Naval Architecture & Ocean Engineering

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Engineering Mathematics 2

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Naval Architecture & Ocean Engineering

Fourier Series(3)
: Bessel and Legendre Series

Fourier-Bessel Series

Fourier-Legendre Series

Bessel and Legendre Series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\right)$$

- **Fourier Series**

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{p}x$$

- **Fourier Cosine Series**

$$f(x)=\sum_{n=1}^{\infty}\left(b_n\sin\frac{n\pi}{p}x\right)$$

- **Fourier Sine Series**

Three ways of expanding a function in term of orthogonal set of functions

But such *expansion are by no means limited to orthogonal sets of trigonometric functions*

Fourier-Bessel Series

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x)$$

Fourier-Legendre Series

$$f(x)=\sum_{n=0}^{\infty}c_nP_n(x)$$

Recall, orthogonal relation

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx=0,\ \ \lambda_i\neq\lambda_j\ (\lambda=\alpha^2)$$

Recall, orthogonal relation

$$\int_{-1}^{1}1\cdot P_m(x)P_n(x)dx=0,\ \ m\neq n$$

Fourier-Bessel Series

orthogonal relation

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx = 0,\ \lambda_i \neq \lambda_j\ (\lambda = \alpha^2)$$

$$A_2J_n(\alpha b) + B_2\alpha J_n{}'(\alpha b) = 0$$ ⇨

Definition 12.8

Fourier-Bessel Series

The Fourier-Bessel series of a function f defined on the interval $(0,b)$ is given by

1) α_i are defined by $J_n(\alpha_i b) = 0$ **Case i**

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \quad , c_i = \frac{2}{b^2 J^2{}_{n+1}(\alpha_i b)} \int_0^b xJ_n(\alpha_i x) f(x)dx$$

2) α_i are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$ **Case ii**

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \quad , c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2)J_n^2(\alpha_i b)} \int_0^b xJ_n(\alpha_i x) f(x)dx$$

3) α_i are defined by $J_n'(\alpha b) = 0$ **Case iii**

$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_n(\alpha_i x)$$

$$, c_1 = \frac{2}{b^2} \int_0^b xf(x)dx$$

$$, c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b xJ_n(\alpha_i x) f(x)dx$$

Fourier-Bessel Series

Orthogonal series expansion of $f(x)$ (generalized Fourier Series)

$$f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x) \qquad c_n=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\|\phi_n(x)\|^2}$$

$$=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi^2{}_n(x)dx}$$

Bessel function orthogonal relation

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx=0,$$

$$\lambda_i\neq\lambda_j\ (\lambda=\alpha^2)$$

- **Fourier-Bessel Series**

Parametric Bessel equation of order n

$$x^2y''+xy'+(\alpha^2x^2-n^2)y=0,\quad n=0,1,2,\ldots$$

Self-adjoint form

$$\frac{d}{dx}[xy']+(-\frac{n^2}{x}+\alpha^2x)y=0$$

The eigenvalues of the corresponding Sturm-Liouville problem are $\lambda=\alpha_i^2$.

The orthogonal series expansion of a function f defined on the interval $[0,b]$ in terms of this orthogonal set is $f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x)$

$$\uparrow$$

$$c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\|J_n(\alpha_i x)\|^2} \longleftarrow \|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$$

Fourier-Bessel Series

- **Fourier-Bessel Series**

Parametric Bessel equation of order n

$$x^2y''+xy'+(\alpha^2x^2-n^2)y=0,\quad n=0,1,2,\ldots$$

Self-adjoint form

$$\frac{d}{dx}[xy']+(-\frac{n^2}{x}+\alpha^2x)y=0$$

Orthogonal series expansion of (generalized Fourier Series) $f(x)$

$$f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x)\qquad c_n=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\|\phi_n(x)\|^2}=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi^2{}_n(x)dx}$$

Bessel function orthogonal relation

$$\int_0^b xJ_n(\alpha_ix)J_n(\alpha_jx)dx=0,$$

$$\lambda_i\neq\lambda_j\ (\lambda=\alpha^2)$$

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_ix)\qquad[0,b]\qquad\lambda=\alpha_i^2\qquad c_i=\frac{\int_0^b xJ_n(\alpha_ix)f(x)dx}{\|J_n(\alpha_ix)\|^2}$$

