[2008] [11-2]

Engineering Mathematics 2

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Partial Differential Equation(2)
: Classical Equations and BVP

Separable Partial Differential Equations
Classical Equations and Boundary-Value Problems
1-D Wave Equation
1-D Heat Equation
Laplace's Equation
Nonhomogeneous Equations and Boundary Conditions
Orthogonal Series Expansions
Fourier Series in Two Variables

Separable Partial Differential Equations

Separable Partial Differential Equations

☑ Linear Partial Differential Equation

General form of a linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

$G(x, y)=0$ **Equation is said to be 'Homogeneous'**

Otherwise, nonhomogeneous

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Homogeneous

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}=xy$$

Nonhomogeneous

Separable Partial Differential Equations

☑ Solution of Linear P.D.E

- **Solution of linear PDE is function $u(x, y) = 0$ of two independent variables**

- **Our focus throughout will be on finding *particular solutions* of some of the important linear PDEs(Heat eq, Wave eq, Laplace's Eq, etc.)**

- **Notation**

$$\frac{\partial u}{\partial x} = u_x,\ \frac{\partial u}{\partial y} = u_y,\ \frac{\partial^2 u}{\partial x^2} = u_{xx},\ \frac{\partial^2 u}{\partial y \partial x} = u_{xy}$$

Separable Partial Differential Equations

☑ Separation of Variables

- To find particular solution, we will use separation of variables which is form of *product* of a function of x and a function of y

$$u(x,y) = X(x)Y(y)$$

With this assumption, it is *sometimes* possible to reduce a linear PDE in two variables to two ODEs

$$\frac{\partial u}{\partial x} = X'Y,\quad \frac{\partial u}{\partial y} = XY',\quad \frac{\partial^2 u}{\partial x^2} = X''Y,\quad \frac{\partial^2 u}{\partial y^2} = XY''$$

Separable Partial Differential Equations

☑ Separation of Variables

- **Find product solution**

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

When substituting $u(x, y) = X(x)Y(y)$ **into PDE,**

$$X''Y = 4XY'$$

Separation constant (real)

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

$$X'' + 4\lambda X = 0 \;\; and \;\; Y' + \lambda Y = 0$$

Separable Partial Differential Equations

☑ Separation of Variables

- Find product solution

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

$$X'' + 4\lambda X = 0 \;\; and \;\; Y' + \lambda Y = 0$$

There are three cases for λ: zero, negative, or positive

$$Case I(\lambda = 0): \qquad X'' = 0 \;\; and \;\; Y' = 0$$

$$Case II(\lambda = -\alpha^2): \quad X'' - 4\alpha^2 X = 0 \;\; and \;\; Y' - \alpha^2 Y = 0$$

$$Case III(\lambda = \alpha^2): \quad X'' + 4\alpha^2 X = 0 \;\; and \;\; Y' + \alpha^2 Y = 0$$

Separable Partial Differential Equations

☑ Separation of Variables

- **Find product solution**

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

$$Case I(\lambda = 0):\ X'' = 0 \ \ and \ \ Y' = 0$$

By integration,

$$X = c_1 + c_2 x \quad and \quad Y = c_3$$

$$u = XY = (c_1 + c_2 x)c_3 = A_1 + B_1 x$$

Separable Partial Differential Equations

☑ Separation of Variables

- **Find product solution**

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

$$Case II(\lambda = -\alpha^2): \quad X'' - 4\alpha^2 X = 0 \;\; and \;\; Y' - \alpha^2 Y = 0$$

$$X = e^{mx}$$

$$m^2 - 4\alpha^2 = 0, \; m_1 = 2\alpha, \; m_2 = -2\alpha$$

$$X = C_1 e^{2\alpha x} + C_2 e^{-2\alpha x}$$

$$Y = e^{m'y}$$

$$m' = \alpha^2$$

$$Y = C_3 e^{\alpha^2 y}$$

$$u = XY = (C_1 e^{2\alpha x} + C_2 e^{-2\alpha x}) C_3 e^{\alpha^2 y}$$

$$= (C_1' e^{2\alpha x} + C_2' e^{-2\alpha x}) e^{\alpha^2 y}$$

Separable Partial Differential Equations

☑ Separation of Variables

- **Find product solution**

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

$$Case III(\lambda = \alpha^2): \quad X'' + 4\alpha^2 X = 0 \;\; and \;\; Y' + \alpha^2 Y = 0$$

$$X = e^{mx}$$

$$m^2 + 4\alpha^2 = 0,\; m_1 = 2\alpha i,\;\; m_2 = -2\alpha i$$

$$X = C_7 \cos 2\alpha x + C_8 \sin 2\alpha x$$

$$Y = e^{m'y}$$

$$m' = -\alpha^2$$

$$Y = C_3 e^{-\alpha^2 y}$$

$$u = XY = (C_1 \cos 2\alpha x + C_2 \sin 2\alpha x) C_3 e^{-\alpha^2 y}$$

$$= (C_1' \cos 2\alpha x + C_2' \sin 2\alpha x) e^{-\alpha^2 y}$$

Separable Partial Differential Equations

☑ Superposition Principle

Theorem 13.1 **Superposition Principle**

If $u_1, u_2, \ldots, u_k$ are solution of a homogeneous <u>linear</u> partial differential equation, then the linear combination

$$u = c_1 u_1 + c_2 u_2 + \cdots c_k u_k,$$

Where, the $c_i,\ i = 1, 2, \ldots, k$ are constants, is also a solution.

When we have infinite solution set $u_1, u_2, \ldots$ of homogeneous linear equation, we can construct another solution

$$u = \sum_{k=1}^{\infty} c_k u_k$$

Separable Partial Differential Equations

☑ Classification of Equations

Definition 13.1

Classification of Equations

The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$$

Where A,B,C,D,E and F are real constants, is said to be

Hyperbolic if $B^2-4AC>0$

Parabolic if $B^2-4AC=0$

Elliptic if $B^2-4AC<0$

Separable Partial Differential Equations

☑ Classification of Equations

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Example)

$$3\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Separable Partial Differential Equations

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Example)

$$3\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial y}$$

$$A=3, B=0, C=0$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Separable Partial Differential Equations

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Parabolic

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Example)

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$$A=3, B=0, C=0$$

$$B^2-4AC=0$$

Parabolic

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$$

$$A=1, B=0, C=-1$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Separable Partial Differential Equations

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Parabolic

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$$

$$A=1, B=0, C=-1$$

$$B^2-4AC=4>0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Separable Partial Differential Equations

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Definition 13.1 — **Classification of Equations**

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Example)

$$3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$$

$$A = 3, B = 0, C = 0$$

$$B^2 - 4AC = 0$$

Parabolic

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$$A = 1, B = 0, C = -1$$

$$B^2 - 4AC = 4 > 0$$

Hyperbolic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Separable Partial Differential Equations

☑ Classification of Equations

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Hyperbolic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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Separable Partial Differential Equations

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Hyperbolic

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$A=1, B=0, C=1$$

$$B^2-4AC=-4<0$$

Elliptic

Classical Equations and Boundary Value Problems

Classical Equations and B.V.P

☑ Introduction

(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ 1-D Wave Equation (Hyperbolic Type)

(2) $k \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$ 1-D Heat Equation (Parabolic Type)

(3) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ 2-D Laplace's Equation (Elliptic Type)

비교*)

$\dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \Leftrightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ (Hyperbolic Curve)

$\dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} \Leftrightarrow y = ax^2$ (Parabolic Curve)

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \Leftrightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (Elliptic Curve)

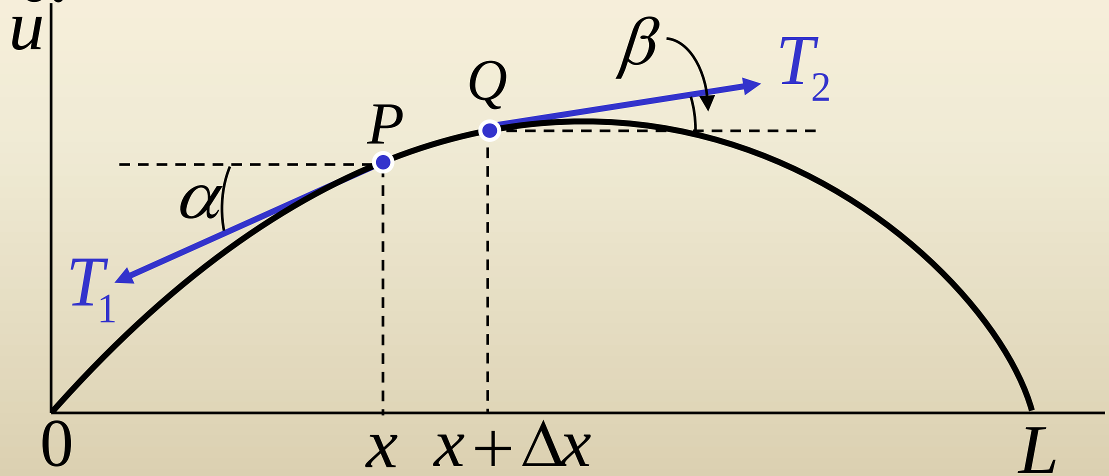
*김찬중, 길잡이 공업수학, 제4판, 문운당, p451

Classical Equations and B.V.P

☑ 1-D Wave Equation

We place the string along the x-axis, stretch it to length L, and fasten it at the ends $x = 0$ and $x = L$.

The problem is to determine the vibration of the string, that is to find its deflection $u(x,t)$ at any point x and at any time $t > 0$.

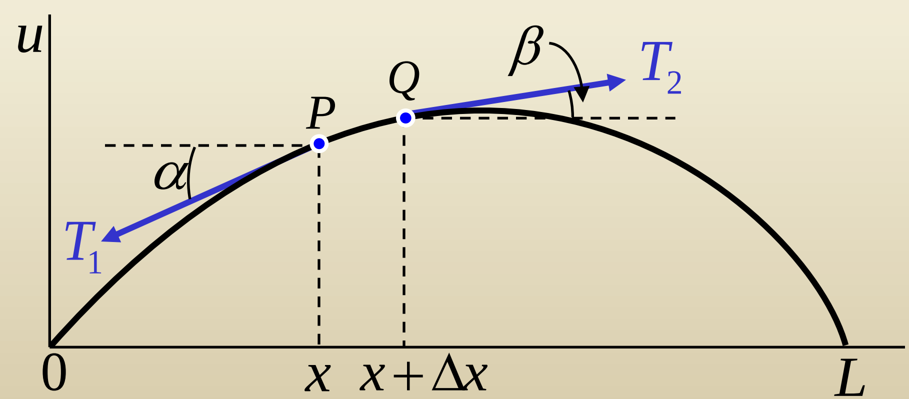


Classical Equations and B.V.P

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

☑ 1-D Wave Equation

Physical Assumptions

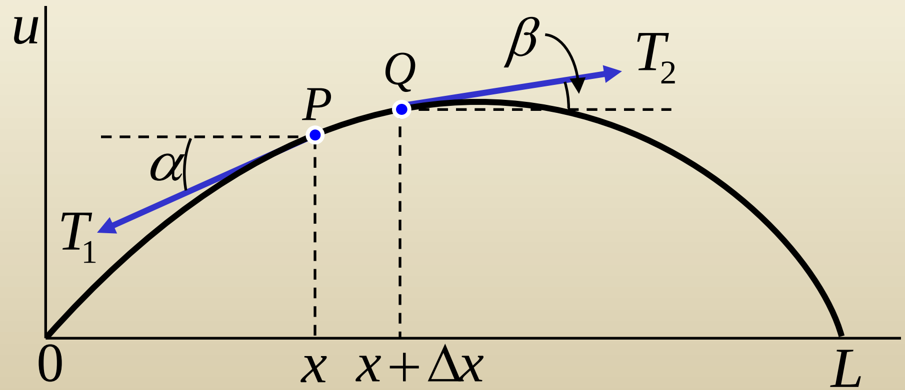


Classical Equations and B.V.P

☑ 1-D Wave Equation

Physical Assumptions

1. The mass of the string per unit length is constant (“homogeneous string). The string is perfectly elastic and does not offer any resistance to bending.



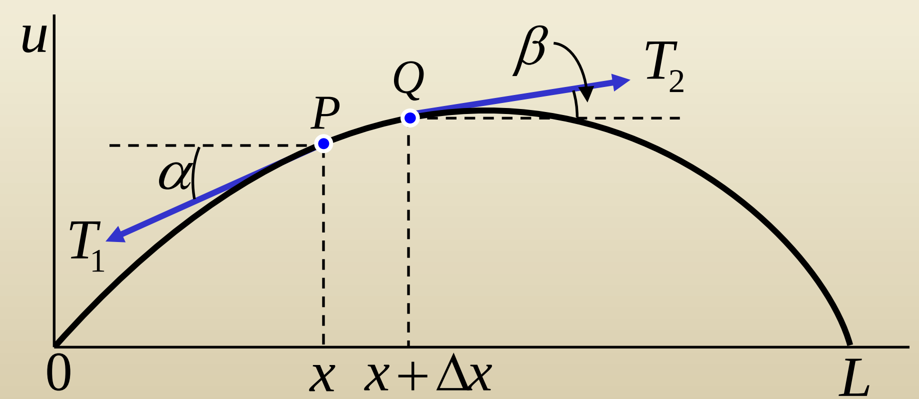
Classical Equations and B.V.P

☑ 1-D Wave Equation

Physical Assumptions

1. The mass of the string per unit length is constant (“homogeneous string). The string is perfectly elastic and does not offer any resistance to bending.

2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.



Classical Equations and B.V.P

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

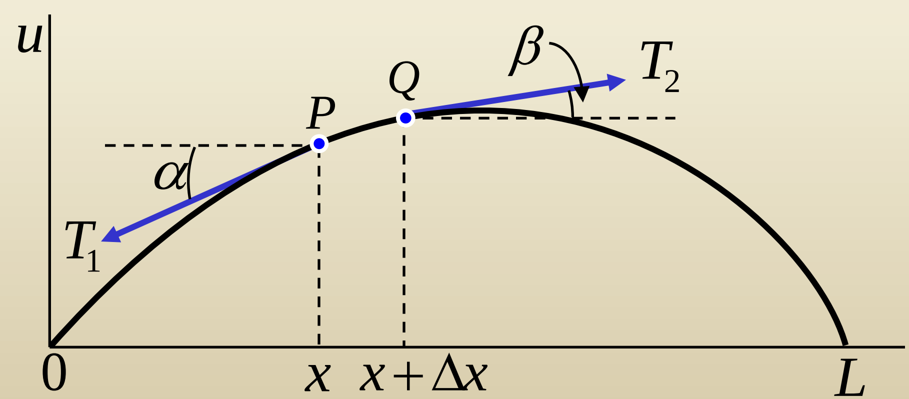
☑ 1-D Wave Equation

Physical Assumptions

1. The mass of the string per unit length is constant ("homogeneous string). The string is perfectly elastic and does not offer any resistance to bending.

2. The tension caused by stretching the string before fastening it at the ends is so large that the action of the gravitational force on the string (trying to pull the string down a little) can be neglected.

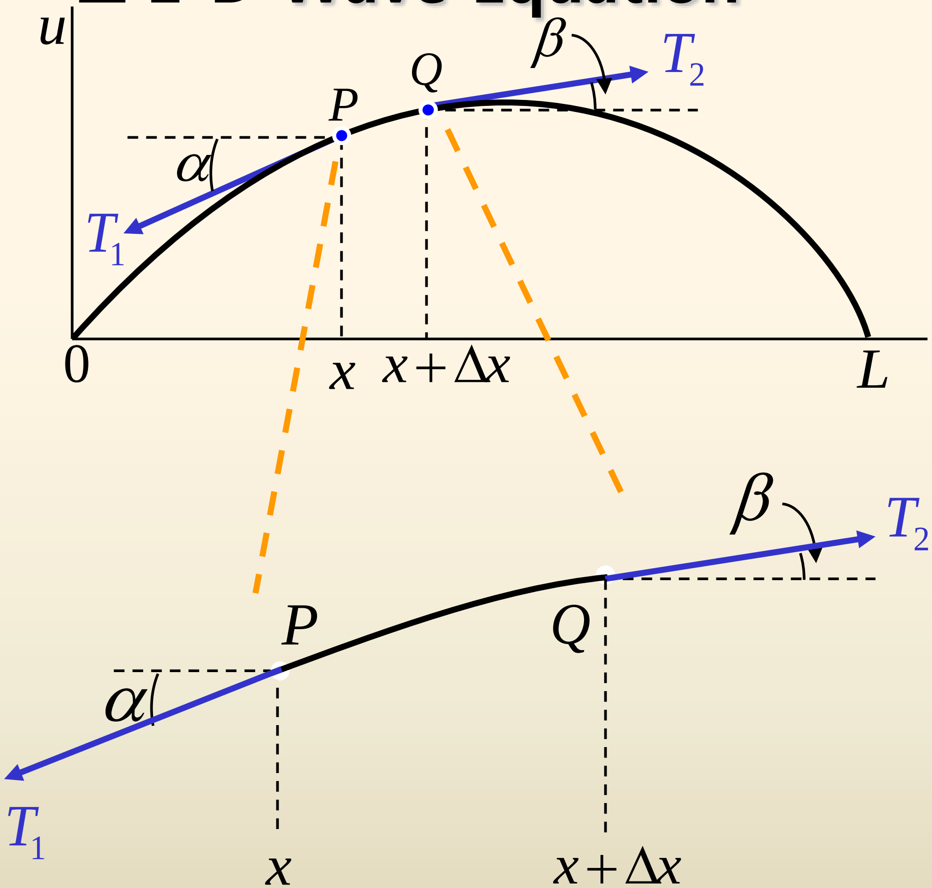
3. The string performs small transverse motions in a vertical plane; that is, every particle of the string moves strictly vertically and so that the deflection and the slope at every point of the string always remain small in absolute value



Classical Equations and B.V.P

$$(1) \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

☑ 1-D Wave Equation



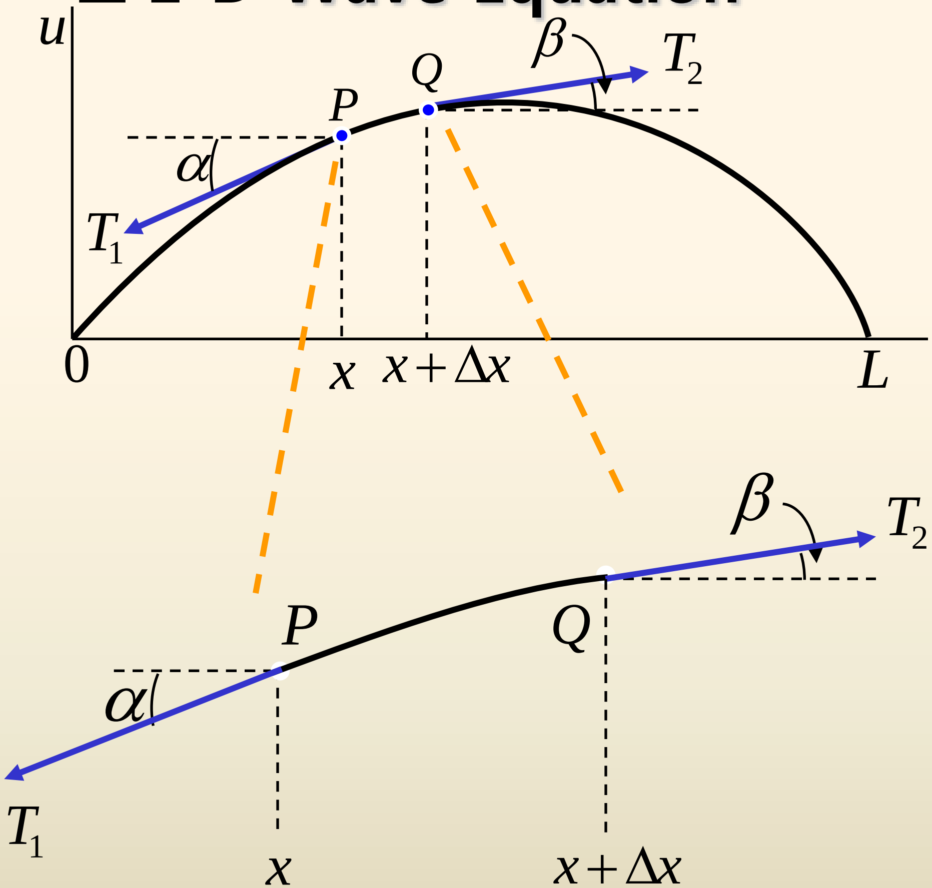
T_1, T_2 : tensions at the end point P, Q and they are directed along the tangents at the points

Forces acting on a small portion of the string

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Classical Equations and B.V.P

☑ 1-D Wave Equation



T_1, T_2 : tensions at the end point P, Q and they are directed along the tangents at the points

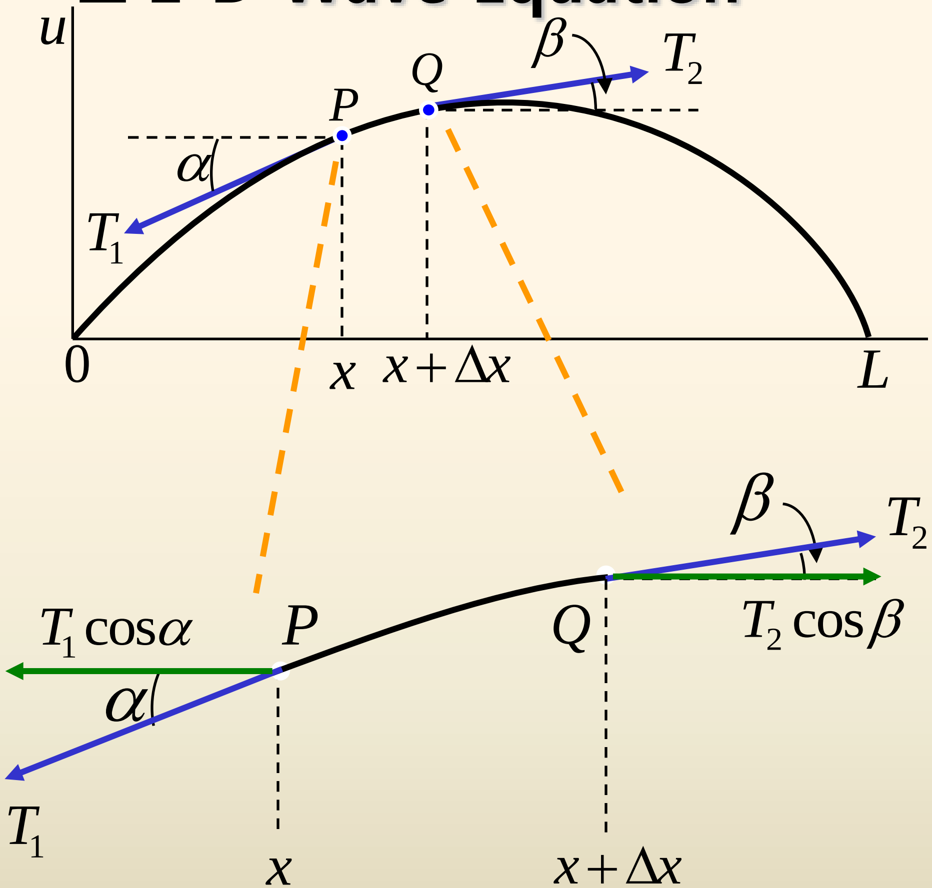
Forces acting on a small portion of the string

- **Point of the string move vertically. No motion in the horizontal direction**

Classical Equations and B.V.P

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

☑ 1-D Wave Equation



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Classical Equations and B.V.P

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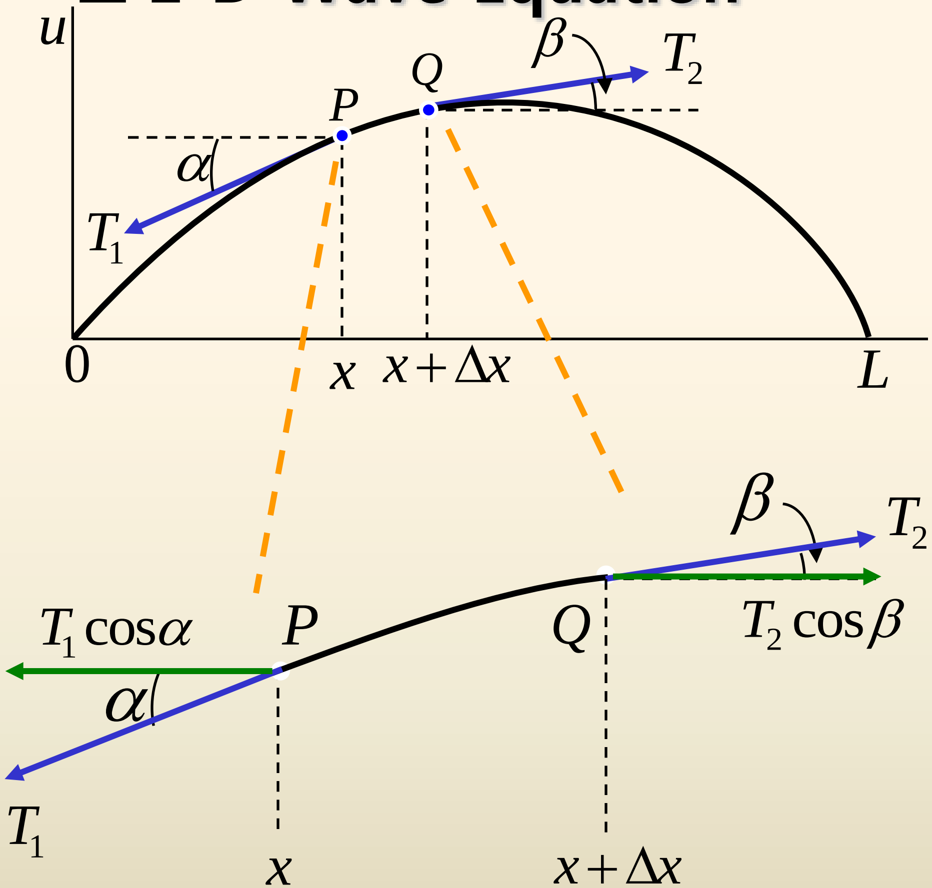


T_1, T_2 : tensions at the end point P, Q and they are directed along the tangents at the points

Forces acting on a small portion of the string

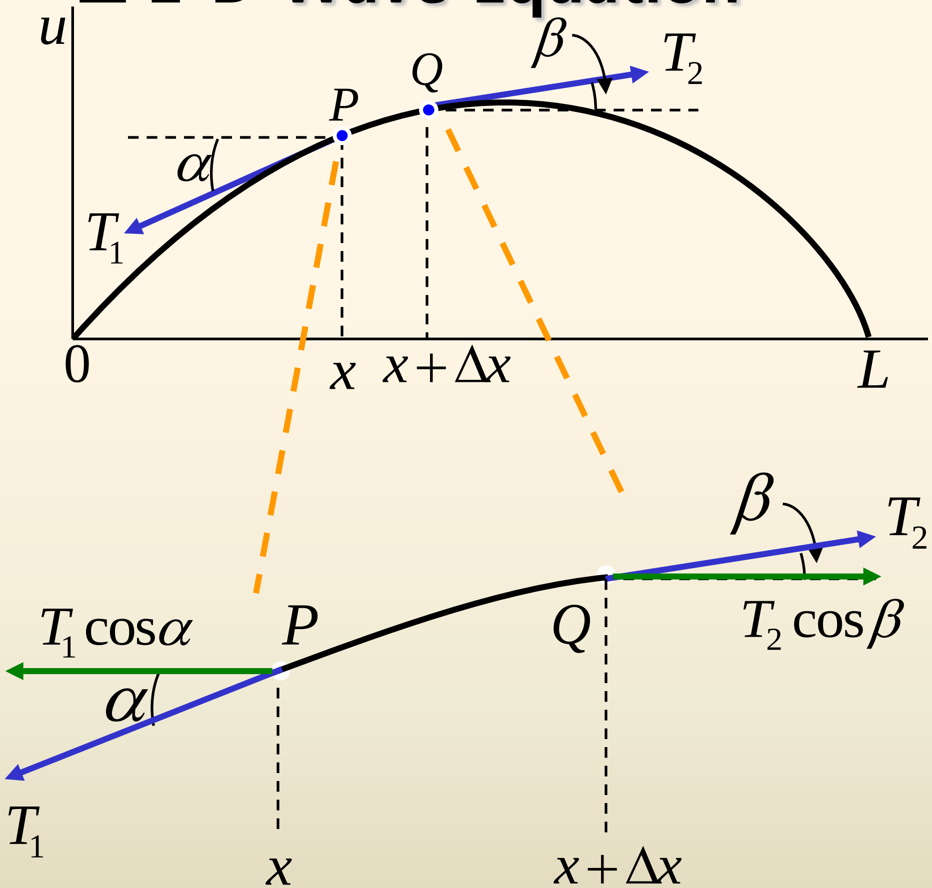
- **Point of the string move vertically. No motion in the horizontal direction**

$$\therefore T_2 \cos\beta - T_1 \cos\alpha = 0$$

Classical Equations and B.V.P

(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

☑ 1-D Wave Equation



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$$\therefore T_1 \cos\alpha = T_2 \cos\beta$$

Classical Equations and B.V.P

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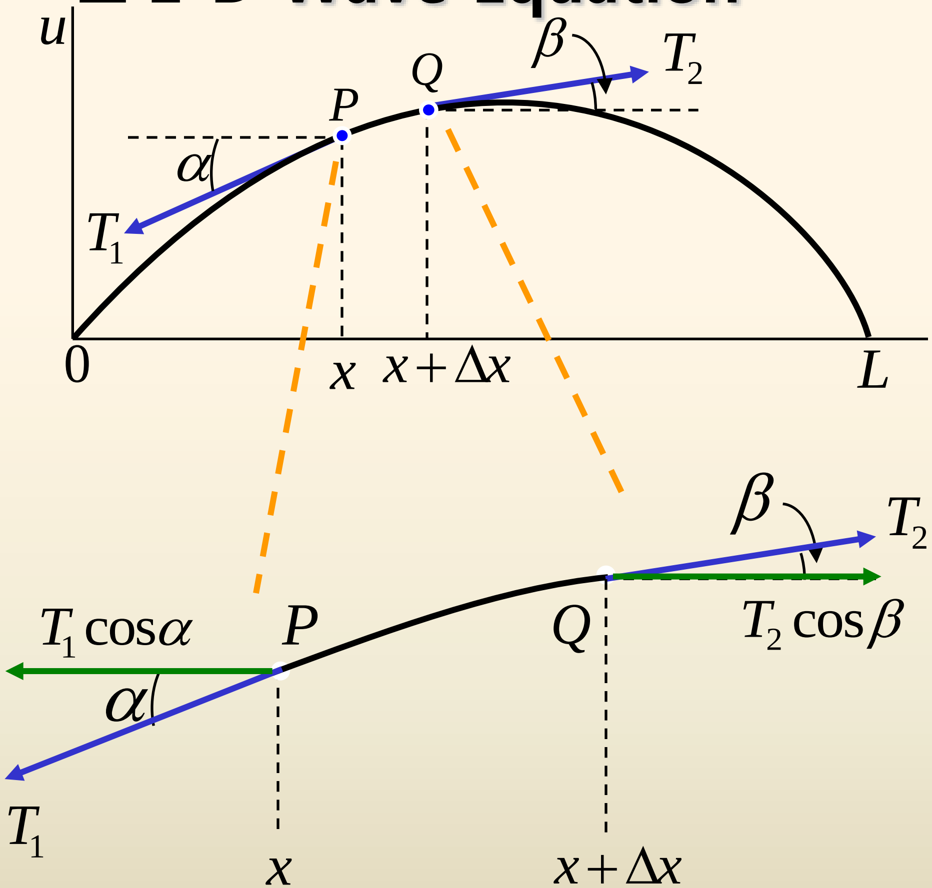

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Forces acting on a small portion of the string

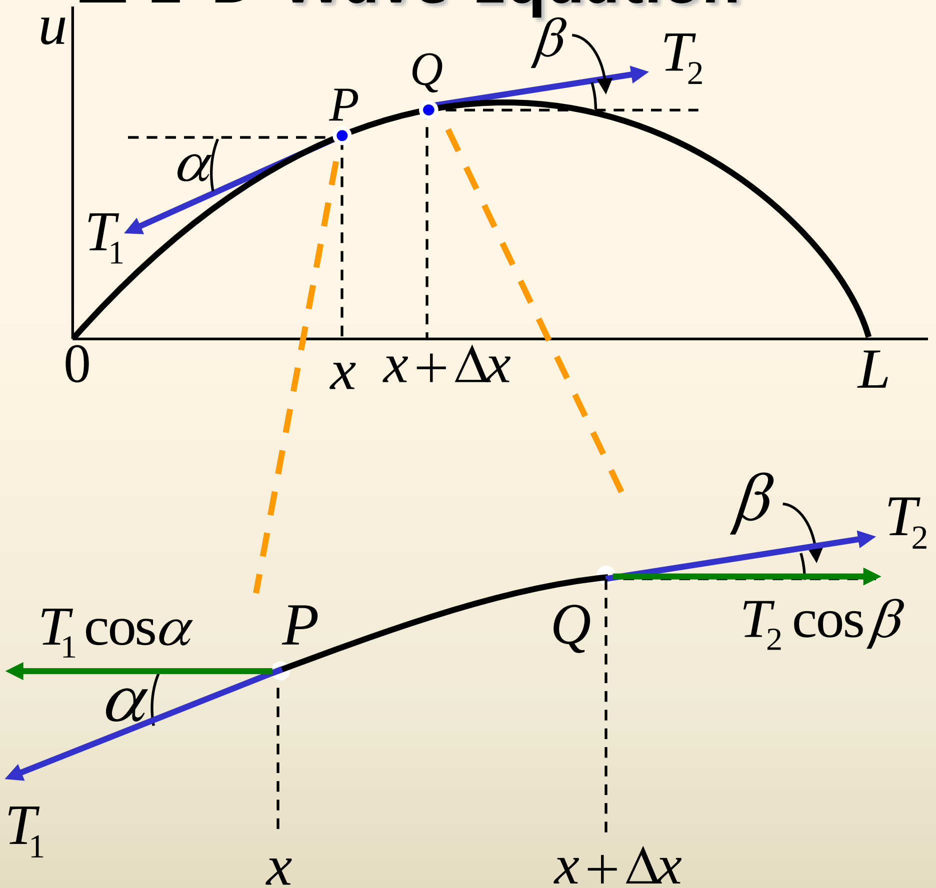
- **Point of the string move vertically. No motion in the horizontal direction**

$$\therefore T_2\cos\beta - T_1\cos\alpha = 0$$

$$\therefore T_1\cos\alpha = T_2\cos\beta$$
$$= T = \text{constant}\cdots(1)$$

Classical Equations and B.V.P

☑ 1-D Wave Equation



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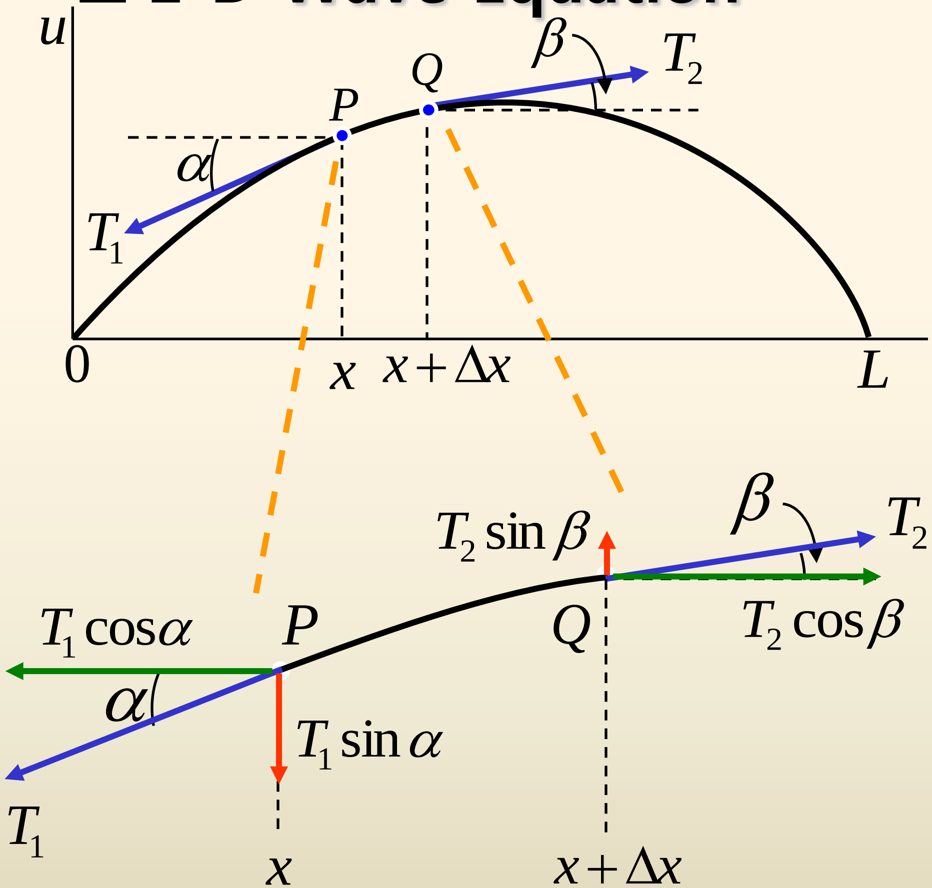
$$= T = \text{constant} \cdots (1)$$

- **Net force acting on a small portion of the string**

Classical Equations and B.V.P

☑ 1-D Wave Equation



T_1, T_2 : tensions at the end point P, Q and they are directed along the tangents at the points

Forces acting on a small portion of the string

- **Point of the string move vertically. No motion in the horizontal direction**

$$\therefore T_2 \cos\beta - T_1 \cos\alpha = 0$$

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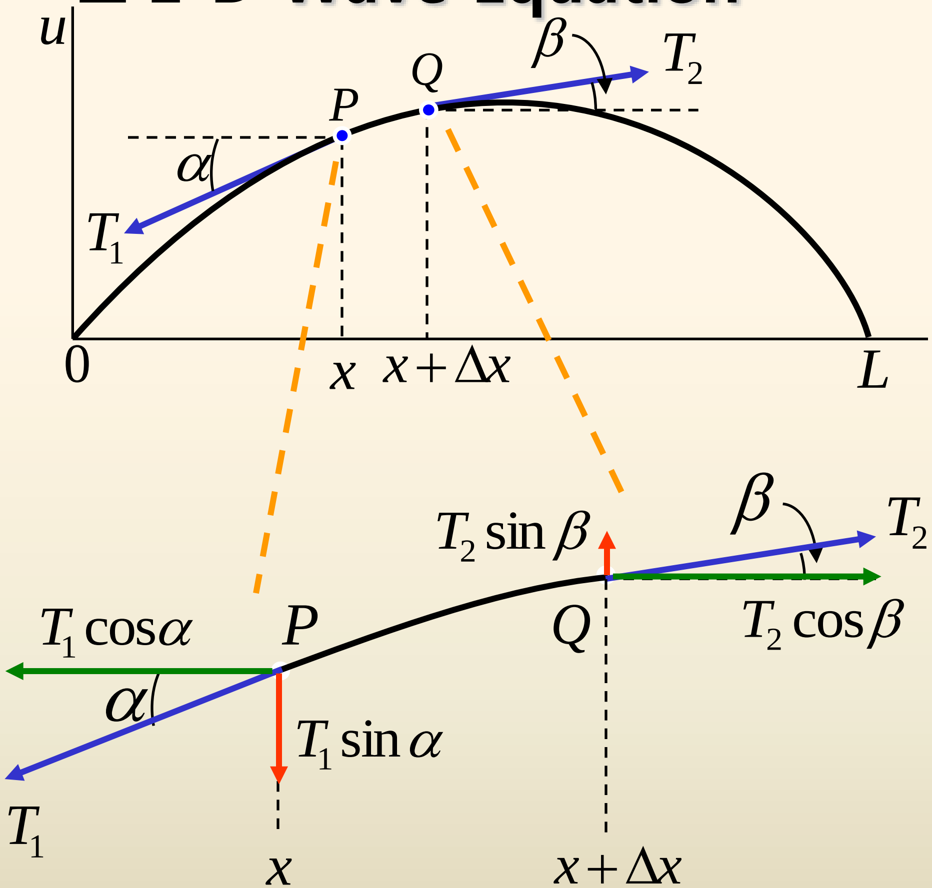

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$$\text{net force} = T_2 \sin\beta - T_1 \sin\alpha$$

Classical Equations and B.V.P

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

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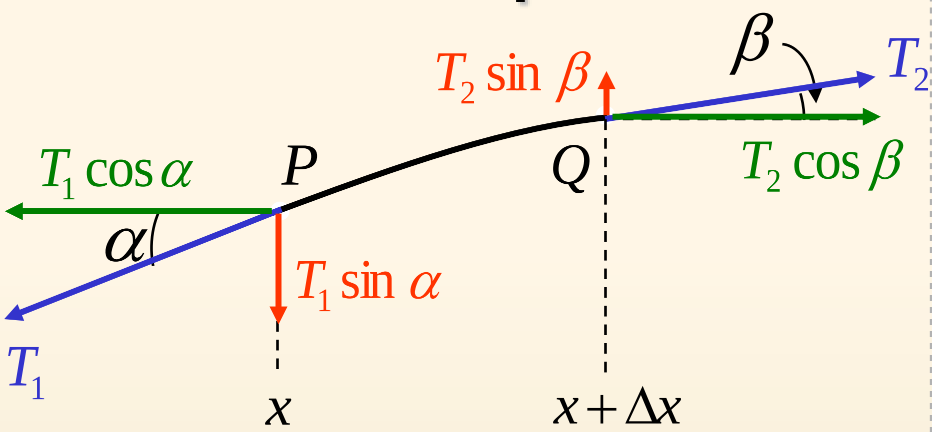

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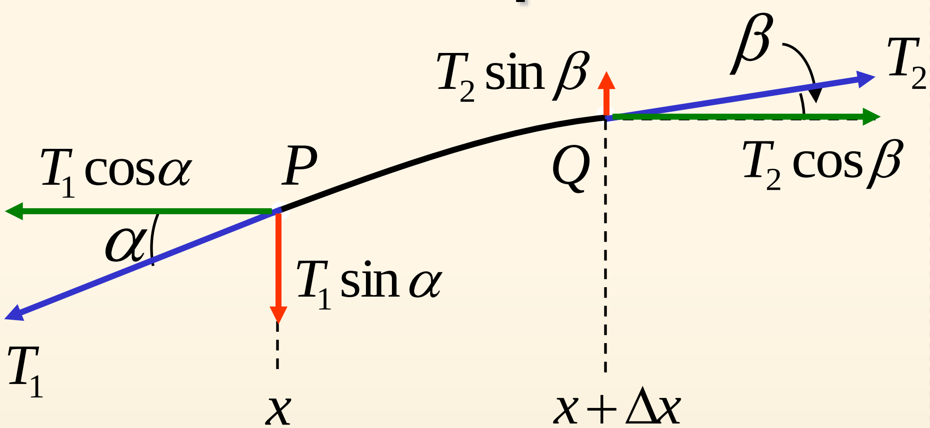

- the mass of the small portion

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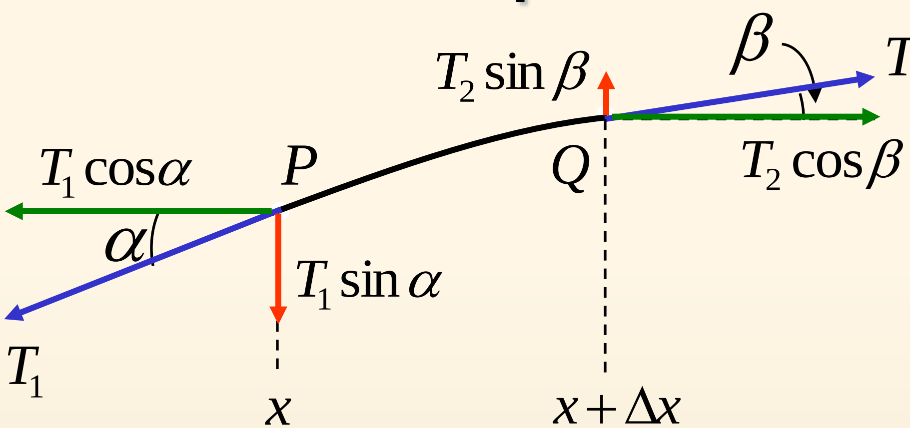

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$$\Delta m = \rho \Delta x$$

ρ : mass of the undeflected string per unit length

Δx :length of the portion of the undeflected string

Classical Equations and B.V.P

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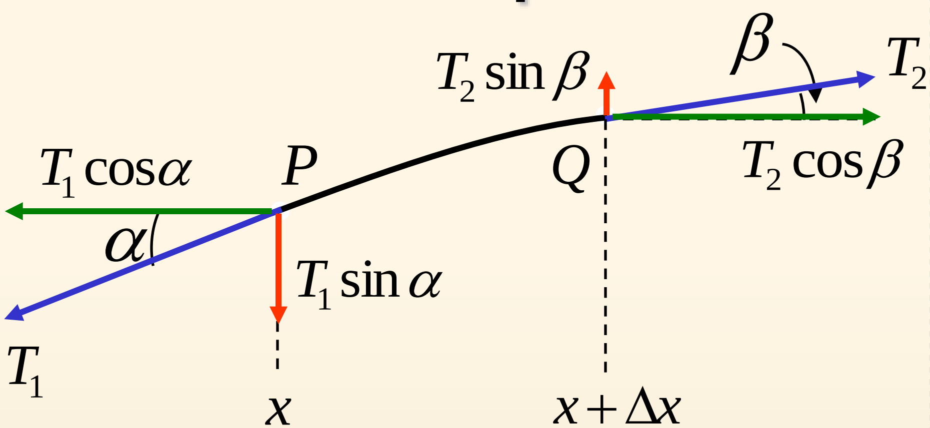

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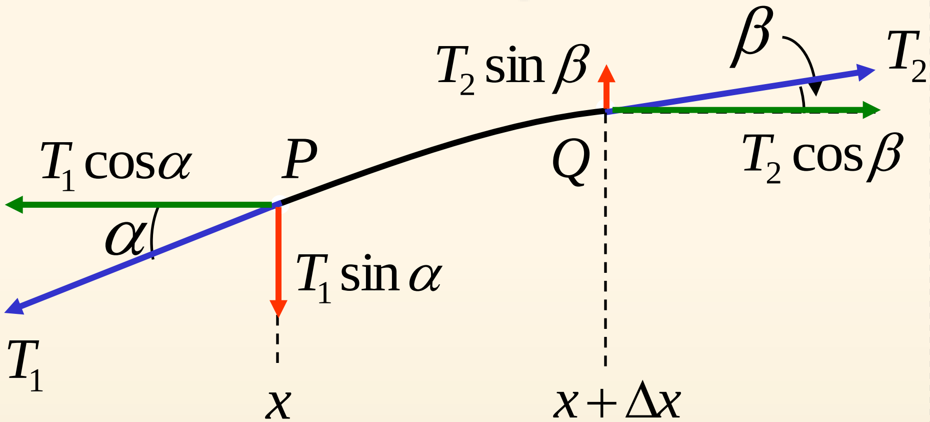

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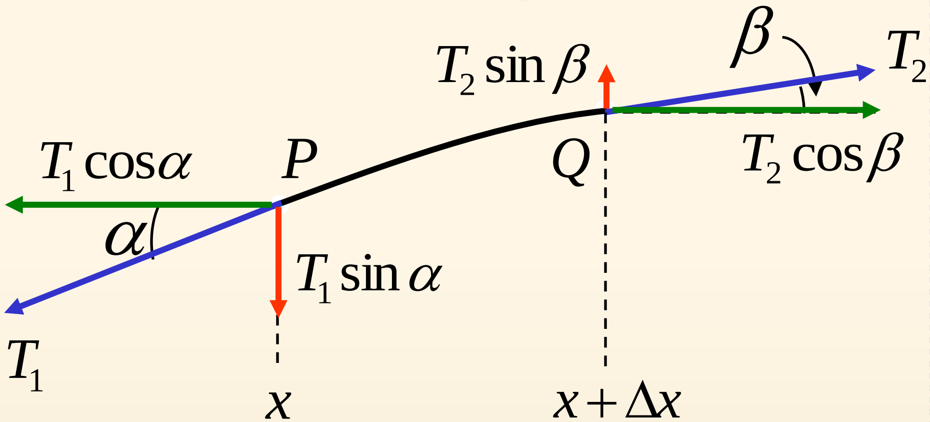

