$$W = W_1 + W_2 = -\mu mg(x_2 - x_1) + \mu mg(x_1 - x_2) = -2\mu mg(x_2 - x_1) \neq 0$$

닫힌 경로를 따라 마찰력이 한 일이 0이 아니므로 마찰력은 비보존력이다.

(참고) 보존력과 역학적 에너지 보존



다음과 같이 힘 $\mathbf{F_g}$ 를 받는 중력장 (→보존장) 속에서 각각 경로 1,2,3를 따라 물체를 움직일 때 한 일을 구해보자.

일의 정의에 따라 힘과 수직으로 움직인 경로의 일은 0이다.

경로1: $W_1 = \int \mathbf{F}_g \cdot d\mathbf{r}$

$$= F_g(z_2 - z_0) + 0$$
$$= F_g(z_2 - z_0)$$

경로2: $W_2 = \int \mathbf{F}_g \cdot d\mathbf{r}$

$$= 0 + F_g(z_1 - z_0) + 0 + F_g(z_2 - z_1)$$
$$= F_g(z_2 - z_0)$$

경로3: $W_3 = \int \mathbf{F}_g \cdot d\mathbf{r}$

$$= \int_{\mathbf{r}_i}^{\mathbf{r}_f} \left(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}\right) \cdot \left(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}\right)$$
$$= \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz$$
$$= 0 + 0 + \int_{z_0}^{z_2} F_g dz$$
$$= F_g(z_2 - z_0)$$

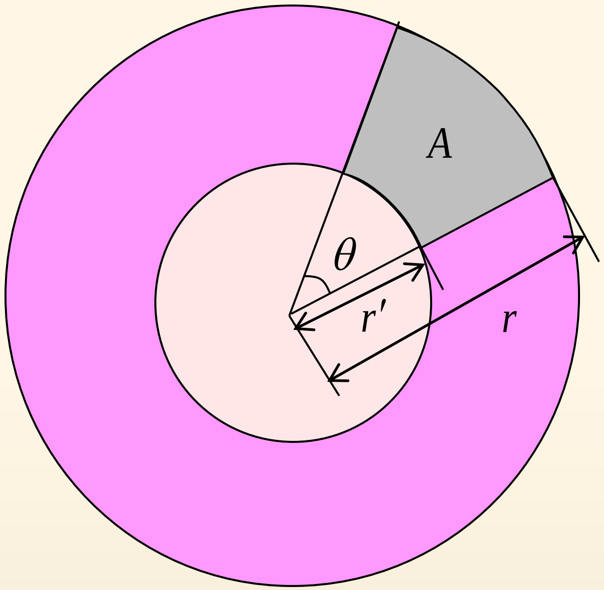
$$F_x, F_y = 0$$

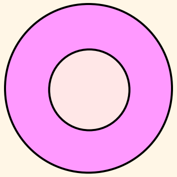
따라서 보존력인 한 일은 경로에 무관함

Reference slides

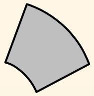
Area of polar rectangles

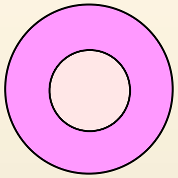
Area of polar rectangles



Area of outer circle minus inner circle: 

$$\pi r^2 - \pi r'^2$$

Area of polar rectangle A : 

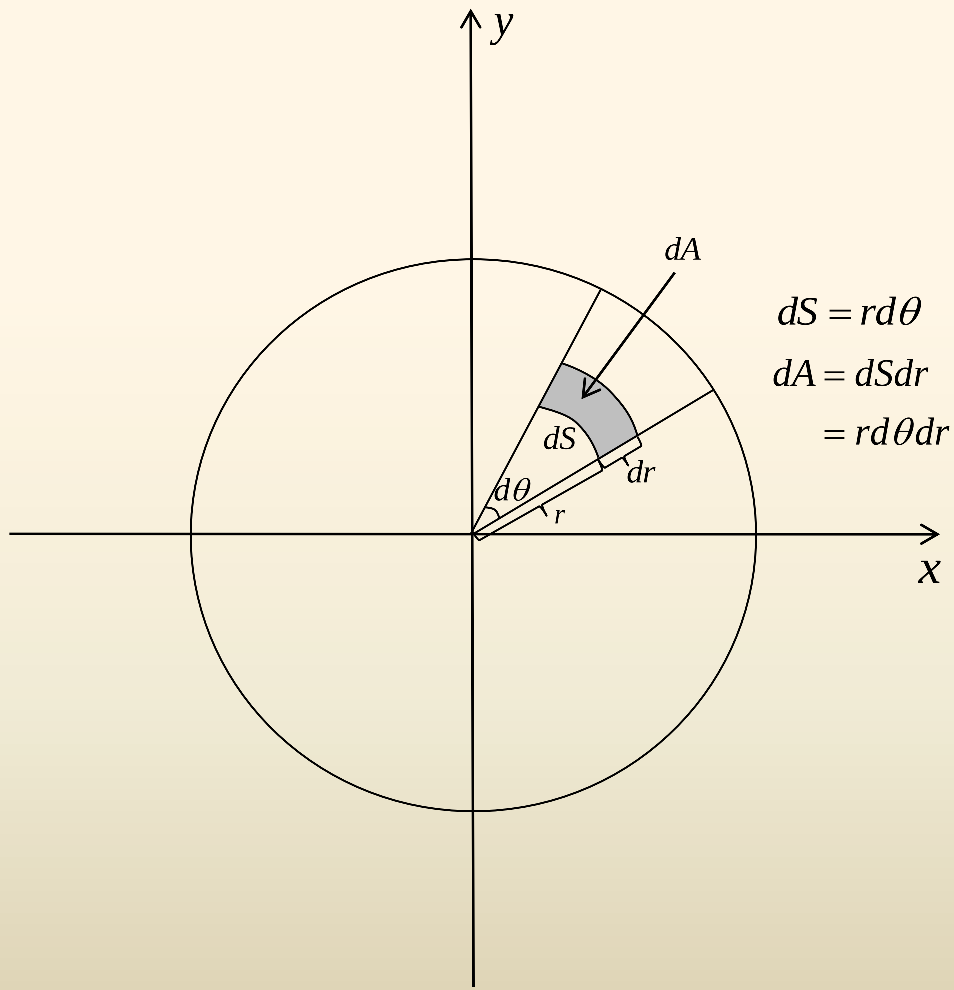
 :  = $2\pi : \theta$

$$(\pi r^2 - \pi r'^2) : A = 2\pi : \theta$$

$$A = (\pi r^2 - \pi r'^2)\frac{\theta}{2\pi}$$

$$= \frac{1}{2} r^2 \theta - \frac{1}{2} r'^2 \theta$$

Double Integrals in Polar Coordinates



$$dS = r d\theta$$

$$dA = dS dr$$

$$= r d\theta dr$$