- **Recall Differential Recurrence Relations***

$$\frac{d}{dx}\left[x^nJ_n(x)\right]=x^nJ_{n-1}(x)$$

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$$

- **Derivatives**

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$$

$$-nx^{-n-1}J_n(x)+x^{-n}J_n'(x)=-x^{-n}J_{n+1}(x)$$

$$-nJ_n(x)+xJ_n'(x)=-xJ_{n+1}(x)$$

$$\therefore xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x)$$

*Kreyszig E. ,Advanced Engineering Mathematics, 9th edition, Willey, 2006, p194

Fourier-Bessel Series

Orthogonal series expansion of $f(x)$
(generalized Fourier Series)

$$f(x)=\sum_{n=0}^{\infty}c_n\phi_n(x) \qquad c_n=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\|\phi_n(x)\|^2}$$

$$=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi^2{}_n(x)dx}$$

Bessel function orthogonal relation

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx=0,$$

$$\lambda_i\neq\lambda_j\ (\lambda=\alpha^2)$$

- **Fourier-Bessel Series**

Parametric Bessel equation of order n

$$x^2y''+xy'+(\alpha^2x^2-n^2)y=0,\quad n=0,1,2,\ldots$$

Self-adjoint form

$$\frac{d}{dx}[xy']+(-\frac{n^2}{x}+\alpha^2x)y=0$$

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x) \qquad [0,b] \qquad \lambda=\alpha_i^2 \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\|J_n(\alpha_i x)\|^2}}$$

- **Square Norm** $\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$\frac{d}{dx}[xy']+(\alpha^2x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty}c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\left\|J_n(\alpha_i x)\right\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0 \xrightarrow{\qquad} 2xy'\frac{d}{dx}[xy']+2xy'(\alpha^2 x-\frac{n^2}{x})y=0$$

Multiply by $2xy'$

$$\frac{d}{dx}[xy']^2+2y'(\alpha^2x^2-n^2)y=0$$

$$\frac{d}{dx}[xy']^2+(\alpha^2x^2-n^2)\frac{d}{dx}[y]^2=0$$

Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty}c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\left\|J_n(\alpha_i x)\right\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$\frac{d}{dx}[xy']^2+(\alpha^2 x^2-n^2)\frac{d}{dx}[y]^2=0$$

$$\frac{d}{dx}[xy']^2+\alpha^2 x^2\frac{d}{dx}[y]^2-n^2\frac{d}{dx}[y]^2=0$$

$$\frac{d}{dx}[xy']^2+\alpha^2 x^2\frac{d}{dx}[y]^2=n^2\frac{d}{dx}[y]^2$$

Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty}c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\left\|J_n(\alpha_i x)\right\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$\frac{d}{dx}[xy']^2+\alpha^2x^2\frac{d}{dx}[y]^2=n^2\frac{d}{dx}[y]^2$$

Integration by part

$$\int_0^b\frac{d}{dx}[xy']^2dx+\underline{\int_0^b\alpha^2x^2\frac{d}{dx}[y]^2dx}=\int_0^b n^2\frac{d}{dx}[y]^2dx$$

$$\int_0^b\frac{d}{dx}[xy']^2dx+\underline{\left[\alpha^2x^2y^2\right]_0^b-\int_0^b\alpha^2 2xy^2dx}=\int_0^b n^2\frac{d}{dx}[y]^2dx$$

Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\left\|J_n(\alpha_i x)\right\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$\int_0^b \frac{d}{dx}[xy']^2 dx+\left[\alpha^2x^2y^2\right]_0^b-\int_0^b \alpha^2 2xy^2dx=\int_0^b n^2\frac{d}{dx}[y]^2dx$$

$$\int_0^b \alpha^2 2xy^2dx=\int_0^b \frac{d}{dx}[xy']^2 dx+\left[\alpha^2x^2y^2\right]_0^b-\int_0^b n^2\frac{d}{dx}[y]^2dx$$

$$2\alpha^2\int_0^b xy^2dx=\left[(xy')^2\right]_0^b+\left[\alpha^2x^2y^2\right]_0^b-\left[\left(n^2y^2\right)\right]_0^b$$

$$2\alpha^2\int_0^b xy^2dx=\left[(xy')^2+\left(\alpha^2x^2-n^2\right)y^2\right]_0^b$$