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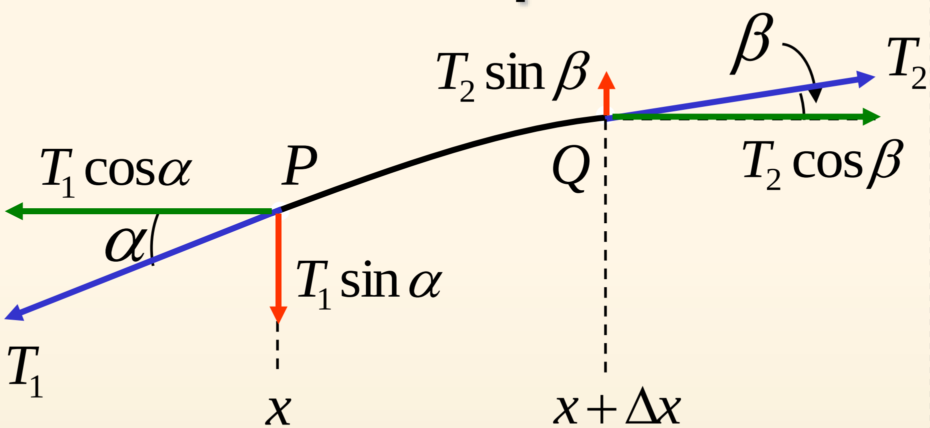

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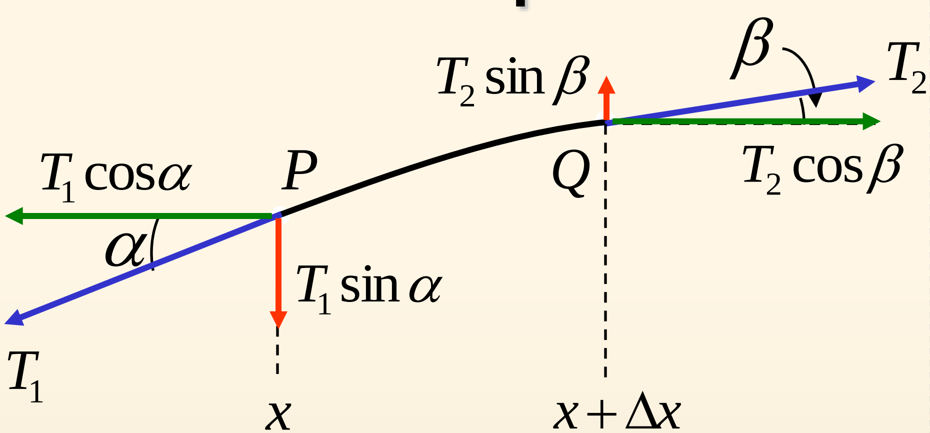

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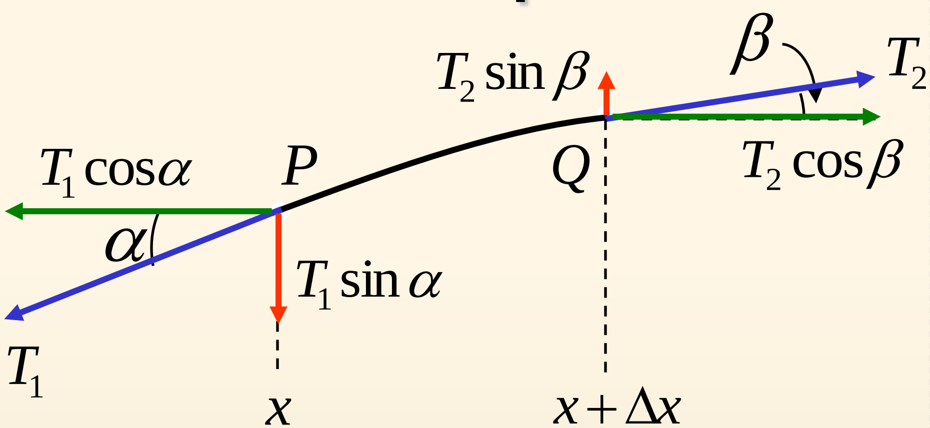

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$$\therefore T_2\sin\beta - T_1\sin\alpha = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$

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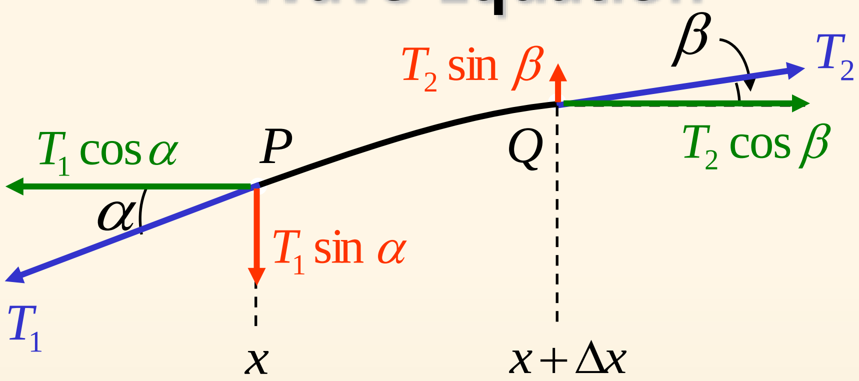

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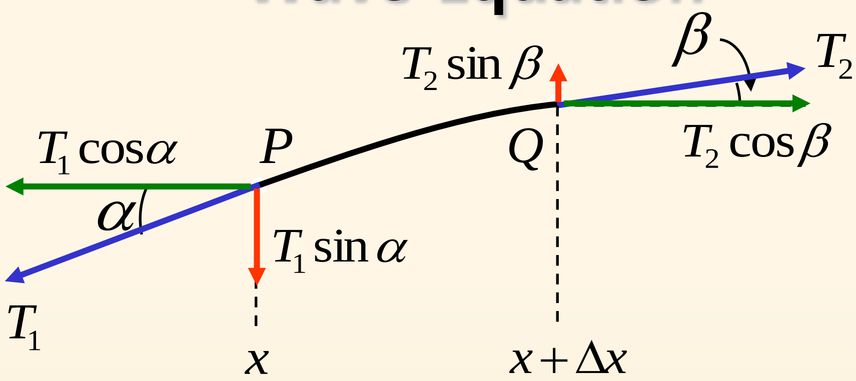

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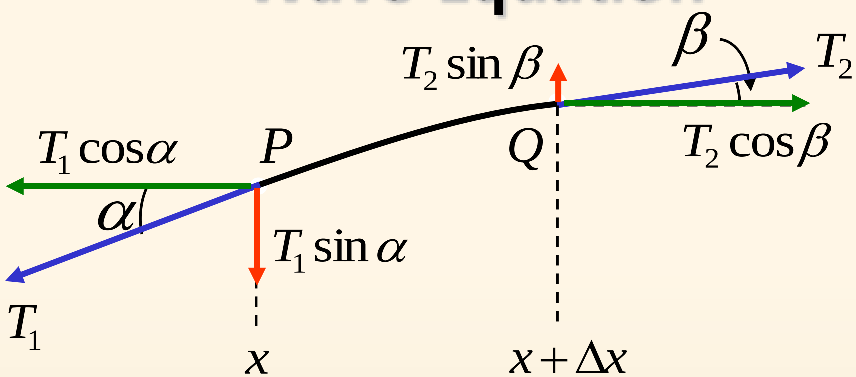

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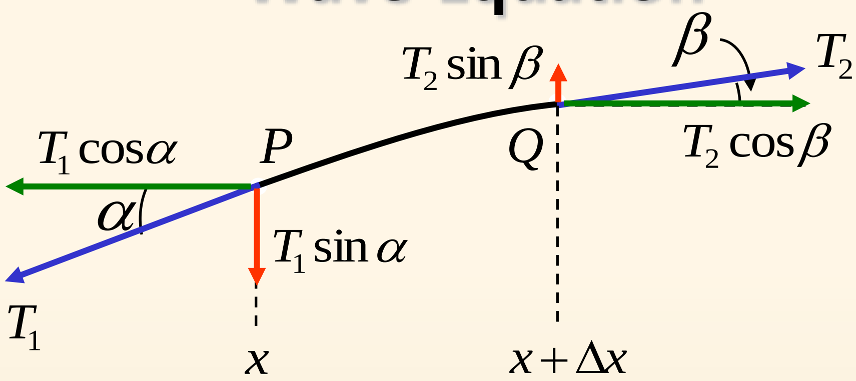

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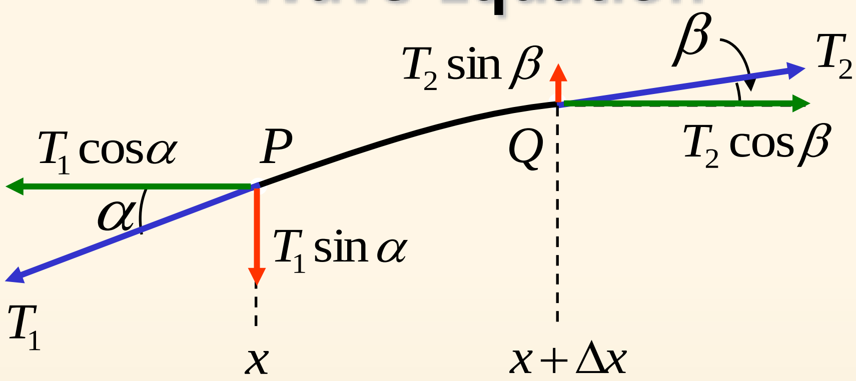

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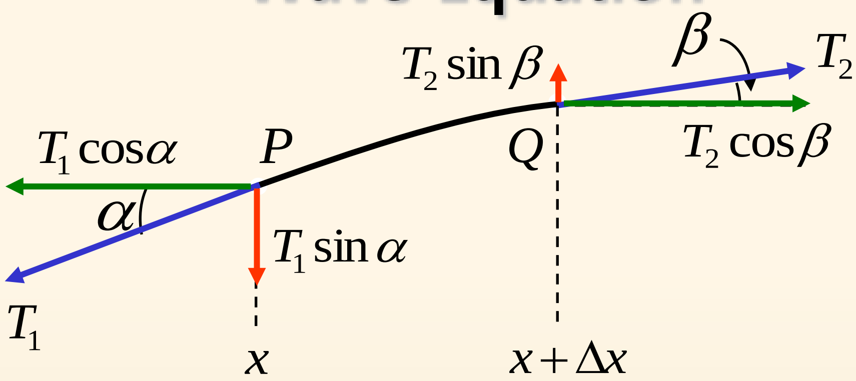

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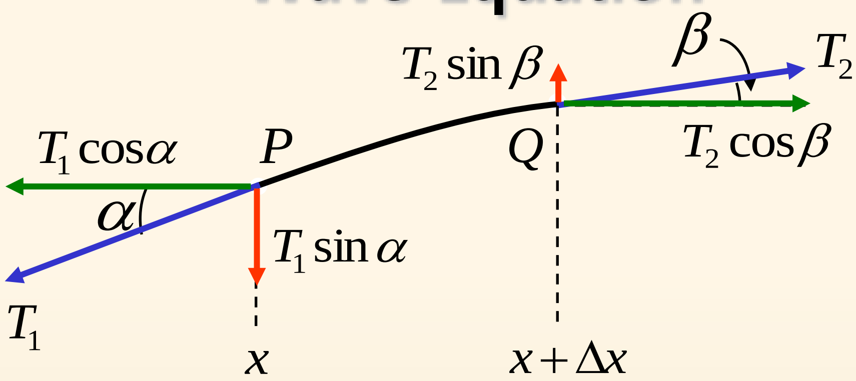

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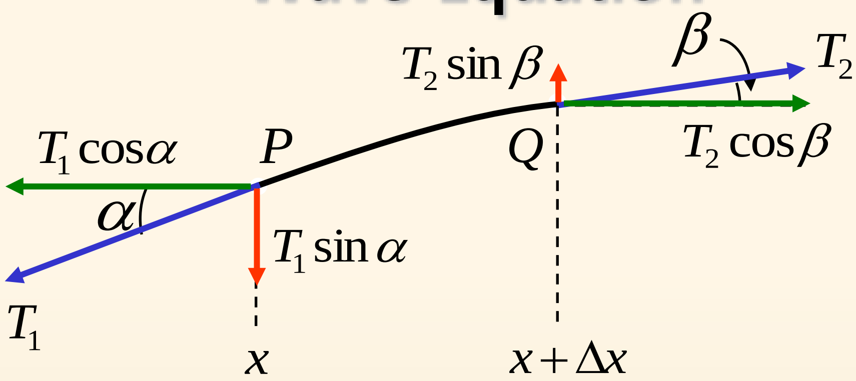

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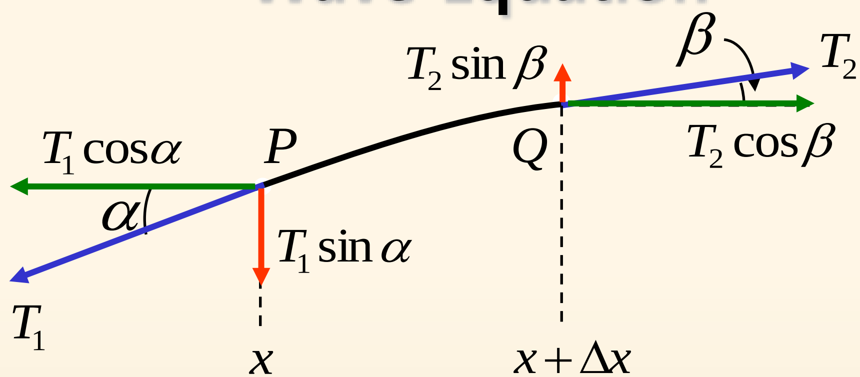

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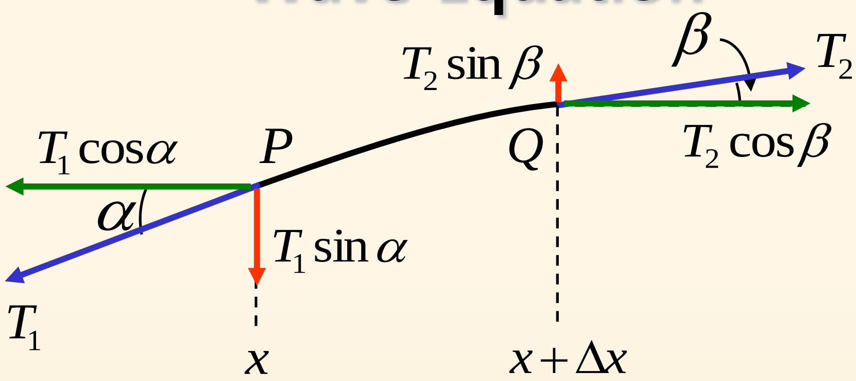

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Letting $\Delta x\to 0$, weget

$$\frac{\partial^2 u}{\partial x^2}=\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

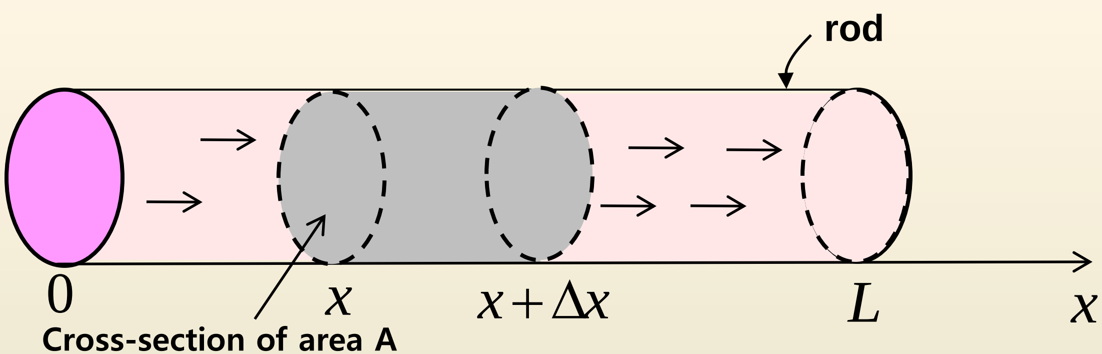
$$\boxed{\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}}\ \text{where } c^2=\frac{T}{\rho}$$

Classical Equations and B.V.P

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},k>0$$

☑ 1-D Heat Equation*

- **Assumptions**
- The rod is made of a single homogeneous conducting material
- The rod is laterally insulated (heat flows only x-direction)
- The rod is thin (the temperature at all points of a cross section is constant)

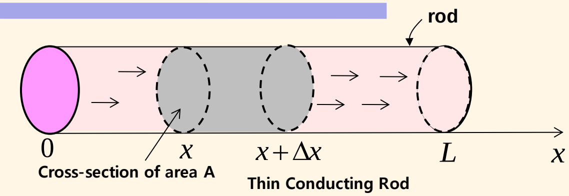


Thin Conducting Rod

*Farlow S.J., Partial Differential Equations for Scientists and Engineers, Dover, 1982, Lesson 4

Classical Equations and B.V.P

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},k>0$$



☑ 1-D Heat Equation*

■ Conservation of heat

Total heat inside $[x,x+\Delta x]$ $=\int_x^{x+\Delta x} c\rho A u(s,t)\,ds$

c : thermal capacity of the rod (measures the ability of the rod to store heat)**

ρ: density of the rod

A: cross-section area of the rod

Net change of heat inside $[x,x+\Delta x]$ = Net flux of heat across the boundaries + Total heat generated inside $[x,x+\Delta x]$

We can write the conservation of energy equation via calculus as

$$\frac{d}{dt}\int_x^{x+\Delta x} c\rho A u(s,t)\,ds = c\rho A\int_x^{x+\Delta x} u_t(s,t)\,ds = kA[u_x(x+\Delta x,t)-u_x(x,t)] + A\int_x^{x+\Delta x} f(s,t)ds$$

k : thermal conductivity of the rod (measure the ability to conduct heat)***

$f(x,t)$: external heat source

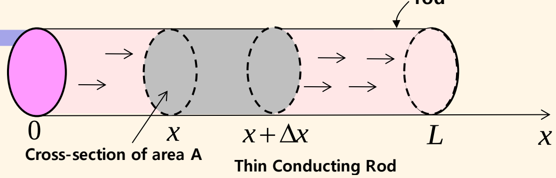
*Farlow S.J., Partial Differential Equations for Scientists and Engineers, Dover, 1982, Lesson 4

** Oxtoby, Principles of Modern Chemistry, Sixth Edition, Thomson, Index 1.25, “Specific heat capacity : The amount of heat required to raise the temperature of one gram of a substance by one kelvin at constant pressure”

*** 여상도, 열역학 개념의 해설, 청문각, 2006, p18 “온도가 동일한 두 물체와 우리의 손이 닿았을 때 그 차갑고 뜨거운 정도가 다른 이유는, 두 물체의 온도가 다르기 때문이 아니라 우리 손에서 물체로 이동하는 열의 전달 속도가 다르기 때문이다. 열전도도가 큰 철판이 열전도도가 작은 나무판에 비해 훨씬 빨리 손으로부터 열을 빼앗아 간다.”

Classical Equations and B.V.P

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},k>0$$



c : thermal capacity of the rod (measures the ability of the rod to store heat)
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☑ 1-D Heat Equation*

■ Conservation of heat

Total heat inside $[x,x+\Delta x]$ $=\int_x^{x+\Delta x} c\rho A u(s,t)\,ds$

Net change of heat inside $[x,x+\Delta x]$ **=** Net flux of heat across the boundaries **+** Total heat generated inside $[x,x+\Delta x]$

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'Mean Value Theorem'
If $f(x)$ is a continuous function on [a,b], then there exists at least one number

$$\int_a^b f(x)\,dx = f(\xi)(b-a)\ ,a<\xi<b$$

$$c\rho A u_t(\xi_1,t)\Delta x = kA[u_x(x+\Delta x,t)-u_x(x,t)] + Af(\xi_2,t)\Delta t \quad ,x<\xi_1,\xi_2<x+\Delta x$$

or divide by $\frac{c\rho\Delta x}{A}$, $\quad u_t(\xi_1,t)=\frac{k}{c\rho}\frac{[u_x(x+\Delta x,t)-u_x(x,t)]}{\Delta x}+\frac{1}{c\rho}f(x,t)$

finally letting, $\Delta x\to 0$ $\quad u_t(x,t)=\frac{k}{c\rho}u_{tt}+\frac{1}{c\rho}f(\xi_2,t)$

2008_P.D.E(2)

*Farlow S.J., Partial Differential Equations for Scientists and Engineers, Dover, 1982, Lesson 4

Classical Equations and B.V.P

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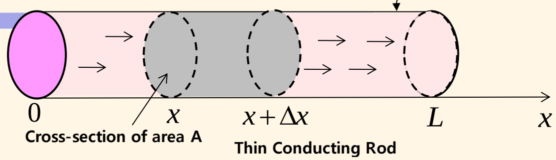

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or divide by $\dfrac{c\rho\Delta x}{A}$, and letting, $\Delta x \to 0$ $\quad u_t(x,t) = \dfrac{k}{c\rho}u_{tt} + \dfrac{1}{c\rho}f(x,t)$

$$u_t(x,t) = \alpha^2 u_{tt} + F(x,t)$$

$\alpha^2 = \dfrac{k}{c\rho}$: called the diffusivity of the rod

$F(x,t) = \dfrac{1}{c\rho}f(x,t)$: heat source density

2008_P.D.E(2)

*Farlow S.J., Partial Differential Equations for Scientists and Engineers, Dover, 1982, Lesson 4

Classical Equations and B.V.P

☑ Laplace's Equation

- **Laplace's equation occurs in time-independent problems (steady-state) involving *potentials* such as electrostatic, gravitational, and *velocity in fluid mechanics*.**

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\nabla^2 u=0$$

$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Classical Equations and B.V.P

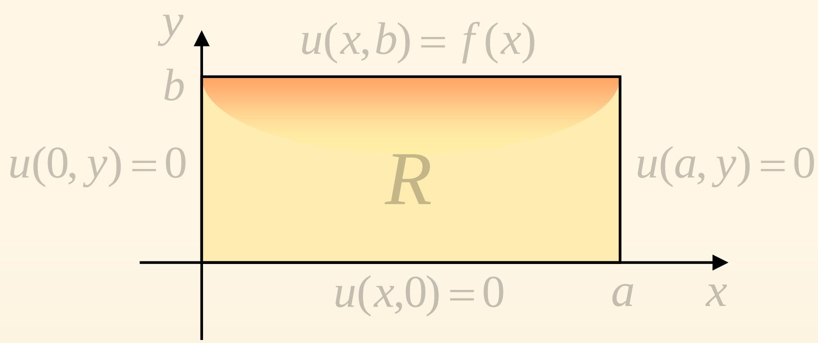
☑ Examples of Laplace's Equation

공학 수학 Chapter 12.5[1)]
(2-D Heat equation of Steady state)

① Governing Equation :

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

② Boundary condition : (Dirichlet B.C.[1)])



동일한 방정식을 푸는데 각각 다른 경계 조건이 적용됨

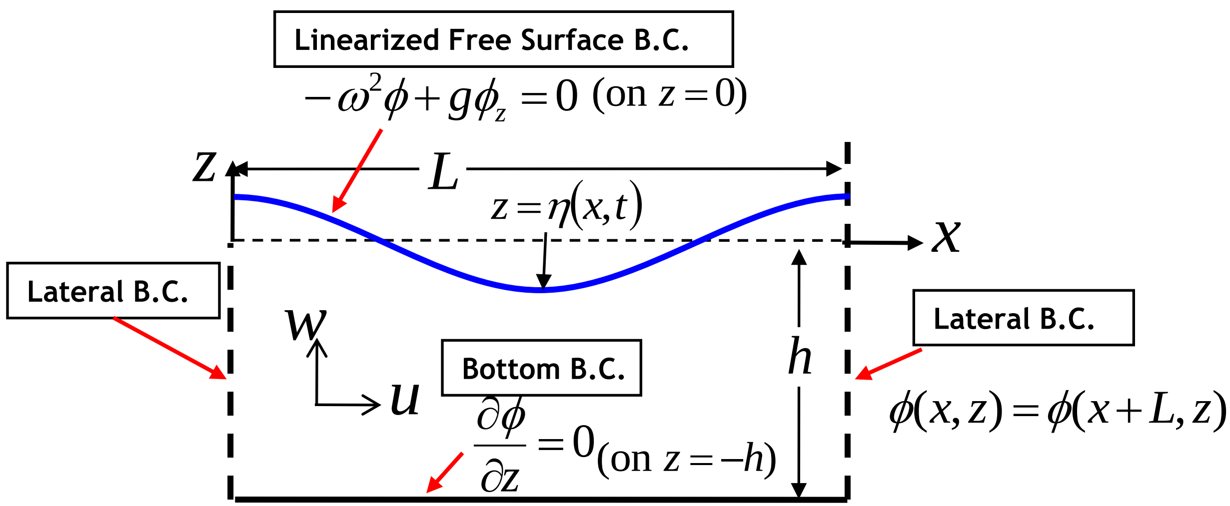
3학년 해양파 역학
(Wave Equation)

① Governing Equation :

$$\nabla^2 \phi=0$$

$$\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0\right)$$

② Boundary condition(B.C.) : (Robin B.C.[1)])



1) Erwin Kreyszig, Advanced Engineering Mathematics 9th ,Wiley,2005,Ch12.5(p552~562)

Classical Equations and B.V.P

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},k>0$$

$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

☑ Initial Conditions

- Related to time(t)
- Since solution of equation (1) and (2) *depend on time t*, we can prescribe what happens at t=0, that is, we can give <u>initial conditions</u>

$$u(x,0)=f(x),\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)$$

Classical Equations and B.V.P

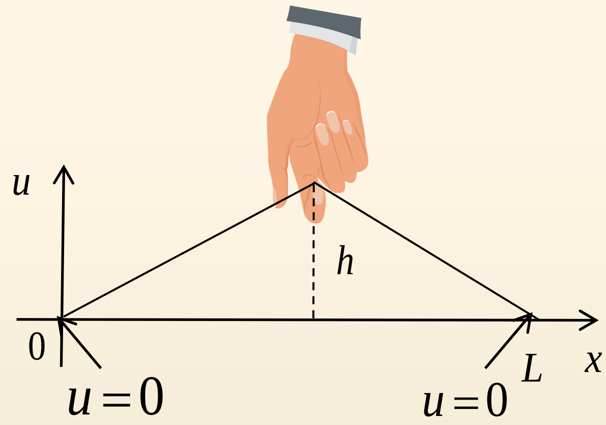
(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

(2) $k \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$

(3) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

☑ Boundary Conditions

- **Related to position (*x*)**



$$u(0,t) = 0, \quad u(L,t) = 0, \; t > 0$$

Classical Equations and B.V.P

(1) $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

(2) $k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, k > 0$

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

☑ Boundary Conditions

- **There are three types of boundary conditions associated with equation (1),(2), and (3).**

$(i)\ u$

(Dirichlet condition)

$(ii)\ \frac{\partial u}{\partial n}$

(Newman condition)

$(iii)\ \frac{\partial u}{\partial n} + hu$

h : a constant

(Robin condition)

Classical Equations and B.V.P

(1) $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

(2) $k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, k > 0$

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☑ Boundary Conditions

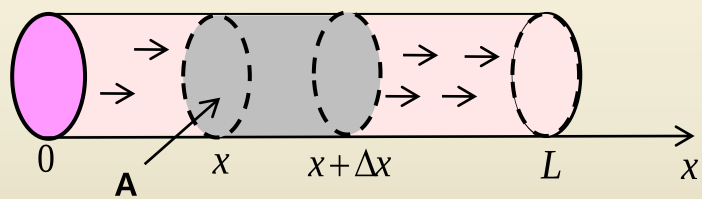
- **There are three types of boundary conditions associated with equation (1),(2), and (3).**

$(i)\ u$ *(Dirichlet condition)*

$(ii)\ \frac{\partial u}{\partial n}$ *(Newman condition)*

$(iii)\ \frac{\partial u}{\partial n} + hu$, h: a constant *(Robin condition)*

Example)



For a rod in figure,

$$(i)\ u(L,t) = u_0 \qquad (ii)\ \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0 \qquad (iii)\ \left.\frac{\partial u}{\partial x}\right|_{x=L} = -h(u(L,t) - u_m)$$

Classical Equations and B.V.P

(1) $a^2\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}$

(2) $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t},k>0$

(3) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

☑ Boundary-Value Problems

Problems such as

Solve : $$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2},\quad 0<x<L,\ t>0$$

Subject to: $$(BC)\ \ u(0,t)=0,\ u(L,t)=0,\ t>0$$
$$(IC)\ \ u(x,0)=f(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x),\ \ 0<x<L$$

Solve : $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ \ 0<x<a,\ 0<y<b$$

Subject to: $$(BC)\begin{cases}\left.\dfrac{\partial u}{\partial x}\right|_{x=0}=0,\ \left.\dfrac{\partial u}{\partial x}\right|_{x=a}=0, & 0<y<b\\ u(x,0)=0,\ u(x,b)=f(x), & 0<x<a\end{cases}$$

Classical Equations and B.V.P

(1) $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

(2) $k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, k > 0$

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

☑ Variations

- **The PDE (1),(2),and (3) must be modified to take into consideration internal or external influences action on the physical system.**

external influences

$$k^2 \frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_x) = \frac{\partial u}{\partial t}$$

Heat equation

heat transfer from the lateral surface of a rod into a surrounding medium that is held at a constant temperature u_m

$$\Rightarrow \quad k^2 \frac{\partial^2 u}{\partial x^2} - h(u - u_m) = \frac{\partial u}{\partial t}$$

external influences

$$a^2 \frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_x) = \frac{\partial^2 u}{\partial t^2}$$

Wave equation

external force, damping, restoring force

$$\Rightarrow \quad a^2 \frac{\partial^2 u}{\partial x^2} + \underbrace{f(x,t) - c\frac{\partial u}{\partial t} - ku}_{F(x,t,u,u_t)} = \frac{\partial^2 u}{\partial t^2}$$

1-D Wave Equation

1-D Wave Equation

$$(1) \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

- **Method of separating variables (or product method)**

Although there are several methods that can be tried to find particular solutions of a linear PDE, in the method of separation of variables we seek to find a particular solution of the form of a product of a function of *x* and a function of *t*,

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

which are a product of two functions, each depending only on one of the variables *x* and *t*.

1-D Wave Equation

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$$\therefore \frac{\partial^2 u}{\partial t^2}=X\ddot{T},\quad \frac{\partial^2 u}{\partial x^2}=X''T \qquad \text{where } \ddot{\mathrm{T}}=\frac{d^2T}{dt^2},\quad X''=\frac{d^2X}{dx^2}$$

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$$\therefore \frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \quad\Rightarrow\quad X\ddot{T}=c^2X''T$$

1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

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Dividing by $c^2 XT$,

1-D Wave Equation

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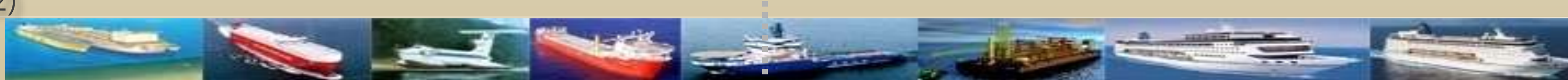
$$\frac{X\ddot{T}}{c^2 XT} = \frac{c^2 X'' T}{c^2 XT}$$



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Left side depending on only on *t* and right side depending on only on *x*.

1-D Wave Equation

$$\boxed{u(x,t)=X(x)T(t)\ldots(1)}$$

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

$$X\ddot{T}=c^2X''T$$

Dividing by c^2XT,

$$\frac{X\ddot{T}}{c^2XT}=\frac{c^2X''T}{c^2XT}$$

$$\therefore\frac{\ddot{T}}{c^2T}=\frac{X''}{X}$$

Left side depending on only on *t* and right side depending on only on *x*.

For example, Let's consider

$$\frac{\ddot{T}}{c^2T}=a_1t+b_1,\quad \frac{X''}{X}=a_2x+b_2$$

1-D Wave Equation

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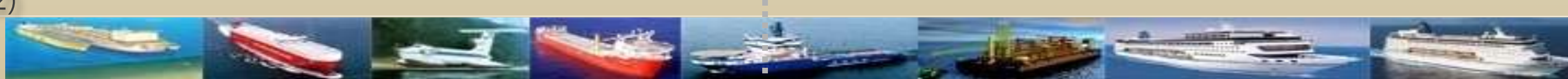

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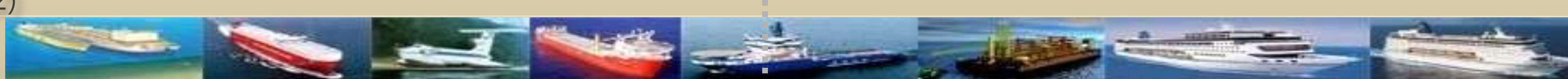
The variables *t* and *x* are linearly independent. So, a_1 and a_2 must be zero to satisfy the equation with respect to all *t* and *x*.



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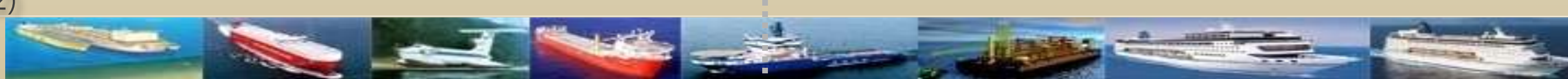

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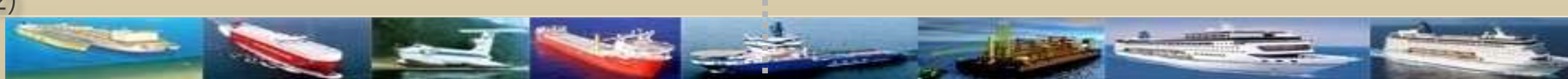
$$\therefore b_1 = b_2$$

So, both sides must be constant



1-D Wave Equation

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$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$$

Both sides must be constant

1-D Wave Equation

$$\boxed{u(x,t)=X(x)T(t)\ldots(1)}$$

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\ddot{T}}{c^2T}=\frac{X''}{X}$$

Both sides must be constant

$$\therefore \frac{\ddot{T}}{c^2T}=\frac{X''}{X}=-\lambda$$

1-D Wave Equation

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$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$$

Both sides must be constant

$$\therefore \frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda$$

Multiplying by the denominators gives two ordinary differential equations

1-D Wave Equation

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$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$$

Both sides must be constant

$$\therefore \frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda$$

Multiplying by the denominators gives two ordinary differential equations

$$\boxed{X'' + \lambda X = 0 \cdots (2)}$$

$$\boxed{\ddot{T} + c^2 \lambda T = 0 \cdots (3)}$$

$\Rightarrow$ **Solve 2^{nd} order ODEs**

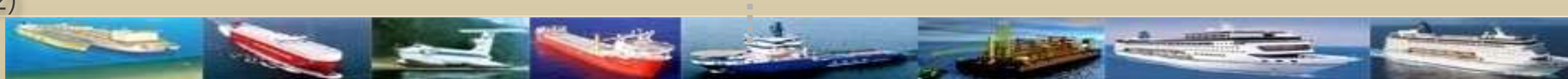
1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$



1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

$$(1) \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$

- **Boundary condition**

1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$

- **Boundary condition**

$$u(0,t) = X(0)T(t) = 0,$$

1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

$$(1) \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

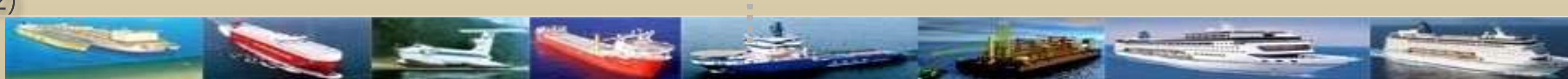
$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$

- **Boundary condition**

$$u(0,t) = X(0)T(t) = 0,$$

$$u(L,t) = X(L)T(t) = 0,$$



1-D Wave Equation

$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$

(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$

- **Boundary condition**

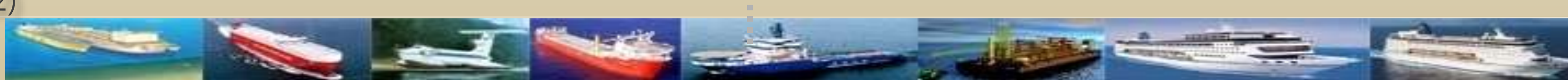
$$u(0,t) = X(0)T(t) = 0,$$

$$u(L,t) = X(L)T(t) = 0,$$

$T(t) \neq 0$

(otherwise , $u(x,t) = 0$

trivial solution→ no interest)



1-D Wave Equation

$$u(x,t) = X(x)T(t)\ldots(1)$$

(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$

- **Boundary condition**

$$u(0,t) = X(0)T(t) = 0,$$

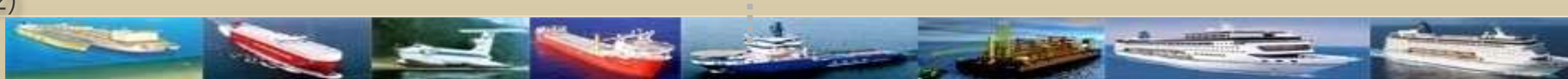
$$u(L,t) = X(L)T(t) = 0,$$

$T(t) \neq 0$

(otherwise , $u(x,t) = 0$

trivial solution→ no interest)

$$\therefore X(0) = X(L) = 0$$



1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$

- **Boundary condition**

$$u(0,t) = X(0)T(t) = 0,$$

$$u(L,t) = X(L)T(t) = 0,$$

$$T(t) \neq 0$$

(otherwise , $u(x,t) = 0$
trivial solution→ no interest)

$$\therefore X(0) = X(L) = 0$$

- **three cases according to *λ***

1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t) \ldots (1)}$$

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$

- **Boundary condition**

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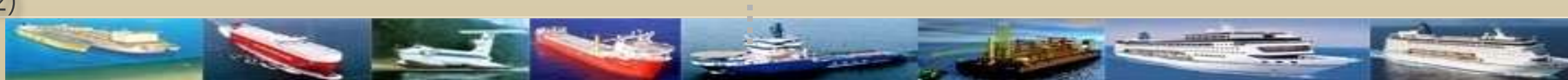
trivial solution→ no interest)

$$\therefore X(0) = X(L) = 0$$

- **three cases according to *λ***

1) *λ* = 0



1-D Wave Equation

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$$X'' + \lambda X = 0 \cdots (2)$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$

- **Boundary condition**

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$$T(t) \neq 0$$

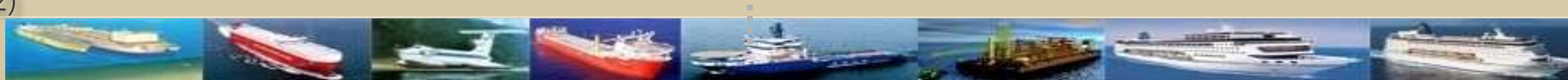
(otherwise , $u(x,t) = 0$
trivial solution→ no interest)

$$\therefore X(0) = X(L) = 0$$

- **three cases according to *λ***

1) *λ* = 0

$$X'' = 0$$



1-D Wave Equation

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- **Boundary condition**

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$$u(L,t) = X(L)T(t) = 0,$$

$$T(t) \neq 0$$

(otherwise , $u(x,t) = 0$
trivial solution→ no interest)

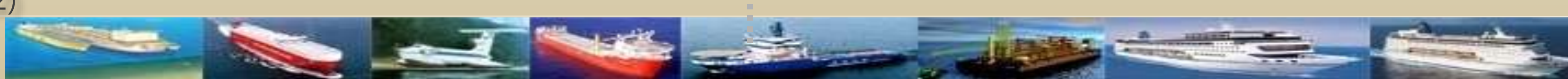
$$\therefore X(0) = X(L) = 0$$

- **three cases according to *λ***

1) *λ* = 0

$$X'' = 0$$

$$X(x) = Ax + B$$



1-D Wave Equation

$$\boxed{u(x,t)=X(x)T(t)\ldots(1)}$$

(1) $a^2\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}$

$$X''+\lambda X=0\cdots(2)$$

$$\ddot{T}+a^2\lambda T=0\cdots(3)$$

- **Boundary condition**

$$u(0,t)=X(0)T(t)=0,$$

$$u(L,t)=X(L)T(t)=0,$$

$T(t)\neq 0$

(otherwise , $u(x,t)=0$
trivial solution→ no interest)

$$\therefore X(0)=X(L)=0$$

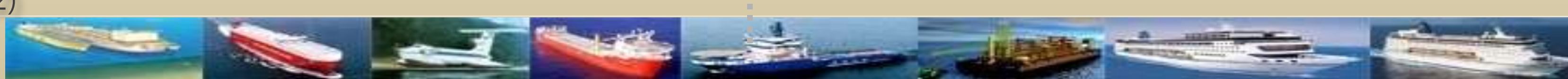
- **three cases according to *λ***

1) *λ* = 0

$$X''=0$$

$$X(x)=Ax+B$$

$$\left.\begin{array}{l}\text{Boundarycondition}\\ F(0)=A\cdot 0+B=0\\ F(L)=A\cdot L+B=0\end{array}\right\}A=B=0$$



1-D Wave Equation

$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$X'' + \lambda X = 0\cdots(2)$$

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$$u(0,t) = X(0)T(t) = 0,$$

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(otherwise , $u(x,t) = 0$
trivial solution→ no interest)

$$\therefore X(0) = X(L) = 0$$

- **three cases according to *λ***

1) *λ* = 0

$$X'' = 0$$

$$X(x) = Ax + B$$

$$\left.\begin{array}{l}\text{Boundarycondition}\\ F(0) = A\cdot 0 + B = 0\\ F(L) = A\cdot L + B = 0\end{array}\right\} A = B = 0$$

$$\therefore X(x) = 0$$

1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t) \ldots (1)}$$

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$X'' + \lambda X = 0 \cdots (2)$$

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trivial solution→ no interest)

$$\therefore X(0) = X(L) = 0$$

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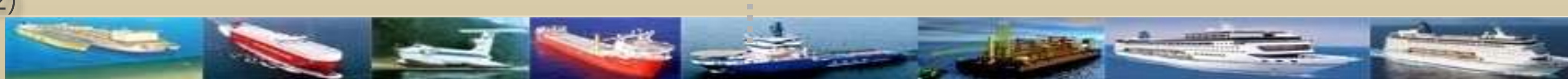
$$\therefore X(x) = 0$$

trivial solution. no interest



1-D Wave Equation

$$u(x,t) = X(x)T(t) \ldots (1)$$

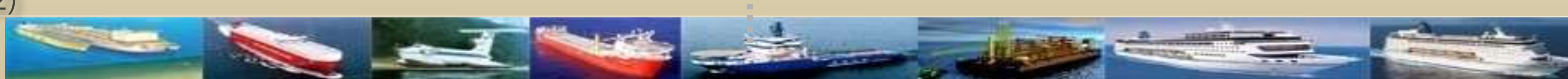
$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (1)$$

Boundary condition

$$X(0) = X(L) = 0$$

$$X'' + \lambda X = 0$$

- **three cases according to *λ***



1-D Wave Equation

$$u(x,t)=X(x)T(t)\ldots(1)$$

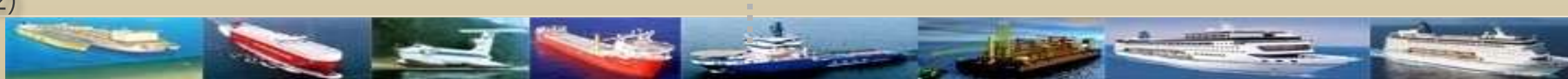
$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

Boundary condition $X(0)=X(L)=0$

$$X''+\lambda X=0$$

- **three cases according to *λ***

2) $\lambda = -\alpha^2 < 0$



1-D Wave Equation

$$u(x,t)=X(x)T(t)\ldots(1)$$

(1) $$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

Boundary condition

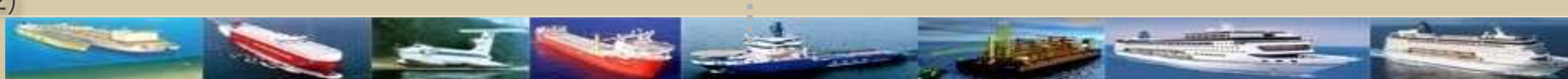
$$X(0)=X(L)=0$$

$$X''+\lambda X=0$$

- **three cases according to *λ***

2) $\lambda = -\alpha^2 < 0$

$$X''-\alpha^2 X=0$$



1-D Wave Equation

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \quad (1)$$

Boundary condition $X(0)=X(L)=0$

$$X''+\lambda X=0$$

- **three cases according to *λ***

2) *λ* = -*α*^{2} < 0

$$X''-\alpha^2 X=0$$

$$X(x)=Ae^{\alpha x}+Be^{-\alpha x}$$

1-D Wave Equation

$$u(x,t) = X(x)T(t) \cdots (1)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \cdots (1)$$

Boundarycondition

$$X(0) = X(L) = 0$$

$$X'' + \lambda X = 0$$

- **three cases according to *λ***

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

$$X(0) = A + B = 0, \quad B = -A$$

1-D Wave Equation

$$u(x,t) = X(x)T(t) \cdots (1) \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Boundarycondition

$X(0) = X(L) = 0$

$$X'' + \lambda X = 0$$

- **three cases according to *λ***

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

$X(0) = A + B = 0, \quad B = -A$

$$\therefore X(x) = A(e^{\alpha x} - e^{-\alpha x})$$

1-D Wave Equation

$$u(x,t) = X(x)T(t) \cdots (1)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (1)$$

Boundary condition
$X(0) = X(L) = 0$

$$X'' + \lambda X = 0$$

- **three cases according to *λ***

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$$X'' - \alpha^2 X = 0$$

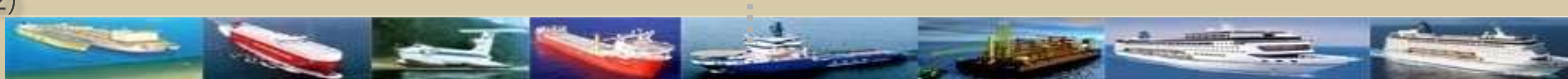
$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

$$X(0) = A + B = 0, \quad B = -A$$

$$\therefore X(x) = A(e^{\alpha x} - e^{-\alpha x})$$

$$X(L) = A(e^{\alpha L} - e^{-\alpha L}) = 0$$



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$$X(0) = X(L) = 0$$

$$X'' + \lambda X = 0$$

- **three cases according to *λ***

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

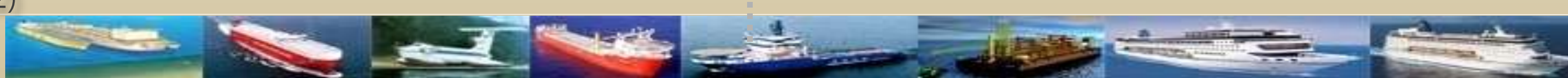
$$X(0) = A + B = 0, \quad B = -A$$

$$\therefore X(x) = A(e^{\alpha x} - e^{-\alpha x})$$

$$X(L) = A(e^{\alpha L} - e^{-\alpha L}) = 0$$

if $A = 0 \rightarrow B = 0$ ⇨ $X(x) = 0$

trivial solution → no interest



1-D Wave Equation

$$u(x,t) = X(x)T(t) \ldots (1)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (1)$$