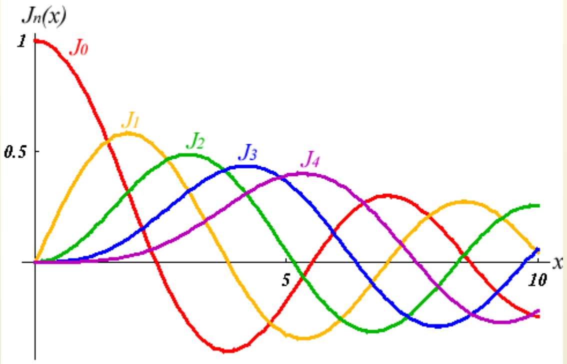
Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\left\|J_n(\alpha_i x)\right\|^2}}$$

$$\frac{d}{dx}\left[xy'\right]+(\alpha^2x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*



To fine square norm $\left\|J_n(\alpha_i x)\right\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$2\alpha^2\int_0^b xy^2dx=\left[(xy')^2+\left(\alpha^2x^2-n^2\right)y^2\right]_0^b$$

since $y=J_n(\alpha x),\ \ y'=\alpha J_n'(\alpha x),\ \ J_n(0)=0$

$$2\alpha^2\int_0^b xJ_n^2(\alpha x)dx=\alpha^2b^2\left[J_n'(\alpha b)\right]^2+\left(\alpha^2b^2-n^2\right)\left[J_n(\alpha b)\right]^2$$

$$\therefore\int_0^b xJ_n^2(\alpha x)dx=\frac{1}{2\alpha^2}\left\{\alpha^2b^2\left[J_n'(\alpha b)\right]^2+\left(\alpha^2b^2-n^2\right)\left[J_n(\alpha b)\right]^2\right\}$$

Fourier-Bessel Series

- **Square Norm**

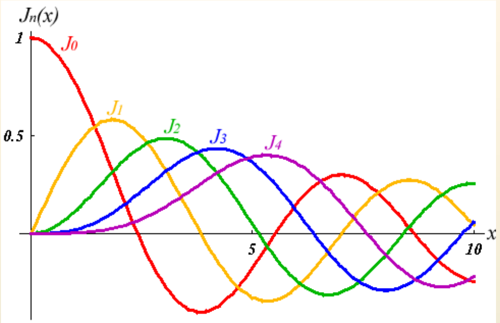
$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\|J_n(\alpha_i x)\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

To fine square norm $\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$\therefore \int_0^b xJ_n^2(\alpha x)dx=\frac{1}{2\alpha^2}\left\{\alpha^2 b^2[J_n'(\alpha b)]^2+(\alpha^2 b^2-n^2)[J_n(\alpha b)]^2\right\}$$



$\lambda=\alpha^2$
Eigenvalues from this equation of condition according to the cases (for nontrivial solutions)

Consider boundary condition

$$A_2 J_n(\alpha b)+\alpha B_2 J_n'(\alpha b)=0$$

$$A_1 y(a)+B_1 y'(a)=0$$

$$\boxed{A_2 y(b)+B_2 y'(b)=0}$$

Case I $A_2=1, B_2=0$

Case II $A_2=h\geq 0,\ B_2=b$

Case III $h=0,\ n=0$

Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x)\qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\|J_n(\alpha_i x)\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{1}{2\alpha_i^2}\left\{\alpha_i^2b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i^2b^2-n^2\right)\left[J_n(\alpha_i b)\right]^2\right\}$$

Consider boundary condition $A_2J_n(\alpha b)+\alpha B_2J_n'(\alpha b)=0$

α_i **are defined from the root of**

$$J_n(\alpha_i b)=0$$

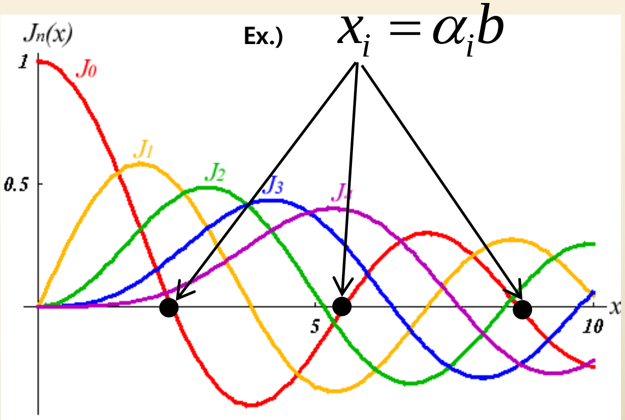
Case I $A_2=1, B_2=0$

then $\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$=\frac{1}{2\alpha_i^2}\left\{\alpha_i^2b^2\left[J_n'(\alpha_i b)\right]^2+\left(\alpha_i^2b^2-n^2\right)\left[\cancelto{\text{zero}}{J_n(\alpha_i b)}\right]^2\right\}$$

$$=\frac{b^2}{2}\left[J_n'(\alpha_i b)\right]^2=\frac{b^2}{2}\left[J_{n+1}(\alpha_i b)\right]^2$$

$$\because xJ_n'(x)=n\cancelto{\text{zero}}{J_n(x)}-xJ_{n+1}(x)$$



Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\|J_n(\alpha_i x)\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{1}{2\alpha_i^2}\left\{\alpha_i^2 b^2[J_n'(\alpha_i b)]^2+\left(\alpha_i^2 b^2-n^2\right)[J_n(\alpha_i b)]^2\right\}$$

Consider boundary condition $A_2J_n(\alpha b)+\alpha B_2J_n'(\alpha b)=0$

α_i **are defined from the root of**

Case I $A_2=1, B_2=0 \longrightarrow J_n(\alpha_i b)=0$

then $\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx$

$$=\frac{1}{2\alpha_i^2}\left\{\alpha_i^2 b^2[J_n'(\alpha_i b)]^2+\left(\alpha_i^2 b^2-n^2\right)[\overset{\text{zero}}{\cancel{J_n(\alpha_i b)}}]^2\right\}$$

$$=\frac{b^2}{2}[J_n'(\alpha_i b)]^2=\frac{b^2}{2}[J_{n+1}(\alpha_i b)]^2$$

Recall, ch.5.3
differential recurrence relations

$$\frac{d}{dx}[x^{-n}J_n(x)]=-x^{-n}J_{n+1}(x)$$

$$-nx^{-n-1}J_n(x)+x^{-n}J_n'(x)=-x^{-n}J_{n+1}(x)$$

$$-nJ_n(x)+xJ_n'(x)=-xJ_{n+1}(x)$$

$$\therefore xJ_n'(x)=nJ_n(x)-xJ_{n+1}(x)$$

$$xJ_n'(\alpha b)=n\cancel{J_n(\alpha b)}-xJ_{n+1}(\alpha b)$$

$$J_n'(\alpha b)=-J_{n+1}(\alpha b)$$

Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i = \frac{\int_0^b x J_n(\alpha_i x) f(x)dx}{\boxed{\|J_n(\alpha_i x)\|^2}}$$

$$\frac{d}{dx}[xy'] + (\alpha^2 x - \frac{n^2}{x})y = 0$$

The value of the square norm depends on how the eigenvalues $\lambda = \alpha_i^2$ *are defined*

$$\|J_n(\alpha_i x)\|^2 = \int_0^b x J_n^2(\alpha_i x)dx = \frac{1}{2\alpha^2}\left\{\alpha_i^{\,2} b^2 [J_n'(\alpha_i b)]^2 + \left(\alpha_i^{\,2} b^2 - n^2\right)[J_n(\alpha_i b)]^2\right\}$$

Consider boundary condition $A_2 J_n(\alpha b) + \alpha B_2 J_n'(\alpha b) = 0$

α_i **are defined from the root of**

Case II $A_2 = h \geq 0,\ B_2 = b \longrightarrow \alpha_i b J_n'(\alpha_i b) = -h J_n(\alpha_i b)$

then $\|J_n(\alpha_i x)\|^2 = \int_0^b x J_n^2(\alpha_i x)dx$

$$= \frac{1}{2\alpha_i^{\,2}}\left\{\alpha_i^{\,2} b^2 [J_n'(\alpha_i b)]^2 + \left(\alpha_i^{\,2} b^2 - n^2\right)[J_n(\alpha_i b)]^2\right\} = \frac{1}{2\alpha_i^{\,2}}\left\{h^2 [J_n(\alpha_i b)]^2 + \left(\alpha_i^{\,2} b^2 - n^2\right)[J_n(\alpha_i b)]^2\right\}$$