Boundary condition $X(0) = X(L) = 0$

$$X'' + \lambda X = 0$$

- **three cases according to λ**

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

$$X(0) = A + B = 0, \quad B = -A$$

$$\therefore X(x) = A(e^{\alpha x} - e^{-\alpha x})$$

$$X(L) = A(e^{\alpha L} - e^{-\alpha L}) = 0$$

if $A = 0 \rightarrow B = 0 \Rightarrow X(x) = 0$

trivial solution → no interest

if $(e^{\alpha L} - e^{-\alpha L}) = 0$

$$\rightarrow e^{2\alpha L} = 1,$$

$$\rightarrow \alpha = 0 \ (\because L \neq 0) \Rightarrow X(x) = 0$$

trivial solution → no interest

1-D Wave Equation

$u(x,t) = X(x)T(t) \cdots (1)$

$a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} \cdots (1)$

Boundary condition $X(0) = X(L) = 0$

$X'' + \lambda X = 0$

- **three cases according to *λ***

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

$$X(0) = A + B = 0, \quad B = -A$$

$$\therefore X(x) = A(e^{\alpha x} - e^{-\alpha x})$$

$$X(L) = A(e^{\alpha L} - e^{-\alpha L}) = 0$$

if $A = 0 \rightarrow B = 0 \Rightarrow X(x) = 0$

trivial solution → no interest

if $(e^{\alpha L} - e^{-\alpha L}) = 0$

$$\rightarrow e^{2\alpha L} = 1,$$

$$\rightarrow \alpha = 0 \ (\because L \neq 0) \Rightarrow X(x) = 0$$

trivial solution → no interest

3) $\lambda = \alpha^2 > 0$

1-D Wave Equation

$u(x,t) = X(x)T(t) \ldots (1)$ $\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ (1)

Boundary condition $X(0) = X(L) = 0$

$$X'' + \lambda X = 0$$

- **three cases according to λ**

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

$$X(0) = A + B = 0, \quad B = -A$$

$$\therefore X(x) = A(e^{\alpha x} - e^{-\alpha x})$$

$$X(L) = A(e^{\alpha L} - e^{-\alpha L}) = 0$$

if $A = 0 \rightarrow B = 0$ ⇨ $X(x) = 0$

trivial solution → no interest

if $(e^{\alpha L} - e^{-\alpha L}) = 0$

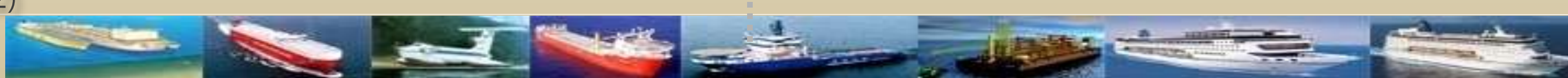
$$\rightarrow e^{2\alpha L} = 1,$$

$$\rightarrow \alpha = 0 \ (\because L \neq 0)$$ ⇨ $X(x) = 0$

trivial solution → no interest

3) $\lambda = \alpha^2 > 0$

$$X'' + \alpha^2 X = 0$$



1-D Wave Equation

$$u(x,t)=X(x)T(t) \ldots (1) \qquad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \ (1)$$

Boundary condition

$$X(0)=X(L)=0$$

$$X''+\lambda X=0$$

- **three cases according to *λ***

2) $\lambda = -\alpha^2 < 0$

$$X''-\alpha^2 X=0$$

$$X(x)=Ae^{\alpha x}+Be^{-\alpha x}$$

Boundary condition

$$X(0)=A+B=0, \ \ B=-A$$

$$\therefore X(x)=A(e^{\alpha x}-e^{-\alpha x})$$

$$X(L)=A(e^{\alpha L}-e^{-\alpha L})=0$$

if $A=0 \rightarrow B=0$ ⇨ $X(x)=0$

trivial solution → no interest

if $(e^{\alpha L}-e^{-\alpha L})=0$

$$\rightarrow e^{2\alpha L}=1,$$

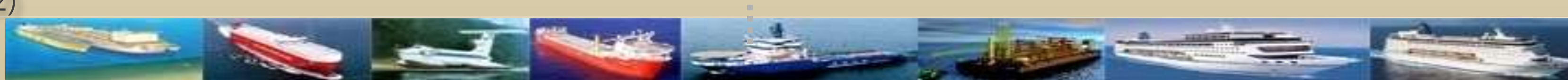
$$\rightarrow \alpha=0 \ (\because L\neq 0)$$ ⇨ $X(x)=0$

trivial solution → no interest

3) $\lambda = \alpha^2 > 0$

$$X''+\alpha^2 X=0$$

$$X(x)=c_1\cos\alpha x+c_2\sin\alpha x$$



1-D Wave Equation

$u(x,t)=X(x)T(t)\cdots(1)$ $\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}\cdots(1)$

Boundary condition $X(0)=X(L)=0$

$X''+\lambda X=0$

- **three cases according to *λ***

2) $\lambda = -\alpha^2 < 0$

$$X''-\alpha^2 X=0$$

$$X(x)=Ae^{\alpha x}+Be^{-\alpha x}$$

Boundary condition

$$X(0)=A+B=0,\ \ B=-A$$

$$\therefore X(x)=A(e^{\alpha x}-e^{-\alpha x})$$

$$X(L)=A(e^{\alpha L}-e^{-\alpha L})=0$$

if $A=0\rightarrow B=0$ ⇨ $X(x)=0$

trivial solution → no interest

if $(e^{\alpha L}-e^{-\alpha L})=0$

$$\rightarrow e^{2\alpha L}=1,$$

$$\rightarrow \alpha=0\ (\because L\neq 0)$$ ⇨ $X(x)=0$

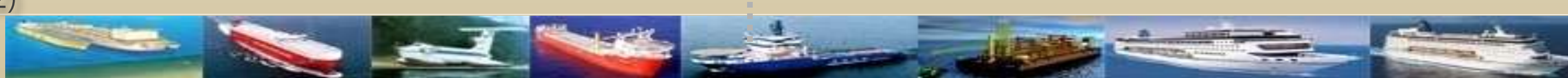
trivial solution → no interest

3) $\lambda = \alpha^2 > 0$

$$X''+\alpha^2 X=0$$

$$X(x)=c_1\cos\alpha x+c_2\sin\alpha x$$

Boundary condition



1-D Wave Equation

$$u(x,t) = X(x)T(t) \cdots (1)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (1)$$

Boundarycondition

$X(0) = X(L) = 0$

$$X'' + \lambda X = 0$$

- **three cases according to λ**

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

$$X(0) = A + B = 0, \quad B = -A$$

$$\therefore X(x) = A(e^{\alpha x} - e^{-\alpha x})$$

$$X(L) = A(e^{\alpha L} - e^{-\alpha L}) = 0$$

if $A = 0 \rightarrow B = 0 \Rightarrow X(x) = 0$

trivial solution → no interest

if $(e^{\alpha L} - e^{-\alpha L}) = 0$

$\rightarrow e^{2\alpha L} = 1,$

$\rightarrow \alpha = 0 \ (\because L \neq 0) \Rightarrow X(x) = 0$

trivial solution → no interest

3) $\lambda = \alpha^2 > 0$

$$X'' + \alpha^2 X = 0$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

Boundary condition

$$X(0) = c_1 = 0$$

1-D Wave Equation

$$u(x,t)=X(x)T(t)\ldots(1)\qquad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}\ (1)$$

Boundary condition $X(0)=X(L)=0$

$$X''+\lambda X=0$$

- **three cases according to λ**

2) $\lambda = -\alpha^2 < 0$

$$X''-\alpha^2 X=0$$

$$X(x)=Ae^{\alpha x}+Be^{-\alpha x}$$

Boundary condition

$$X(0)=A+B=0,\ \ B=-A$$

$$\therefore X(x)=A(e^{\alpha x}-e^{-\alpha x})$$

$$X(L)=A(e^{\alpha L}-e^{-\alpha L})=0$$

if $A=0\rightarrow B=0 \Rightarrow X(x)=0$

trivial solution → no interest

if $(e^{\alpha L}-e^{-\alpha L})=0$

$$\rightarrow e^{2\alpha L}=1,$$

$$\rightarrow \alpha=0\ (\because L\neq 0) \Rightarrow X(x)=0$$

trivial solution → no interest

3) $\lambda = \alpha^2 > 0$

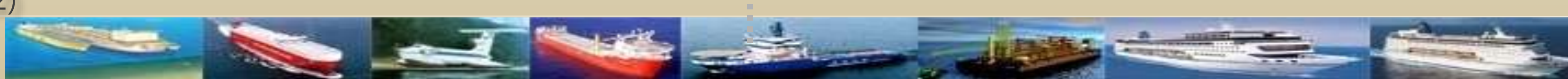
$$X''+\alpha^2 X=0$$

$$X(x)=c_1\cos\alpha x+c_2\sin\alpha x$$

Boundary condition

$$X(0)=c_1=0$$

$$\therefore X(x)=c_2\sin\alpha x$$



1-D Wave Equation

$$u(x,t) = X(x)T(t) \quad \cdots (1)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (1)$$

Boundary condition $X(0) = X(L) = 0$

$$X'' + \lambda X = 0$$

- **three cases according to *λ***

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$$

Boundary condition

$$X(0) = A + B = 0, \quad B = -A$$

$$\therefore X(x) = A(e^{\alpha x} - e^{-\alpha x})$$

$$X(L) = A(e^{\alpha L} - e^{-\alpha L}) = 0$$

if $A = 0 \to B = 0 \Rightarrow X(x) = 0$

trivial solution → no interest

if $(e^{\alpha L} - e^{-\alpha L}) = 0$

$$\to e^{2\alpha L} = 1,$$

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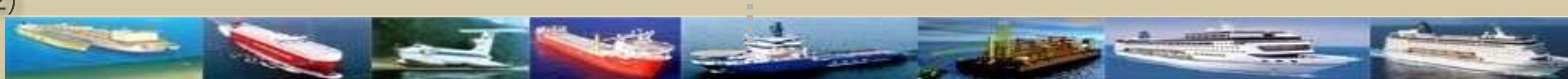

1-D Wave Equation

$u(x,t)=X(x)T(t)\ldots(1)$ $\quad a^2\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}$ $\quad$ Boundary condition $X(0)=X(L)=0$

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Boundary condition

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if $A=0\rightarrow B=0$ ⇨ $X(x)=0$

trivial solution → no interest

if $(e^{\alpha L}-e^{-\alpha L})=0$

$$\rightarrow e^{2\alpha L}=1,$$

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$$X''+\alpha^2 X=0$$

$$X(x)=c_1\cos\alpha x+c_2\sin\alpha x$$

Boundary condition

$$X(0)=c_1=0$$

$$\therefore X(x)=c_2\sin\alpha x$$

$$X(L)=c_2\sin\alpha L=0$$

$c_2\neq 0$ **(otherwise** $X(x)=0$ **: trivial solution → no interest).**

So, $\sin\alpha L = 0$

1-D Wave Equation

$$u(x,t) = X(x)T(t) \cdots (1)$$

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \cdots (1)$$

Boundary condition
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if $A = 0 \to B = 0 \Rightarrow X(x) = 0$

trivial solution → no interest

if $(e^{\alpha L} - e^{-\alpha L}) = 0$

$$\to e^{2\alpha L} = 1,$$

$$\to \alpha = 0 \;(\because L \neq 0) \Rightarrow X(x) = 0$$

trivial solution → no interest

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$$X'' + \alpha^2 X = 0$$

$$X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$$

Boundary condition

$$X(0) = c_1 = 0$$

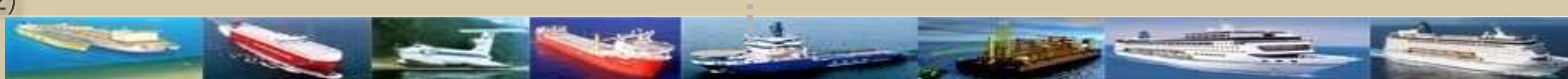
$$\therefore X(x) = c_2 \sin\alpha x$$

$$X(L) = c_2 \sin\alpha L = 0$$

$c_2 \neq 0$ **(otherwise** $X(x) = 0$ **: trivial solution → no interest).**

So, $\sin\alpha L = 0$

$$\therefore \alpha L = n\pi \;, (n = 1, 2, \cdots)$$



1-D Wave Equation

(1) $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

Boundarycondition

$X(0) = X(L) = 0$

$$u(x,t) = X(x)T(t)\ldots(1)$$

$$X'' + \lambda X = 0$$

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

Boundarycondition

$$X(0)=X(L)=0$$

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$X''+\lambda X=0$$

$$X(x)=c_2\sin\alpha x$$

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

Boundarycondition

$$X(0)=X(L)=0$$

$$\boxed{u(x,t)=X(x)T(t)\ldots(1)}$$

$$\boxed{X''+\lambda X=0}$$

$$X(x)=c_2\sin\alpha x$$

$$\alpha=\frac{n\pi}{L}\;,\;(n=1,2,\cdots)$$

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

Boundarycondition

$$X(0)=X(L)=0$$

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$X''+\lambda X=0$$

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We obtain many solutions $X(x)=X_n(x)$, where

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

Boundarycondition

$$X(0)=X(L)=0$$

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$X''+\lambda X=0$$

$$X(x)=c_2\sin\alpha x$$

$$\alpha=\frac{n\pi}{L}\ ,\ (n=1,2,\cdots)$$

We obtain many solutions *X*(*x*) = X_n(*x*) , where

$$X_n(x)=c_2\sin\frac{n\pi}{L}x,\ \ (n=1,2,\cdots)$$

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

Boundarycondition
$X(0)=X(L)=0$

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$X''+\lambda X=0$$

$$X(x)=c_2\sin\alpha x$$

$$\alpha=\frac{n\pi}{L}\ ,\ (n=1,2,\cdots)$$

We obtain many solutions $X(x)=X_n(x)$, where

$$X_n(x)=c_2\sin\frac{n\pi}{L}x,\ \ (n=1,2,\cdots)$$

For negative integer *n* , we obtain essentially the same solutions, except for a minus sign, because sin(-α) = -sin(α). And if *n* is zero, F_n becomes zero, so it can't be a basis.

1-D Wave Equation

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$X_n(x) = A_n \sin \frac{n\pi}{L} x$$

$$\lambda = \alpha^2 > 0$$

$$\ddot{T} + a^2 \lambda T = 0 \cdots (3)$$

$$u(0,t) = 0,\ \ u(L,t) = 0, \text{for all } t$$

$$u(x,t) = X(x)T(t) \ldots (1)$$

$$\alpha = \frac{n\pi}{L}\ , (n = 1, 2, \cdots)$$

1-D Wave Equation

$$(1)\quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$X_n(x) = A_n \sin \frac{n\pi}{L} x$$

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$$u(x,t) = X(x)T(t) \ldots (1)$$

$$\alpha = \frac{n\pi}{L}\ , (n = 1, 2, \cdots)$$

$$\lambda_n = {\alpha_n}^2 = \frac{n^2 \pi^2}{L^2}$$

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

$$X_n(x)=A_n\sin\frac{n\pi}{L}x$$

$$\lambda=\alpha^2>0$$

$$\ddot{T}+a^2\lambda T=0\cdots(3)$$

$$u(0,t)=0,\ \ u(L,t)=0,\ \text{for all}\ \ t$$

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$\alpha=\frac{n\pi}{L}\ ,\ (n=1,2,\cdots)$$

$$\lambda_n=\alpha_n{}^2=\frac{n^2\pi^2}{L^2}$$

$$\ddot{T}+a^2\lambda T=0$$

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$ $$X_n(x)=A_n\sin\frac{n\pi}{L}x$$

$$\lambda=\alpha^2>0$$ $$\ddot{T}+a^2\lambda T=0\cdots(3)$$

$$u(0,t)=0,\ \ u(L,t)=0,\ \text{for all}\ \ t$$

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$\alpha=\frac{n\pi}{L}\ ,\ (n=1,2,\cdots)$$

$$\lambda_n=\alpha_n{}^2=\frac{n^2\pi^2}{L^2}$$

$$\ddot{T}+a^2\lambda T=0$$

$$T_n(t)=c_3\cos\frac{n\pi}{L}at+c_4\sin\frac{n\pi}{L}at$$

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

$$X_n(x)=A_n\sin\frac{n\pi}{L}x$$

$$\lambda=\alpha^2>0$$

$$\ddot{T}+a^2\lambda T=0\cdots(3)$$

$$u(0,t)=0,\ \ u(L,t)=0, \text{for all}\ \ t$$

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Hence solutions of (1) satisfying (2) are $u_n(x,t)=X_n(x)T_n(t)=T_n(t)X_n(x)$, written out

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

$$X_n(x)=A_n\sin\frac{n\pi}{L}x$$

$$\lambda=\alpha^2>0$$

$$\ddot{T}+a^2\lambda T=0\cdots(3)$$

$$u(0,t)=0,\ \ u(L,t)=0, \text{for all } \ t$$

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$\alpha=\frac{n\pi}{L}\ ,\ (n=1,2,\cdots)$$

$$\lambda_n=\alpha_n{}^2=\frac{n^2\pi^2}{L^2}$$

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Hence solutions of (1) satisfying (2) are $u_n(x,t)$ = $X_n(x)T_n(t)$ = $T_n(t)X_n(x)$, written out

$$u_n=(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t)\cdot\sin\frac{n\pi}{L}x$$

1-D Wave Equation

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}\qquad X_n(x)=A_n\sin\frac{n\pi}{L}x$$

$$\lambda=\alpha^2>0\qquad \ddot{T}+a^2\lambda T=0\cdots(3)$$

$$u(0,t)=0,\ \ u(L,t)=0,\ \text{for all}\ \ t$$

$$u(x,t)=X(x)T(t)\ldots(1)$$

$$\alpha=\frac{n\pi}{L}\ ,\ (n=1,2,\cdots)$$

$$\lambda_n=\alpha_n{}^2=\frac{n^2\pi^2}{L^2}$$

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$$u_n=(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t)\cdot\sin\frac{n\pi}{L}x$$

$$u(x,t)=\sum_{n=1}^{\infty}(A_n\cos\frac{n\pi a}{L}t+B_n\sin\frac{n\pi a}{L}t)\cdot\sin\frac{n\pi}{L}x$$

1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

$$(1) \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

One-dimensional wave equation : $u_{tt} = c^2 u_{xx}$

General Solution : $u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t) \sin \frac{n\pi}{L} x$

Initial condition : $u(x,0) = f(x), \dot{u}(x,0) = g(x)$

1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

$$(1)\quad a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

One-dimensional wave equation : $u_{tt} = c^2 u_{xx}$

$$\Downarrow$$

General Solution : $u(x,t) = \sum_{n=1}^{\infty}(A_n \cos\frac{n\pi a}{L}t + B_n \sin\frac{n\pi a}{L}t)\sin\frac{n\pi}{L}x$

Initial condition : $u(x,0) = f(x), \dot{u}(x,0) = g(x)$

$$u(x,0) = \underbrace{\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x}_{\text{Fourier sine series of } f(x)} = f(x)$$

$$\therefore A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,$$

1-D Wave Equation

$$\boxed{u(x,t) = X(x)T(t)\ldots(1)}$$

$$(1) \quad a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

One-dimensional wave equation : $u_{tt} = c^2 u_{xx}$

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General Solution : $u(x,t) = \sum_{n=1}^{\infty} (A_n \cos\frac{n\pi a}{L} t + B_n \sin\frac{n\pi a}{L} t) \sin\frac{n\pi}{L} x$

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$$u(x,0) = \underline{\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L} x} = f(x)$$

Fourier sine series of ***f*** **(*x*)**

$$\therefore A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty}(-A_n\frac{n\pi a}{L}\sin\frac{n\pi a}{L}t + B_n\frac{n\pi a}{L}\cos\frac{n\pi a}{L}t)\sin\frac{n\pi}{L}x\right]_{t=0}$$

$$= \sum_{n=1}^{\infty}\underline{(B_n\frac{n\pi a}{L})\sin\frac{n\pi}{L}x} = g(x)$$

Fourier sine series of ***g*****(*x*)**

$$\therefore B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}dx,$$

1-D Heat Equation

1-D Heat Equation

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\, k>0$$

$$\boxed{k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\cdots(1)}$$

Boundary conditions

$$u(0,t)=0,\quad u(L,t)=0$$

for all t

initial conditions

$$u(x,0)=f(x)$$

$f(x)$ is given

$$u(x,t)=X(x)T(t)$$

1-D Heat Equation

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},k>0$$

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$f(x)$ is given

$$u(x,t)=X(x)T(t)$$

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

$$k\frac{\partial^2 X(x)}{\partial x^2}T(t)=X(x)\frac{\partial T(t)}{\partial t}$$

$$kX''(x)T(t)=X(x)\dot{T}(t)$$

1-D Heat Equation

$$(2)\quad k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\, k>0$$

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separating variable,

1-D Heat Equation

$$\boxed{k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\cdots(1)}$$

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separating variable,

$$\frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{kT(t)}=-\lambda$$

1-D Heat Equation

$$\boxed{k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\cdots(1)}$$

Boundary conditions

$$u(0,t)=0,\quad u(L,t)=0$$

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$$u(x,0)=f(x)$$

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$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

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$$kX''(x)T(t)=X(x)\dot{T}(t)$$

separating variable,

$$\frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{kT(t)}=-\lambda$$

If $-\lambda\geq 0$, $X(x)$ becomes trivial solution by satisfying boundary condition. It is no interest. So, $-\lambda$ must be negative.
(ref. 1-D Wave Equation)

1-D Heat Equation

initial condition
$u(x,0) = f(x)$

$$(2) \quad k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, k > 0$$

Boundaryconditions
$u(0,t) = 0, \quad u(L,t) = 0$

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{kT(t)} = -\lambda$$

1-D Heat Equation

initial condition
$u(x,0) = f(x)$

(2) $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$

Boundary conditions
$u(0,t) = 0, \quad u(L,t) = 0$

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{kT(t)} = -\lambda$$

Step 1. Two ODEs

1-D Heat Equation

initial condition
$u(x,0) = f(x)$

$(2) \quad k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$

Boundaryconditions
$u(0,t) = 0, \quad u(L,t) = 0$

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{kT(t)} = -\lambda$$

Step 1. Two ODEs

$$X''(x) + \lambda X(x) = 0 \cdots (2)$$

$$T'(t) + k\lambda T(t) = 0 \cdots (3)$$

1-D Heat Equation

initial condition
$u(x,0)=f(x)$

(2) $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}, k>0$

Boundaryconditions
$u(0,t)=0, \quad u(L,t)=0$

$$\frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{kT(t)}=-\lambda$$

Step 1. Two ODEs

$$X''(x)+\lambda X(x)=0\cdots(2)$$

$$T'(t)+k\lambda T(t)=0\cdots(3)$$

Step 2. Satisfying the boundary conditions

1-D Heat Equation

initial condition
$u(x,0) = f(x)$

$(2) \quad k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$

Boundary conditions
$u(0,t) = 0, \quad u(L,t) = 0$

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{kT(t)} = -\lambda$$

Step 1. Two ODEs

$$X''(x) + \lambda X(x) = 0 \cdots (2)$$

$$T'(t) + k\lambda T(t) = 0 \cdots (3)$$

Step 2. Satisfying the boundary conditions

- **general solution of (2)**

$$X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x, \lambda = \alpha^2$$

1-D Heat Equation

initial condition
$u(x,0) = f(x)$

(2) $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$

Boundaryconditions
$u(0,t) = 0, \quad u(L,t) = 0$

$$\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{kT(t)} = -\lambda$$

Step 1. Two ODEs

$$X''(x) + \lambda X(x) = 0 \cdots (2)$$

$$T'(t) + k\lambda T(t) = 0 \cdots (3)$$

Step 2. Satisfying the boundary conditions

- **general solution of (2)**

$$X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x, \lambda = \alpha^2$$

•boundary conditions

$$u(0,t) = X(0) \cdot T(t) = 0$$

$$u(L,t) = X(L) \cdot T(t) = 0$$

1-D Heat Equation

initial condition
$u(x,0)=f(x)$

(2) $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}, k>0$

Boundary conditions
$u(0,t)=0,\quad u(L,t)=0$

$$\frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{kT(t)}=-\lambda$$

Step 1. Two ODEs

$$X''(x)+\lambda X(x)=0\cdots(2)$$

$$T'(t)+k\lambda T(t)=0\cdots(3)$$

Step 2. Satisfying the boundary conditions

- **general solution of (2)**

$$X(x)=c_1\cos\alpha x+c_2\sin\alpha x, \lambda=\alpha^2$$

- **boundary conditions**

$$u(0,t)=X(0)\cdot T(t)=0$$

$$u(L,t)=X(L)\cdot T(t)=0$$

if $T(t)\equiv 0$, u is a trivial solution.

1-D Heat Equation

initial condition $u(x,0)=f(x)$

(2) $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}, k>0$

Boundary conditions $u(0,t)=0,\ \ u(L,t)=0$

$$\frac{X''(x)}{X(x)}=\frac{\dot{T}(t)}{kT(t)}=-\lambda$$

Step 1. Two ODEs

$$X''(x)+\lambda X(x)=0\cdots(2)$$

$$T'(t)+k\lambda T(t)=0\cdots(3)$$

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$$\therefore X(0)=c_1=0$$

$$X(x)=c_2\sin\alpha x$$

1-D Heat Equation

initial condition
$u(x,0) = f(x)$

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Boundaryconditions
$u(0,t) = 0, \quad u(L,t) = 0$

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$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x, \lambda = \alpha^2$$

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$$\therefore X(0) = c_1 = 0$$

$$X(x) = c_2 \sin\alpha x$$

$$X(L) = c_2 \sin\alpha L = 0$$

$c_2 \neq 0$ **(otherwise** $X(x) = 0$ **: trivial solution → no interest).**

So, $\sin\alpha L = 0$

1-D Heat Equation

initial condition
$u(x,0) = f(x)$

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So, $\sin\alpha L = 0$

$$\therefore \alpha L = n\pi, (n = 1, 2, \cdots)$$

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initial condition
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So, $\sin\alpha L=0$

$$\therefore \alpha L=n\pi \ ,(n=1,2,\cdots)$$

$$\therefore \alpha=\frac{n\pi}{L}, \ (n=1,2,\cdots)$$

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initial condition
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(2) $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, k > 0$

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$u(0,t) = 0, \quad u(L,t) = 0$

$T'(t) + k\lambda T(t) = 0 \cdots (3)$

$$X(x) = c_2 \sin \alpha x$$

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Setting c_2 = 1, we obtain many solutions $X(x) = X_n(x)$, where

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- **general solution of above ODE**

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- **general solution of above ODE**

$$T(t) = T_n(t) = c_3 e^{-k\frac{n^2\pi^2}{L^2}t}, (n = 1, 2, \cdots)$$

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$$T'(t)+k\lambda T(t)=0$$

- **general solution of above ODE**

$$T(t)=T_n(t)=c_3e^{-k\frac{n^2\pi^2}{L^2}t},(n=1,2,\cdots)$$

$$\therefore u_n(x,t)=X_n(x)T_n(t)$$

$$=A_n\sin\frac{n\pi x}{L}e^{-k\frac{n^2\pi^2}{L^2}t}$$

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Step 3. Solution of the entire problem

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$$(n=1,2,\cdots),$$

Step 3. Solution of the entire problem

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)$$

$$=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}e^{-k\frac{n^2\pi^2}{L^2}t}$$

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Fourier sine series of $f(x)$

1-D Heat Equation

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$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

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Fourier sine series of *f* (*x*)

$$\therefore A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

1-D Heat Equation

initial condition $u(x,0)=f(x)$	(2) $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}, k>0$

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Fourier sine series of *f* (*x*)

$$\therefore A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$\boxed{\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}e^{-k\frac{n^2\pi^2}{L^2}t}\\ A_n&=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx\end{aligned}}$$

Eigenfunction Expansion*

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*Tyn Myint-U, Partial Differential Equations of Mathematical Physics, Second Edition, Elsevier North Holland, 1980, p179~185

Eigenfunction Expansion*

$$u(x,0) = f(x)$$

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Eigenfunction Expansion*

$$u(x,0) = f(x)$$

$$\sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}x = f(x) \Leftarrow \therefore \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

Eigenfunction Expansion*

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Eigenfunction Expansion*

$$u(x,0)=f(x)$$

$$\sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}\,x = f(x) \quad \Leftarrow \quad \therefore \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

Eigenfunction Expansion*

A real-valued functions $\phi(x)$ is said to be *square-integrable* with respect to a weight function $\rho(x)>0$, if on an interval I,

$$\int_I \phi^2(x)\rho(x)dx < +\infty$$

Let $\{\phi_n(x)\}$ for a positive integer n, be an orthogonal set of square-integrable functions with a positive weight function $\rho(x)$ on an interval I,

Let $f(x)$ be a given function that can be represented by a uniformly convergent series of the form

$$f(x)=\sum_{n=1}^{\infty} c_n\phi_n(x)$$

Where the coefficients c_n are constants. Now multiplying both sides by $\phi_m(x)\rho(x)$ and integrating term by term over the interval I

$$\int_I f(x)\phi_m(x)\rho(x)dx = \sum_{n=1}^{\infty}\int_I c_n\phi_n(x)\phi_m(x)\rho(x)dx$$

and hence $$\int_I f(x)\phi_n(x)\rho(x)dx = c_n\int_I \phi_n^{\,2}(x)\rho(x)dx$$

Thus $$c_n = \frac{\int_I f\phi_n\rho dx}{\int_I \phi_n^{\,2}\rho dx}$$

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Eigenfunction Expansion*

$$u(x,0) = f(x) \sim \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}\, x$$

Eigenfunction Expansion*

A real-valued functions $\phi(x)$ is said to be *square-integrable* with respect to a weight function $\rho(x) > 0$, if on an interval *I*,

$$\int_I \phi^2(x)\rho(x)dx < +\infty$$

Theorem 7.3.1 If f is represented by a uniformly convergent series

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

on an interval *I*, where $\phi_n(x)$ are square-integrable functions orthogonal with respect to a positive weight function $\rho(x)$

then, c_n are determined by

Thus $$c_n = \frac{\int_I f\phi_n \rho dx}{\int_I \phi_n{}^2 \rho dx}$$

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Eigenfunction Expansion*

$$u(x,0) = f(x) \sim \sum_{n=1}^{\infty} A_n \sin \sqrt{\lambda_n}\, x$$

Eigenfunction Expansion*

When every continuous square-integrable function *f*(*x*) can be expanded into an infinite series

$$f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

The sequence of continuous square-integrable functions $\{\phi_k\}$ orthogonal with respect to the weight function ρ is said to be complete

Theorem 7.5.1

The eigenfunctions of any regular Sturm-Liouville system are complete in the space of functions that are piecewise continuous on the interval [a,b] with respect to the weight function *s(x)*.

Moreover, any piecewise smooth function on [a,b] that satisfies the end condition of the regular Sturm-Liouville system can be expanded in an absolutely and uniformly convergent series

$$f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x)$$

Where c_k are given by

$$c_k = \frac{\int_a^b f \phi_k s\,dx}{\int_a^b \phi_k^{\,2} s\,dx}$$

*Tyn Myint-U, Partial Differential Equations of Mathematical Physics, Second Edition, Elsevier North Holland, 1980, p179~185

Eigenfunction Expansion*

$$u(x,0) = f(x) \sim \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}\,x$$

Eigenfunction Expansion*

If assume that f is a piecewise smooth function on [a,b], then by theorem 7.5.1 we can expand $f(x)$ in terms of eigenfunctions, and formally we write

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}\,x$$

With the coefficient A_n is given by $A_n = \dfrac{\int_0^L f(x)\sin\sqrt{\lambda_n}\,x\,dx}{\int_0^L \sin^2\sqrt{\lambda_n}\,x\,dx}$

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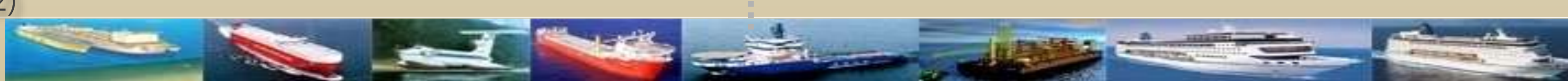
Laplace's Equation

Laplace's Equation

$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)



Laplace's Equation

$$(3)\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

Two-Dimensional Heat Problem

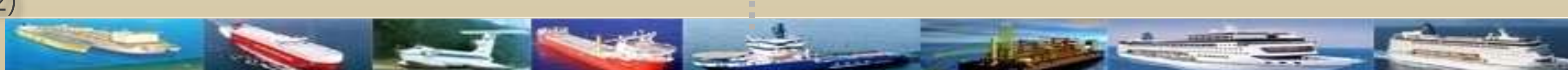
Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$



Laplace's Equation

$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Steady $\Downarrow$ $\dfrac{\partial u}{\partial t}=0$

Laplace's Equation

- **Laplace's equation**

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Two-Dimensional Heat Problem

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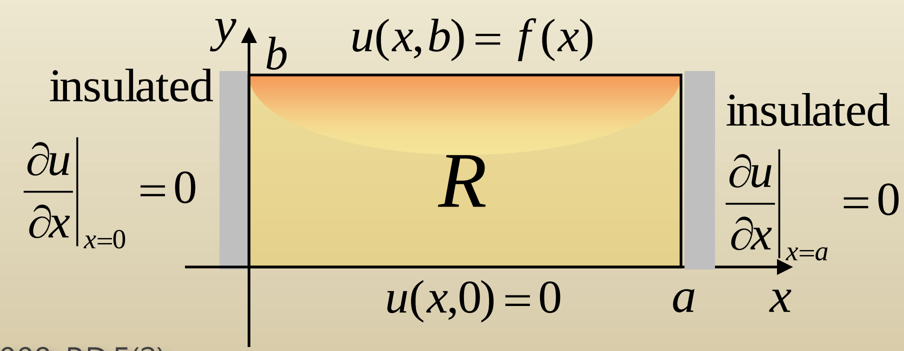

Laplace's Equation

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(<u>Steady</u> Two-Dimensional Heat Problem)

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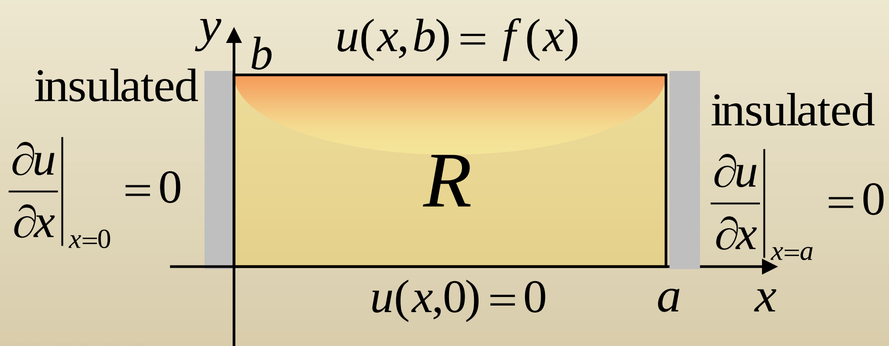
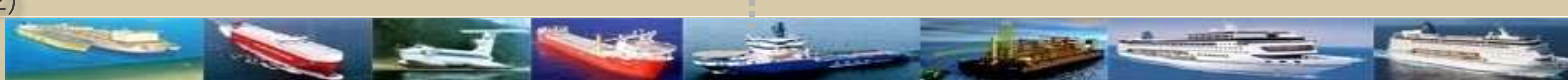


$$u(x,y)=X(x)Y(y)$$



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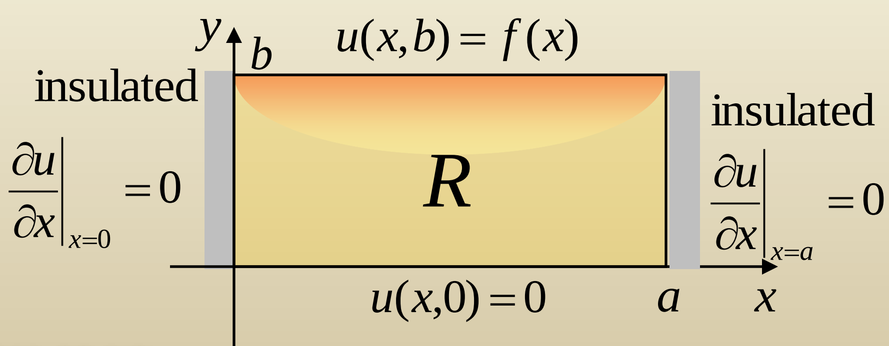

$$u(x,y)=X(x)Y(y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 X}{\partial x^2}Y(y)+X(x)\frac{\partial^2 Y}{\partial y^2}=0$$

Laplace's Equation

$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

- Laplace's equation

(Steady Two-Dimensional Heat Problem)

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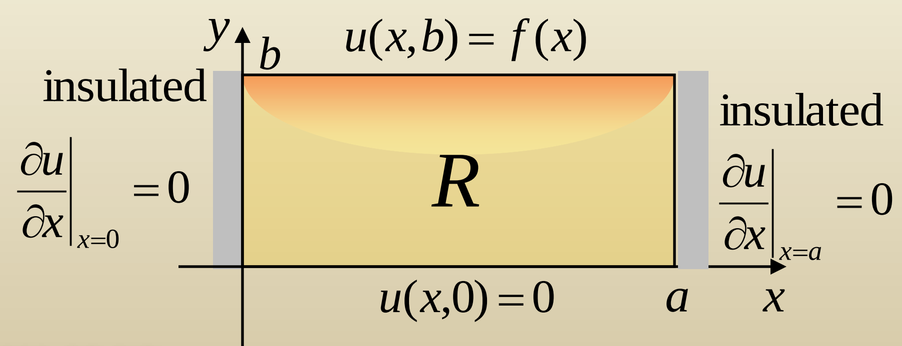

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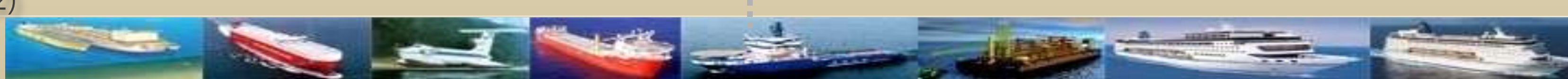
separating variable,



Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

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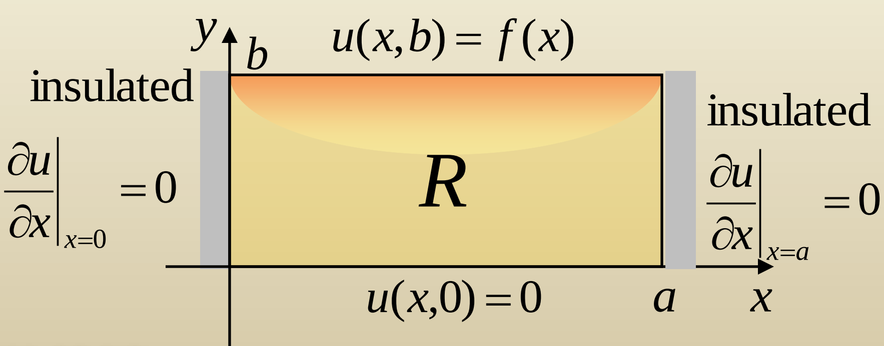

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separating variable,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-\lambda<0$$

Laplace's Equation

$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

Two-Dimensional Heat Problem

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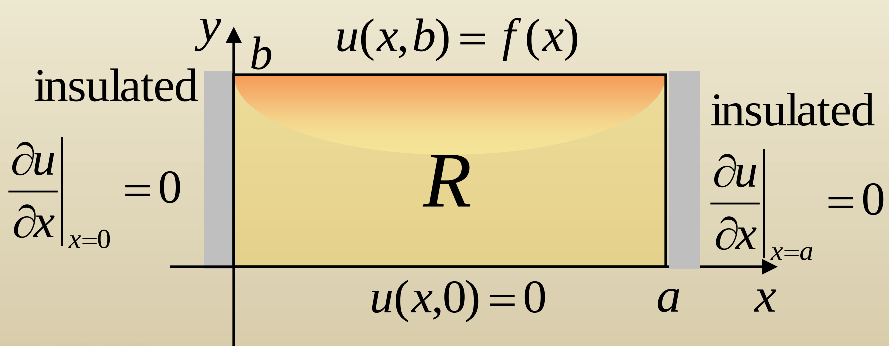

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Two ODEs

Laplace's Equation

$$(3)\quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

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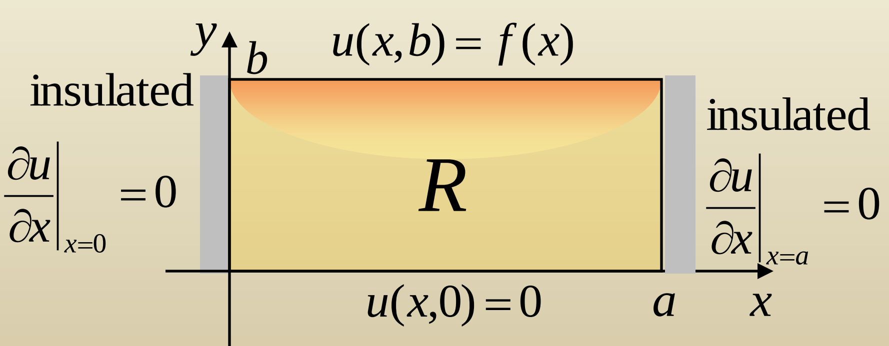

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separating variable,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-\lambda\ <0$$

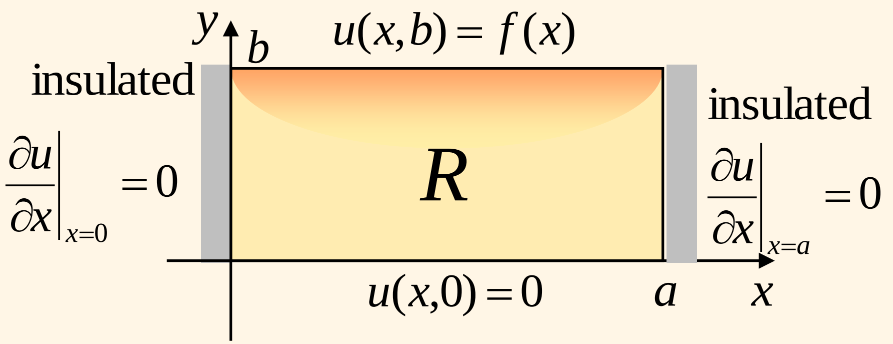
Two ODEs

$$X''+\lambda X=0$$

$$Y''-\lambda Y=0$$

$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$

Laplace's Equation

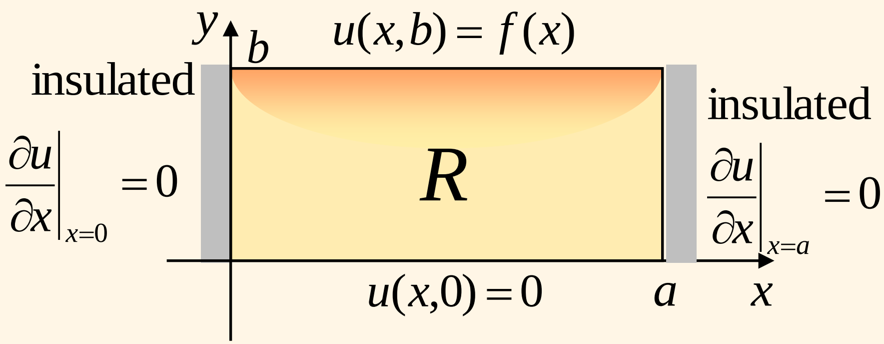


$$X'' + \lambda X = 0$$

c_2

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

Laplace's Equation



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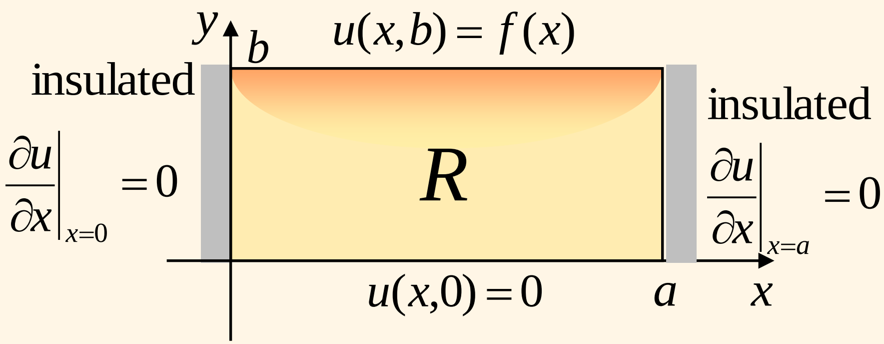


$$X'' + \lambda X = 0$$

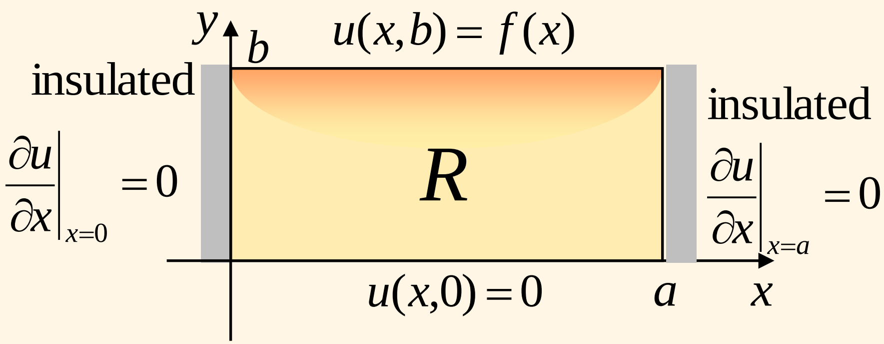
1) $\lambda = 0$

$$X'' = 0 \;\Rightarrow X(x) = c_1 x + c_2$$

c_2

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

Laplace's Equation



$$X'' + \lambda X = 0$$

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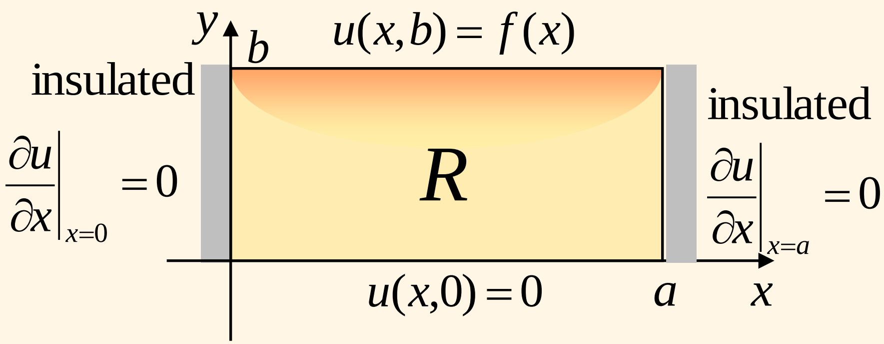
- **boundary condition**

c_2

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

Laplace's Equation



$$X'' + \lambda X = 0$$

1) λ = 0

$$X'' = 0 \;\Rightarrow X(x) = c_1 x + c_2$$

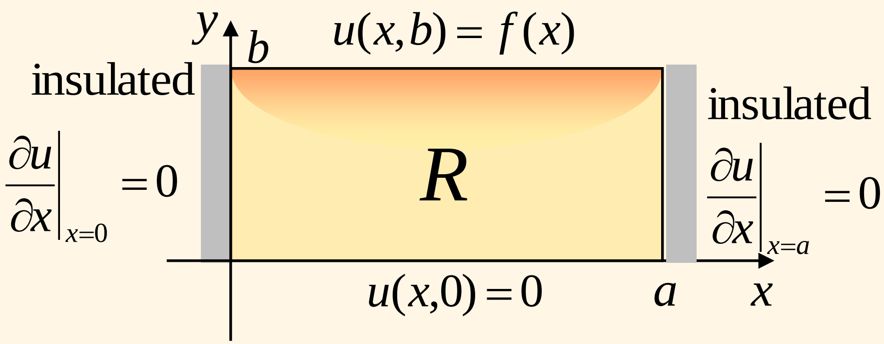
- **boundary condition**

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = X'(0)Y(y) = 0 \;, \left.\frac{\partial u}{\partial x}\right|_{x=a} = X'(a)Y(y) = 0$$

$$c_2$$

$$X''+\lambda X=0,\quad Y''-\lambda Y=0$$

Laplace’s Equation



$$X''+\lambda X=0$$

1) λ = 0

$$X''=0 \;\Rightarrow X(x)=c_1x+c_2$$

- **boundary condition**

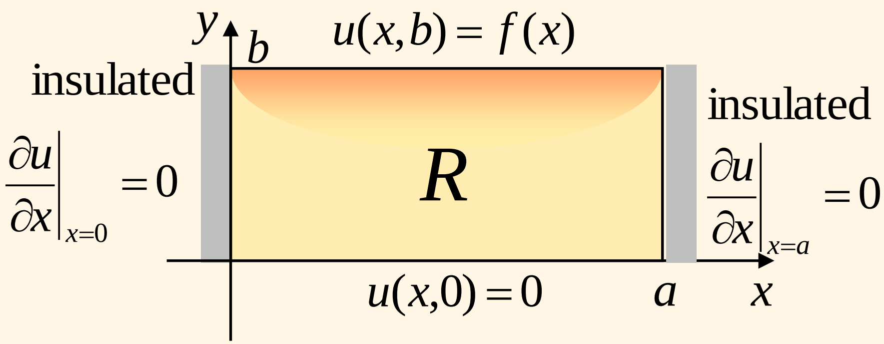
$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=X'(0)Y(y)=0\ ,\ \left.\frac{\partial u}{\partial x}\right|_{x=a}=X'(a)Y(y)=0$$

$$\therefore X'(0)=X'(a)=0$$

c_2

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

Laplace's Equation



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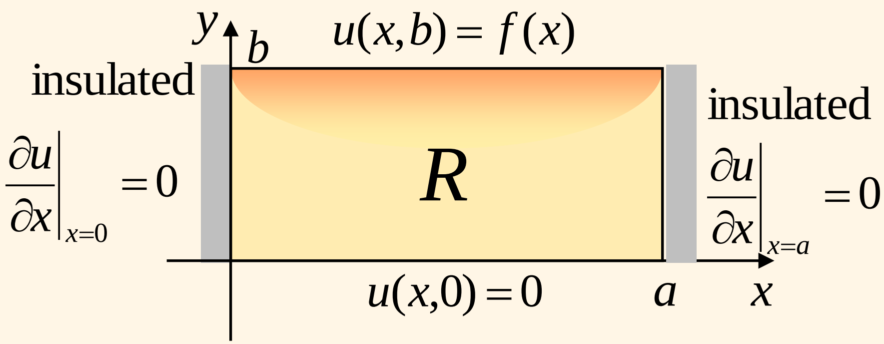
$$X'(0) = c_1 = 0$$

$$c_2$$

Laplace's Equation



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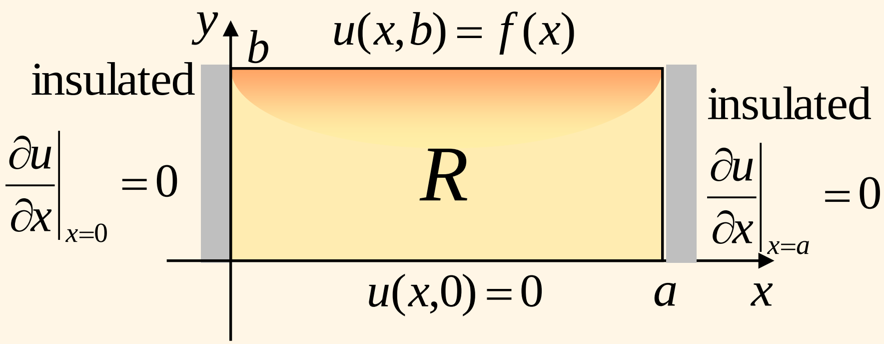
$$X'(0)=c_1=0$$

For any c_2 **,the second b/c** $X'(a)=0$ **is satisfied**

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

Laplace's Equation



$$X'' + \lambda X = 0$$

1) $\lambda = 0$

$$X'' = 0 \Rightarrow X(x) = c_1 x + c_2$$

- **boundary condition**

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = X'(0)Y(y) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = X'(a)Y(y) = 0$$

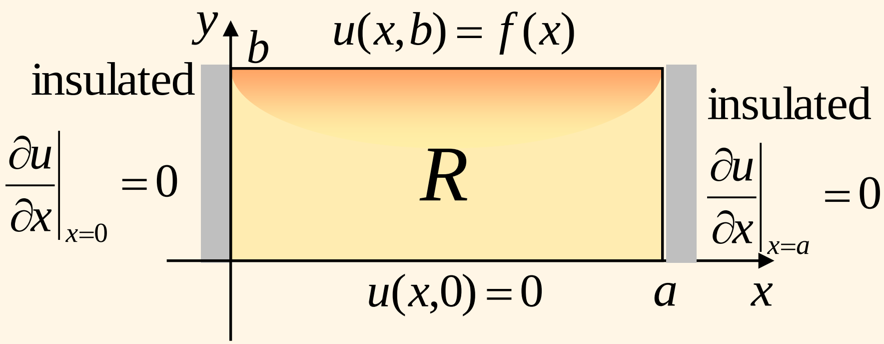
$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = c_1 = 0$$

For any c_2 **,the second b/c** $X'(a) = 0$ **is satisfied**

So for $c_2 \neq 0$, $X(x) = c_2$ **:** **nontrivial solution!**

Laplace's Equation



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1) λ = 0

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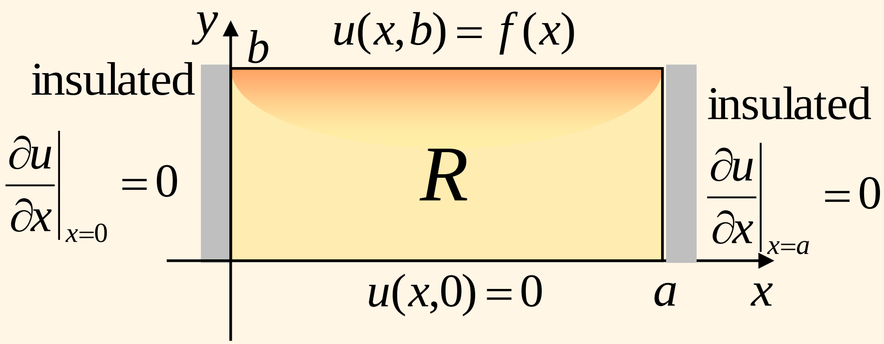
For any c_2 ,the second b/c $X'(a) = 0$ is satisfied

So for $c_2 \neq 0$, $X(x) = c_2$: **nontrivial solution!**

2) $\lambda = -\alpha^2 < 0$

Laplace's Equation



$$X'' + \lambda X = 0$$

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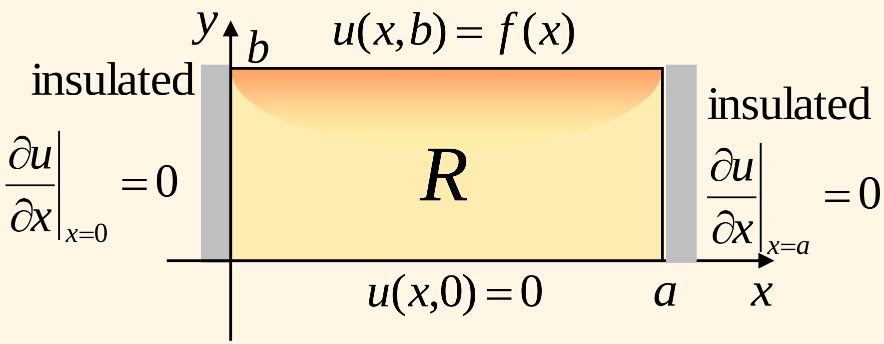
So for $c_2 \neq 0$, $X(x) = c_2$: nontrivial solution!

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

Laplace's Equation



$$X'' + \lambda X = 0$$

1) $\lambda = 0$

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- **boundary condition**

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = X'(0)Y(y) = 0\ , \left.\frac{\partial u}{\partial x}\right|_{x=a} = X'(a)Y(y) = 0$$

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$$X'(0) = c_1 = 0$$

For any c_2 ,the second b/c $X'(a) = 0$ is satisfied

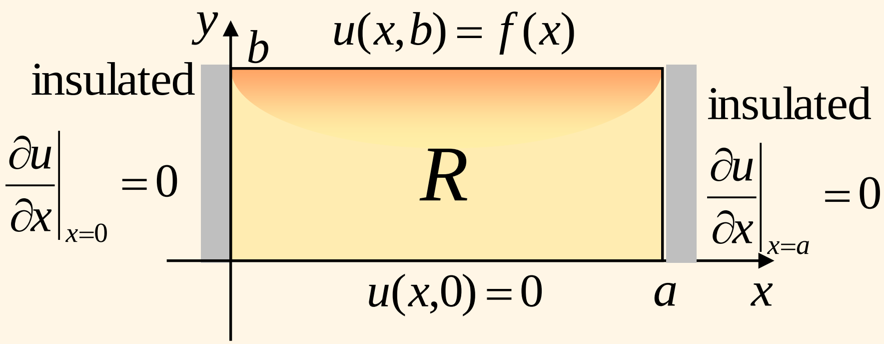
So for $c_2 \neq 0$, $X(x) = c_2$: nontrivial solution!

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = c_3 e^{\alpha x} + c_4 e^{-\alpha x}$$

Laplace's Equation



$$X'' + \lambda X = 0$$

1) λ = 0

$$X'' = 0 \Rightarrow X(x) = c_1 x + c_2$$

- boundary condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = X'(0)Y(y) = 0 \;, \left.\frac{\partial u}{\partial x}\right|_{x=a} = X'(a)Y(y) = 0$$

$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = c_1 = 0$$

For any c_2 ,the second b/c $X'(a) = 0$ is satisfied

So for $c_2 \neq 0$, $X(x) = c_2$: nontrivial solution!

2) λ = -α² < 0

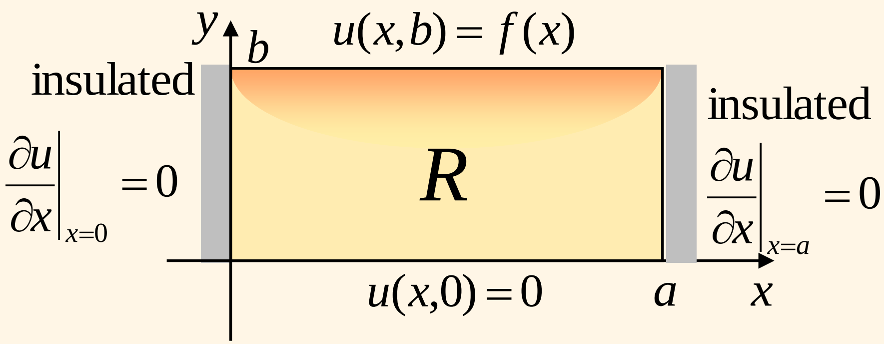
$$X'' - \alpha^2 X = 0$$

$$X(x) = c_3 e^{\alpha x} + c_4 e^{-\alpha x}$$

- boundary condition

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

Laplace's Equation



$$X'' + \lambda X = 0$$

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$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = c_1 = 0$$

For any c_2 **,the second b/c** $X'(a) = 0$ **is satisfied**

So for $c_2 \neq 0$ **,** $X(x) = c_2$ **:** **nontrivial solution!**

2) $\lambda = -\alpha^2 < 0$

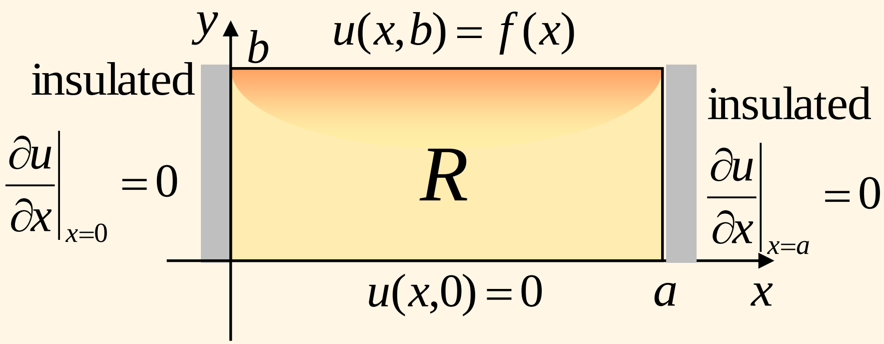
$$X'' - \alpha^2 X = 0$$

$$X(x) = c_3 e^{\alpha x} + c_4 e^{-\alpha x}$$

- **boundary condition**

$$\therefore X'(0) = X'(a) = 0$$

Laplace's Equation



$$X''+\lambda X=0$$

1) λ = 0

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- boundary condition

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$$\therefore X'(0)=X'(a)=0$$

$$X'(0)=c_1=0$$

For any C_2 ,the second b/c $X'(a)=0$ is satisfied

So for $c_2\neq 0$, $X(x)=c_2$: nontrivial solution!

2) λ = -α² < 0

$$X''-\alpha^2X=0$$

$$X(x)=c_3e^{\alpha x}+c_4e^{-\alpha x}$$

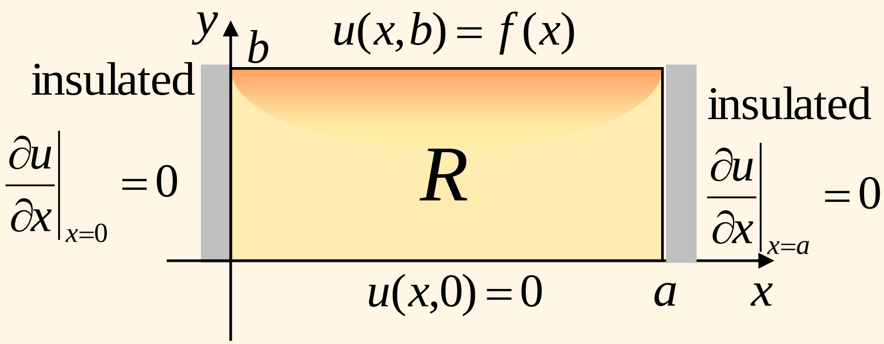
- boundary condition

$$\therefore X'(0)=X'(a)=0$$

$$X'(0)=(c_3-c_4)\alpha=0\ \therefore c_3=c_4$$

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

Laplace's Equation



$$X'' + \lambda X = 0$$

1) $\lambda = 0$

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- **boundary condition**

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$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = c_1 = 0$$

For any c_2 ,the second b/c $X'(a) = 0$ is satisfied

So for $c_2 \neq 0$, $X(x) = c_2$: nontrivial solution!

2) $\lambda = -\alpha^2 < 0$

$$X'' - \alpha^2 X = 0$$

$$X(x) = c_3 e^{\alpha x} + c_4 e^{-\alpha x}$$

- **boundary condition**

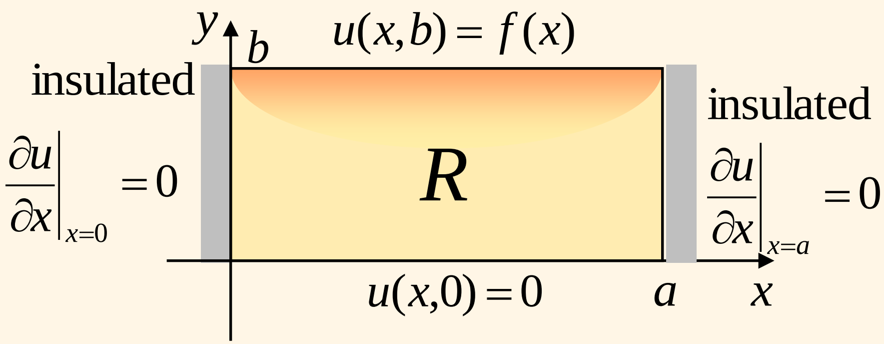
$$\therefore X'(0) = X'(a) = 0$$

$$X'(0) = (c_3 - c_4)\alpha = 0 \;\; \therefore c_3 = c_4$$

$$X'(a) = c_3 \alpha (e^{\alpha x} - e^{-\alpha x}) = 0$$

$$X''+\lambda X=0,\quad Y''-\lambda Y=0$$

Laplace's Equation



$$X''+\lambda X=0$$

1) λ = 0

$$X''=0 \;\Rightarrow X(x)=c_1x+c_2$$

- **boundary condition**

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$$\therefore X'(0)=X'(a)=0$$

$$X'(0)=c_1=0$$

For any c_2 ,the second b/c $X'(a)=0$ is satisfied

So for $c_2\neq 0$, $X(x)=c_2$: nontrivial solution!

2) λ = -α² < 0

$$X''-\alpha^2X=0$$

$$X(x)=c_3e^{\alpha x}+c_4e^{-\alpha x}$$

- **boundary condition**

$$\therefore X'(0)=X'(a)=0$$

$$X'(0)=(c_3-c_4)\alpha=0\;\;\therefore c_3=c_4$$

$$X'(a)=c_3\alpha(e^{\alpha x}-e^{-\alpha x})=0$$

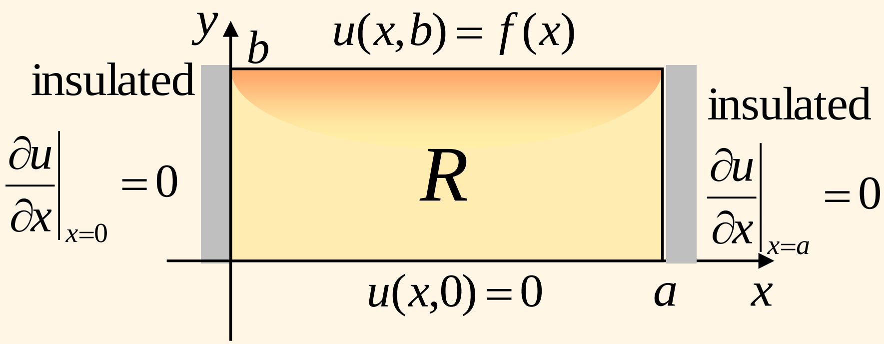
if $c_3=0\rightarrow c_4=0 \;\Rightarrow\; X(x)=0$

trivial solution → no interest

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

Laplace's Equation

When $\lambda = 0$, $X(x) = c_2$: **nontrivial solution!**



$$X'' + \lambda X = 0$$

Laplace's Equation

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

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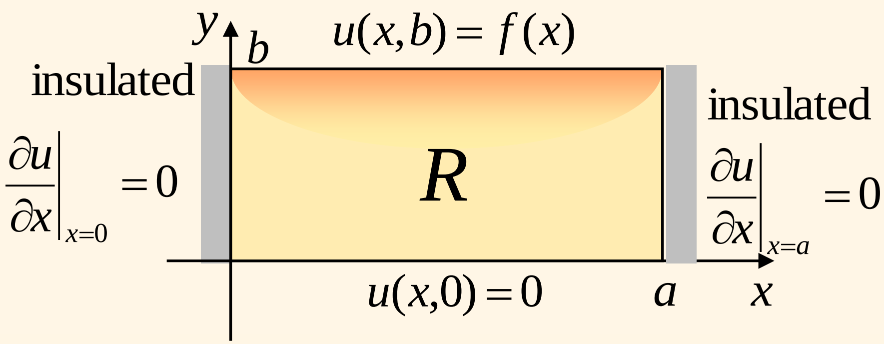

$$X'' + \lambda X = 0$$

3) $\lambda = \alpha^2 > 0$

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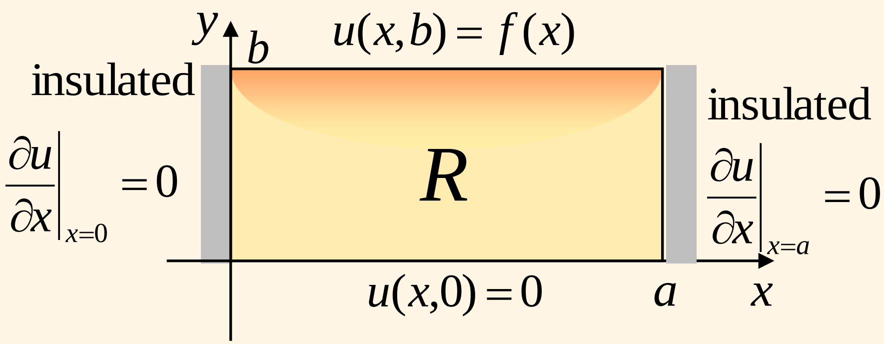

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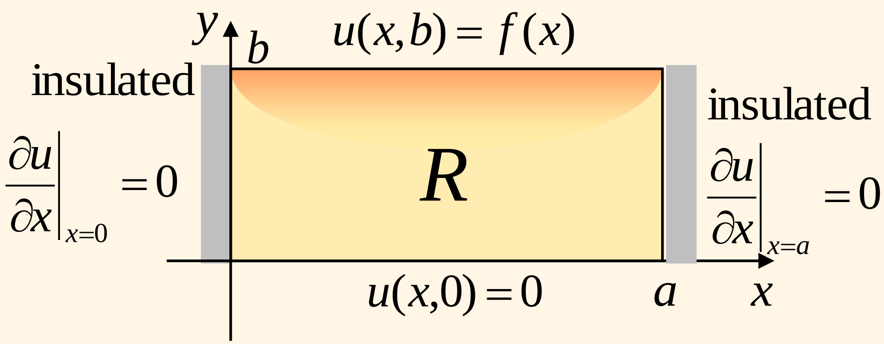

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$$X(x) = c_5 \cos \alpha x + c_6 \sin \alpha x$$

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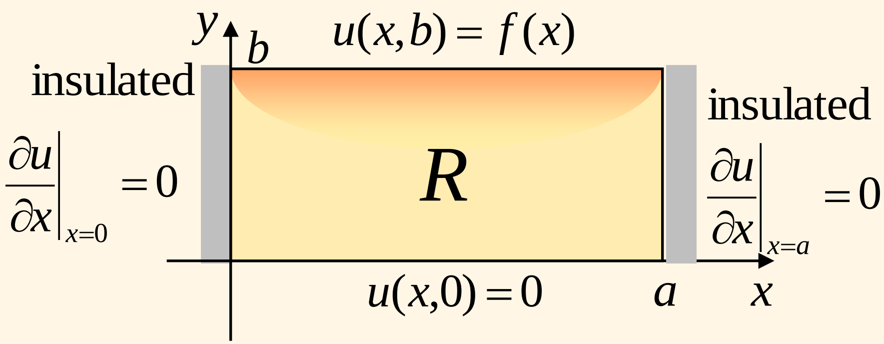

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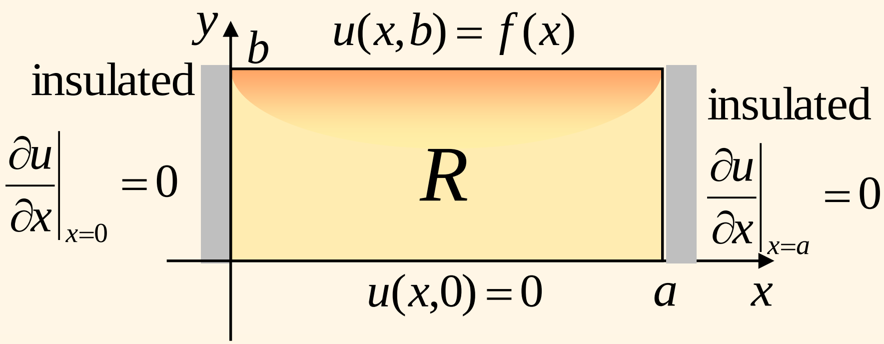
When $\lambda = 0$ **,** $X(x) = c_2$ **: nontrivial solution!**



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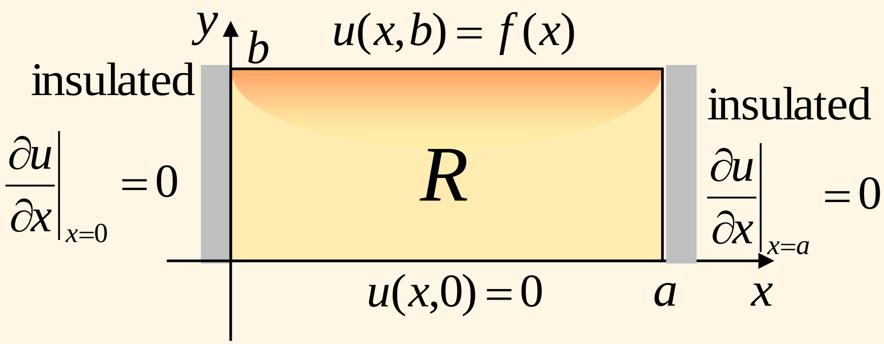

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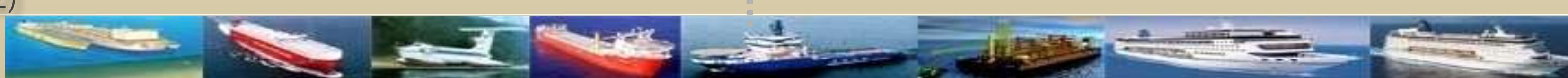

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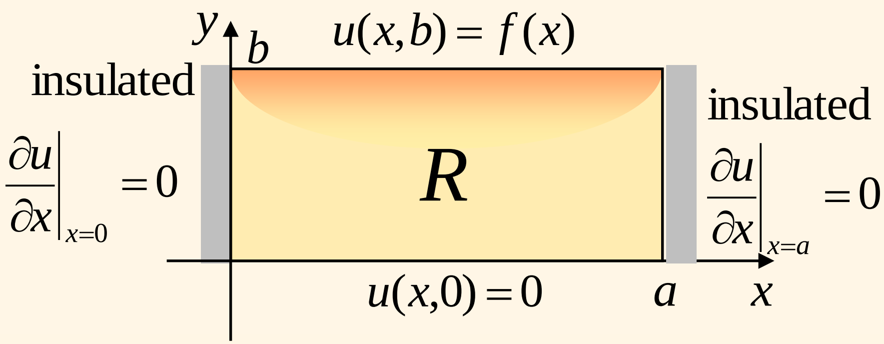

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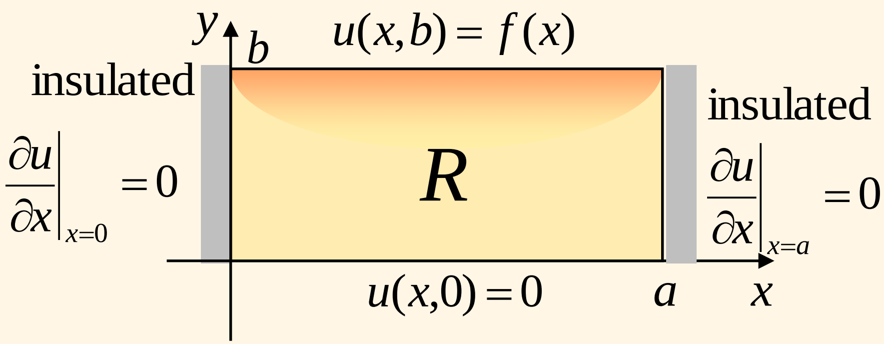

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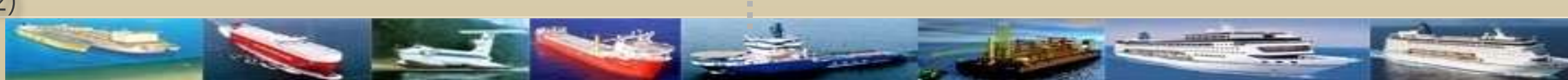

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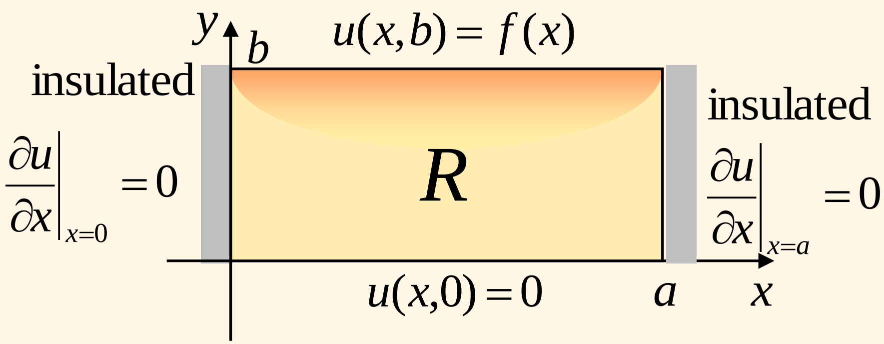

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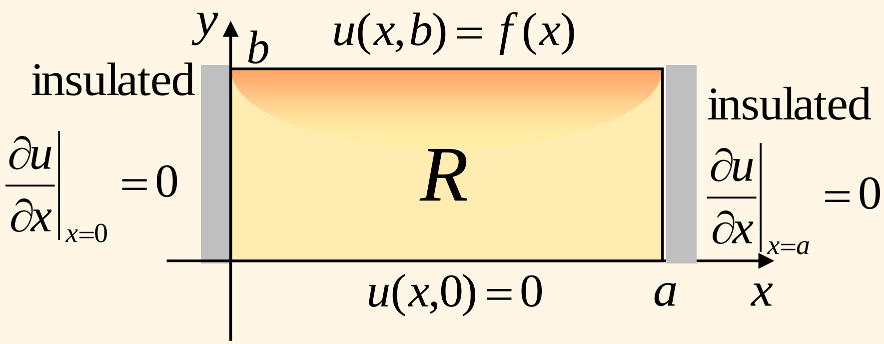

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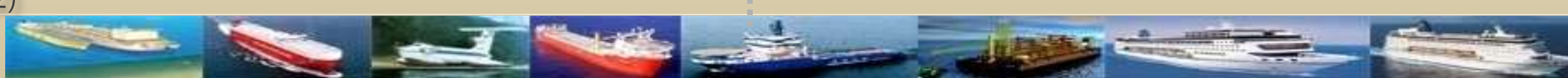

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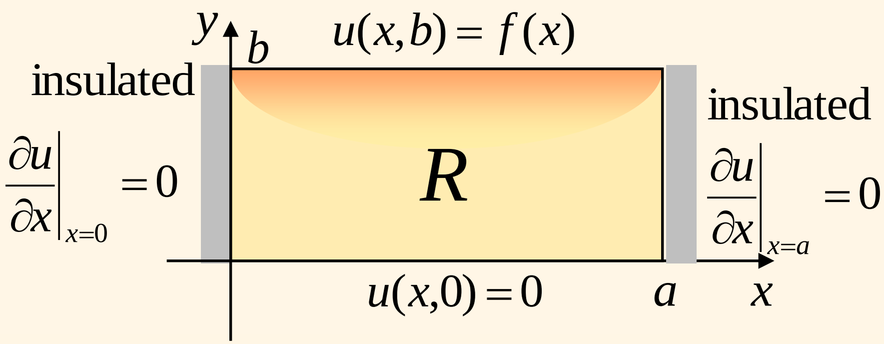

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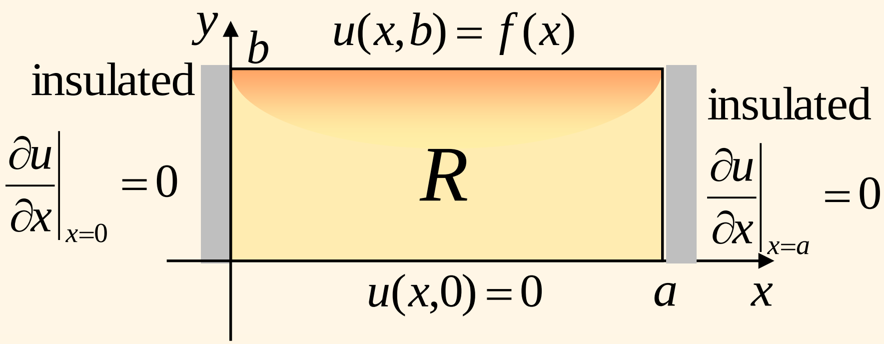

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$$a\sqrt{\lambda}=n\pi,$$

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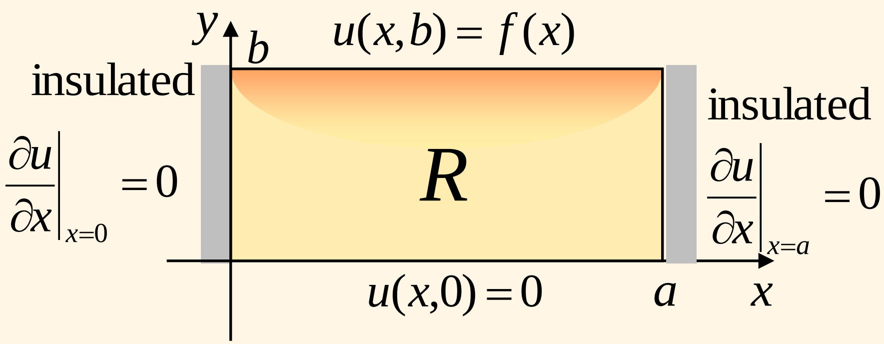

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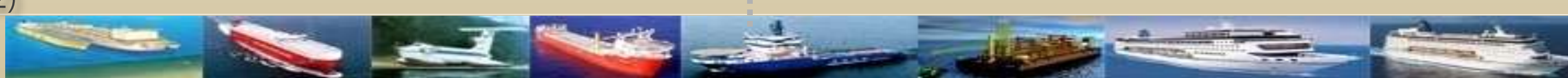

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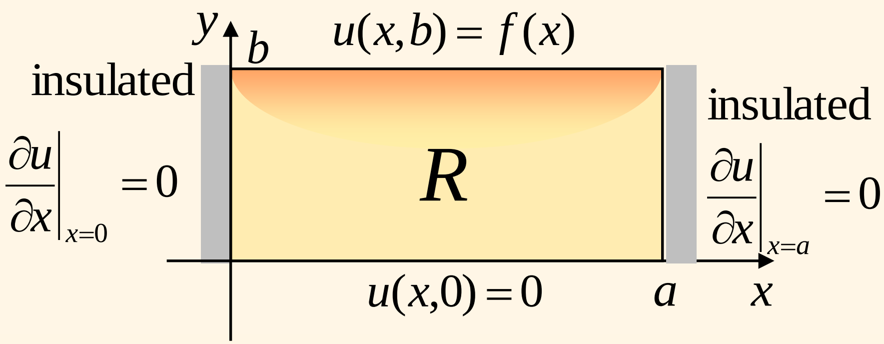

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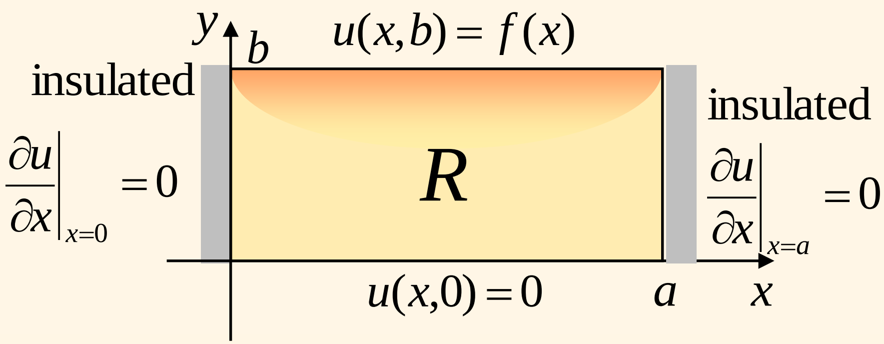

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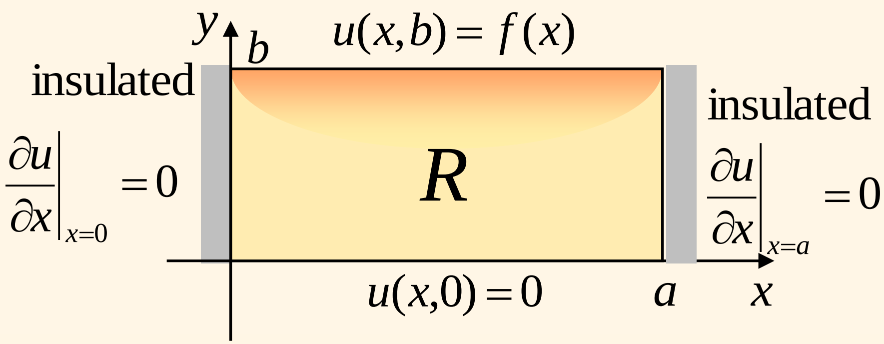
When $\lambda = 0$, $X(x) = c_2$ **: nontrivial solution!**



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$$\therefore X(x) = X_n(x) = c_5 \cos \frac{n\pi}{a} x, \quad (n = 1, 2, \cdots)$$

by corresponding $\lambda_0 = 0$ with $n = 0$

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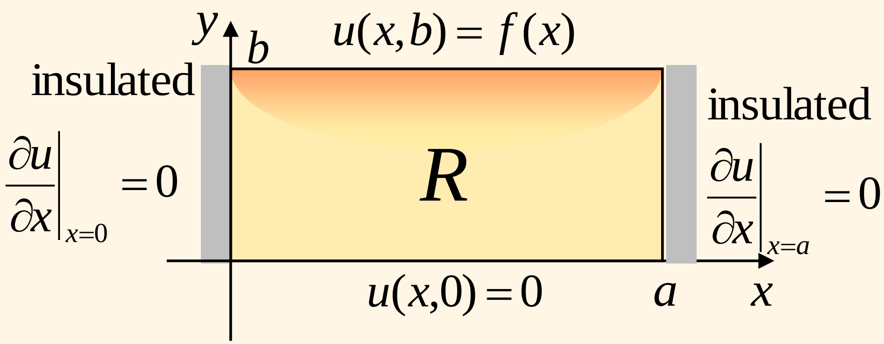
When $\lambda=0$, $X(x)=c_2$: **nontrivial solution!**



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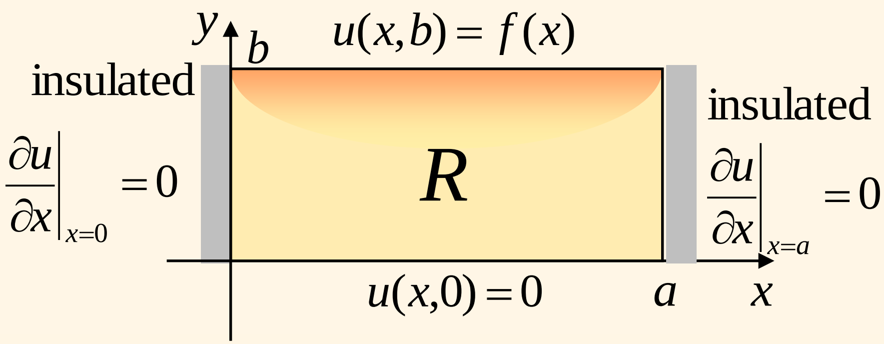
by corresponding $\lambda_0=0$ with $n=0$

$$\begin{cases} X(x)=c_2,\ n=0 \\ X_n(x)=c_5\cos\dfrac{n\pi}{a}x,\quad (n=1,2,\cdots)\end{cases}$$

Laplace's Equation

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots) \quad \begin{cases} X(x) = c_2, \ n = 0 \\ X_n(x) = \cos\frac{n\pi}{a}x, \quad (n = 1, 2, \cdots) \end{cases}$$



$$Y'' - \lambda Y = 0$$

First, for $\lambda_0 = 0 \ (n = 0)$

$$Y'' = 0 \Rightarrow Y(x) = c_7 y + c_8$$

- **boundary condition**

$$u(x,0) = X(x)Y(0) = 0$$

$$\therefore Y(0) = 0$$

$$Y(0) = c_8 = 0$$

$$\therefore Y(x) = c_7 y$$: nontrivial solution!

Second, for $\lambda_n = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$

$$Y'' - \frac{n^2\pi^2}{a^2}Y = 0$$

- **general solution**

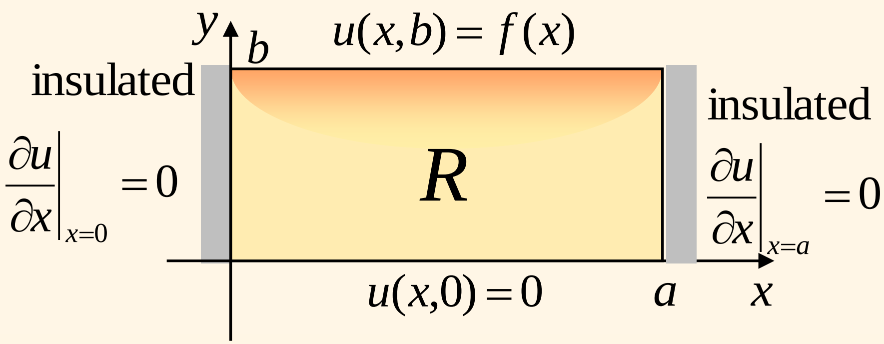
$$Y(y) = Y_n(y) = c_9 e^{n\pi y/a} + c_{10} e^{-n\pi y/a}$$

$$Y(0) = c_9 + c_{10} = 0 \qquad \therefore c_{10} = -c_9$$

$$\therefore Y_n(y) = c_9(e^{n\pi y/a} - e^{-n\pi y/a}) = 2c_9 \sinh\frac{n\pi y}{a}$$

$$\begin{cases} Y(y) = c_7 y, \ n = 0 \\ Y_n(y) = c_9^* \sinh\frac{n\pi y}{a}, \quad (n = 1, 2, \cdots), c_9^* = 2c_9 \end{cases}$$

Laplace's Equation



$$\boxed{X''+\lambda X=0, \quad Y''-\lambda Y=0}$$

$$\begin{cases} X(x)=c_2,\ n=0 \\ X_n(x)=c_5\cos\dfrac{n\pi}{a}x, \quad (n=1,2,\cdots) \end{cases} \qquad \begin{cases} Y(y)=c_7 y,\ n=0 \\ Y_n(y)=c_9^*\sinh\dfrac{n\pi y}{a}, \quad (n=1,2,\cdots) \end{cases}$$

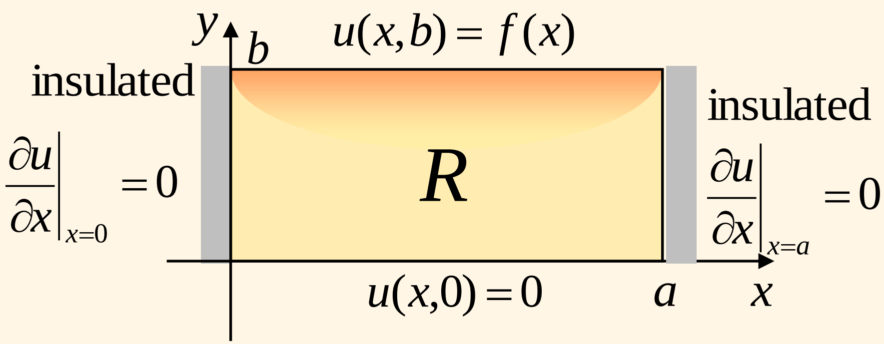
$$\therefore u_n(x,y)=X_n(x)Y_n(y)=\begin{cases} A_0^* y, \quad (n=0) \\ A_n^*\sinh\dfrac{n\pi y}{a}\cos\dfrac{n\pi x}{a}, \quad (n=1,2,\ldots) \end{cases}$$

$$where,\ A_0^*=c_2c_7,\ A_n^*=c_5c_9^*$$

Laplace's Equation

Fourier cosine series where, $(-p, p)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x \qquad a_0 = \frac{2}{p}\int_0^p f(x)dx, \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$



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Fourier cosine series $\quad$ *where*, $(-p, p)$

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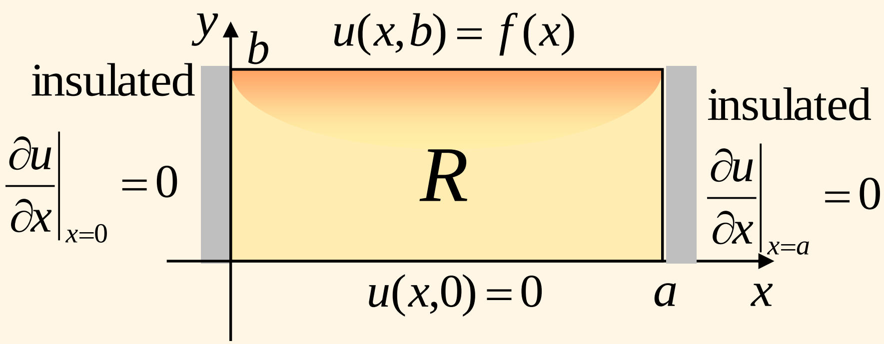

$$u_n(x, y) = X_n(x)Y_n(y)$$

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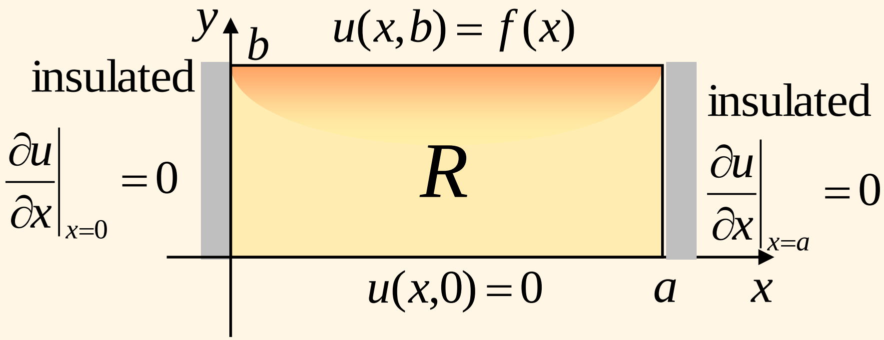

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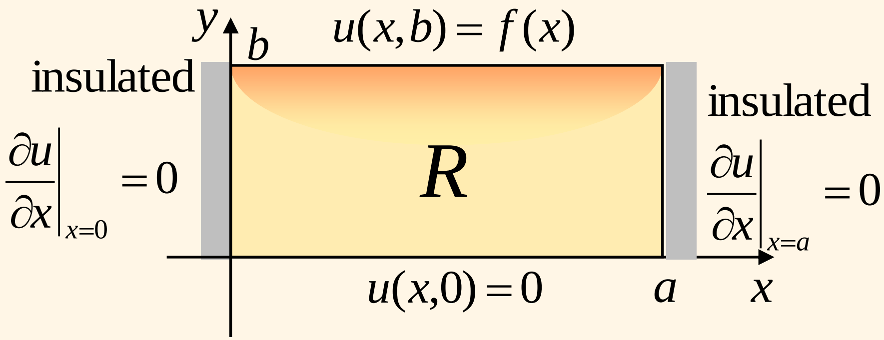
By Superposition

Laplace's Equation

Fourier cosine series *where,* $(-p,p)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{p}x \quad a_0=\frac{2}{p}\int_0^p f(x)dx,\quad a_n=\frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$



$$u_n(x,y)=X_n(x)Y_n(y)$$

$$=\begin{cases} A_0^* y, & (n=0) \\ A_n^*\sinh\dfrac{n\pi y}{a}\cos\dfrac{n\pi x}{a}, & (n=1,2,\ldots) \end{cases}$$

By Superposition

$$u(x,t)=A_0^* y+\sum_{n=1}^{\infty}u_n(x,y)$$

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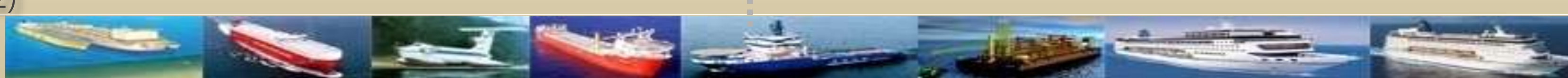
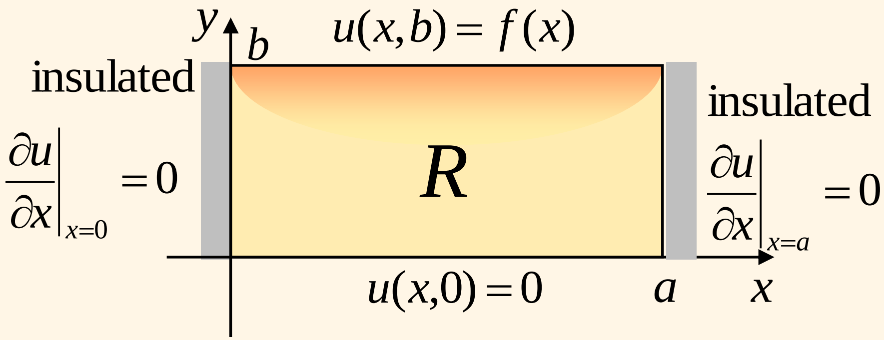