$$= \frac{\alpha_i^2 b^2 - n^2 + h^2}{2\alpha_i^2}[J_n(\alpha_i b)]^2$$

Fourier-Bessel Series

▪**Square Norm**

$$f(x)=\sum_{i=1}^{\infty}c_iJ_n(\alpha_i x)\qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\|J_n(\alpha_i x)\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{1}{2\alpha^2}\left\{\alpha_i{}^2b^2[J_n'(\alpha_i b)]^2+\left(\alpha_i{}^2b^2-n^2\right)[J_n(\alpha_i b)]^2\right\}$$

Consider boundary condition $A_2J_n(\alpha b)+\alpha B_2J_n'(\alpha b)=0$

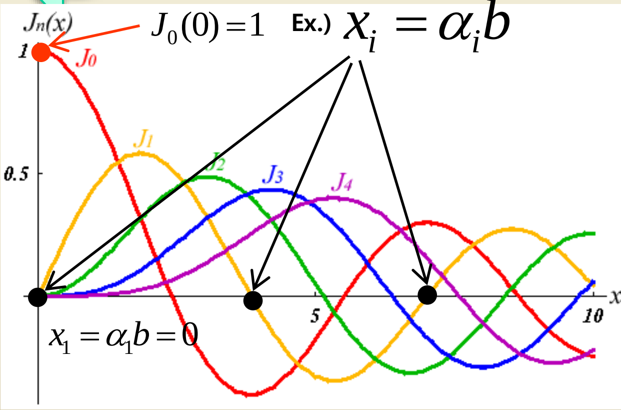
α_i **are defined from the root of**

Case III $A_2=h=0,\ n=0 \longrightarrow J_0'(\alpha b)=0$

Then ***Case III-1)*** $\alpha_i=0\ ,(or\ \alpha_1=0,\ \lambda=0)$

from $J_n'(\alpha b)=-J_{n+1}(\alpha b) \Rightarrow J_0'(\alpha_i b)=-J_1(\alpha_i b)=0$

$J_0(0)=1$ **: nontrivial solution! so λ=0 is also eigenvalue**



$$\therefore\|J_0(0)\|^2=\|1\|^2=\int_0^b xJ_0^2(0)dx=\frac{b^2}{2}$$

Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\|J_n(\alpha_i x)\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{1}{2\alpha^2}\left\{\alpha_i{}^2b^2[J_n'(\alpha_i b)]^2+\left(\alpha_i{}^2b^2-n^2\right)[J_n(\alpha_i b)]^2\right\}$$

Consider boundary condition $A_2J_n(\alpha b)+\alpha B_2J_n'(\alpha b)=0$

α_i **are defined from the root of**

Case III $A_2=h=0,\ n=0 \longrightarrow J_0'(\alpha b)=0$

Case III-2) $\alpha_i>0$

$$\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{\alpha_i^2b^2-n^2+h^2}{2\alpha_i^2}[J_n(\alpha_i b)]^2=\frac{b^2}{2}[J_n(\alpha_i b)]^2$$

$$\Downarrow$$

$$\|J_0(\alpha_i x)\|^2=\int_0^b xJ_0^2(\alpha_i x)dx=\frac{b^2}{2}[J_0(\alpha_i b)]^2$$

Fourier-Bessel Series

- **Square Norm**

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad c_i=\frac{\int_0^b xJ_n(\alpha_i x)f(x)dx}{\boxed{\|J_n(\alpha_i x)\|^2}}$$

$$\frac{d}{dx}[xy']+(\alpha^2 x-\frac{n^2}{x})y=0$$

The value of the square norm depends on how the eigenvalues $\lambda=\alpha_i^2$ *are defined*

$$\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{1}{2\alpha^2}\left\{\alpha_i^2 b^2[J_n'(\alpha_i b)]^2+\left(\alpha_i^2 b^2-n^2\right)[J_n(\alpha_i b)]^2\right\}$$