Laplace's Equation

Fourier cosine series *where*, $(-p, p)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x \quad a_0 = \frac{2}{p}\int_0^p f(x)dx, \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$



$$u_n(x, y) = X_n(x)Y_n(y)$$

$$= \begin{cases} A_0^* y, & (n = 0) \\ A_n^* \sinh\dfrac{n\pi y}{a}\cos\dfrac{n\pi x}{a}, & (n = 1, 2, \ldots) \end{cases}$$

By Superposition

$$u(x,t) = A_0^* y + \sum_{n=1}^{\infty} u_n(x, y)$$

$$= A_0^* y + \sum_{n=1}^{\infty} A_n^* \sinh\frac{n\pi y}{a}\cos\frac{n\pi x}{a}$$

- **boundary condition**

Laplace's Equation

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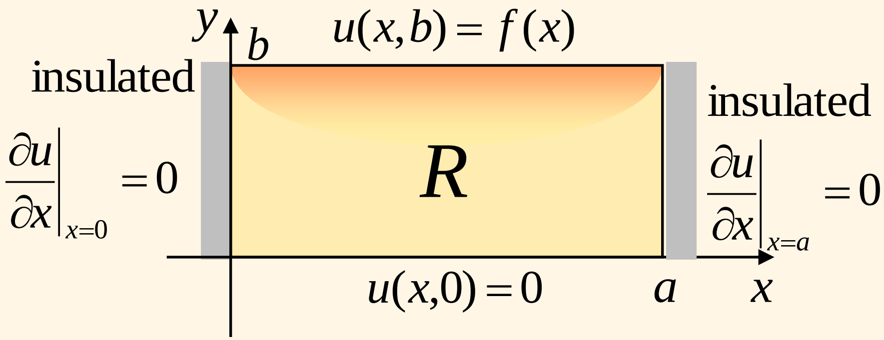

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$$u(x,b)=X(x)Y(b)=f(x)$$

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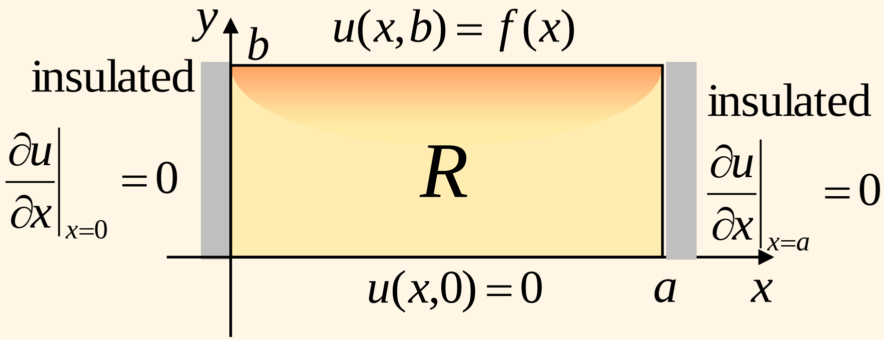

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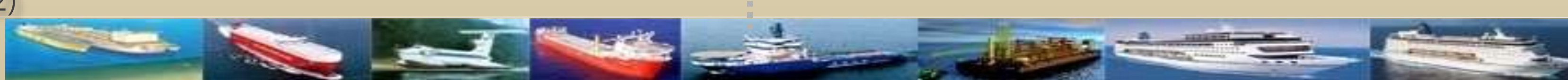

Laplace's Equation

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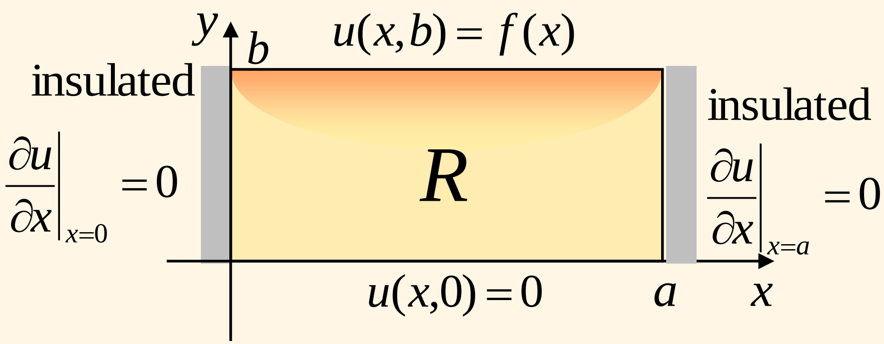

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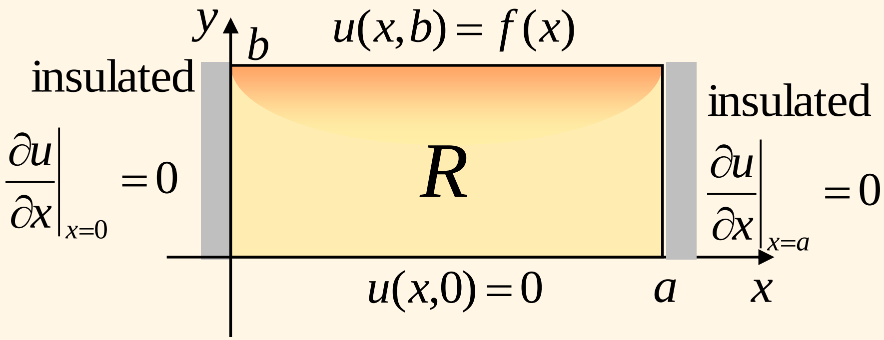

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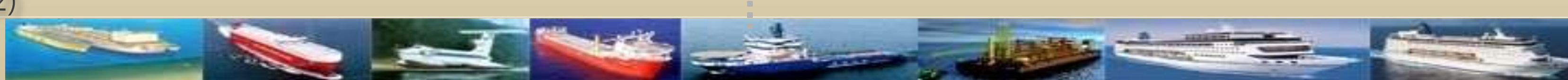
Fourier cosine series of $f(x)$



Laplace's Equation

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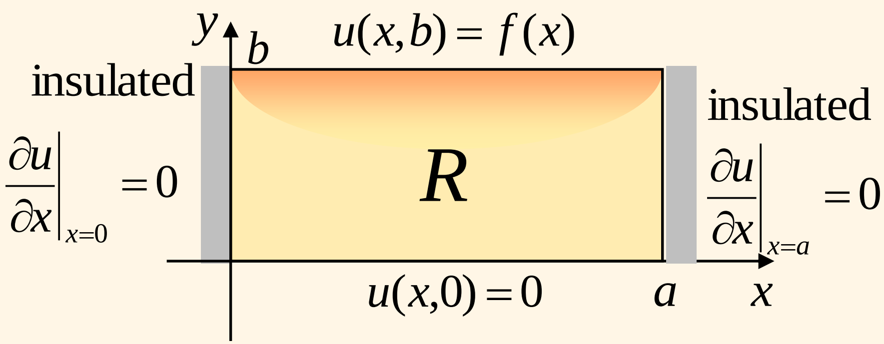

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Fourier cosine series of $f(x)$

$$A_0^*b=\frac{1}{2}a_0\quad\left(a_n=A_n^*\sinh\frac{n\pi b}{a}\right)$$

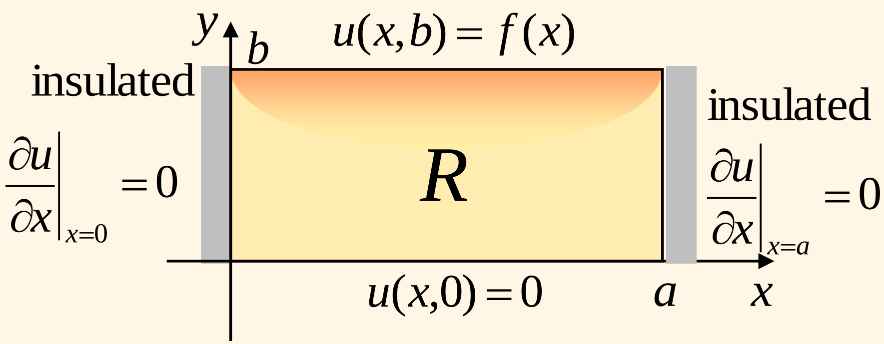
$$2A_0^*b=\frac{2}{a}\int_0^a f(x)dx$$

$$A_0^*=\frac{1}{ab}\int_0^a f(x)dx$$

Laplace's Equation

Fourier cosine series *where,* $(-p, p)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{p}x \qquad a_0=\frac{2}{p}\int_0^p f(x)dx,\quad a_n=\frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$



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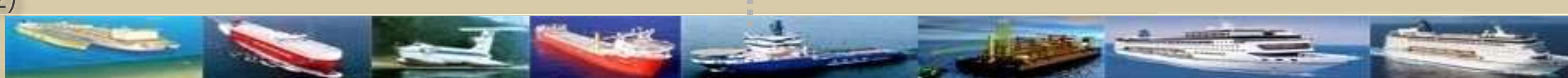
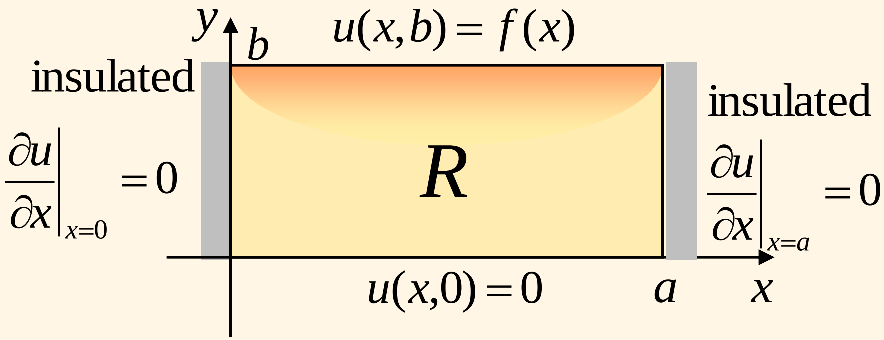


Laplace's Equation

Fourier cosine series *where,* $(-p, p)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{p}x \qquad a_0=\frac{2}{p}\int_0^p f(x)dx,\quad a_n=\frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$



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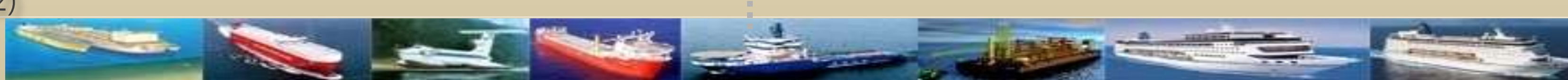
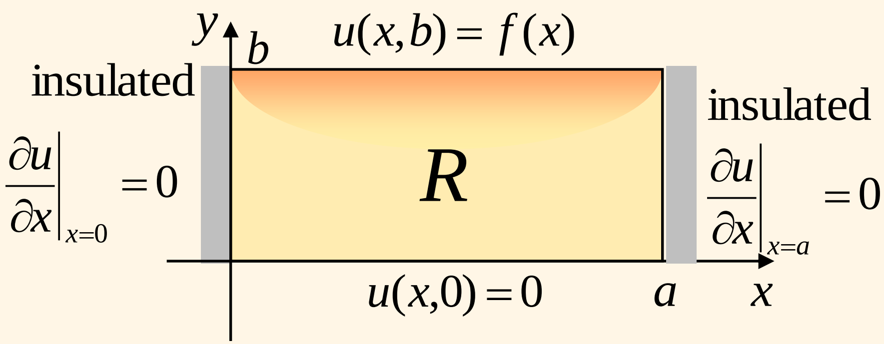
$$A_0^* b=\frac{1}{ab}\int_0^a f(x)dx$$



Laplace's Equation

Fourier cosine series where, $(-p, p)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi}{p}x \qquad a_0=\frac{2}{p}\int_0^p f(x)dx,\quad a_n=\frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}xdx$$



$$u_n(x,y)=X_n(x)Y_n(y)$$

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Laplace's Equation

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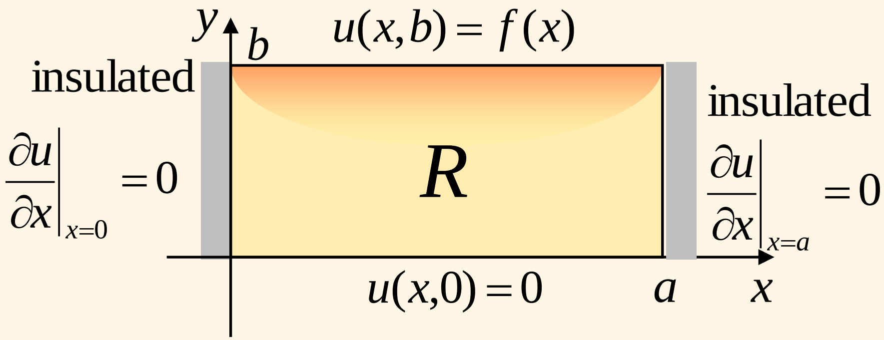

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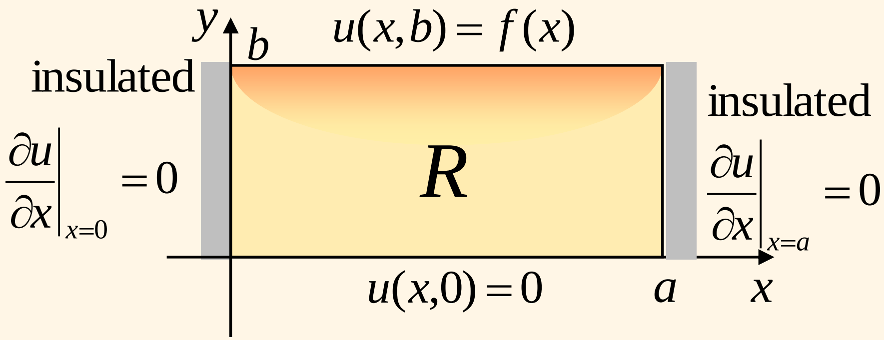
$$\boxed{\therefore A_n^*=\frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\cos\frac{n\pi x}{a}dx}$$

Laplace's Equation



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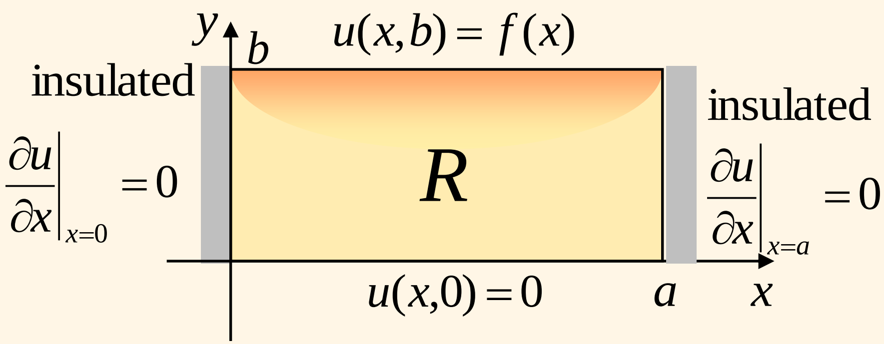
Laplace's Equation



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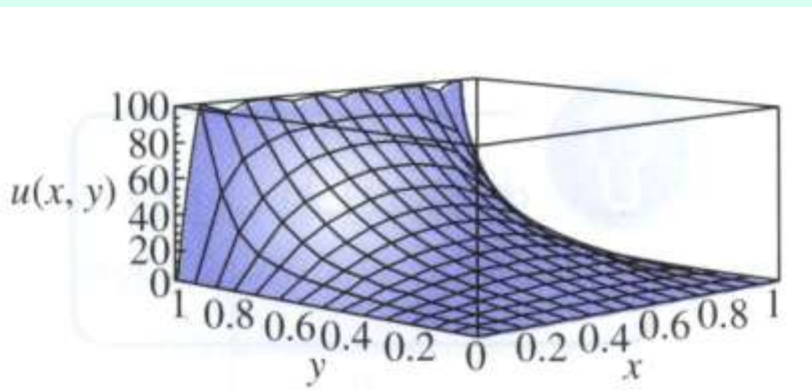
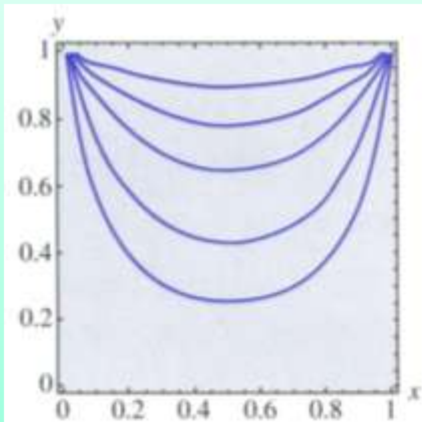
Ex) f(x)=100, a=1, b=1



(a) 곡면



(b) 등온선

Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

Laplace's Equation

- ## Laplace's equation

(Steady Two-Dimensional Heat Problem)

Two-Dimensional Heat Problem

Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Steady $\Downarrow$ $\dfrac{\partial u}{\partial t} = 0$

Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

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$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Steady $\Downarrow$ $\quad \frac{\partial u}{\partial t} = 0$

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Laplace's Equation

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(Steady Two-Dimensional Heat Problem)

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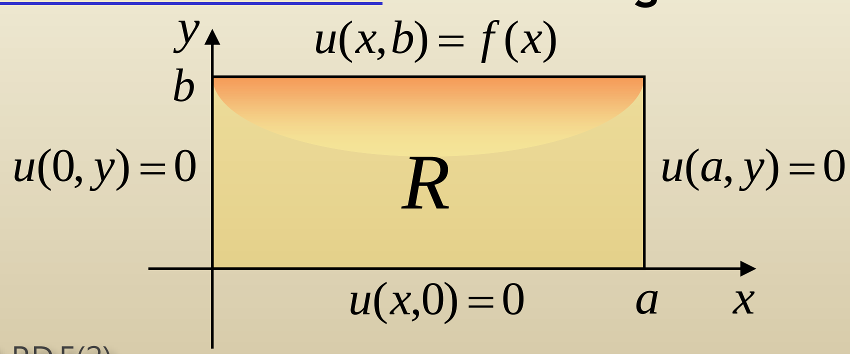
Dirichlet Problem in a Rectangle *R*



Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

$$u(x,y) = F(x)G(y)$$

Two-Dimensional Heat Problem

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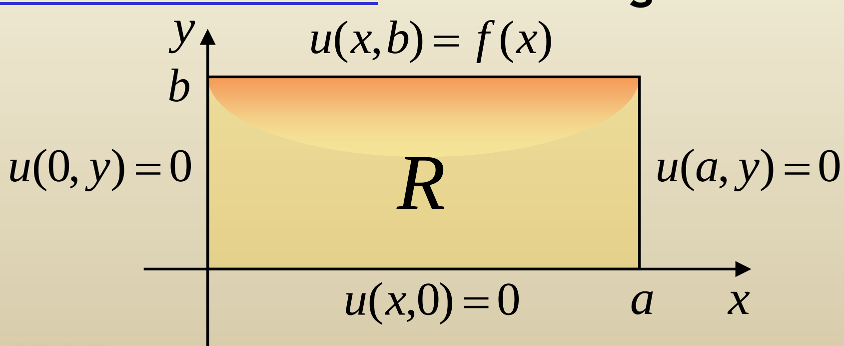


Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

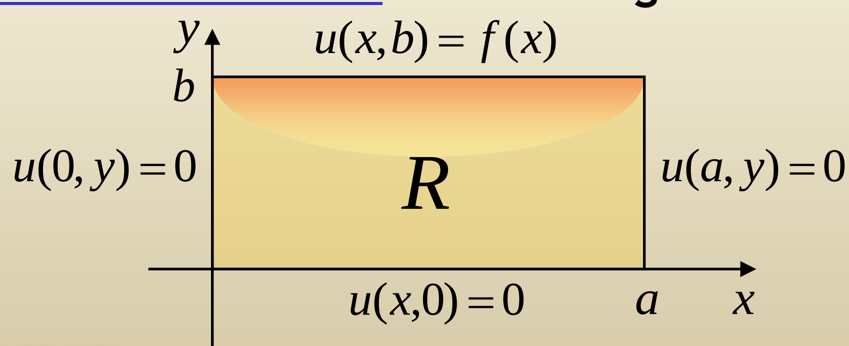
Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Steady $\quad \frac{\partial u}{\partial t}=0$

$$\boxed{\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0}$$

Dirichlet Problem in a Rectangle *R*



$$u(x,y)=F(x)G(y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 F}{\partial x^2}G(y)+F(x)\frac{\partial^2 G}{\partial y^2}=0$$

Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

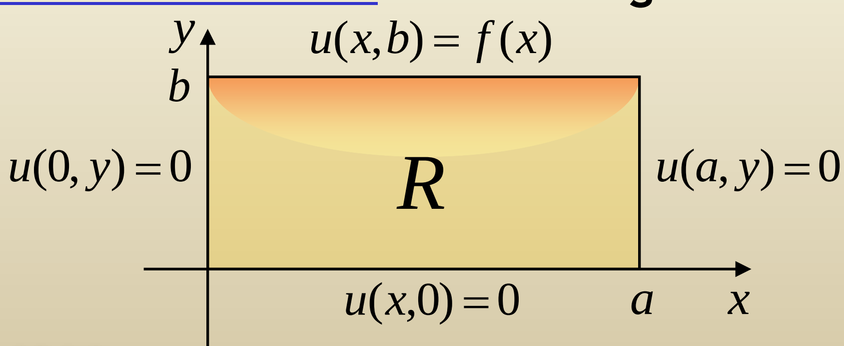
Two-Dimensional Heat Problem

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Steady $\Downarrow$ $\frac{\partial u}{\partial t} = 0$

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Dirichlet Problem in a Rectangle *R*



$$u(x,y) = F(x)G(y)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 F}{\partial x^2}G(y) + F(x)\frac{\partial^2 G}{\partial y^2} = 0$$

separating variable,

Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

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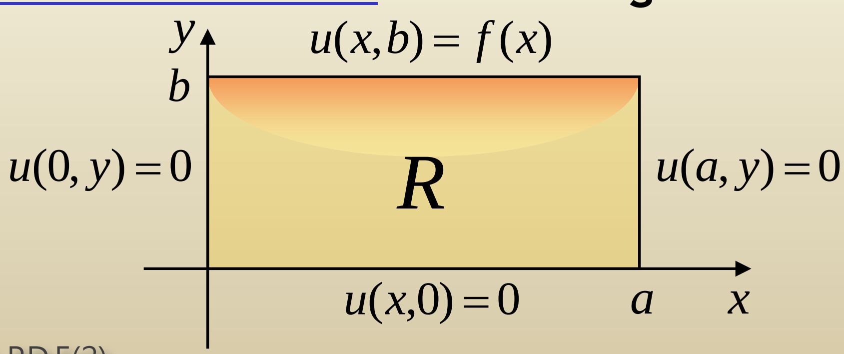

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separating variable,

$$\frac{1}{F} \frac{\partial^2 F}{\partial x^2} = -\frac{1}{G} \frac{\partial^2 G}{\partial y^2} = -\lambda < 0$$

Laplace's Equation

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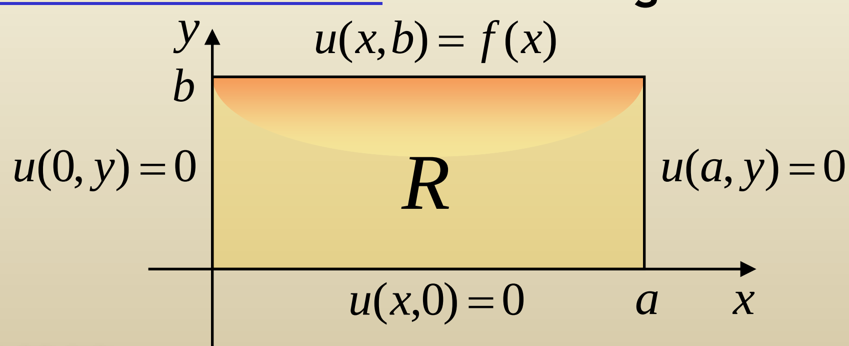

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Two ODEs

Laplace's Equation

- **Laplace's equation**

(Steady Two-Dimensional Heat Problem)

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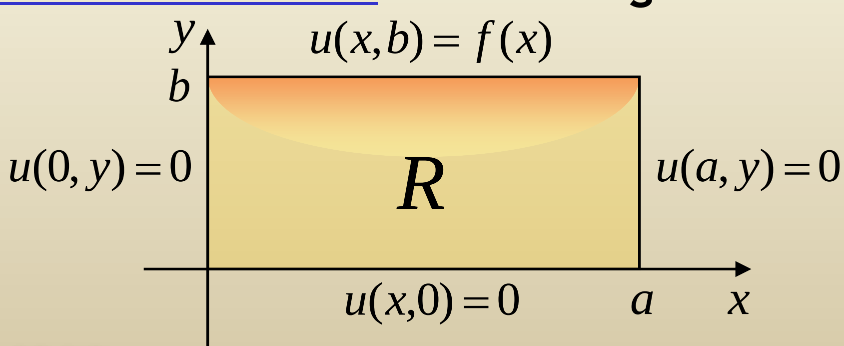
Dirichlet Problem in a Rectangle *R*



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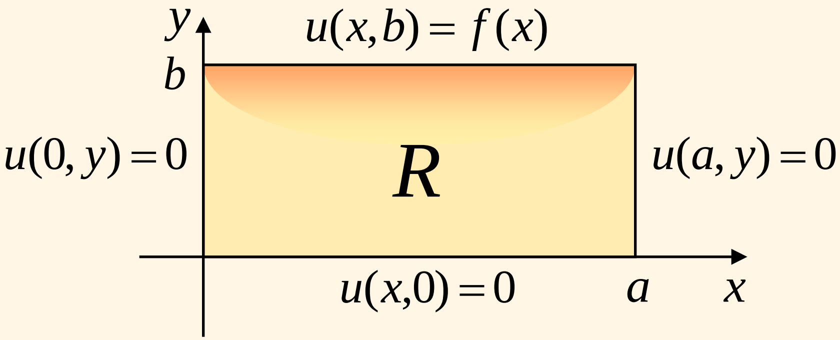
$$\frac{1}{F}\frac{\partial^2 F}{\partial x^2} = -\frac{1}{G}\frac{\partial^2 G}{\partial y^2} = -\lambda < 0$$

Two ODEs

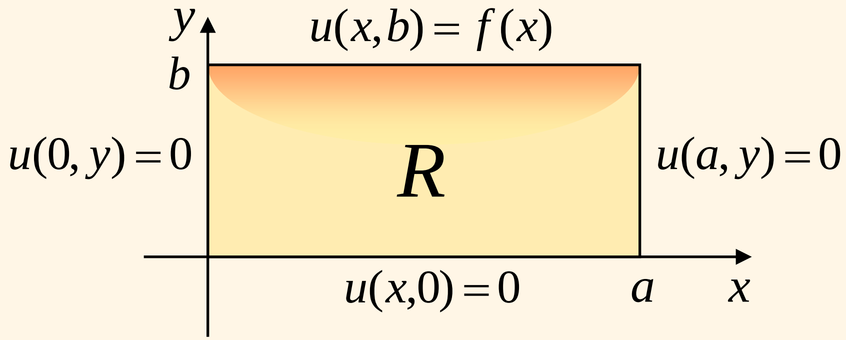
$$F'' + \lambda F = 0$$
$$G'' - \lambda G = 0$$

Laplace's Equation



$$F'' + \lambda F = 0$$

Laplace's Equation

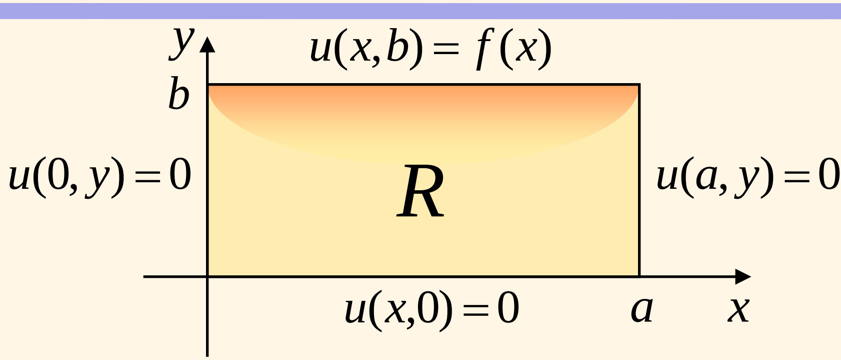


$$F'' + \lambda F = 0$$

- **general solution**

$$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$$

Laplace's Equation



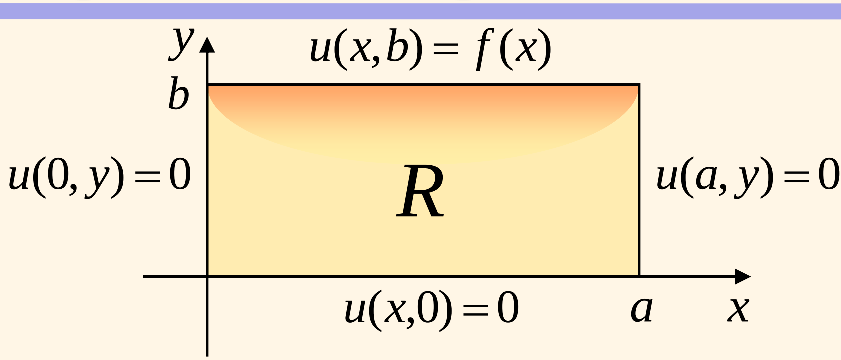
$$F'' + \lambda F = 0$$

- **general solution**

$$F(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$$F''+\lambda F=0, \quad G''-\lambda G=0$$

Laplace's Equation



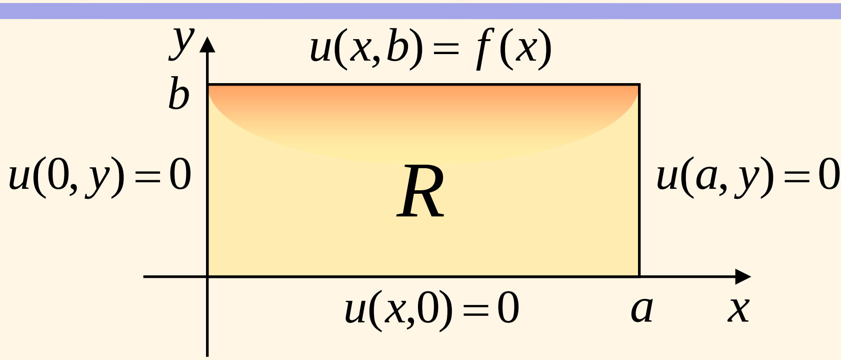
$$F''+\lambda F=0$$

- **general solution**

$$F(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$$

- **boundary condition**

Laplace's Equation



$$F'' + \lambda F = 0$$

- **general solution**

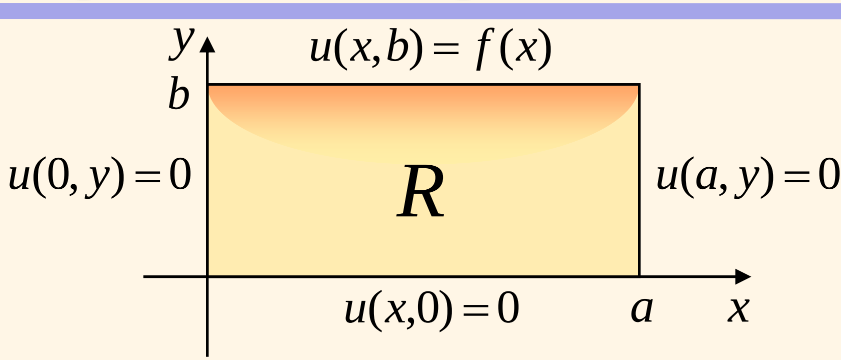
$$F(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

- **boundary condition**

$$u(0,y) = F(0)G(y) = 0$$

$$\boxed{F'' + \lambda F = 0, \quad G'' - \lambda G = 0}$$

Laplace's Equation



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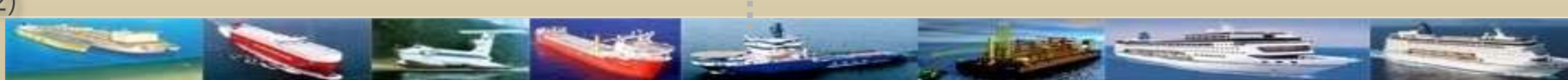
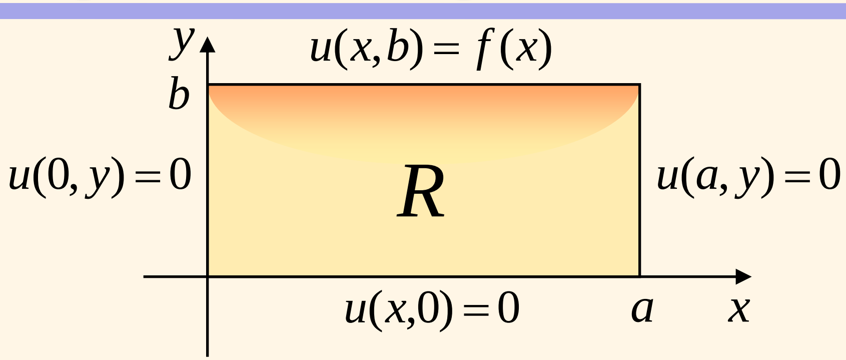


Laplace's Equation



$$F'' + \lambda F = 0$$

- **general solution**

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$$u(0,y) = F(0)G(y) = 0$$

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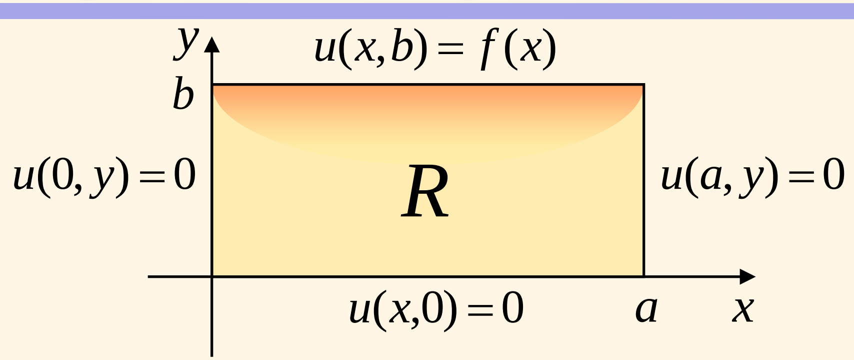
$$\therefore F(0) = F(a) = 0$$

Laplace's Equation



$$F'' + \lambda F = 0$$

- **general solution**

$$F(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

- **boundary condition**

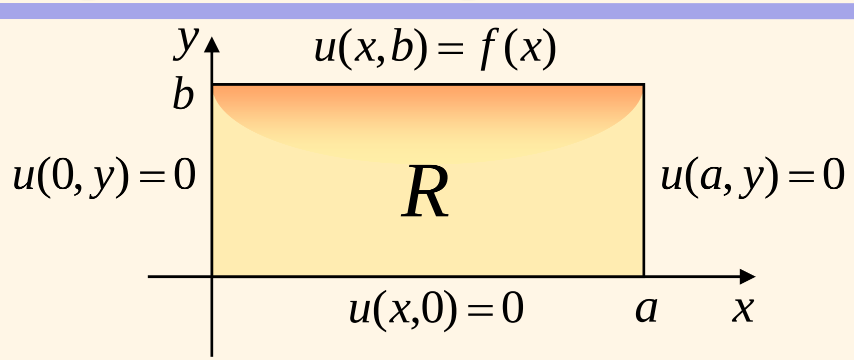
$$u(0,y) = F(0)G(y) = 0$$

$$u(a,y) = F(a)G(y) = 0$$

$$\therefore F(0) = F(a) = 0$$

$$F(0) = A = 0$$

Laplace's Equation



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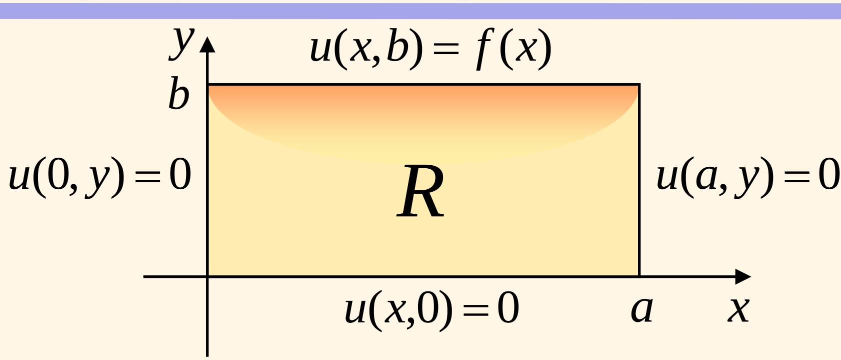
$$F(0) = A = 0$$

$$\therefore F(x) = B\sin\sqrt{\lambda}x$$

$\boxed{F'' + \lambda F = 0, \quad G'' - \lambda G = 0}$

Laplace's Equation



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$$\therefore F(x) = B\sin\sqrt{\lambda}x$$

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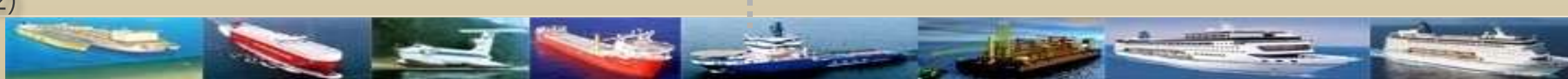

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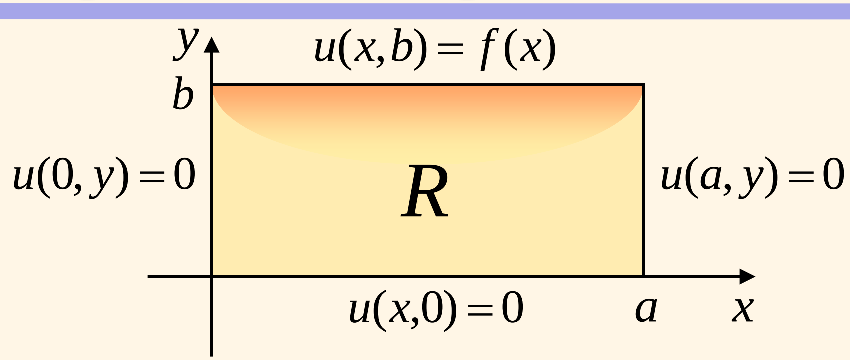

$$F''+\lambda F=0$$

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$$\therefore F(0)=F(a)=0$$

$$F(0)=A=0$$

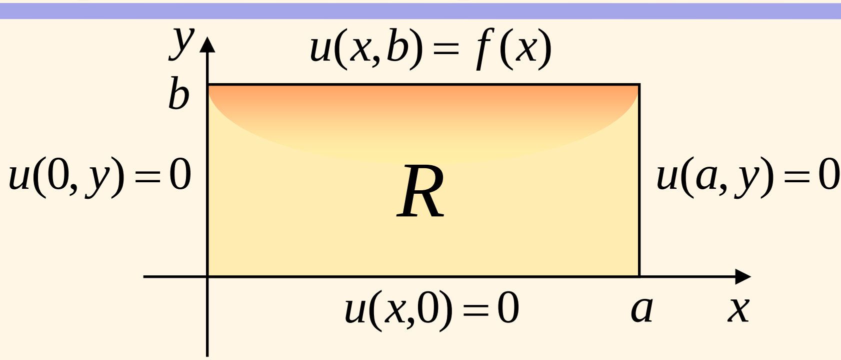
$$\therefore F(x)=B\sin\sqrt{\lambda}x$$

$$F(a)=B\sin\sqrt{\lambda}a=0$$

$$\therefore \sin\sqrt{\lambda}a=0$$

$$\boxed{F'' + \lambda F = 0, \quad G'' - \lambda G = 0}$$

Laplace's Equation



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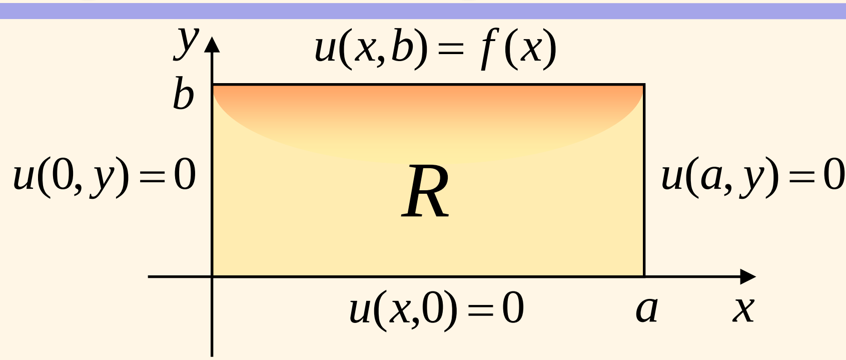
$$F(a) = B\sin\sqrt{\lambda}a = 0$$

$$\therefore \sin\sqrt{\lambda}a = 0$$

$$a\sqrt{\lambda} = n\pi,$$

$$\boxed{F'' + \lambda F = 0, \quad G'' - \lambda G = 0}$$

Laplace's Equation



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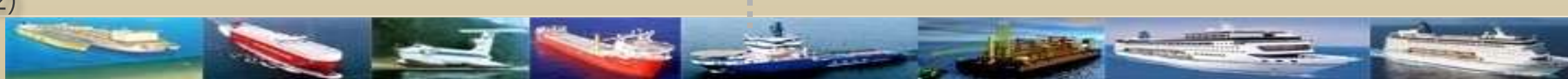
$$a\sqrt{\lambda} = n\pi, \quad \sqrt{\lambda} = \frac{n\pi}{a}, \quad (n = 1, 2, \cdots)$$



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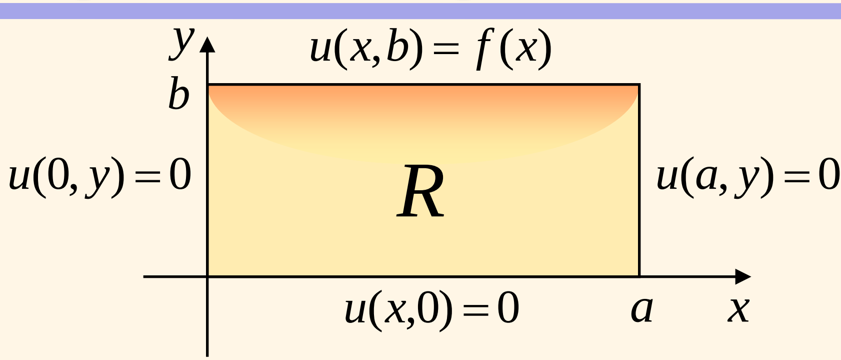

$$F'' + \lambda F = 0$$

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$$u(0,y) = F(0)G(y) = 0$$

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$$F(0) = A = 0$$

$$\therefore F(x) = B\sin\sqrt{\lambda}x$$

$$F(a) = B\sin\sqrt{\lambda}a = 0$$

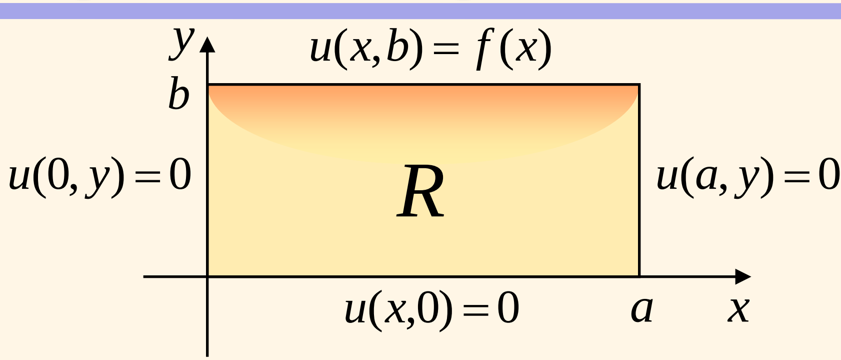
$$\therefore \sin\sqrt{\lambda}a = 0$$

$$a\sqrt{\lambda} = n\pi, \quad \sqrt{\lambda} = \frac{n\pi}{a}, \ (n = 1, 2, \cdots)$$

$$\therefore \lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

$$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$$

Laplace's Equation



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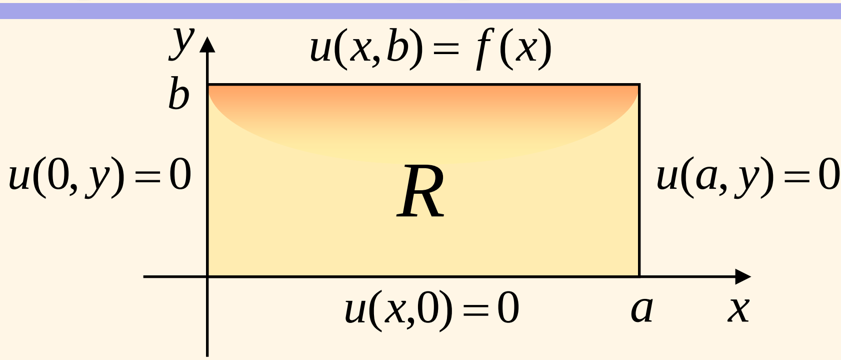
$$\therefore \lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

Setting $B = 1$,

$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$

Laplace's Equation



$$F'' + \lambda F = 0$$

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$$a\sqrt{\lambda} = n\pi, \quad \sqrt{\lambda} = \frac{n\pi}{a}, \ (n = 1, 2, \cdots)$$

$$\therefore \lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

Setting $B = 1$,

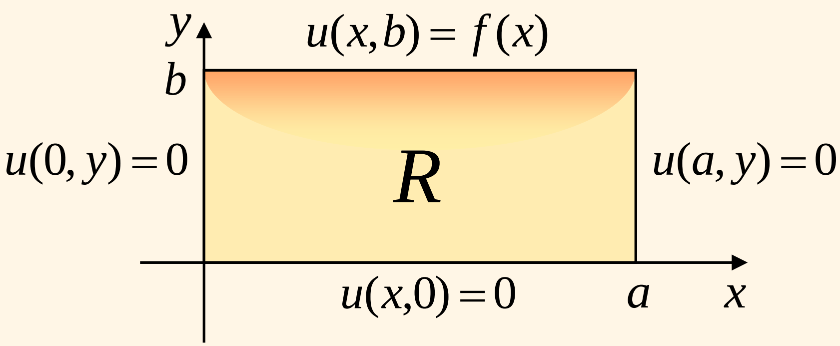
$$\therefore F(x) = F_n(x)$$

$$= \sin\frac{n\pi}{a}x, \quad (n = 1, 2, \cdots)$$

$$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots) \qquad F_n(x) = \sin\frac{n\pi}{a}x$$

Laplace's Equation



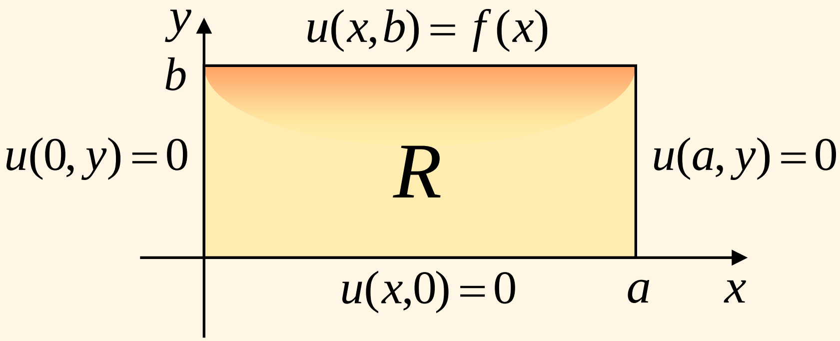
$$G'' - \lambda G = 0, \qquad k = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

$$\therefore G'' - \frac{n^2\pi^2}{a^2} G = 0$$

Laplace's Equation

$$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots) \quad F_n(x) = \sin\frac{n\pi}{a}x$$



$$G'' - \lambda G = 0, \qquad k = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

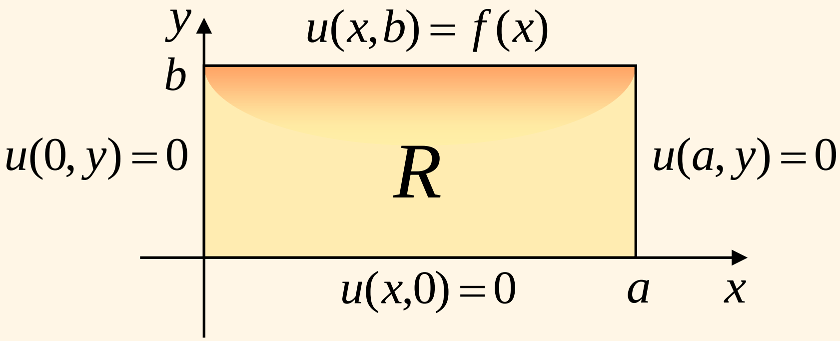
$$\therefore G'' - \frac{n^2\pi^2}{a^2}G = 0$$

- **general solution**

$$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots) \qquad F_n(x) = \sin\frac{n\pi}{a}x$$

Laplace's Equation



$$G'' - \lambda G = 0, \qquad k = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

$$\therefore G'' - \frac{n^2\pi^2}{a^2} G = 0$$

- **general solution**

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$$

$$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$$

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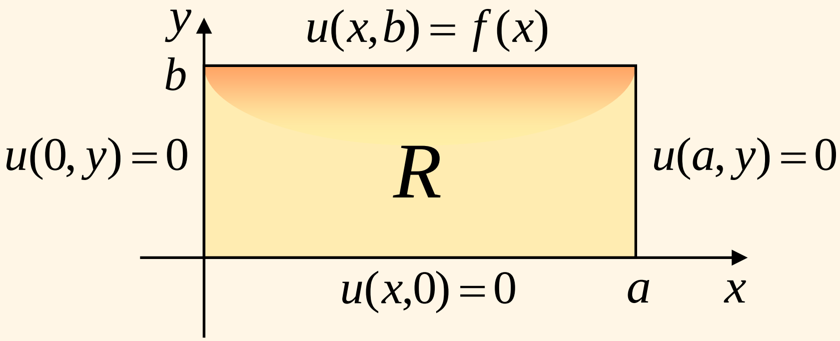
Laplace's Equation



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$$\therefore G'' - \frac{n^2\pi^2}{a^2} G = 0$$

- **general solution**

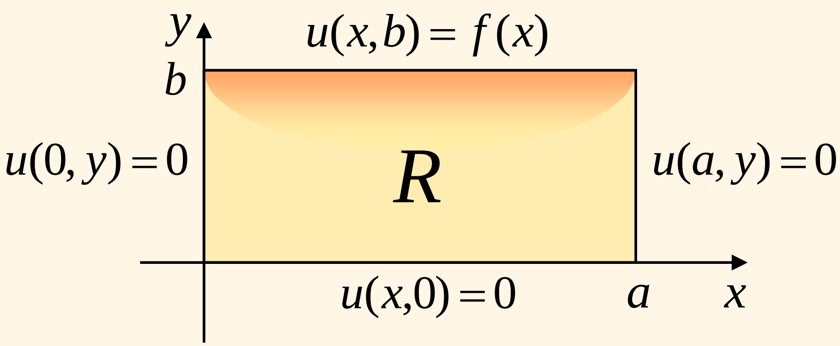
$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$$

- **boundary condition**

$$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots) \quad F_n(x) = \sin\frac{n\pi}{a}x$$

Laplace's Equation



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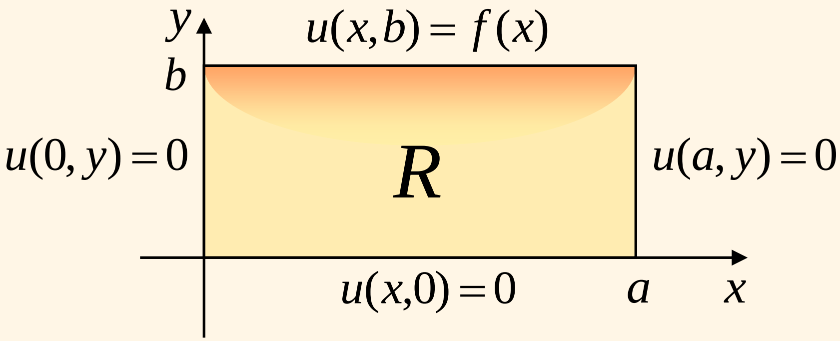
- **general solution**

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$$

- **boundary condition**

$$u(x,0) = F(x)G(0) = 0$$

Laplace's Equation



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- **general solution**

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$$

- **boundary condition**

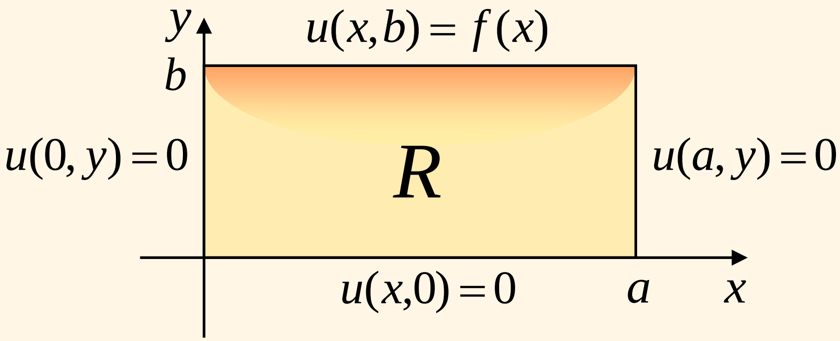
$$u(x,0) = F(x)G(0) = 0$$

$$\therefore G(0) = A_n + B_n = 0$$

$$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots) \quad F_n(x) = \sin\frac{n\pi}{a}x$$

Laplace's Equation



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$$\therefore G'' - \frac{n^2\pi^2}{a^2}G = 0$$

- **general solution**

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$$

- **boundary condition**

$$u(x,0) = F(x)G(0) = 0$$

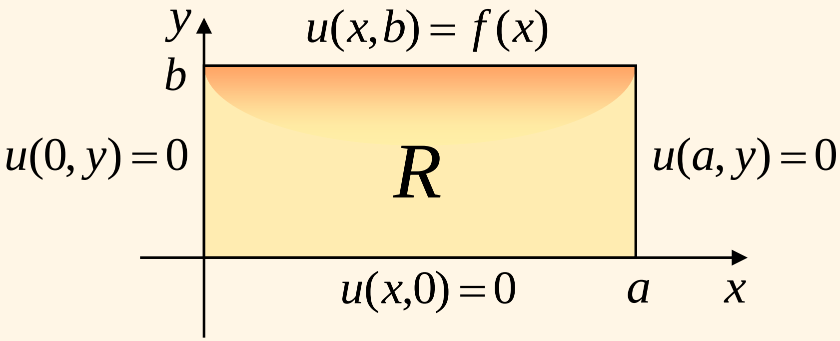
$$\therefore G(0) = A_n + B_n = 0$$

$$\therefore B_n = -A_n$$

$$\boxed{F'' + \lambda F = 0, \quad G'' - \lambda G = 0}$$

$$\boxed{\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)} \ \boxed{F_n(x) = \sin\frac{n\pi}{a}x}$$

Laplace's Equation



$$G'' - \lambda G = 0, \qquad k = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

$$\therefore G'' - \frac{n^2\pi^2}{a^2}G = 0$$

- **general solution**

$$G(y) = G_n(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$$

- **boundary condition**

$$u(x,0) = F(x)G(0) = 0$$

$$\therefore G(0) = A_n + B_n = 0$$

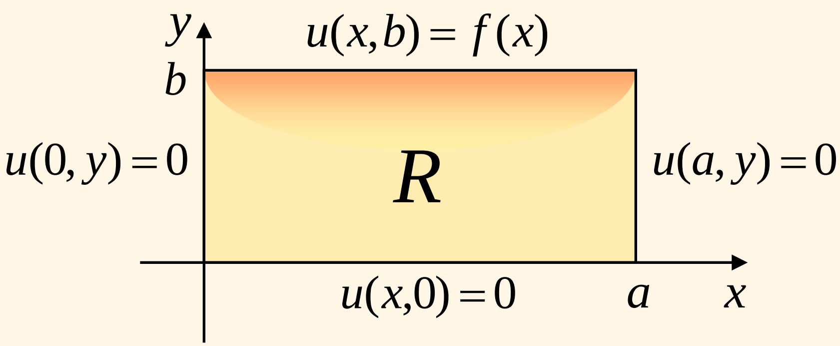
$$\therefore B_n = -A_n$$

$$\therefore G_n(y) = A_n(e^{n\pi y/a} - e^{-n\pi y/a})$$

Laplace's Equation

$$F'' + \lambda F = 0, \quad G'' - \lambda G = 0$$

$$\lambda = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots) \qquad F_n(x) = \sin\frac{n\pi}{a}x$$



$$G'' - \lambda G = 0, \qquad k = \left(\frac{n\pi}{a}\right)^2, \ (n = 1, 2, \cdots)$$

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- **boundary condition**

$$u(x,0) = F(x)G(0) = 0$$

$$\therefore G(0) = A_n + B_n = 0$$

$$\therefore B_n = -A_n$$

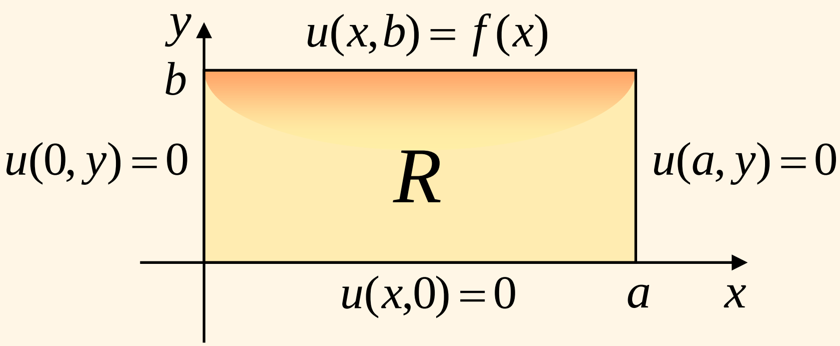
$$\therefore G_n(y) = A_n(e^{n\pi y/a} - e^{-n\pi y/a})$$

$$= 2A_n \sinh\frac{n\pi y}{a}$$

Laplace's Equation

$$F''+\lambda F=0,\quad G''-\lambda G=0$$

$$\lambda=\left(\frac{n\pi}{a}\right)^2,\ (n=1,2,\cdots)\qquad F_n(x)=\sin\frac{n\pi}{a}x$$



$$G''-\lambda G=0,\qquad k=\left(\frac{n\pi}{a}\right)^2,\ (n=1,2,\cdots)$$

$$\therefore G''-\frac{n^2\pi^2}{a^2}G=0$$

- **general solution**

$$G(y)=G_n(y)=A_n e^{n\pi y/a}+B_n e^{-n\pi y/a}$$

- **boundary condition**

$$u(x,0)=F(x)G(0)=0$$

$$\therefore G(0)=A_n+B_n=0$$

$$\therefore B_n=-A_n$$

$$\therefore G_n(y)=A_n(e^{n\pi y/a}-e^{-n\pi y/a})$$

$$=2A_n\sinh\frac{n\pi y}{a}$$

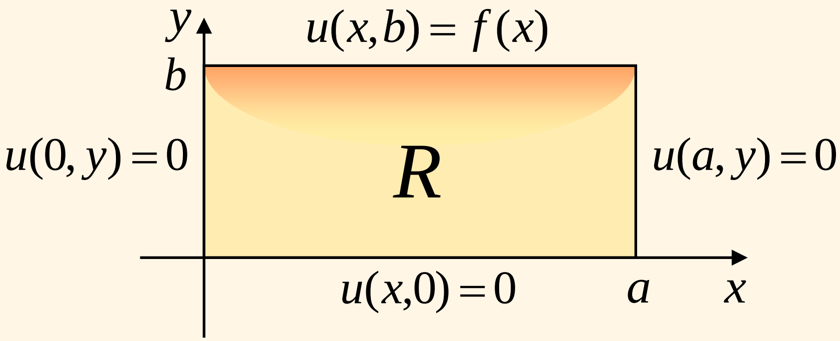
$$\therefore G_n(y)=A_n^*\sinh\frac{n\pi y}{a}$$

$$\text{where}\ A_n^*=2A_n$$

Laplace's Equation

$$F''+\lambda F=0,\quad G''-\lambda G=0$$

$$\boxed{\lambda=\left(\frac{n\pi}{a}\right)^2,\ (n=1,2,\cdots)}\ \boxed{F_n(x)=\sin\frac{n\pi}{a}x}$$



$$G''-\lambda G=0,\qquad k=\left(\frac{n\pi}{a}\right)^2,\ (n=1,2,\cdots)$$

$$\therefore G''-\frac{n^2\pi^2}{a^2}G=0$$

- **general solution**

$$G(y)=G_n(y)=A_ne^{n\pi y/a}+B_ne^{-n\pi y/a}$$

- **boundary condition**

$$u(x,0)=F(x)G(0)=0$$

$$\therefore G(0)=A_n+B_n=0$$

$$\therefore B_n=-A_n$$

$$\therefore G_n(y)=A_n(e^{n\pi y/a}-e^{-n\pi y/a})$$

$$=2A_n\sinh\frac{n\pi y}{a}$$

$$\therefore G_n(y)=A_n^*\sinh\frac{n\pi y}{a}$$

$$\text{where}A_n^*=2A_n$$

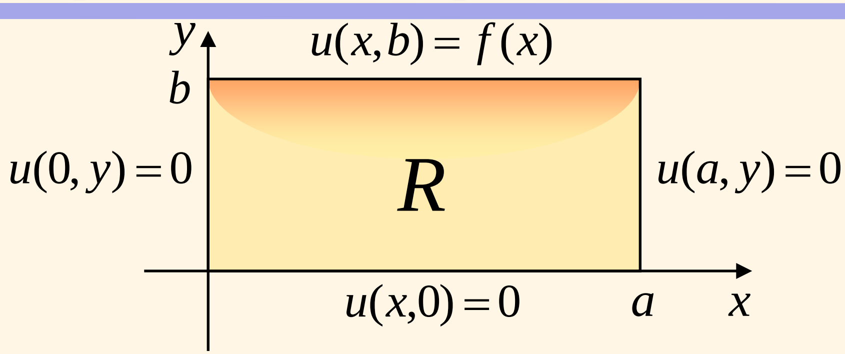
$$\therefore u_n(x,y)=F_n(x)G_n(y)$$

$$=A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

Laplace's Equation

Fourier sine series *where*, $(-p, p)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \qquad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$



$$u_n(x,y) = A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

Laplace's Equation

Fourier sine series *where,* $(-p, p)$

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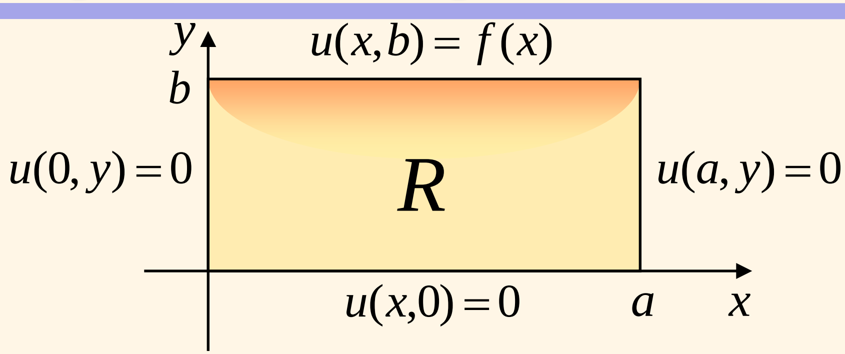


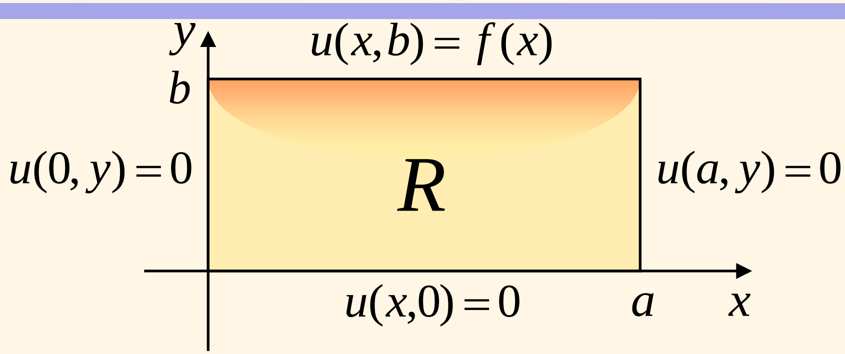
$$u_n(x,y) = A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

By Superposition

Laplace's Equation

Fourier sine series where, $(-p, p)$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \qquad b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$



$$u_n(x,y)=A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

By Superposition

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,y)$$

$$=\sum_{n=1}^{\infty}A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

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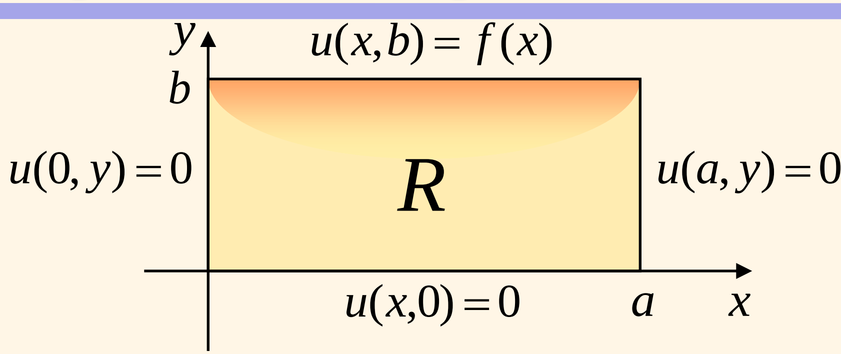


$$u_n(x,y)=A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

By Superposition

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,y)$$

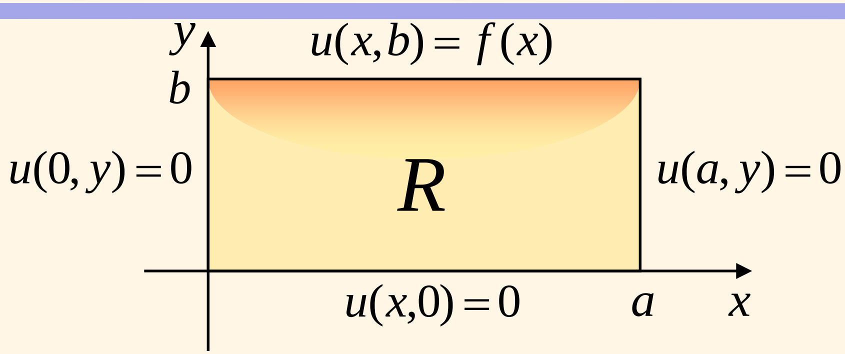
$$=\sum_{n=1}^{\infty}A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

- **boundary condition**

Laplace's Equation

Fourier sine series where, $(-p, p)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \qquad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$



$$u_n(x,y) = A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

By Superposition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,y)$$

$$= \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

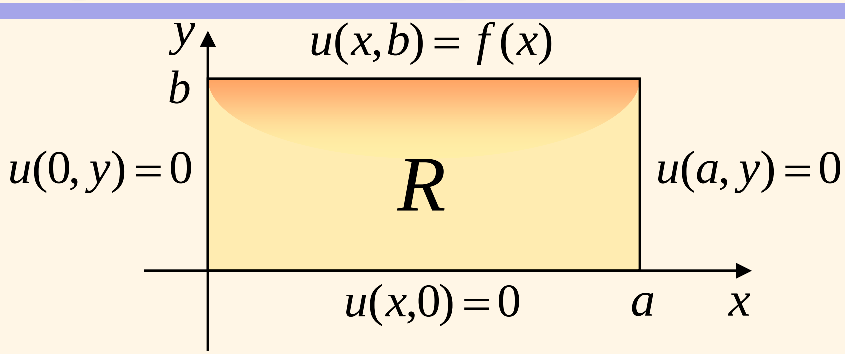
- **boundary condition**

$$u(x,b) = F(x)G(b) = f(x)$$

Laplace's Equation

Fourier sine series where, $(-p, p)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \qquad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$



$$u_n(x,y) = A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

By Superposition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,y)$$

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$$u(x,b) = F(x)G(b) = f(x)$$

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Laplace's Equation

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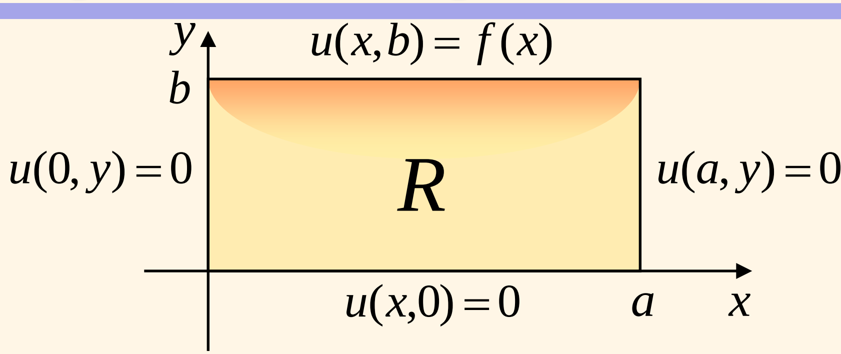


$$u_n(x,y)=A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

By Superposition

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,y)$$

$$=\sum_{n=1}^{\infty}A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

- **boundary condition**

$$u(x,b)=F(x)G(b)=f(x)$$

$$u(x,b)=f(x)$$

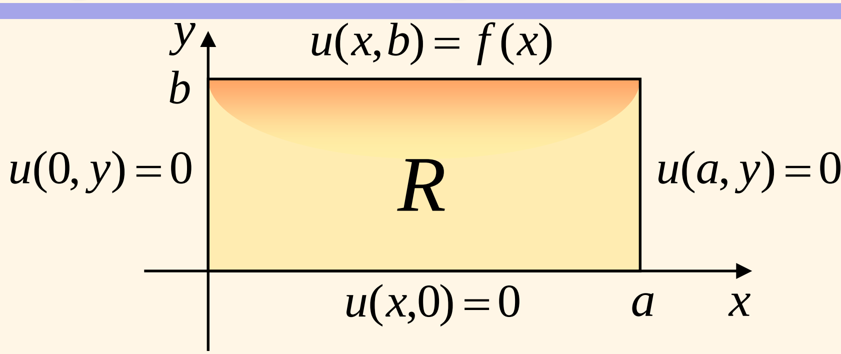
$$=\sum_{n=1}^{\infty}A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

$$=\sum_{n=1}^{\infty}\left(A_n^*\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$$

Laplace's Equation

Fourier sine series where, $(-p, p)$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \qquad b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$



$$u_n(x,y)=A_n^* \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

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$$u(x,t)=\sum_{n=1}^{\infty} u_n(x,y)$$

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Fourier sine series of $f(x)$

Laplace's Equation

Fourier sine series where, $(-p,p)$

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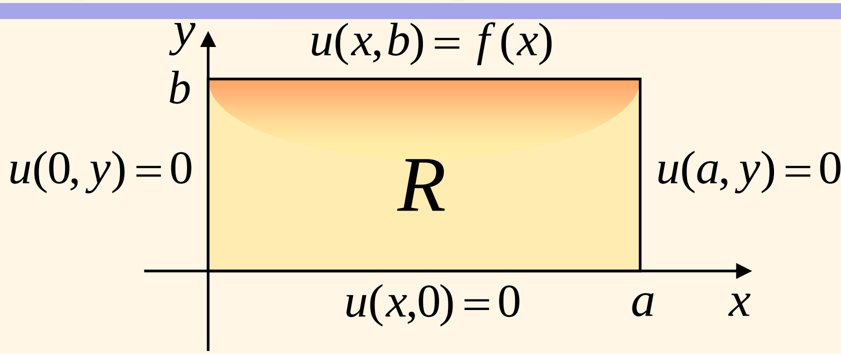

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$$u(x,b)=F(x)G(b)=f(x)$$

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$$=\sum_{n=1}^{\infty}A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a}$$

$$=\sum_{n=1}^{\infty}\left(A_n^*\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$$

Fourier sine series of $f(x)$

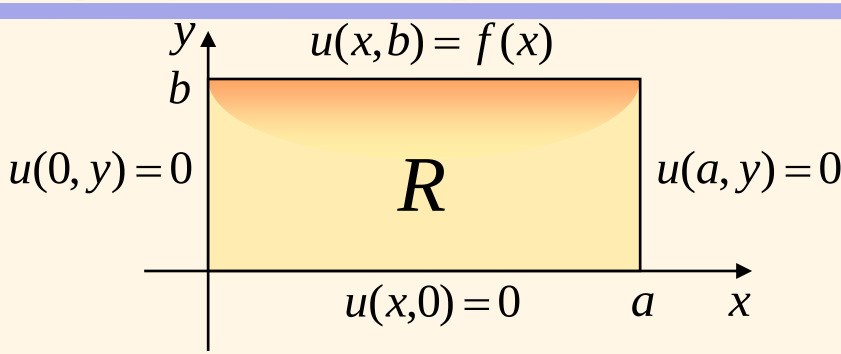
$$\therefore A_n^*\sinh\frac{n\pi b}{a}$$

$$=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

Laplace's Equation

Fourier sine series where, $(-p, p)$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x \qquad b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$



$$u_n(x,y)=A_n^*\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$

By Superposition

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,y)$$

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$$u(x,b)=F(x)G(b)=f(x)$$

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$$=\sum_{n=1}^{\infty}\left(A_n^*\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$$

Fourier sine series of $f(x)$

$$\therefore A_n^*\sinh\frac{n\pi b}{a}$$

$$=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

$$\therefore A_n^*=\frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

Laplace's Equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad u(x,0)=0,\ \ u(x,b)=f(x)$$
$$u(0,y)=0,\ \ u(a,y)=0$$

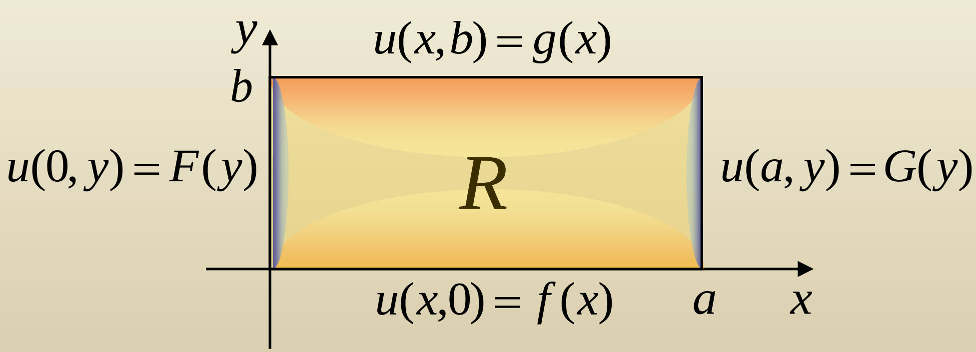
Superposition

The method of separation of variables is not applicable to a Dirichelt problem when the boundary conditions on all four sides of the rectangle are non-homogeneous.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0<x<a, \quad 0<y<b$$

$$u(0,y)=F(y),\ \ u(a,y)=G(y),\ \ 0<y<b$$

$$u(x,0)=f(x),\ \ u(x,b)=g(x),\ \ 0<x<a$$



Laplace's Equation

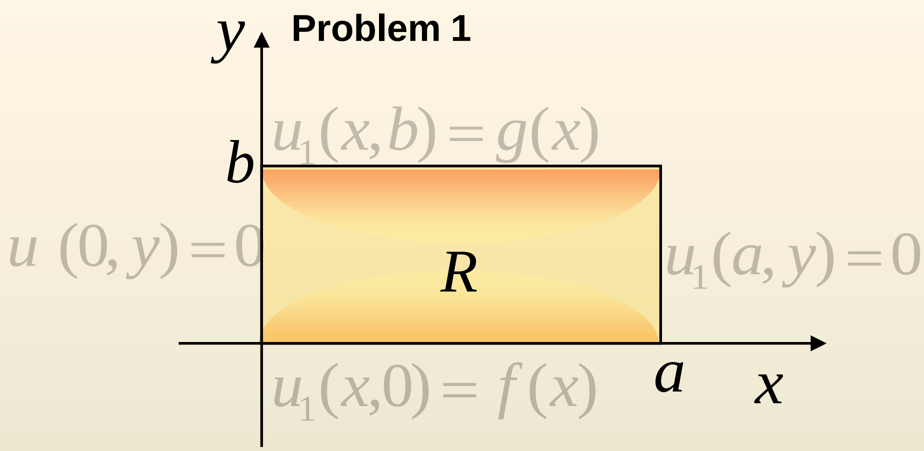
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

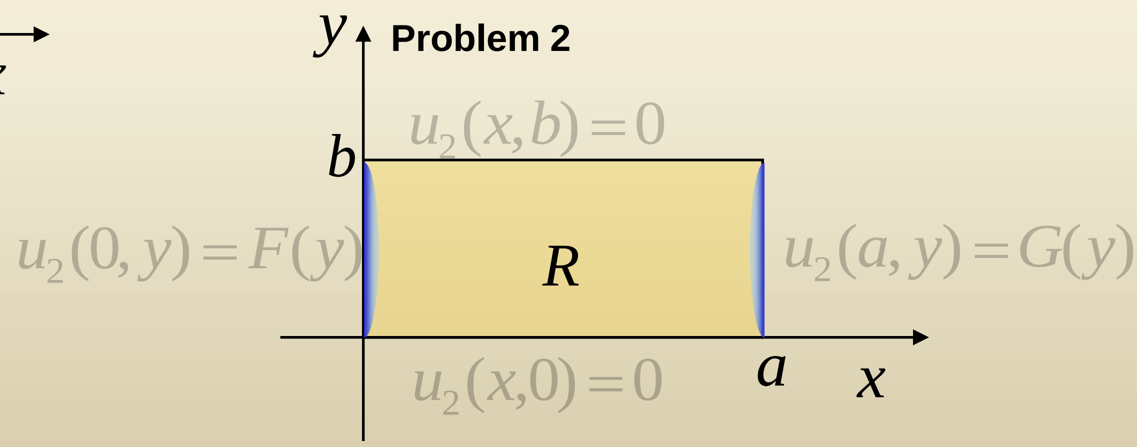
$$u(0,y)=F(y),\ \ u(a,y)=G(y),\ \ 0<y<b$$

$$u(x,0)=f(x),\ \ u(x,b)=g(x),\ \ 0<x<a$$

<u>Superposition</u>

To get around this difficulty we break the problem into two problems, each of which has homogeneous boundary conditions on parallel boundaries, as shown.





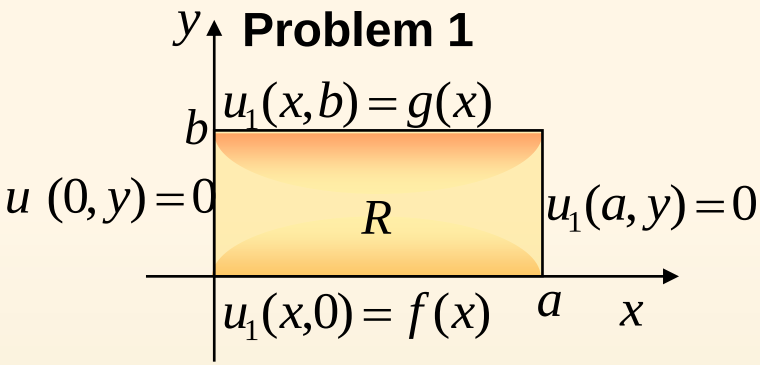
Laplace’s Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

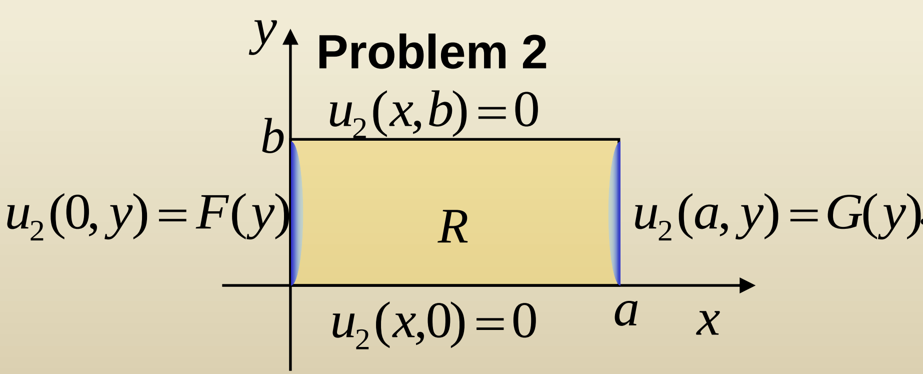
$$u(0,y)=F(y),\ u(a,y)=G(y),\ 0<y<b$$

$$u(x,0)=f(x),\ \ u(x,b)=g(x),\ \ 0<x<a$$

Superposition



$$\begin{cases}\dfrac{\partial^2 u_1}{\partial x^2}+\dfrac{\partial^2 u_1}{\partial y^2}=0, & 0<x<a,\quad 0<y<b\\ u_1(0,y)=0, \qquad u_1(a,y)=0, & 0<y<b\\ u_1(x,0)=f(x),\quad u_1(x,b)=g(x), & 0<x<a\end{cases}$$



$$\begin{cases}\dfrac{\partial^2 u_2}{\partial x^2}+\dfrac{\partial^2 u_2}{\partial y^2}=0, & 0<x<a,\quad 0<y<b\\ u_2(0,y)=F(y),\quad u_2(a,y)=G(y), & 0<y<b\\ u_2(x,0)=0, \qquad u_2(x,b)=0, & 0<x<a\end{cases}$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u(0,y)=F(y),\ u(a,y)=G(y),\ 0<y<b$$

$$u(x,0)=f(x),\ \ u(x,b)=g(x),\ \ 0<x<a$$

Superposition

Problem 1

$$\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u_1(0,y)=0,\qquad u_1(a,y)=0,\ 0<y<b$$

$$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\ \ 0<x<a$$

Problem 2

$$\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u_2(0,y)=F(y),\ u_2(a,y)=G(y),\ 0<y<b$$

$$u_2(x,0)=0,\qquad u_2(x,b)=0,\ 0<x<a$$

Suppose u_1 and u_2 are the solutions of Problems 1 and 2, respectively. If we define $u(x,y) = u_1(x,y) + u_2(x,y)$, it is seen that u satisfies all boundary conditions in the original problem above.

$$u(0,y)=u_1(0,y)+u_2(0,y)=0+F(y)=F(y)$$

$$u(a,y)=u_1(a,y)+u_2(a,y)=0+G(y)=G(y)$$

$$u(x,0)=u_1(x,0)+u_2(x,0)=f(x)+0=f(x)$$

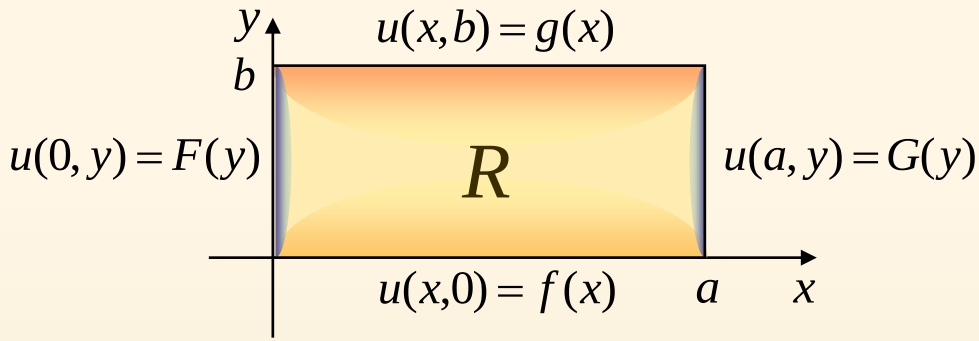
$$u(x,b)=u_1(x,b)+u_2(x,b)=g(x)+0=g(x)$$

Laplace’s Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$
$$u(0,y)=F(y),\ u(a,y)=G(y),\ 0<y<b$$
$$u(x,0)=f(x),\ \ u(x,b)=g(x),\ \ 0<x<a$$

Superposition

$$u(x,y)=u_1(x,y)+u_2(x,y)$$



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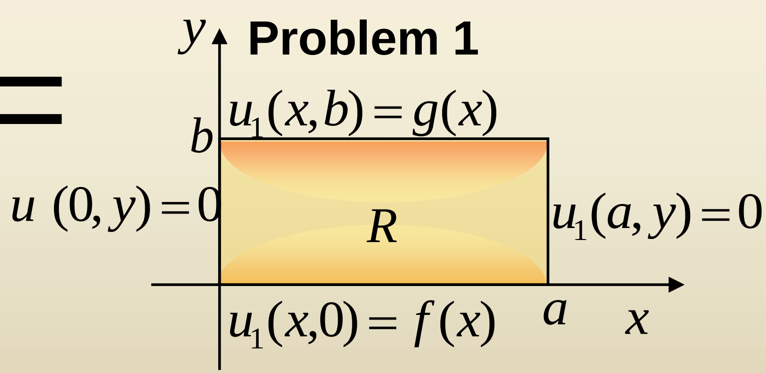

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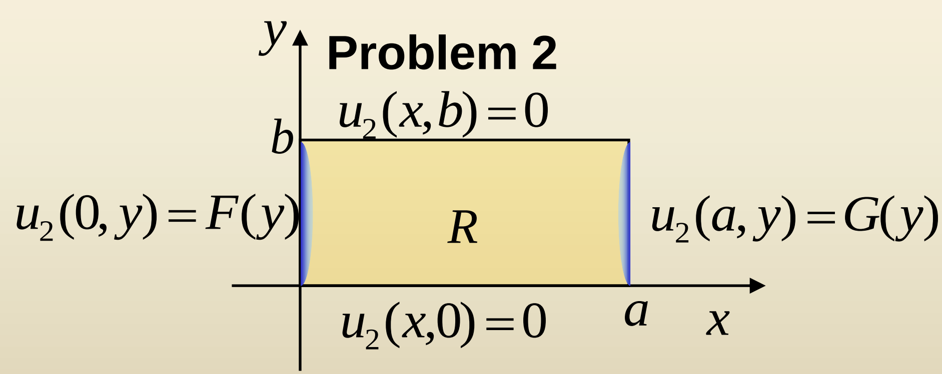
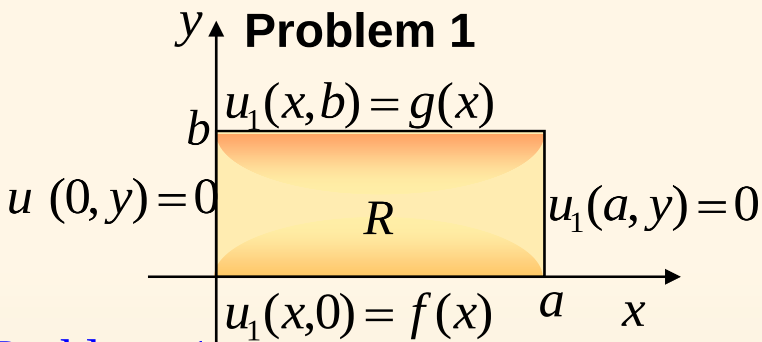


Laplace's Equation

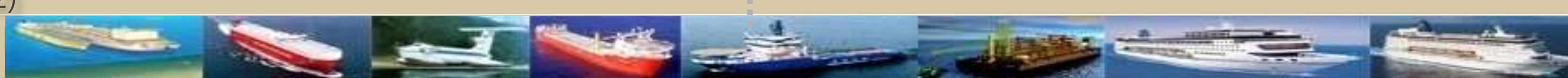
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$
$$u(0,y)=F(y),\ u(a,y)=G(y),\ 0<y<b$$
$$u(x,0)=f(x),\ \ u(x,b)=g(x),\ \ 0<x<a$$

Superposition



Problem 1

$$\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$
$$u_1(0,y)=0,\ u_1(a,y)=0,\ 0<y<b,$$
$$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\qquad 0<x<a$$



Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u(0,y)=F(y),\ u(a,y)=G(y),\ 0<y<b$$

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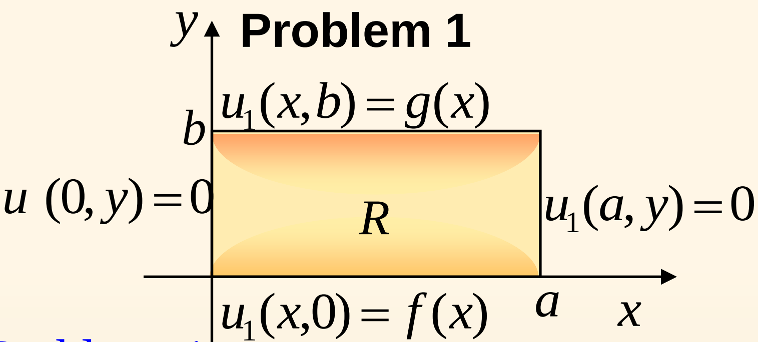
Superposition



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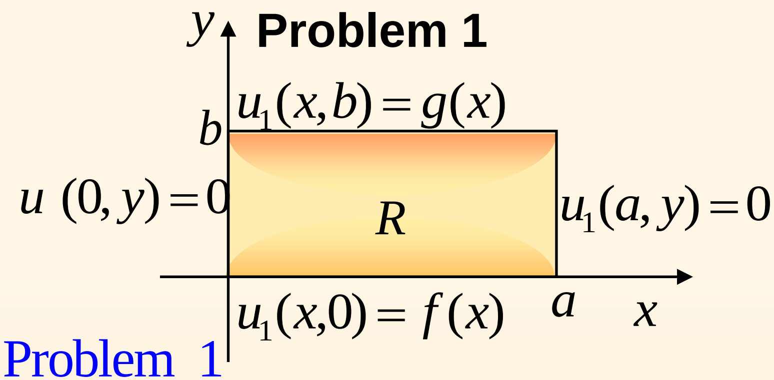

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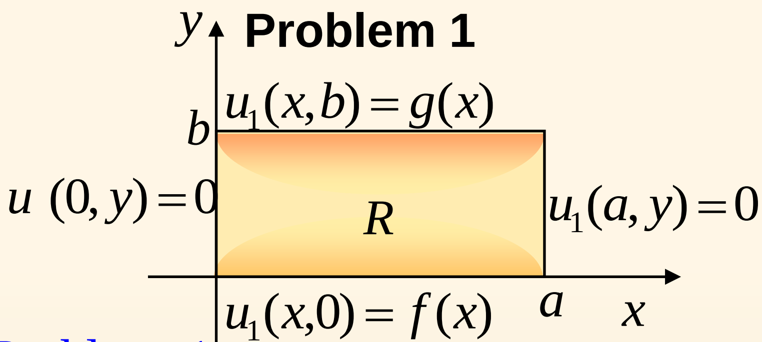

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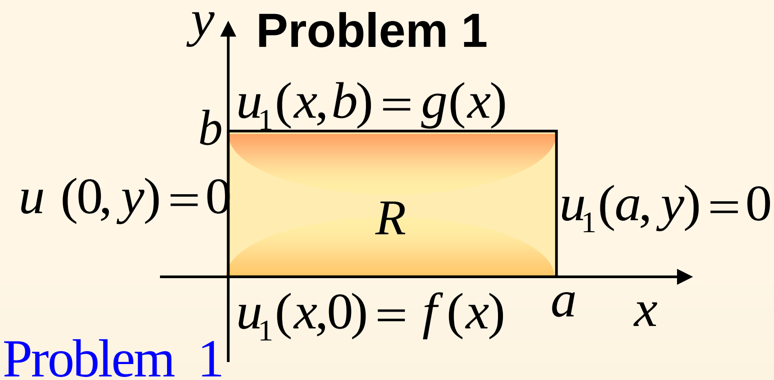

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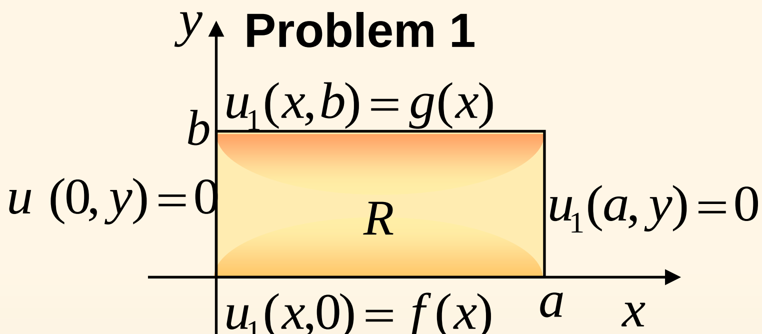

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$$\therefore \frac{d^2G}{dy^2}-\frac{n^2\pi^2}{a^2}G=0$$

$$G_{1n}(y)=A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$
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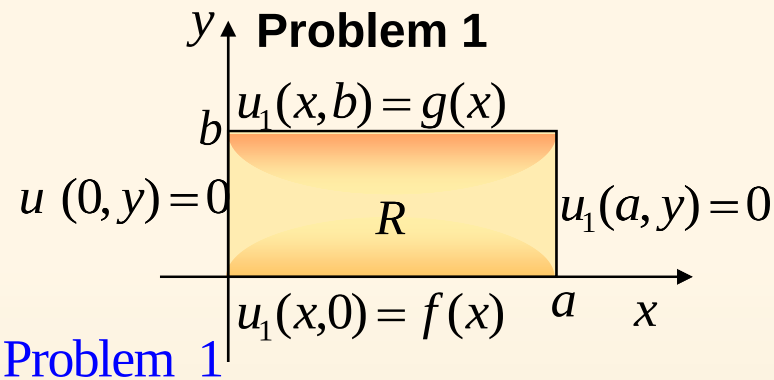
Superposition



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$$G_{1n}(y)=A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y$$

$$\therefore u_{1n}=F_{1n}(x)G_{1n}(y)$$
$$=\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

Laplace's Equation

$$\text{Problem 1}$$
$$u_1(0,y)=0,\ u_1(a,y)=0,\ 0<y<b,$$
$$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\quad 0<x<a$$

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$$u_{1n}=F_{1n}(x)G_{1n}(y)=\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

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- **General solution of the problem 1**

Laplace's Equation

Problem 1

$$u_1(0,y)=0,\ \ u_1(a,y)=0,\ \ 0<y<b,$$

$$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\qquad 0<x<a$$

Superposition

$$u_{1n}=F_{1n}(x)G_{1n}(y)=\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

- **General solution of the problem 1**

$$\therefore u_1(x,y)=\sum_{n=1}^{\infty}u_{1n}=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

Laplace's Equation

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Superposition

$$u_{1n}=F_{1n}(x)G_{1n}(y)=\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

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- **boundary condition to find A_n**

Laplace's Equation

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$$u_1(0,y)=0,\ u_1(a,y)=0,\ 0<y<b,$$

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- **boundary condition to find A_n**

$$u_1(x,0)=f(x)=\sum_{n=1}^{\infty}A_{1n}\sin\frac{n\pi}{a}x$$

Laplace's Equation

Problem 1

$$u_1(0,y)=0,\ \ u_1(a,y)=0,\ \ 0<y<b,$$

$$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\qquad 0<x<a$$

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$$u_{1n}=F_{1n}(x)G_{1n}(y)=\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

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- **boundary condition to find A_n**

$$u_1(x,0)=\underline{f(x)=\sum_{n=1}^{\infty}A_{1n}\sin\frac{n\pi}{a}x}$$

Fourier sine series of $f(x)$

Laplace's Equation

Problem 1

$u_1(0,y)=0,\ u_1(a,y)=0,\ 0<y<b,$

$u_1(x,0)=f(x),\ u_1(x,b)=g(x),\quad 0<x<a$

Superposition

$$u_{1n}=F_{1n}(x)G_{1n}(y)=\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

- **General solution of the problem 1**

$$\therefore u_1(x,y)=\sum_{n=1}^{\infty}u_{1n}=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