Consider boundary condition $A_2 J_n(\alpha b)+\alpha B_2 J_n'(\alpha b)=0$

α_i **are defined from the root of**

Case III $A_2=h=0,\ n=0 \longrightarrow J_0'(\alpha b)=0$

then

$$\|J_n(\alpha_i x)\|^2=\int_0^b xJ_n^2(\alpha_i x)dx=\frac{\alpha_i^2 b^2-n^2+h^2}{2\alpha_i^2}[J_n(\alpha_i b)]^2=\frac{b^2}{2}[J_n(\alpha_i b)]^2$$

$$\|J_0(\alpha_i x)\|^2=\int_0^b xJ_0^2(\alpha_i x)dx=\frac{b^2}{2}[J_0(\alpha_i b)]^2$$

Fourier-Bessel Series

Fourier-Bessel Series

The Fourier-Bessel series of a function f defined on the interval $(0, b)$ is given by

1) α_i are defined by $J_n(\alpha_i b) = 0$

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad , c_i = \frac{2}{b^2 J^2{}_{n+1}(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

2) α_i are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \quad , c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

3) α_i are defined by $J_n'(\alpha b) = 0$

$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_n(\alpha_i x) \qquad , c_1 = \frac{2}{b^2} \int_0^b x f(x) dx$$

$$, c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x) dx$$

Fourier-Bessel Series

- **Convergence of a Fourier-Bessel Series:**

  Sufficient conditions for the convergence of a Fourier-Bessel series are not particularly restrictive

Theorem 12.4 **Conditions for Convergence**

If f and f' are piecewise continuous on the open interval $(0,b)$, then a Fourier-Bessel expansion of f converges to $f(x)$ at any point where f is continuous and to the average $\left[f(x+)+f(x-)\right]/2$ at a point where f is discontinuous.

Fourier-Bessel Series

☑ **Example 1**

Expansion in a Fourier-Bessel Series

Expand f(x)=x, 0<x<3, in a Fourier-Bessel series, using Bessel functions of order one that satisfy the boundary condition $J_1(3\alpha)=0$

$b=3,$

1) α_i are defined by $J_n(\alpha_i b)=0$

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

$$,c_i=\frac{2}{b^2 J^2{}_{n+1}(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx$$

$$c_i=\frac{2}{3^2 J_2{}^2(3\alpha_i)}\int_0^3 x^2 J_1(\alpha_i x)dx$$

$t=\alpha_i x,\ dx=dt/\alpha_i,\ x^2=t^2/\alpha_i{}^2,$

$$c_i=\frac{2}{3^2 J_2{}^2(3\alpha_i)}\int_0^{3\alpha_i}\frac{1}{\alpha_i^3}t^2 J_1(t)dt$$

$$\frac{d}{dt}[t^2 J_2(t)]=t^2 J_1(t)$$

recall $\frac{d}{dx}\left[x^n J_n(x)\right]=x^n J_{n-1}(x)$

$$c_i=\frac{2}{9J_2{}^2(3\alpha_i)}\frac{1}{\alpha_i^3}\int_0^{3\alpha_i}\frac{d}{dt}[t^2 J_2(t)]dt=\frac{2}{9J_2{}^2(3\alpha_i)}\frac{1}{\alpha_i^3}9\alpha_i^2 J_2(3\alpha_i)=\frac{2}{\alpha_i J_2(3\alpha_i)}$$

$$\therefore f(x)=2\sum_{i=1}^{\infty} c_i J_1(\alpha_i x)=2\sum_{i=1}^{\infty}\frac{1}{\alpha_i J_2(3\alpha_i)}J_1(\alpha_i x)$$

Fourier-Bessel Series

☑ Example 2

Expansion in a Fourier-Bessel Series

Expand f(x)=x, 0<x<3, in a Fourier-Bessel series, using Bessel functions of order one that satisfy the boundary condition $J_1(3\alpha)+\alpha J_1'(3\alpha)=0$

2) α_i are defined by $hJ_n(\alpha b)+\alpha b J_n'(\alpha b)=0$

$$f(x)=\sum_{i=1}^{\infty} c_i J_n(\alpha_i x)$$

$$,c_i=\frac{2\alpha_i^2}{(\alpha_i^2 b^2-n^2+h^2)J_n^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx$$

the only thing that changes in the expansion is the value of the square norm.