- **boundary condition to find A_n**

$$u_1(x,0)=\underline{f(x)=\sum_{n=1}^{\infty}A_{1n}\sin\frac{n\pi}{a}x}$$

Fourier sine series of $f(x)$

$$\therefore A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

Laplace's Equation

Problem 1

$$u_1(0,y)=0,\ \ u_1(a,y)=0,\ \ 0<y<b,$$

$$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\qquad 0<x<a$$

Superposition

$$u_1(x,y)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

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Superposition

$$u_1(x,y)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

$$A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

- **boundary condition to find B_n**

Laplace's Equation

Problem 1

$$u_1(0,y)=0,\ \ u_1(a,y)=0,\ \ 0<y<b,$$

$$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\qquad 0<x<a$$

Superposition

$$u_1(x,y)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

$$A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}xdx$$

- **boundary condition to find B_n**

$$u_1(x,b)=g(x)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi b}{a}+B_{1n}\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi}{a}x$$

Laplace's Equation

Problem 1

$$u_1(0,y)=0,\ \ u_1(a,y)=0,\ \ 0<y<b,$$

$$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\qquad 0<x<a$$

Superposition

$$u_1(x,y)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

$$A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

- **boundary condition to find B_n**

$$u_1(x,b)=g(x)=\underline{\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi b}{a}+B_{1n}\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi}{a}x}$$

Fourier sine series of $g(x)$

Laplace's Equation

Problem 1

$u_1(0,y)=0,\ u_1(a,y)=0,\ 0<y<b,$

$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\qquad 0<x<a$

Superposition

$$u_1(x,y)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

$$A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}xdx$$

- **boundary condition to find B_n**

$$u_1(x,b)=g(x)=\underbrace{\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi b}{a}+B_{1n}\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi}{a}x}_{\text{Fourier sine series of } g(x)}$$

Fourier sine series of g (x)

$$\therefore A_{1n}\cosh\frac{n\pi b}{a}+B_{1n}\sinh\frac{n\pi b}{a}=\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}xdx$$

Laplace's Equation

Problem 1

$$u_1(0,y)=0,\ u_1(a,y)=0,\ 0<y<b,$$

$$u_1(x,0)=f(x),\ \ u_1(x,b)=g(x),\qquad 0<x<a$$

Superposition

$$u_1(x,y)=\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi}{a}y+B_{1n}\sinh\frac{n\pi}{a}y\right)\sin\frac{n\pi}{a}x$$

$$A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}xdx$$

- **boundary condition to find B_n**

$$u_1(x,b)=g(x)=\underline{\sum_{n=1}^{\infty}\left(A_{1n}\cosh\frac{n\pi b}{a}+B_{1n}\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi}{a}x}$$

Fourier sine series of $g(x)$

$$\therefore A_{1n}\cosh\frac{n\pi b}{a}+B_{1n}\sinh\frac{n\pi b}{a}=\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}xdx$$

$$\therefore B_{1n}=\frac{1}{\sinh(n\pi b/a)}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}xdx-A_{1n}\cosh\frac{n\pi}{a}b\right)$$

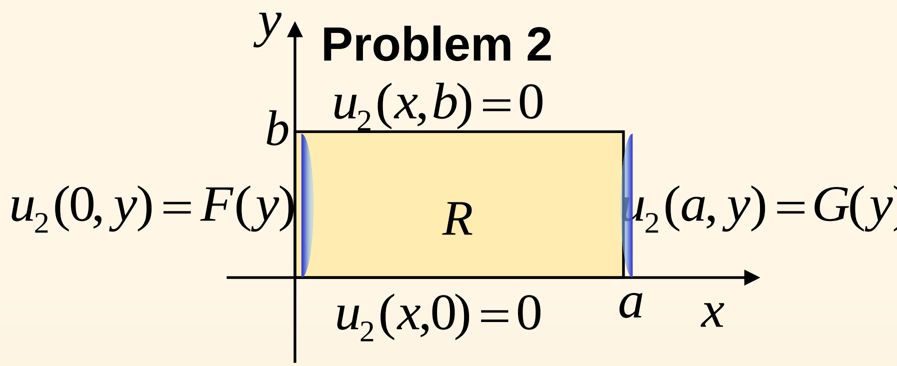
Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$
$$u(0,y)=F(y),\ u(a,y)=G(y),\ 0<y<b$$
$$u(x,0)=f(x),\ u(x,b)=g(x),\ 0<x<a$$

Superposition

Problem 2

$$\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$
$$u_2(0,y)=F(y),\ u_2(a,y)=G(y),\ 0<y<b,$$
$$u_2(x,0)=0,\ u_2(x,b)=0,\quad 0<x<a$$



Laplace's Equation

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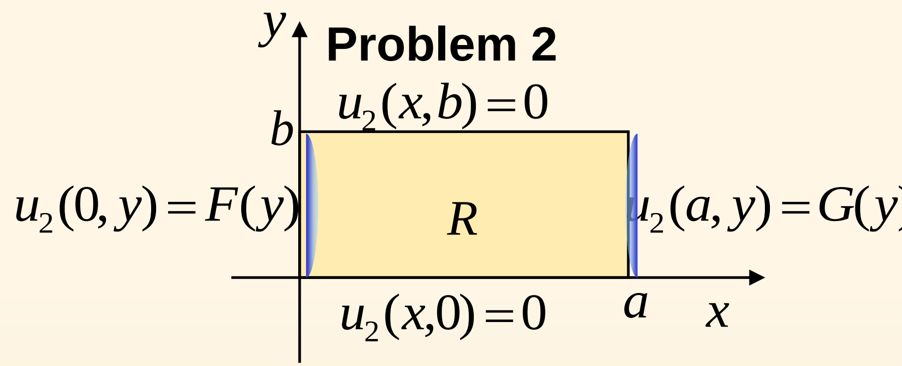

$$u_2(x,y)=\sum_{n=1}^{\infty}\left\{A_{2n}\cosh\frac{n\pi}{b}x+B_{2n}\sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

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$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

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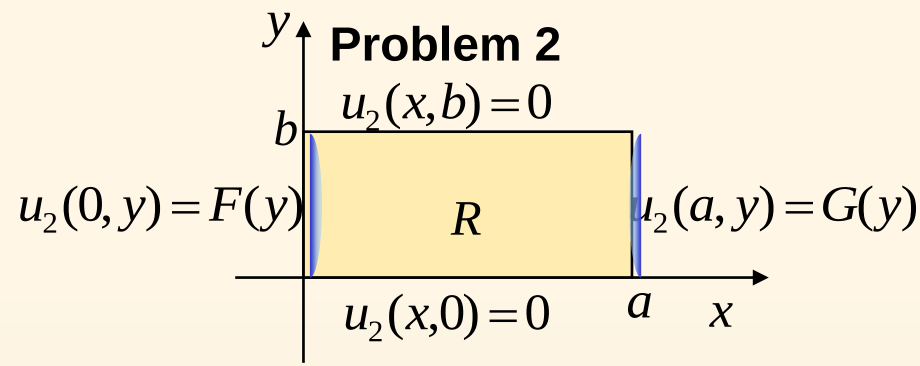

$$u_2(x,y)=\sum_{n=1}^{\infty}\left\{A_{2n}\cosh\frac{n\pi}{b}x+B_{2n}\sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

$$A_{2n}=\frac{2}{b}\int_0^b F(y)\sin\frac{n\pi}{b}y\,dy$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

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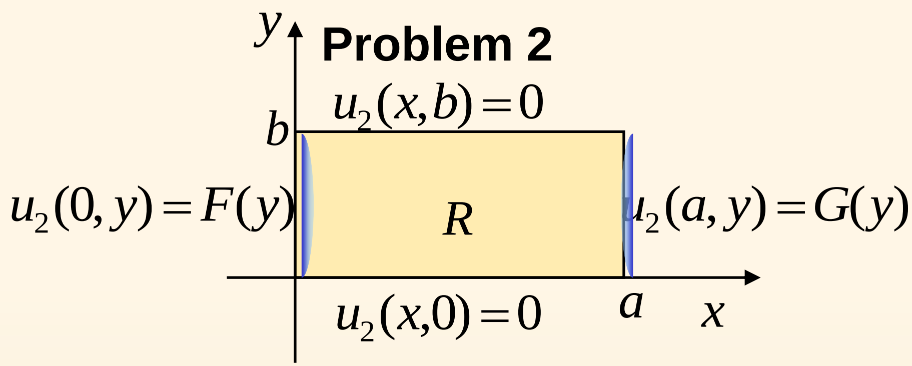

$$u_2(x,y)=\sum_{n=1}^{\infty}\left\{A_{2n}\cosh\frac{n\pi}{b}x+B_{2n}\sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

$$A_{2n}=\frac{2}{b}\int_0^b F(y)\sin\frac{n\pi}{b}y\,dy$$

$$B_{2n}=\frac{1}{\sinh\frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^b G(y)\sin\frac{n\pi}{b}y\,dy-A_{2n}\cosh\frac{n\pi}{b}a\right)$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u(0,y)=F(y),\ \ u(a,y)=G(y),\ \ 0<y<b$$

$$u(x,0)=f(x),\ \ u(x,b)=g(x),\ \ 0<x<a$$

Superposition

$$\boxed{u(x,y)=u_1(x,y)+u_2(x,y)}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u(0,y)=F(y),\ u(a,y)=G(y),\ 0<y<b$$

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Laplace's Equation

Superposition

$$\boxed{u(x,y)=u_1(x,y)+u_2(x,y)}$$

$$u_1(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{a}y+B_n\sinh\frac{n\pi}{a}y\right\}\sin\frac{n\pi}{a}x$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u(0,y)=F(y),\ \ u(a,y)=G(y),\ \ 0<y<b$$

$$u(x,0)=f(x),\ \ u(x,b)=g(x),\ \ 0<x<a$$

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$$A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx,$$

Laplace's Equation

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$$A_{1n}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx,\qquad B_{1n}=\frac{1}{\sinh\frac{n\pi}{a}b}\left(\frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx-A_{1n}\cosh\frac{n\pi}{a}b\right)$$

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$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

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$$u_2(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{b}x+B_n\sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u(0,y)=F(y),\ u(a,y)=G(y),\ 0<y<b$$

$$u(x,0)=f(x),\ \ u(x,b)=g(x),\ \ 0<x<a$$

Superposition

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$$A_{2n}=\frac{2}{b}\int_0^b F(y)\sin\frac{n\pi}{b}y\,dy,$$

Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<a,\quad 0<y<b$$

$$u(0,y)=F(y),\;\; u(a,y)=G(y),\;\; 0<y<b$$

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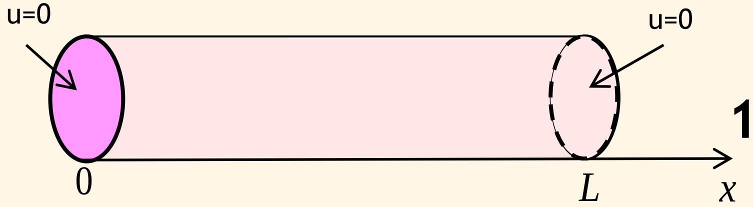
$$A_{2n}=\frac{2}{b}\int_0^b F(y)\sin\frac{n\pi}{b}y\,dy,\qquad B_{2n}=\frac{1}{\sinh\frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^b G(y)\sin\frac{n\pi}{b}y\,dy-A_{2n}\cosh\frac{n\pi}{b}a\right)$$

Nonhomogeneous BVPs

Nonhomogeneous BVPs

Homogeneous Boundary value problem



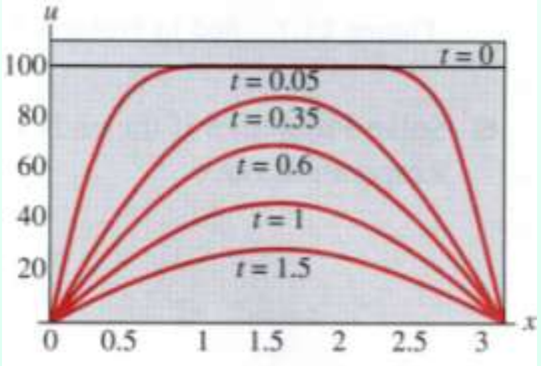
1-D heat equation

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$

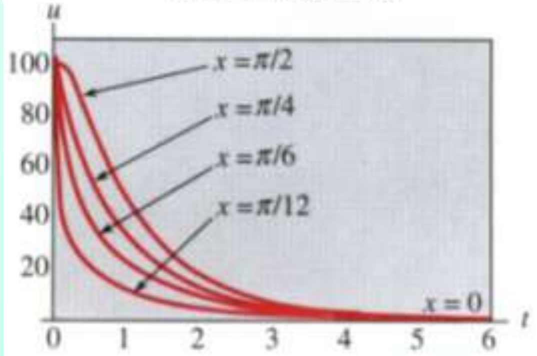
$$u(0,t)=0,\ u(L,t)=0,\ t>0$$

$$u(x,0)=f(x),\ \ 0<x<L$$

$Ex.)\,u(x,0)=100,\ L=\pi,\ k=1$



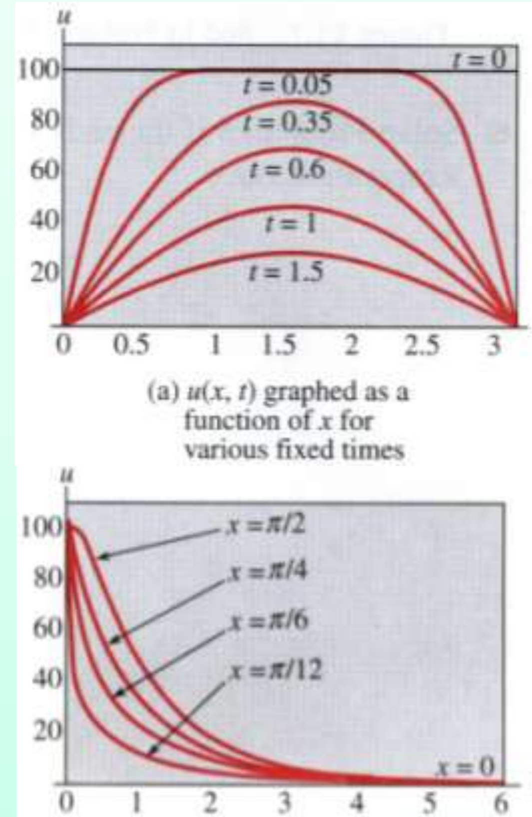
(a) $u(x, t)$ graphed as a function of x for various fixed times



(b) $u(x, t)$ graphed as a function of t for various fixed positions

Nonhomogeneous BVPs

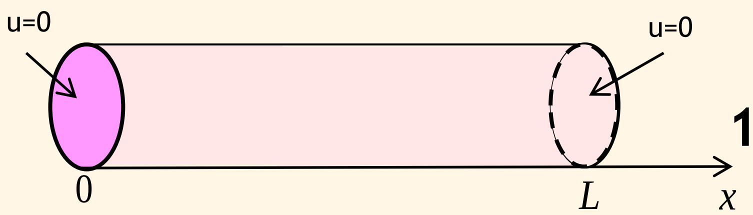
$Ex.)\, u(x,0) = 100,\ L = \pi,\ k = 1$



(a) $u(x, t)$ graphed as a function of x for various fixed times

(b) $u(x, t)$ graphed as a function of t for various fixed positions

Homogeneous Boundary value problem



1-D heat equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0$$ **: homogeneous PDE**

$$u(0,t) = 0,\ u(L,t) = 0,\ t > 0$$

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Nonhomogeneous BVPs

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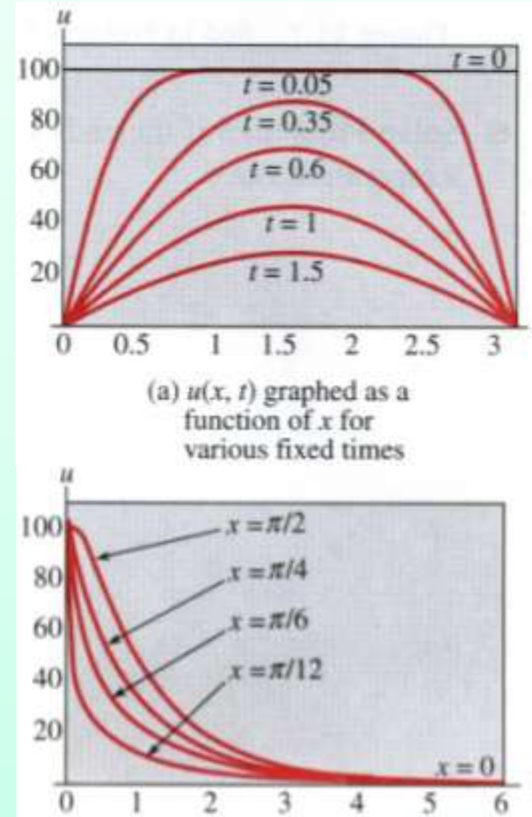

(a) $u(x, t)$ graphed as a function of x for various fixed times

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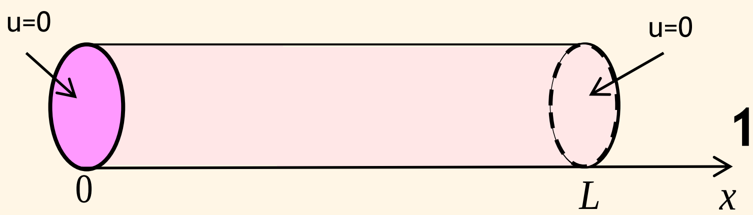
Homogeneous Boundary value problem



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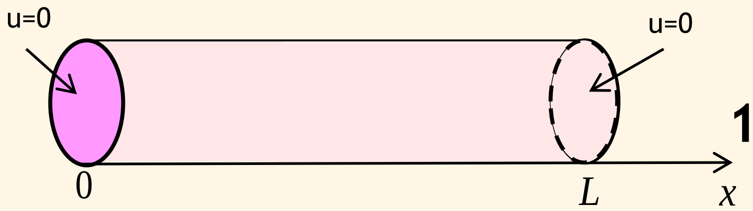
Nonhomogeneous BVPs

Homogeneous Boundary value problem



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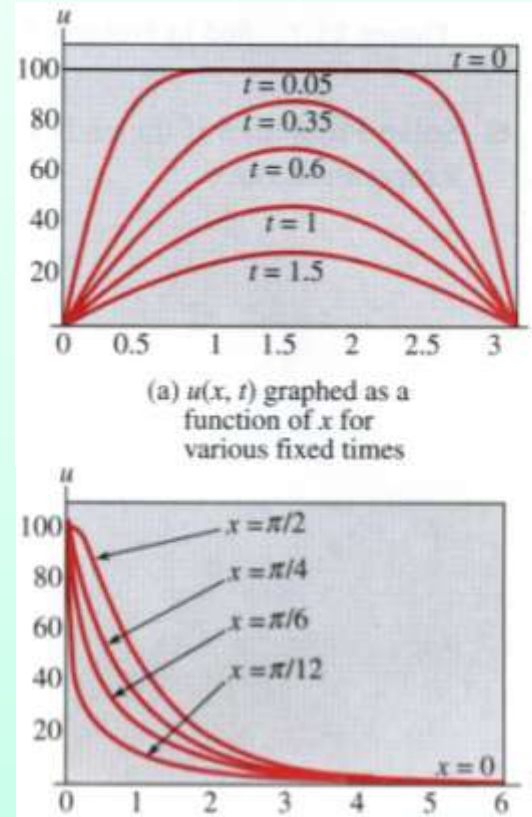

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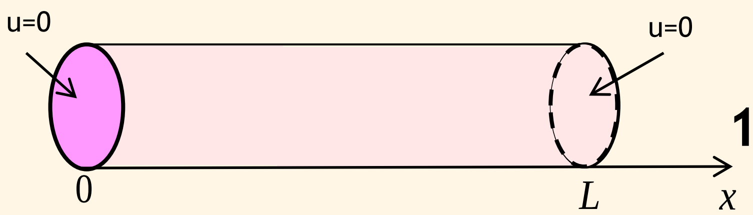
(b) $u(x, t)$ graphed as a function of t for various fixed positions

Nonhomogeneous BVPs

Homogeneous Boundary value problem



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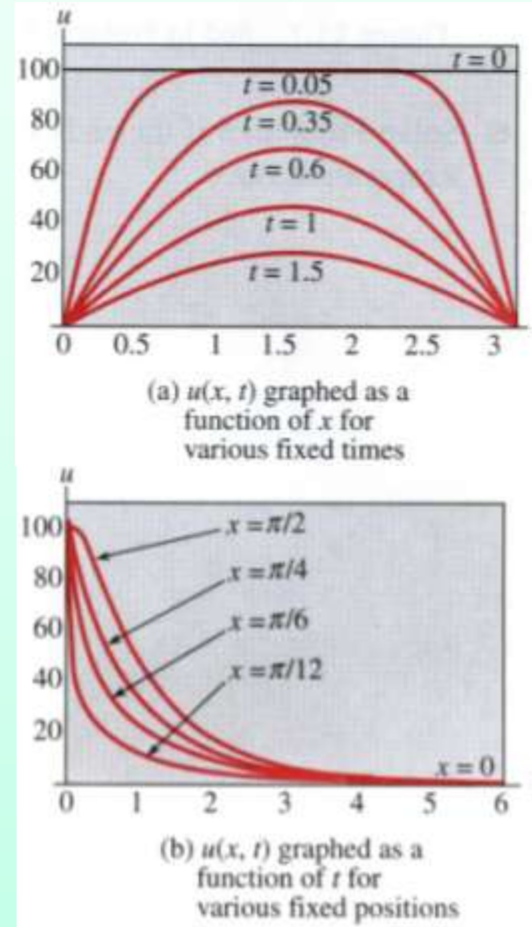
Can be solved by Separation of variable

$Ex.)\,u(x,0)=100,\ L=\pi,\ k=1$



(a) $u(x, t)$ graphed as a function of x for various fixed times

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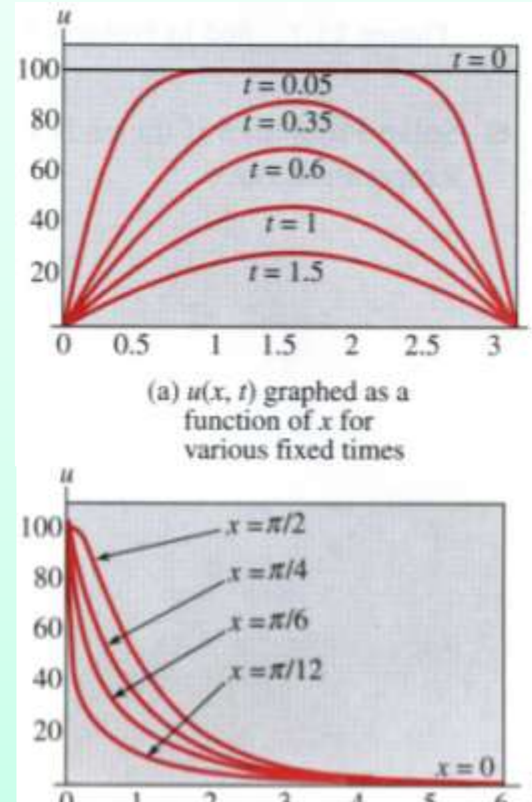
Nonhomogeneous BVPs

$Ex.)\, u(x,0)=100,\ L=\pi,\ k=1$



(a) $u(x, t)$ graphed as a function of x for various fixed times

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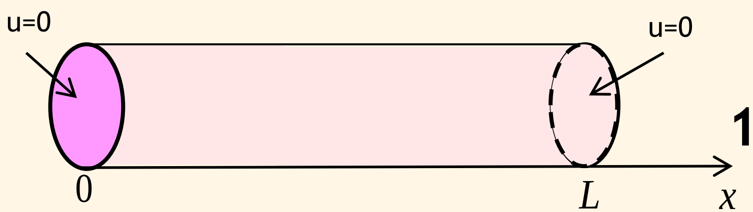
Homogeneous Boundary value problem



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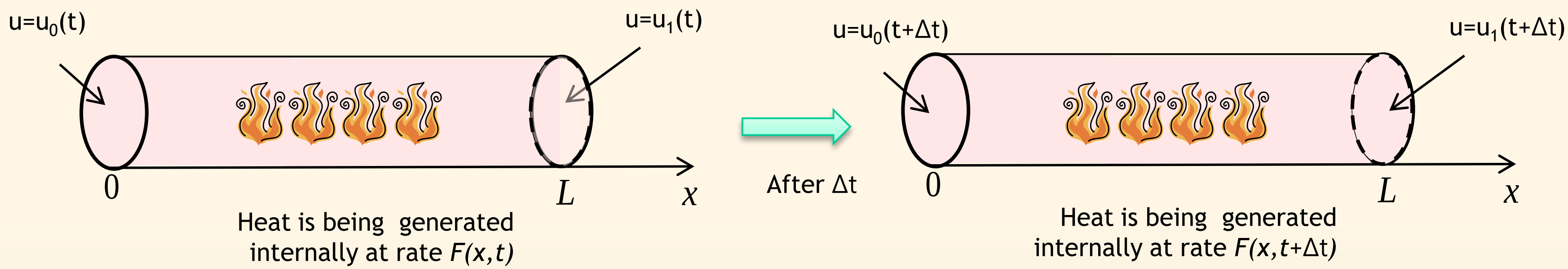
$$u(x,0)=f(x),\ \ 0<x<L$$ **: Initial condition**

Can be solved by Separation of variable

$$u(x,t)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L}e^{-k\frac{n^2\pi^2}{L^2}t}\quad \left(A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx\right)$$

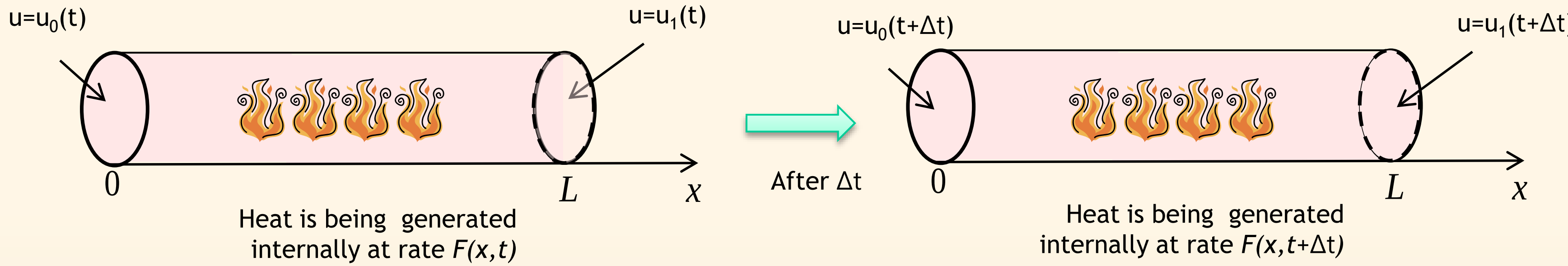
Nonhomogeneous BVPs

Nonhomogeneous Boundary value problem



Nonhomogeneous BVPs

Nonhomogeneous Boundary value problem



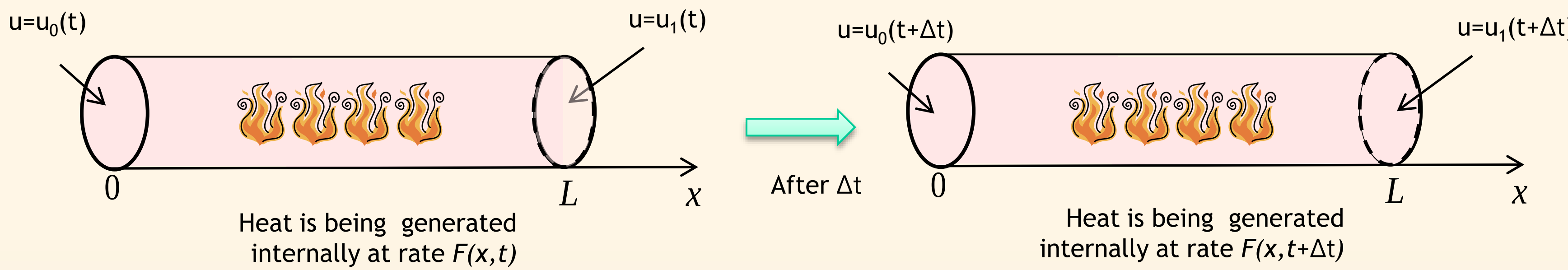
$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$

$$u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0$$

$$u(x,0)=f(x),\ \ 0<x<L$$

Nonhomogeneous BVPs

Nonhomogeneous Boundary value problem



Nonhomogeneous term

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$

$$u(0,t)=\boxed{u_0(t)},\ u(L,t)=\boxed{u_1(t)},\ t>0$$

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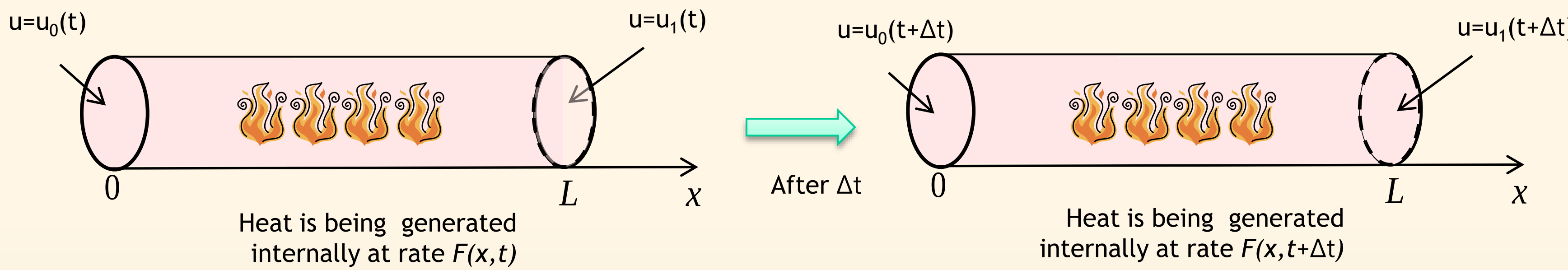
Nonhomogeneous BVPs

Nonhomogeneous Boundary value problem



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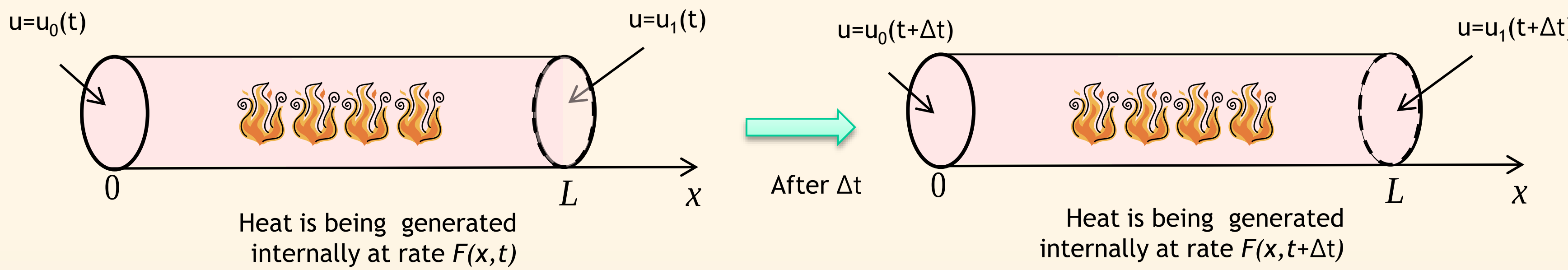
Nonhomogeneous BVPs

Nonhomogeneous Boundary value problem



Nonhomogeneous term

$$k\frac{\partial^2 u}{\partial x^2} + \boxed{F(x,t)} = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0$$ **: Nonhomogeneous PDE**

$$u(0,t) = \boxed{u_0(t)},\ u(L,t) = \boxed{u_1(t)},\ t > 0$$

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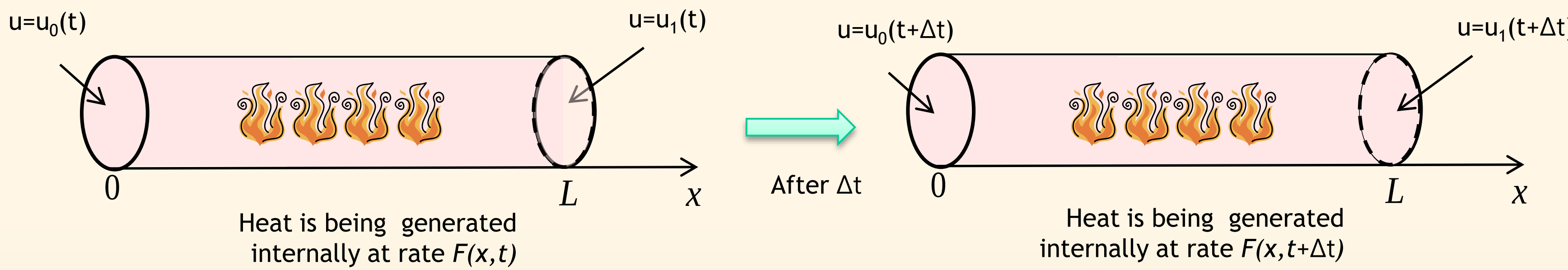
Nonhomogeneous BVPs

Nonhomogeneous Boundary value problem



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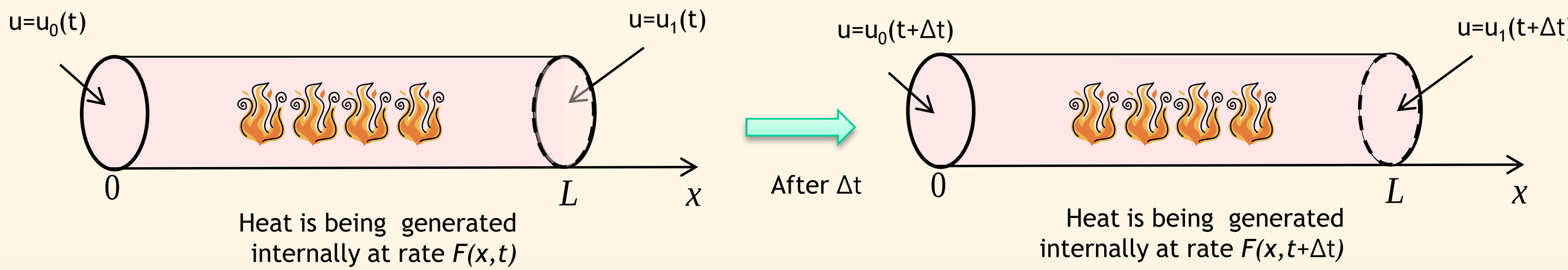
Nonhomogeneous BVPs

Nonhomogeneous Boundary value problem



Nonhomogeneous term

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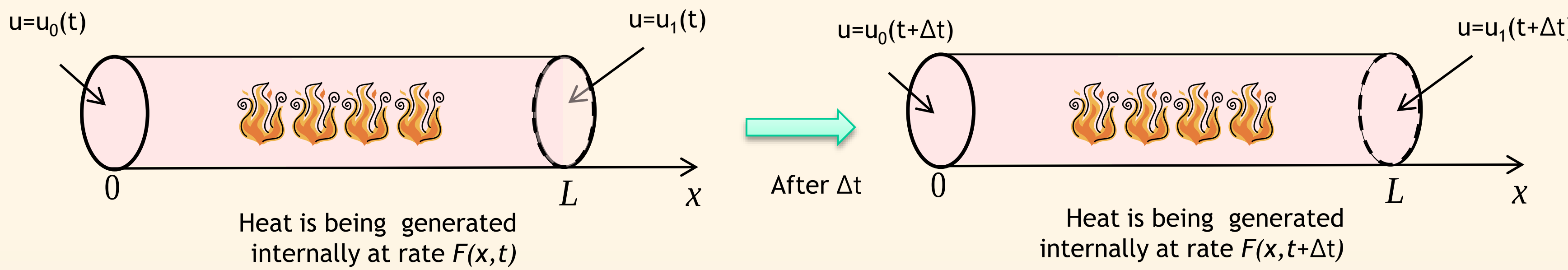
Nonhomogeneous BVPs

Nonhomogeneous Boundary value problem



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$$k\frac{\partial^2 u}{\partial x^2} + \boxed{F(x,t)} = \frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$ **: Nonhomogeneous PDE**

$$u(0,t) = \boxed{u_0(t)},\ u(L,t) = \boxed{u_1(t)},\ t>0$$ **: Nonhomogeneous B.C.**

$$u(x,0) = f(x),\ \ 0<x<L$$ **: Initial condition**

When either PDE or Boundary conditions are nonhomogeneous,
A boundary-value problem is said to be nonhomogeneous.

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$

$$u(0,t)=\boxed{u_0(t)},\ u(L,t)=\boxed{u_1(t)},\ t>0$$

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Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$

$$u(0,t)=\boxed{u_0(t)},\ u(L,t)=\boxed{u_1(t)},\ t>0$$

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Nonhomogeneous Boundary value problem

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$

$$u(0,t)=\boxed{u_0(t)},\ u(L,t)=\boxed{u_1(t)},\ t>0$$

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Nonhomogeneous Boundary value problem

(A) In case of nonhomogeneous PDE

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=\boxed{u_0(t)},\ u(L,t)=\boxed{u_1(t)},\ t>0$$
$$u(x,0)=f(x),\ \ 0<x<L$$

Nonhomogeneous Boundary value problem

(A) In case of nonhomogeneous PDE

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0\ \ \cdots(1)$$

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$

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Nonhomogeneous Boundary value problem

(A) In case of nonhomogeneous PDE

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0\ \cdots(1)$$

If $F(x,t)=\sin(kx-\omega t)$, Equation (1) is not separable,

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2} + \boxed{F(x,t)} = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0$$
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Nonhomogeneous Boundary value problem

(A) In case of nonhomogeneous PDE

$$k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0\ \ \cdots(1)$$

If $F(x,t) = \sin(kx - \omega t)$, Equation (1) is not separable,

(B) In case of nonhomogeneous Boundary Conditions

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
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Nonhomogeneous Boundary value problem

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(B) In case of nonhomogeneous Boundary Conditions

$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$ Subject to $u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0$

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
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Nonhomogeneous Boundary value problem

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Even though substitution $u(x,t)=X(x)T(t),$

Nonhomogeneous BVPs

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Nonhomogeneous Boundary value problem

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$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$ Subject to $u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0$

Even though substitution $u(x,t)=X(x)T(t)$,

Boundary condition $u(0,t)=X(0)T(t)=u_0(t),\ \ u(L,t)=X(L)T(t)=u_1(t)$

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=\boxed{u_0(t)},\ u(L,t)=\boxed{u_1(t)},\ t>0$$
$$u(x,0)=f(x),\ \ 0<x<L$$

Nonhomogeneous Boundary value problem

(A) In case of nonhomogeneous PDE

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0\ \cdots(1)$$

If $F(x,t)=\sin(kx-\omega t)$, Equation (1) is not separable,

(B) In case of nonhomogeneous Boundary Conditions

$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$ Subject to $u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0$

Even though substitution $u(x,t)=X(x)T(t)$,

Boundary condition $u(0,t)=X(0)T(t)=u_0(t),\ \ u(L,t)=X(L)T(t)=u_1(t)$

cannot determine $X(0), X(L)$ because we don't know about $T(t)$

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+\boxed{F(x,t)}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
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(B) In case of nonhomogeneous Boundary Conditions

$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$ Subject to $u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0$

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cannot determine $X(0), X(L)$ because we don't know about $T(t)$

➡ **Nonhomogeneous BVPs can't be solved by separation of variables!!**

Nonhomogeneous BVPs

Time Independent PDE and Boundary Conditions

$$k\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=u_0,\ u(L,t)=u_1,\ t>0$$
$$u(x,0)=f(x),\ \ 0<x<L$$

$$k\frac{\partial^2 v(x,t)}{\partial x^2}+k\psi''(x)+F(x)=\frac{\partial v(x,t)}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=v(0,t)+\psi(0)=u_0,$$
$$u(L,t)=v(L,t)+\psi(L)=u_1,\ t>0$$
$$u(x,0)=v(x,0)+\psi(x)=f(x),\ \ 0<x<L$$

Change of Dependent Variables

$$u(x,t)=v(x,t)+\psi(x)$$

Solution u=Solution ψ of (1) +Solution v of (2)

(1) $k\psi''+F(x)=0,\ \psi(0)=u_0,\ \psi(L)=u_1$ → **Solved by integration**

(2) $$\begin{cases} k\dfrac{\partial^2 v}{\partial x^2}=\dfrac{\partial v}{\partial t}, \\ v(0,t)=0,\ \ v(L,t)=0 \\ v(x,0)=f(x)-\psi(x) \end{cases}$$

→ **Solved by separation of variables**

Nonhomogeneous BVPs

Time Independent PDE and Boundary Conditions

$$k\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=u_0,\ u(L,t)=u_1,\ t>0$$
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Change of Dependent Variables

$$u(x,t)=v(x,t)+\psi(x)$$

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(1) $\boxed{k\psi''+F(x)=0},\ \psi(0)=u_0,\ \psi(L)=u_1$ **Solved by integration**

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Nonhomogeneous BVPs

Time Independent PDE and Boundary Conditions

$$k\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=u_0,\ u(L,t)=u_1,\ t>0$$
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$$u(L,t)=v(L,t)+\psi(L)=u_1,\ t>0$$
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Change of Dependent Variables

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Nonhomogeneous BVPs

Time Independent PDE and Boundary Conditions

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$$k\frac{\partial^2 v(x,t)}{\partial x^2}+\boxed{k\psi''(x)+F(x)}=\frac{\partial v(x,t)}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=v(0,t)+\boxed{\psi(0)=u_0,}$$
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Change of Dependent Variables

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Solved by separation of variables

Nonhomogeneous BVPs

Time Independent PDE and Boundary Conditions

$$k\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
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Change of Dependent Variables

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Solved by separation of variables

Nonhomogeneous BVPs

Time Independent PDE and Boundary Conditions

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Nonhomogeneous BVPs

Time Independent PDE and Boundary Conditions

$$k\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
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$$u(x,0)=v(x,0)+\psi(x)=f(x),\ \ 0<x<L$$

Change of Dependent Variables

$$u(x,t)=v(x,t)+\psi(x)$$

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$\boxed{v(0,t)=0},\ \boxed{v(L,t)=0}$ **Solved by separation of variables**
$v(x,0)=f(x)-\psi(x)$

Nonhomogeneous BVPs

Time Independent PDE and Boundary Conditions

$$k\frac{\partial^2 u}{\partial x^2}+F(x)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
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Nonhomogeneous BVPs

(1) $k\psi'' + F(x) = 0,$
$\psi(0) = u_0(t),\ \psi(L) = u_1(t)$

(2) $k\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t},$
$v(0,t) = 0,\ \ v(L,t) = 0$
$v(x,0) = f(x) - \psi(x)$

☑ Example 1 Time Independent PDE and BCs

Solve equation $$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}$$

subject to $$u(0,t) = 0,\ u(1,t) = u_0,\ t > 0$$
$$u(x,0) = f(x),\ 0 < x < 1$$

Nonhomogeneous BVPs

$$(1)\ \ k\psi'' + F(x) = 0,$$
$$\psi(0) = u_0(t),\ \psi(L) = u_1(t)$$

$$(2)\ k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t},$$
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Solution) Change of Dependent Variables

Nonhomogeneous BVPs

$$(1)\ \ k\psi'' + F(x) = 0,$$
$$\psi(0) = u_0(t),\ \psi(L) = u_1(t)$$

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$u(x,0) = f(x),\ 0 < x < 1$

Solution) Change of Dependent Variables

Let $u(x,t) = v(x,t) + \psi(x)$

Nonhomogeneous BVPs

$(1)\ k\psi'' + F(x) = 0,$
$\psi(0) = u_0(t),\ \psi(L) = u_1(t)$

$(2)\ k\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t},$
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☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}$

subject to $u(0,t) = 0,\ u(1,t) = u_0,\ t > 0$

$u(x,0) = f(x),\ 0 < x < 1$

Solution) **Change of Dependent Variables**

Let $u(x,t) = v(x,t) + \psi(x)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \psi'' \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}$$

Nonhomogeneous BVPs

$$(1)\ \ k\psi'' + F(x) = 0,$$
$$\psi(0) = u_0(t),\ \psi(L) = u_1(t)$$

$$(2)\ k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t},$$
$$v(0,t) = 0,\ \ v(L,t) = 0$$
$$v(x,0) = f(x) - \psi(x)$$

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}$

subject to $u(0,t) = 0,\ u(1,t) = u_0,\ t > 0$

$u(x,0) = f(x),\ 0 < x < 1$

Solution) **Change of Dependent Variables**

Let $u(x,t) = v(x,t) + \psi(x)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \psi'' \quad and \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}$$

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' + r = \frac{\partial v}{\partial t}$$

$$(1)\ k\psi''+F(x)=0,$$
$$\psi(0)=u_0(t),\ \psi(L)=u_1(t)$$
$$(2)\ k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t},$$
$$v(0,t)=0,\ \ v(L,t)=0$$
$$v(x,0)=f(x)-\psi(x)$$

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $$k\frac{\partial^2 u}{\partial x^2}+r=\frac{\partial u}{\partial t}$$

subject to $$u(0,t)=0,\ u(1,t)=u_0,\ t>0$$
$$u(x,0)=f(x),\ 0<x<1$$

Solution) **Change of Dependent Variables**

Let $u(x,t)=v(x,t)+\psi(x)$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x^2}+\psi'' \quad and \quad \frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}$$

$$k\frac{\partial^2 v}{\partial x^2}+k\psi''+r=\frac{\partial v}{\partial t} \Rightarrow$$

$$1)\ \ k\psi''+r=0\ \ or\ \psi''=-\frac{r}{k}$$

$$2)\ \ k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}\ \ 0<x<1,\ t>0$$

Nonhomogeneous BVPs

$$u(0,t)=v(0,t)+\psi(0)=0$$
$$u(1,t)=v(1,t)+\psi(1)=u_0$$
$$u(x,0)=v(x,0)+\psi(x)=f(x)$$

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2}+r=\dfrac{\partial u}{\partial t}$

subject to $u(0,t)=0,\ u(1,t)=u_0,\ t>0$

$u(x,0)=f(x),\ 0<x<1$

$u(x,t)=v(x,t)+\psi(x)$

$$\boxed{\begin{gathered}\psi''=-\frac{r}{k}\\ \psi(0)=0\\ \psi(1)=u_0\end{gathered}} + \boxed{\begin{gathered}k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}\\ v(0,t)=0\\ v(1,t)=0\\ v(x,0)=f(x)-\psi(x)\end{gathered}}$$

$$\boxed{1)\ \ \psi''=-\frac{r}{k}}$$

$$u(0,t) = v(0,t) + \psi(0) = 0$$
$$u(1,t) = v(1,t) + \psi(1) = u_0$$
$$u(x,0) = v(x,0) + \psi(x) = f(x)$$

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}$$

subject to $$u(0,t) = 0,\ u(1,t) = u_0,\ t > 0$$
$$u(x,0) = f(x),\ 0 < x < 1$$

$$u(x,t) = v(x,t) + \psi(x) \Rightarrow$$

$$\boxed{\begin{gathered}\psi'' = -\frac{r}{k}\\ \psi(0) = 0\\ \psi(1) = u_0\end{gathered}} + \boxed{\begin{gathered}k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}\\ v(0,t) = 0\\ v(1,t) = 0\\ v(x,0) = f(x) - \psi(x)\end{gathered}}$$

$$\boxed{1)\ \psi'' = -\frac{r}{k}}$$

$$\psi(x) = -\frac{r}{2k}x^2 + c_1 x + c_2$$

$$u(0,t)=v(0,t)+\psi(0)=0$$
$$u(1,t)=v(1,t)+\psi(1)=u_0$$
$$u(x,0)=v(x,0)+\psi(x)=f(x)$$

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2}+r=\dfrac{\partial u}{\partial t}$

subject to $u(0,t)=0,\ u(1,t)=u_0,\ t>0$

$u(x,0)=f(x),\ 0<x<1$

$u(x,t)=v(x,t)+\psi(x)$ ⇨

$$\boxed{\begin{gathered}\psi''=-\frac{r}{k}\\ \psi(0)=0\\ \psi(1)=u_0\end{gathered}}\ +\ \boxed{\begin{gathered}k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}\\ v(0,t)=0\\ v(1,t)=0\\ v(x,0)=f(x)-\psi(x)\end{gathered}}$$

$$\boxed{1)\ \ \psi''=-\frac{r}{k}}$$

$$\psi(x)=-\frac{r}{2k}x^2+c_1x+c_2$$

$$\psi(0)=c_2=0$$

$$\psi(1)=-\frac{r}{2k}+c_1=u_0,\ \ c_1=\frac{r}{2k}+u_0$$

Nonhomogeneous BVPs

$$u(0,t)=v(0,t)+\psi(0)=0$$
$$u(1,t)=v(1,t)+\psi(1)=u_0$$
$$u(x,0)=v(x,0)+\psi(x)=f(x)$$