$$J_1(3\alpha)+\alpha J_1'(3\alpha)=0$$

Multiplying the boundary condition by 3 gives

$$3J_1(3\alpha)+3\alpha J_1'(3\alpha)=0$$ **→h=3, b=3, and n=1.**

$$c_i=\frac{18\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2+8)J_1{}^2(3\alpha_i)},$$

$$c_i=\frac{2\alpha_i^2}{(\alpha_i^2 b^2-n^2+h^2)J_n^2(\alpha_i b)}\int_0^b xJ_n(\alpha_i x)f(x)dx=\frac{2\alpha_i^2}{(\alpha_i^2 3^2-1^2+3^2)J_1^2(3\alpha_i)}\int_0^3 x^2J_1(\alpha_i x)dx$$

$$=\frac{2\alpha_i^2}{(\alpha_i^2 3^2-1^2+3^2)J_1^2(3\alpha_i)}\frac{1}{\alpha_i^3}9\alpha_i^2 J_2(3\alpha_i)=\frac{18\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2+8)J_1^2(3\alpha_i)}$$

$$\therefore f(x)=18\sum_{i=1}^{\infty}\frac{\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2+8)J_1^2(3\alpha_i)}J_1(\alpha_i x)$$

Fourier-Legendre Series

Definition 12.9

Fourier-Legendre Series

The Fourier-Legendre series of a function f defined on the interval $(-1,1)$ is given by

$$f(x)=\sum_{n=0}^{\infty}c_nP_n(x)$$

$$c_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx$$

- **In an alternative form, let** $x=\cos\theta$

$$F(\theta)=\sum_{n=0}^{\infty}c_nP_n(\cos\theta)$$

$$c_n=\frac{2n+1}{2}\int_{0}^{\pi}F(\theta)P_n(\cos\theta)\sin\theta d\theta$$

$$F(\theta)=f(\cos\theta)$$

Fourier-Legendre Series

▪**Convergence of a Fourier-Legendre Series:**

Theorem 12.5 **Conditions for Convergence**

If f and f' are piecewise continuous on the open interval $(0, b)$, then a Fourier-Bessel expansion of f converges to $f(x)$ at any point where f is continuous and to the average $[f(x+) + f(x-)]/2$ at a point where f is discontinuous.

Fourier-Legendre Series

$$f(x)=\sum_{n=0}^{\infty}c_nP_n(x)$$

$$c_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx$$

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\frac{1}{2}(3x^2-1)$$

$$P_3(x)=\frac{1}{2}(5x^3-3x)$$

$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$$

☑ Example 3

Expansion in a Fourier-Bessel Series

Write out the first four nonzero terms in the Fourier-Legendre expansion of

$$f(x)=\begin{cases}0, & -1<x<0\\ 1, & 0\le x<1\end{cases}$$

$$c_0=\frac{1}{2}\int_{-1}^{1}f(x)P_0(x)dx=\frac{1}{2}\int_{0}^{1}1\cdot 1dx=\frac{1}{2}$$

$$c_1=\frac{3}{2}\int_{-1}^{1}f(x)P_1(x)dx=\frac{3}{2}\int_{0}^{1}1\cdot xdx=\frac{3}{4}$$

$$c_3=\frac{7}{2}\int_{-1}^{1}f(x)P_3(x)dx=\frac{5}{2}\int_{0}^{1}1\cdot\frac{1}{2}(5x^3-3x)dx=-\frac{7}{16}$$

$$c_5=\frac{11}{2}\int_{-1}^{1}f(x)P_5(x)dx=\frac{11}{2}\int_{0}^{1}1\cdot\frac{1}{8}(63x^5-70x^3+15x)dx=\frac{11}{32}$$

$$c_2=\frac{5}{2}\int_{-1}^{1}f(x)P_2(x)dx=\frac{5}{2}\int_{0}^{1}1\cdot\frac{1}{2}(3x^2-1)dx=0$$

$$c_4=\frac{9}{2}\int_{-1}^{1}f(x)P_4(x)dx=\frac{9}{2}\int_{0}^{1}1\cdot\frac{1}{8}(35x^3-30x^2+3)dx=0$$

$$\therefore f(x)=\frac{1}{2}P_0(x)+\frac{3}{4}P_1(x)-\frac{7}{16}P_3(x)+\frac{11}{32}P_5(x)+\cdots$$