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2}+r=\dfrac{\partial u}{\partial t}$

subject to $u(0,t)=0,\ u(1,t)=u_0,\ t>0$

$u(x,0)=f(x),\ 0<x<1$

$$\xrightarrow{u(x,t)=v(x,t)+\psi(x)}\quad \boxed{\begin{array}{c}\psi''=-\dfrac{r}{k}\\ \psi(0)=0\\ \psi(1)=u_0\end{array}}\ +\ \boxed{\begin{array}{c}k\dfrac{\partial^2 v}{\partial x^2}=\dfrac{\partial v}{\partial t}\\ v(0,t)=0\\ v(1,t)=0\\ v(x,0)=f(x)-\psi(x)\end{array}}$$

$$\boxed{1)\ \psi''=-\frac{r}{k}}$$

$$\psi(x)=-\frac{r}{2k}x^2+c_1x+c_2$$

$$\psi(0)=c_2=0$$

$$\psi(1)=-\frac{r}{2k}+c_1=u_0,\ \ c_1=\frac{r}{2k}+u_0$$

$$\therefore \psi(x)=-\frac{r}{2k}x^2+\left(\frac{r}{2k}+u_0\right)x$$

Nonhomogeneous BVPs

$$u(0,t) = v(0,t) + \psi(0) = 0$$
$$u(1,t) = v(1,t) + \psi(1) = u_0$$
$$u(x,0) = v(x,0) + \psi(x) = f(x)$$

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}$

subject to $u(0,t) = 0,\ u(1,t) = u_0,\ t > 0$

$u(x,0) = f(x),\ 0 < x < 1$

$u(x,t) = v(x,t) + \psi(x)$

$$\boxed{\begin{gathered}\psi'' = -\frac{r}{k} \\ \psi(0) = 0 \\ \psi(1) = u_0\end{gathered}} + \boxed{\begin{gathered}k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \\ v(0,t) = 0 \\ v(1,t) = 0 \\ v(x,0) = f(x) - \psi(x)\end{gathered}}$$

$$\boxed{2)\ k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad 0 < x < 1,\ t > 0}$$

$$u(0,t)=v(0,t)+\psi(0)=0$$
$$u(1,t)=v(1,t)+\psi(1)=u_0$$
$$u(x,0)=v(x,0)+\psi(x)=f(x)$$

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2}+r=\dfrac{\partial u}{\partial t}$

subject to $u(0,t)=0,\ u(1,t)=u_0,\ t>0$

$u(x,0)=f(x),\ 0<x<1$

$u(x,t)=v(x,t)+\psi(x)$ ⇨

$$\boxed{\begin{gathered}\psi''=-\frac{r}{k}\\ \psi(0)=0\\ \psi(1)=u_0\end{gathered}}\ +\ \boxed{\begin{gathered}k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}\\ v(0,t)=0\\ v(1,t)=0\\ v(x,0)=f(x)-\psi(x)\end{gathered}}$$

$$\boxed{2)\ \ k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}\quad 0<x<1,\ t>0}$$

$$v(0,t)=0,$$
$$v(1,t)=0,\ t>0$$
$$\begin{aligned}v(x,0)&=f(x)-\psi(x)\\ &=f(x)+\frac{r}{2k}x^2-\left(\frac{r}{2k}+u_0\right)x\ ,\ 0<x<1\end{aligned}$$

Nonhomogeneous BVPs

$$u(0,t)=v(0,t)+\psi(0)=0$$
$$u(1,t)=v(1,t)+\psi(1)=u_0$$
$$u(x,0)=v(x,0)+\psi(x)=f(x)$$

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2}+r=\dfrac{\partial u}{\partial t}$

subject to $u(0,t)=0,\ u(1,t)=u_0,\ t>0$

$u(x,0)=f(x),\ 0<x<1$

$u(x,t)=v(x,t)+\psi(x)$

$$\boxed{\begin{aligned}&\psi''=-\frac{r}{k}\\&\psi(0)=0\\&\psi(1)=u_0\end{aligned}} \quad + \quad \boxed{\begin{aligned}&k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}\\&v(0,t)=0\\&v(1,t)=0\\&v(x,0)=f(x)-\psi(x)\end{aligned}}$$

$$\boxed{2)\ \ k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}\quad 0<x<1,\ t>0}$$

$$v(0,t)=0,$$
$$v(1,t)=0,\ t>0$$
$$v(x,0)=f(x)-\psi(x)$$
$$=f(x)+\frac{r}{2k}x^2-\left(\frac{r}{2k}+u_0\right)x\ ,\ 0<x<1$$

Homogeneous boundary-value problem

Can be solved by separation of variables!!

$$\psi(x) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x$$

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}$

subject to $u(0,t) = 0,\ u(1,t) = u_0,\ t > 0$

$u(x,0) = f(x),\ 0 < x < 1$

$u(x,t) = v(x,t) + \psi(x)$

$$\boxed{\begin{gathered}\psi'' = -\frac{r}{k} \\ \psi(0) = 0 \\ \psi(1) = u_0\end{gathered}} + \boxed{\begin{gathered}k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \\ v(0,t) = 0 \\ v(1,t) = 0 \\ v(x,0) = f(x) - \psi(x)\end{gathered}}$$

$k\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}\quad 0 < x < 1,\ t > 0$ **: 1-D heat equation**

Boundary condition

$v(0,t) = 0\ , v(1,t) = 0$

Initial condition

$$v(x,0) = f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)x\ ,\ 0 < x < 1$$

$$\psi(x) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x$$

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}$

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The solution is $v(x,t) = \displaystyle\sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t}\sin n\pi x$ **(ref. 1-D heat equation*)**

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}$

subject to $u(0,t) = 0,\ u(1,t) = u_0,\ t > 0$

$u(x,0) = f(x),\ 0 < x < 1$

$u(x,t) = v(x,t) + \psi(x)$ ⇒

$$\boxed{\psi'' = -\frac{r}{k},\quad \psi(0) = 0,\quad \psi(1) = u_0} + \boxed{k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t},\quad v(0,t) = 0,\quad v(1,t) = 0,\quad v(x,0) = f(x) - \psi(x)}$$

$k\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}$ $\quad 0 < x < 1,\ t > 0$ **: 1-D heat equation**

Boundary condition

$v(0,t) = 0\ , v(1,t) = 0$

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$$v(x,0) = f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)x\ ,\ 0 < x < 1$$

The solution is $v(x,t) = \displaystyle\sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x$ **(ref. 1-D heat equation*)**

$$, A_n = 2\int_0^1 \left[f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)x \right] \sin n\pi x dx$$ **: Fourier Sine Coefficients**

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2}+r=\dfrac{\partial u}{\partial t}$

subject to $u(0,t)=0,\ u(1,t)=u_0,\ t>0$

$u(x,0)=f(x),\ 0<x<1$

$$u(x,t)=v(x,t)+\psi(x)$$

$$\begin{cases} v(x,t)=\displaystyle\sum_{n=1}^{\infty}A_n e^{-kn^2\pi^2 t}\sin n\pi x \\ \psi(x)=-\dfrac{r}{2k}x^2+\left(\dfrac{r}{2k}+u_0\right)x \end{cases}$$

$$\therefore u(x,t)=\sum_{n=1}^{\infty}A_n e^{-kn^2\pi^2 t}\sin n\pi x+\left(-\frac{r}{2k}x^2+\left(\frac{r}{2k}+u_0\right)x\right)$$

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2}+r=\dfrac{\partial u}{\partial t}$

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$$\therefore u(x,t)=\sum_{n=1}^{\infty}A_n e^{-kn^2\pi^2 t}\sin n\pi x+\left(-\frac{r}{2k}x^2+\left(\frac{r}{2k}+u_0\right)x\right)$$

Nonhomogeneous BVPs

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Solve equation $k\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}$

subject to $u(0,t) = 0,\ u(1,t) = u_0,\ t > 0$

$u(x,0) = f(x),\ 0 < x < 1$

$$u(x,t) = v(x,t) + \psi(x)$$

$$\begin{cases} \boxed{v(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x} \\ \psi(x) = -\dfrac{r}{2k}x^2 + \left(\dfrac{r}{2k} + u_0\right)x \end{cases} \quad as\ \ t \to \infty,\ \begin{cases} u(x,t) \to \psi(x) \\ v(x,t) \to 0 \end{cases}$$

$$\therefore u(x,t) = \underbrace{\boxed{\sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x}}_{\text{transient solution}} + \left(-\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x\right)$$

Nonhomogeneous BVPs

☑ Example 1 Time Independent PDE and BCs

Solve equation $k\dfrac{\partial^2 u}{\partial x^2} + r = \dfrac{\partial u}{\partial t}$

subject to $u(0,t)=0,\ u(1,t)=u_0,\ t>0$

$u(x,0)=f(x),\ 0<x<1$

$$u(x,t)=v(x,t)+\psi(x)$$

$$\begin{cases} \boxed{v(x,t)=\sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t}\sin n\pi x} \\ \boxed{\psi(x)=-\dfrac{r}{2k}x^2+\left(\dfrac{r}{2k}+u_0\right)x} \end{cases} \quad as\ \ t\to\infty,\ \begin{cases} u(x,t)\to\psi(x) \\ v(x,t)\to 0 \end{cases}$$

$$\therefore u(x,t)=\underbrace{\boxed{\sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t}\sin n\pi x}}_{\text{transient solution}}+\underbrace{\boxed{\left(-\frac{r}{2k}x^2+\left(\frac{r}{2k}+u_0\right)x\right)}}_{\text{Steady-state solution}}$$

Nonhomogeneous BVPs

Time Dependent PDE and Boundary Conditions

$$\boxed{\begin{aligned} & k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0 \\ & u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0 \\ & u(x,0)=f(x),\ \ 0<x<L \end{aligned}} \quad \cdots(1)$$

Nonhomogeneous BVPs

Time Dependent PDE and Boundary Conditions

$$\boxed{\begin{aligned} & k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t},\ 0<x<L,\ t>0 \\ & u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0 \\ & u(x,0)=f(x),\ \ 0<x<L \end{aligned}} \quad \cdots(1)$$

Change of Dependent Variables

$$u(x,t) = v(x,t) + \psi(x,t)$$

Nonhomogeneous BVPs

Time Dependent PDE and Boundary Conditions

$$\boxed{\begin{aligned} & k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0 \\ & u(0,t) = u_0(t),\ u(L,t) = u_1(t),\ t > 0 \\ & u(x,0) = f(x),\ \ 0 < x < L \end{aligned}} \cdots (1)$$

Change of Dependent Variables

$$u(x,t) = v(x,t) + \psi(x,t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial \psi}{\partial t} \quad \text{then,}$$

Nonhomogeneous BVPs

Time Dependent PDE and Boundary Conditions

$$\boxed{\begin{aligned} & k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0 \\ & u(0,t) = u_0(t),\ u(L,t) = u_1(t),\ t > 0 \\ & u(x,0) = f(x),\ \ 0 < x < L \end{aligned}} \cdots(1)$$

Change of Dependent Variables

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$$\boxed{\begin{aligned} & k\frac{\partial^2 v}{\partial x^2} + k\frac{\partial^2 \psi}{\partial x^2} + F(x,t) = \frac{\partial v}{\partial t} + \frac{\partial \psi}{\partial t} \cdots(2) \\ & v(0,t) + \psi(0,t) = u_0(t),\ v(L,t) + \psi(L,t) = u_1(t) \\ & v(x,0) = f(x) - \psi(x,0) \end{aligned}}$$

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0 \qquad \cdots(1)$$
$$u(x,0)=f(x),\ \ 0<x<L$$

Time Dependent PDE and Boundary Conditions

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Time Dependent PDE and Boundary Conditions

$$u(x,t)=v(x,t)+\psi(x,t)$$

$$k\frac{\partial^2 v}{\partial x^2}+k\frac{\partial^2 \psi}{\partial x^2}+F(x,t)=\frac{\partial v}{\partial t}+\frac{\partial \psi}{\partial t} \quad \cdots(2)$$
$$v(0,t)+\psi(0,t)=u_0(t),\ v(L,t)+\psi(L,t)=u_1(t)$$
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Time Dependent PDE and Boundary Conditions

$$k\frac{\partial^2 v}{\partial x^2}+k\frac{\partial^2 \psi}{\partial x^2}+F(x,t)=\frac{\partial v}{\partial t}+\frac{\partial \psi}{\partial t} \quad \cdots(2)$$
$$v(0,t)+\psi(0,t)=u_0(t),\ v(L,t)+\psi(L,t)=u_1(t)$$
$$v(x,0)=f(x)-\psi(x,0)$$

A

$$\psi(0,t)=u_0(t),\ \psi(L,t)=u_1(t) \quad \cdots(3)$$

$$k\frac{\partial^2 \psi}{\partial x^2}+F(x,t)=\frac{\partial \psi}{\partial t}\cdots(4)$$

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0 \quad \cdots(1)$$
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$$\psi(0,t)=u_0(t),\ \psi(L,t)=u_1(t)\quad\cdots(3)$$
$$k\frac{\partial^2 \psi}{\partial x^2}+F(x,t)=\frac{\partial \psi}{\partial t}\cdots(4)$$

B

$$k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t}$$
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Nonhomogeneous BVPs

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If **A** can be solved, the original problem can be a homogeneous problem **B** (like example 1)

Nonhomogeneous BVPs

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BUT ! because (4) is nonhomogeneous itself with nonhomogeneous conditions (3), this appoach is unrealistic!

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So, Simply constructing a function ψ that satisfies conditions (3)

One such a function is given by
$$\boxed{\psi(x,t)=u_0(t)+\frac{x}{L}[u_1(t)-u_0(t)]}$$

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0 \qquad \cdots(1)$$
$$u(x,0)=f(x),\ \ 0<x<L$$

Time Dependent PDE and Boundary Conditions

$$u(x,t)=v(x,t)+\psi(x,t)$$

$$k\frac{\partial^2 v}{\partial x^2}+k\frac{\partial^2 \psi}{\partial x^2}+F(x,t)=\frac{\partial v}{\partial t}+\frac{\partial \psi}{\partial t} \quad \cdots(2)$$
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'satisfies conditions (3)'

$$\psi(0,t)=u_0(t)$$
$$\psi(L,t)=u_1(t)$$

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0 \qquad \cdots(1)$$
$$u(x,0)=f(x),\ \ 0<x<L$$

Time Dependent PDE and Boundary Conditions

$$\psi(0,t)=u_0(t),\ \psi(L,t)=u_1(t)\ \ \cdots(3)$$

Simply constructing a function ψ that satisfies conditions (3)

One such a function is given by $\boxed{\psi(x,t)=u_0(t)+\frac{x}{L}[u_1(t)-u_0(t)]}$

Nonhomogeneous BVPs

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Nonhomogeneous BVPs

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Nonhomogeneous BVPs

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$$\Downarrow$$
$$k\frac{\partial^2 v}{\partial x^2}+F(x,t)-\frac{\partial \psi}{\partial t}=\frac{\partial v}{\partial t}$$

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where $\begin{array}{l}\psi(0,t)=u_0(t)\\ \psi(L,t)=u_1(t)\end{array},\ \frac{\partial^2\psi}{\partial x^2}=0$

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Nonhomogeneous BVPs

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$$\boxed{\begin{gathered}k\frac{\partial^2 v}{\partial x^2}+G(x,t)=\frac{\partial v}{\partial t}\\ v(0,t)=0,\ v(L,t)=0\\ v(x,0)=f(x)-\psi(x,0)\end{gathered}}$$

Nonhomogeneous BVPs

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This is a problem that we can solve by a strategy

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
$$u(0,t)=u_0(t),\ u(L,t)=u_1(t),\ t>0 \qquad \cdots(1)$$
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Time Dependent PDE and Boundary Conditions

$$\psi(x,t)=u_0(t)+\frac{x}{L}[u_1(t)-u_0(t)]$$
$$k\frac{\partial^2 v}{\partial x^2}+G(x,t)=\frac{\partial v}{\partial t},\ 0<x<L,\ \ t>0 \quad \cdots(4)$$
$$v(0,t)=0,\ \ v(L,t)=0,\ t>0$$
$$v(x,0)=f(x)-\psi(x,0),\ \ 0<x<L$$

where

$$G(x,t)=F(x,t)-\psi_t$$

$$u(x,t)=v(x,t)+\psi(x,t)$$
$$\downarrow$$
$$u(x,t)=v(x,t)+u_0(t)+\frac{x}{L}[u_1(t)-u_0(t)]$$

It is still nonhomogeneous PDE, but we can solve it by next strategy.

Nonhomogeneous BVPs

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
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Time Dependent PDE and Boundary Conditions

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where $G(x,t)=F(x,t)-\psi_t$

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Make the assumption that time-dependent coefficients $v_n(t)$ and $G_n(t)$ can be found such that both $v(x,t)$ and $G(x,t)$ can be expanded in the series

$$v(x,t)=\sum_{n=1}^{\infty}v_n(t)\sin\frac{n\pi}{L}x \quad and \quad G(x,t)=\sum_{n=1}^{\infty}G_n(t)\sin\frac{n\pi}{L}x$$

Nonhomogeneous BVPs

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$$u(x,0)=f(x),\ 0<x<L$$

Time Dependent PDE and Boundary Conditions

$$\psi(x,t)=u_0(t)+\frac{x}{L}[u_1(t)-u_0(t)]$$
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$$v(0,t)=0,\ v(L,t)=0,\ t>0$$
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where $\sin(n\pi x/L)$ Are the eigenfunctions of

$$X''+\lambda X=0,\ X(0)=0,\ X(L)=0$$

It is obtained from *v(x,t)=X(x)T(t)*

$$k\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t},\quad v(0,t)=0,\ v(L,t)=0,\ 0<x<L,\ t>0$$

(1-D homogeneous heat equation)

Nonhomogeneous BVPs

☑ Example 2 Time Dependent PDE and BCs

Solve equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t},\ 0 < x < 1,\ t > 0$

subject to $u(0,t) = \cos t,\ u(1,t) = 0,\ t > 0$

$u(x,0) = 0,\ 0 < x < 1$

$$k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0$$

$$u(0,t) = u_0(t),\ u(L,t) = u_1(t),\ t > 0 \qquad \cdots(1)$$

$$u(x,0) = f(x),\ 0 < x < L$$

Let $u(x,t) = v(x,t) + \psi(x,t)$

$$\psi(x,t) = u_0(t) + \frac{x}{L}[u_1(t) - u_0(t)]$$

$$k\frac{\partial^2 v}{\partial x^2} + G(x,t) = \frac{\partial v}{\partial t},\ 0 < x < L,\ t > 0$$

where

$$v(0,t) = 0,\ v(L,t) = 0,\ t > 0 \quad G(x,t) = F(x,t) - \psi_t$$

$$v(x,0) = f(x) - \psi(x,0),\ 0 < x < L$$

Nonhomogeneous BVPs

☑ **Example 2** **Time Dependent PDE and BCs**

Solve equation $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ 0<x<1,\ t>0$$

subject to $$u(0,t)=\cos t,\ u(1,t)=0,\ t>0$$
$$u(x,0)=0,\ 0<x<1$$

$$\psi(x,t)=\cos t+x[0-\cos t]=(1-x)\cos t$$

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$
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Let $u(x,t)=v(x,t)+\psi(x,t)$

$$\psi(x,t)=u_0(t)+\frac{x}{L}[u_1(t)-u_0(t)]$$

$$k\frac{\partial^2 v}{\partial x^2}+G(x,t)=\frac{\partial v}{\partial t},\ 0<x<L,\ t>0$$

where

$$v(0,t)=0,\ v(L,t)=0,\ t>0 \qquad G(x,t)=F(x,t)-\psi_t$$
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Nonhomogeneous BVPs

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$$\psi(x,t) = \cos t + x[0 - \cos t] = (1-x)\cos t$$

$$u(x,t) = v(x,t) + (1-x)\cos t$$

$$k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0$$

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Nonhomogeneous BVPs

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$$u(x,t) = v(x,t) + (1-x)\cos t$$

$$\psi_t = -(1-x)\sin t$$

$$k\frac{\partial^2 u}{\partial x^2} + F(x,t) = \frac{\partial u}{\partial t}, \ 0 < x < L, \ t > 0$$

$$u(0,t) = u_0(t), \ u(L,t) = u_1(t), \ t > 0 \qquad \cdots(1)$$

$$u(x,0) = f(x), \ 0 < x < L$$

Let $u(x,t) = v(x,t) + \psi(x,t)$

$$\psi(x,t) = u_0(t) + \frac{x}{L}[u_1(t) - u_0(t)]$$

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$$v(0,t) = 0, \ v(L,t) = 0, \ t > 0$$

where $G(x,t) = F(x,t) - \psi_t$

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Nonhomogeneous BVPs

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$$u(0,t) = u_0(t),\ u(L,t) = u_1(t),\ t > 0 \qquad \cdots(1)$$

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$v(0,t) = 0,\ v(L,t) = 0,\ t > 0$ $\quad G(x,t) = F(x,t) - \psi_t$

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Nonhomogeneous BVPs

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where

$$v(0,t)=0,\ \ v(L,t)=0,\ t>0 \quad G(x,t)=F(x,t)-\psi_t$$

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$$\psi(x,t)=\cos t+x[0-\cos t]=(1-x)\cos t$$

$$u(x,t)=v(x,t)+(1-x)\cos t$$

$$\psi_t=-(1-x)\sin t$$

$$G(x,t)=F(x,t)-\psi_t=0-(-(1-x)\sin t)=(1-x)\sin t$$

$$\boxed{\begin{aligned}&\frac{\partial^2 v}{\partial x^2}+(1-x)\sin t=\frac{\partial v}{\partial t},\ \ 0<x<1,\ t>0\\ &v(0,t)=0, v(1,t)=0,\ t>0\\ &v(x,0)=x-1,\ 0<x<1\end{aligned}}$$

$$u(x,t) = v(x,t) + \psi(x,t)$$

$$\psi(x,t) = (1-x)\cos t$$

Nonhomogeneous BVPs

☑ Example 2 Time Dependent PDE and BCs

Solve equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \ 0<x<1, \ t>0$

subject to $u(0,t) = \cos t, \ u(1,t) = 0, \ t>0$

$u(x,0) = 0, \ 0<x<1$

$$\boxed{\begin{aligned} &\frac{\partial^2 v}{\partial x^2} + (1-x)\sin t = \frac{\partial v}{\partial t}, \ 0<x<1, \ t>0 \\ &v(0,t)=0, v(1,t)=0, \ t>0 \\ &v(x,0) = x-1, \ 0<x<1 \end{aligned}}$$

Find the eigenvalues and eignefunctions of for

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \ 0<x<1, \ t>0$$

Make the assumption that time-dependent coefficients $v_n(t)$ and $G_n(t)$ can be found such that both $v(x,t)$ and $G(x,t)$ can be expanded in the series

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin\frac{n\pi}{L}x \quad and \quad G(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin\frac{n\pi}{L}x$$

where $\sin(n\pi x/L)$ Are the eigenfunctions of

$$X'' + \lambda X = 0, \ X(0)=0, \ X(L)=0$$

It is obtained from *v(x,t)=X(x)T(t)*

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \ \ v(0,t)=0, \ v(L,t)=0, \ 0<x<L, \ t>0$$ (1-D homogeneous heat equation)

$$u(x,t) = v(x,t) + \psi(x,t)$$

$$\psi(x,t) = (1-x)\cos t$$

Nonhomogeneous BVPs

☑ Example 2 Time Dependent PDE and BCs

Solve equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t},\ \ 0 < x < 1,\ t > 0$

subject to $u(0,t) = \cos t,\ u(1,t) = 0,\ t > 0$

$u(x,0) = 0,\ 0 < x < 1$

$$\frac{\partial^2 v}{\partial x^2} + (1-x)\sin t = \frac{\partial v}{\partial t},\ \ 0 < x < 1,\ t > 0$$
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Find the eigenvalues and eignefunctions of for

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t},\ \ 0 < x < 1,\ t > 0$$

(product method)

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Make the assumption that time-dependent coefficients $v_n(t)$ and $G_n(t)$ can be found such that both $v(x,t)$ and $G(x,t)$ can be expanded in the series

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(find eigenvalues)

$$\lambda_n=\alpha_n^2=n^2\pi^2,$$

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$$\sin n\pi x,\ n = 1, 2, 3, \ldots$$

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$$v(x,t)=\sum_{n=1}^{\infty} v_n(t)\sin n\pi x$$

Using same eigenfunctions

$$G(x,t)=(1-x)\sin t=\sum_{n=1}^{\infty} G_n(t)\sin n\pi x$$

Make the assumption that time-dependent coefficients $v_n(t)$ and $G_n(t)$ can be found such that both $v(x,t)$ and $G(x,t)$ can be expanded in the series

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Nonhomogeneous BVPs

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Using <u>same</u> eigenfunctions

$$G(x,t)=(1-x)\sin t=\sum_{n=1}^{\infty}G_n(t)\sin n\pi x$$

Where, (Fourier Sine Series Coefficients)

Make the assumption that time-dependent coefficients $v_n(t)$ and $G_n(t)$ can be found such that both $v(x,t)$ and $G(x,t)$ can be expanded in the series

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$$v(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin n\pi x$$

Using <u>same</u> eigenfunctions

$$G(x,t) = (1-x)\sin t = \sum_{n=1}^{\infty} G_n(t)\sin n\pi x$$

Where, (Fourier Sine Series Coefficients)

$$G_n(t) = \frac{2}{1}\int_0^1 (1-x)\sin t \sin n\pi x dx$$

$$= 2\sin t \int_0^1 (1-x)\sin n\pi x dx = \frac{2}{n\pi}\sin t$$

Make the assumption that time-dependent coefficients $v_n(t)$ and $G_n(t)$ can be found such that both $v(x,t)$ and $G(x,t)$ can be expanded in the series

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$u(x,0)=0,\ 0<x<1$

$$\boxed{\begin{array}{l}\dfrac{\partial^2 v}{\partial x^2}+(1-x)\sin t=\dfrac{\partial v}{\partial t},\ 0<x<1,\ t>0\\ v(0,t)=0, v(1,t)=0,\ t>0\\ v(x,0)=x-1,\ 0<x<1\end{array}}$$

$$v(x,t)=\sum_{n=1}^{\infty}v_n(t)\sin n\pi x$$

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$$G(x,t)=(1-x)\sin t=\sum_{n=1}^{\infty}G_n(t)\sin n\pi x$$

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$$=2\sin t\int_0^1(1-x)\sin n\pi x dx=\frac{2}{n\pi}\sin t$$

$$\therefore G_n(t)=(1-x)\sin t=\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin t\sin n\pi x$$

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Nonhomogeneous BVPs

☑ Example 2 Time Dependent PDE and BCs

Solve equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t},\ 0 < x < 1,\ t > 0$

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Now we have,

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$$(1-x)\sin t=\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin t\sin n\pi x$$

$$\frac{\partial^2 v}{\partial x^2}=\sum_{n=1}^{\infty}v_n(t)(-n^2\pi^2)\sin n\pi x$$

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where $\sin(n\pi x / L)$ Are the eigenfunctions of

$$X'' + \lambda X = 0,\ X(0) = 0,\ X(L) = 0$$

It is obtained from *v(x,t)=X(x)T(t)*

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t},\quad v(0,t) = 0,\ v(L,t) = 0,\ 0 < x < L,\ t > 0$$ (1-D homogeneous heat equation)

Nonhomogeneous BVPs

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☑ Example 2 Time Dependent PDE and BCs

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$$\int_0^1(x-1)\sin n\pi x dx=\int_0^1\left(x\sin n\pi x-\sin n\pi x\right)dx$$

$$=\left[x\left(-\frac{1}{n\pi}\cos n\pi x\right)\right]_0^1-\int_0^1\left(-\frac{1}{n\pi}\cos n\pi x\right)dx-\left[-\frac{1}{n\pi}\cos n\pi x\right]_0^1$$

$$=-\frac{1}{n\pi}\cos n\pi+\frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^1+\frac{1}{n\pi}\cos n\pi-\frac{1}{n\pi}$$

$$=-\frac{1}{n\pi}$$

Nonhomogeneous BVPs

$$u(x,t)=v(x,t)+\psi(x,t)$$

$$\psi(x,t)=(1-x)\cos t$$

☑ Example 2 Time Dependent PDE and BCs

Solve equation $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t},\ 0<x<1,\ t>0$

subject to $u(0,t)=\cos t,\ u(1,t)=0,\ t>0$

$u(x,0)=0,\ 0<x<1$

$$\boxed{\begin{aligned}&\frac{\partial^2 v}{\partial x^2}+(1-x)\sin t=\frac{\partial v}{\partial t},\ 0<x<1,\ t>0\\&v(0,t)=0,\ v(1,t)=0,\ t>0\\&v(x,0)=x-1,\ 0<x<1\end{aligned}}$$

$$v(x,t)=\sum_{n=1}^{\infty}v_n(t)\sin n\pi x$$

$$v(x,t)=\sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\left(\frac{n^2\pi^2\sin t-\cos t}{n^4\pi^4+1}\right)+C_n e^{-n^2\pi^2 t}\right\}\sin n\pi x$$

By Initial Condition, $v(x,0)=x-1$

$$x-1=\sum_{n=1}^{\infty}\left\{\frac{-2}{n\pi(n^4\pi^4+1)}+C_n\right\}\sin n\pi x$$

It is fourier sin series, so (Fourier Sine Series Coefficients)

$$\frac{-2}{n\pi(n^4\pi^4+1)}+C_n=2\int_0^1(x-1)\sin n\pi x\,dx=\frac{-2}{n\pi}$$

$$\therefore C_n=\frac{2}{n\pi(n^4\pi^4+1)}-\frac{2}{n\pi}$$

$$\begin{aligned}\int_0^1(x-1)\sin n\pi x\,dx&=\int_0^1\left(x\sin n\pi x-\sin n\pi x\right)dx\\&=\left[x\left(-\frac{1}{n\pi}\cos n\pi x\right)\right]_0^1-\int_0^1\left(-\frac{1}{n\pi}\cos n\pi x\right)dx-\left[-\frac{1}{n\pi}\cos n\pi x\right]_0^1\\&=-\frac{1}{n\pi}\cos n\pi+\frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^1+\frac{1}{n\pi}\cos n\pi-\frac{1}{n\pi}\\&=-\frac{1}{n\pi}\end{aligned}$$

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$$\Rightarrow\therefore v(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{n^2\pi^2\sin t-\cos t+e^{-n^2\pi^2 t}}{n(n^4\pi^4+1)}-\frac{e^{-n^2\pi^2 t}}{n}\right\}\sin n\pi x$$

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$u(x,0) = 0, \ 0 < x < 1$

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$$v(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{n^2\pi^2 \sin t - \cos t + e^{-n^2\pi^2 t}}{n\pi(n^4\pi^4 + 1)} - \frac{e^{-n^2\pi^2 t}}{n} \right\} \sin n\pi x$$

$$\begin{aligned} u(x,t) &= v(x,t) + \psi(x,t) \\ &= (1-x)\cos t + \frac{2}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{n^2\pi^2 \sin t - \cos t + e^{-n^2\pi^2 t}}{n(n^4\pi^4 + 1)} - \frac{e^{-n^2\pi^2 t}}{n} \right\} \sin n\pi x \end{aligned}$$

Orthogonal Series Expansions

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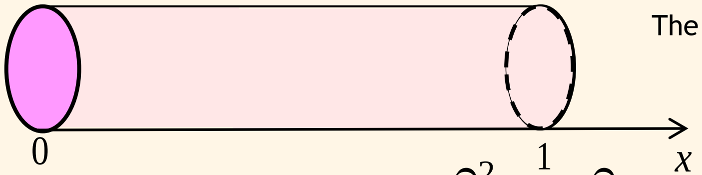
☑ Example 1 Using Orthogonal Series Expansions



The temperature in a rod of unit length in which there is heat transfer from its right boundary into a surrounding medium kept at a constant temperature zero is determined from

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For a certain types of boundary conditions, the method of separation of variables and the superposition principle lead to an expansion of a function in an infinite series that is not a Fourier series

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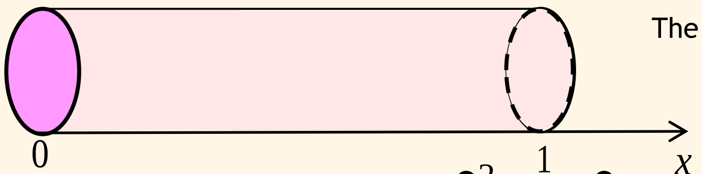

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"Regular Sturm-Liouville Problem"

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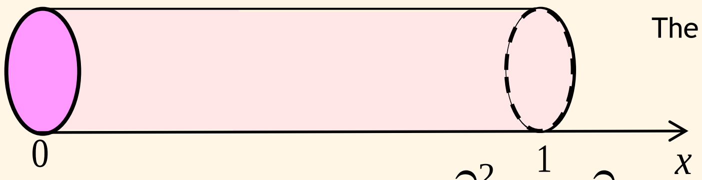

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▪**Regular Sturm-Liouville Problem** B.V.P

Solve : $\frac{d}{dx}[r(x)y']+[q(x)+\lambda p(x)]y=0$

Subject to: $A_1y(a)+B_1y'(a)=0$

$A_2y(b)+B_2y'(b)=0$

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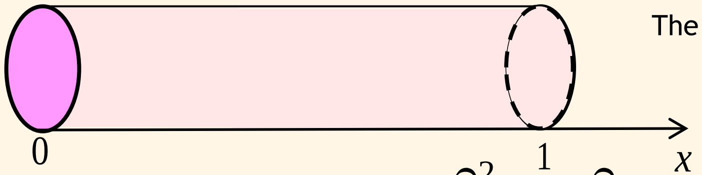

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General solution of (1) is $X(x)=c_1\cos\alpha x+c_2\sin\alpha x,\ \ \lambda_n=\alpha_n^2$

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By $X(0) = 0,\ c_1 = 0$ So, $X(x) = c_2 \sin \alpha x$

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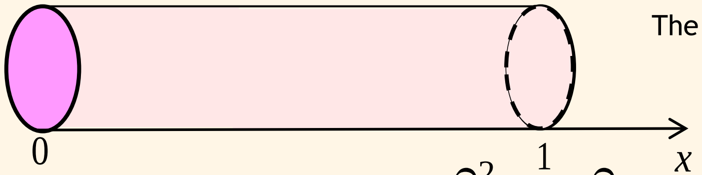

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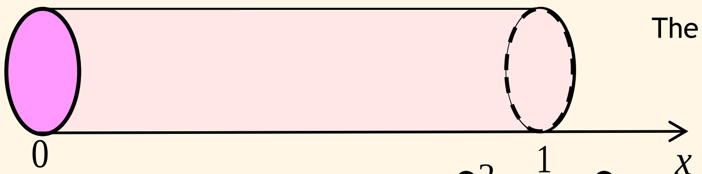

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$$c_2\left(\alpha\cos\alpha x+h\sin\alpha x\right)_{x=1}=c_2\left(\alpha\cos\alpha+h\sin\alpha\right)=0,\ \ or\ \ \tan\alpha=-\frac{\alpha}{h}$$

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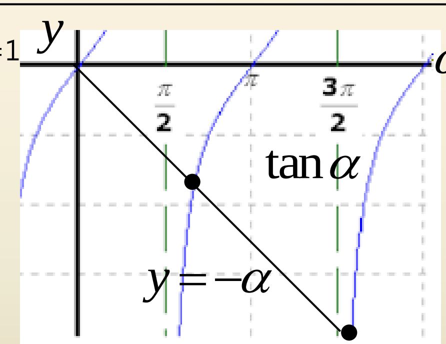
By boundary condition, $\tan\alpha=-\dfrac{\alpha}{h}$

Because the graph of $y=\tan x,\ y=-\dfrac{x}{h},\ h>0$ have an infinite number of intersection points, for x>0

Ex. h=1



If the consecutive positive root are denoted $\alpha_n,\ n=1,2,3,\ldots.$

Eigenvalues of problem are $\lambda_n=\alpha_n^2$

and eigenfunctions are $\sin\alpha_n x,\ n=1,2,3\ldots\Rightarrow X(x)=c_2\sin\alpha_n x$

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And the solution of first order ODE

$T'+k\lambda T=0$ is $T(t)=c_2 e^{-k\alpha_n^2 t}$, $\lambda_n=\alpha_n^2$

$$\therefore u_n=X(x)T(t)=A_n e^{-k\alpha_n^2 t}\sin\alpha_n x \quad \Rightarrow u(x,t)=\sum_{n=1}^{\infty}A_n e^{-k\alpha_n^2 t}\sin\alpha_n x$$

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$T'+k\lambda T=0$ is $T(t)=c_2e^{-k\alpha_n^2 t}$, $\lambda_n=\alpha_n^2$

$$\therefore u_n=X(x)T(t)=A_n e^{-k\alpha_n^2 t}\sin\alpha_n x \quad \Rightarrow u(x,t)=\sum_{n=1}^{\infty}A_n e^{-k\alpha_n^2 t}\sin\alpha_n x$$

Initial condition:

$$u(x,0)=1=\sum_{n=1}^{\infty}A_n\sin\alpha_n x,\ 0<x<1$$

Orthogonal Series Expansions

For a certain types of boundary conditions, the method of separation of variables and the superposition principle lead to an expansion of a function in an infinite series that is not a Fourier series

☑ Example 1 Using Orthogonal Series Expansions

Solve equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t},\ 0 < x < 1,\ t > 0$

subject to $u(0,t) = 0, \left.\dfrac{\partial u}{\partial x}\right|_{x=1} = -hu(1,t),\ h > 0, t > 0$

$u(x,0) = 1,\ 0 < x < 1$

$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t} \sin \alpha_n x$ By initial condition, $1 = \sum_{n=1}^{\infty} A_n \sin \alpha_n x$

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By initial condition, $\boxed{1=\sum_{n=1}^{\infty}A_n\sin\alpha_n x}$

This is not Fourier Sine Series but an expansion of u(x,0)=1 in terms of the orthogonal functions arising from the Sturm-Liouville problem

$$X''+\lambda X=0,$$
$$X(0)=0,\ X'(1)+hX(1)=0$$

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Self-adjoint form of $X''+\lambda X=0$

$$\frac{d}{dx}[r(x)X']+[q(x)+\lambda p(x)]X=0$$

$$r(x)=1, q(x)=0, p(x)=1$$

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$p(x)=1$ **on the interval [0,1].**

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Definition 12.4 Orthogonal Set/Weight Function

A set of real-valued functions $\{\phi_0(x),\phi_1(x),\phi_2(x),\cdots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a,b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx=0,\ m\neq n$$

$$f(x)=c_0\phi_0(x)+c_1\phi_1(x)+\cdots+c_n\phi_n(x)+\cdots$$

$$c_n=\frac{\int_a^b f(x)w(x)\phi_n(x)dx}{\int_a^b w(x)\phi_n^2(x)dx}$$

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$$\therefore A_n = \frac{\int_0^1 1\cdot 1\cdot \sin\alpha_n x dx}{\int_0^1 1\cdot\sin^2\alpha_n x dx}$$

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$u(x,0)=1,\ 0<x<1$

$X(x)=c_2\sin\alpha x$

b/c $X'(1)+hX(1)=0$

$\left[\alpha\cos\alpha x+h\sin\alpha\right]_0^1$

$=\alpha\cos\alpha+h\sin\alpha=0$

$\therefore \sin\alpha=-\dfrac{\alpha\cos\alpha}{h}$

$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-k\alpha_n^2 t}\sin\alpha_n x$

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$$\int_0^1\sin^2\alpha_n x dx=\frac{1}{2}\int_0^1(1-\cos 2\alpha_n x)dx$$

$$=\frac{1}{2}\left(1-\frac{1}{2\alpha_n}\sin 2\alpha_n\right)$$

$$=\frac{1}{2}\left(\frac{2\alpha_n+2\sin\alpha_n\cos\alpha_n}{2\alpha_n}\right)$$

$$=\frac{1}{2}\left(\frac{2\alpha_n+\frac{1}{h}2\alpha_n\cos\alpha_n{}^2}{2\alpha_n}\right)$$

$$=\frac{1}{2h}\left(h+\cos^2\alpha_n\right)$$

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$$\therefore A_n=\frac{\int_0^1 1\cdot 1\cdot\sin\alpha_n x dx}{\int_0^1 1\cdot\sin^2\alpha_n x dx}\longleftarrow \int_0^1\sin^2\alpha_n x dx=\frac{1}{2h}\left(h+\cos^2\alpha_n\right)\longleftarrow$$

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$$\int_0^1 \sin \alpha_n x dx = -\frac{1}{\alpha_n}\cos \alpha_n \Big|_0^1 = \frac{1}{\alpha_n}(1 - \cos \alpha_n)$$

$$\therefore A_n = \frac{\int_0^1 1 \cdot 1 \cdot \sin \alpha_n x dx}{\int_0^1 1 \cdot \sin^2 \alpha_n x dx}$$

$$\int_0^1 \sin^2 \alpha_n x dx = \frac{1}{2h}\left(h + \cos^2 \alpha_n\right)$$

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$$\therefore u(x,t)=\sum_{n=1}^{\infty}\frac{2h(1-\cos\alpha_n)}{\alpha_n(h+\cos^2\alpha_n)}e^{-k\alpha_n^2 t}\sin\alpha_n x$$

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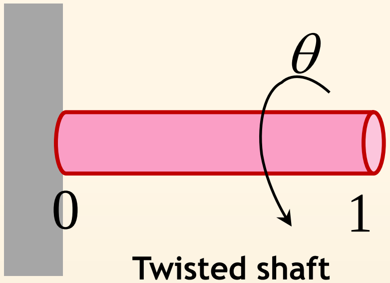
The twist angle $\theta(x,t)$ of a torsionally vibrating shaft of unit length is determined from

Solve equation $a^2 \dfrac{\partial^2\theta}{\partial x^2} = \dfrac{\partial^2\theta}{\partial t^2}, \quad 0<x<1,\ t>0$

subject to $\theta(0,t)=0, \quad \left.\dfrac{\partial\theta}{\partial x}\right|_{x=1} = 0, \quad t>0$

$\theta(x,0)=x, \quad \left.\dfrac{\partial\theta}{\partial t}\right|_{t=0} = 0, \quad 0<x<1$



Twisted shaft

Product Method

$\theta(x,t) = X(x)T(t)$

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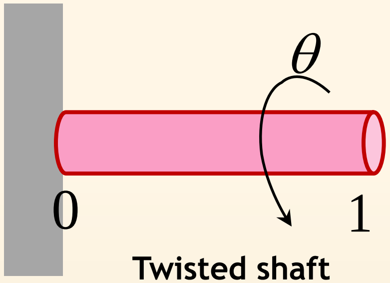

Twisted shaft

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Then,

$$X''+\lambda X=0$$
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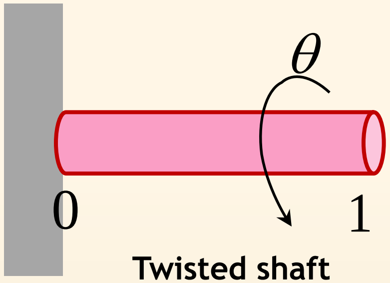

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From $\theta(0,t)=X(0)T(t)=0$

$T(t)\neq 0$ **otherwise θ(x,t) : trivial solution, So**

$$X(0)=0$$

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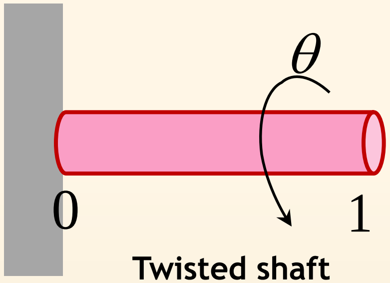

Twisted shaft

Product Method

$\theta(x,t) = X(x)T(t)$

Then,

$X'' + \lambda X = 0$

$T'' + a^2\lambda T = 0$

From $\theta(0,t) = X(0)T(t) = 0$

$T(t) \neq 0$ **otherwise θ(x,t) : trivial solution, So**

$$X(0) = 0$$

From $\theta'(0,t) = X'(0)T(t) = 0$

$T(t) \neq 0$ **otherwise θ(x,t) : trivial solution, So**

$$X'(0) = 0$$

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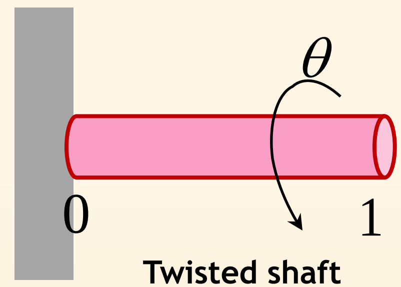

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Product Method

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Then,

$$X''+\lambda X=0$$

$$T''+a^2\lambda T=0$$

$$X(0)=0\ \ and\ \ X'(1)=0$$

From $\theta(0,t)=X(0)T(t)=0$

$T(t)\neq 0$ **otherwise θ(x,t) : trivial solution, So**

$$X(0)=0$$

From$\theta'(0,t)=X'(0)T(t)=0$

$T(t)\neq 0$ **otherwise θ(x,t) : trivial solution, So**

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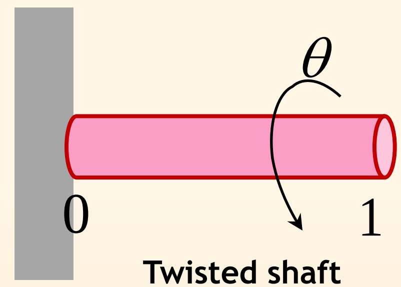

Twisted shaft

Product Method

$$\theta(x,t)=X(x)T(t)$$

Then,

$$X''+\lambda X=0$$

$$T''+a^2\lambda T=0$$

$$X(0)=0\ \ and\ \ X'(1)=0$$

From $\theta(0,t)=X(0)T(t)=0$

$T(t)\neq 0$ **otherwise θ(x,t) : trivial solution, So**

$$X(0)=0$$

From$\theta'(0,t)=X'(0)T(t)=0$

$T(t)\neq 0$ **otherwise θ(x,t) : trivial solution, So**

$$X'(0)=0$$

$$X''+\lambda X=0,\ \ X(0)=0, X'(1)=0$$

Is a regular Sturm-Liouville problem.

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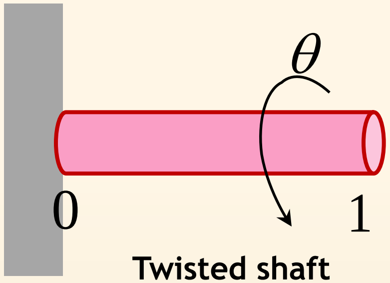

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$$X''+\lambda X=0,\quad X(0)=0,\ X'(1)=0$$

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General solution is $X(x)=c_1\cos\alpha x+c_2\sin\alpha x$

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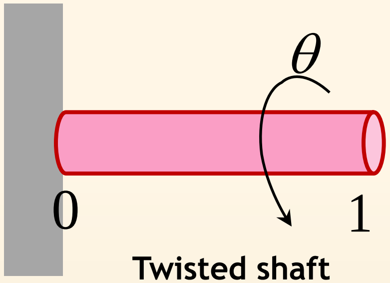

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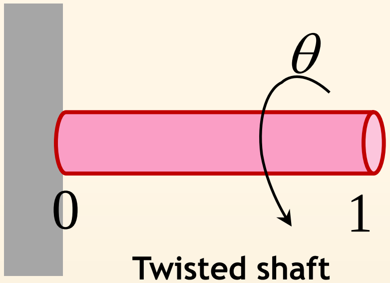

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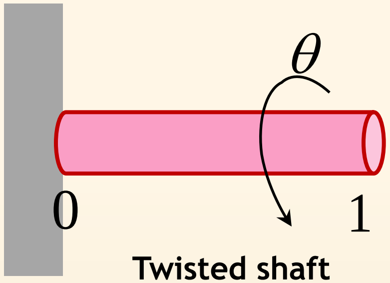

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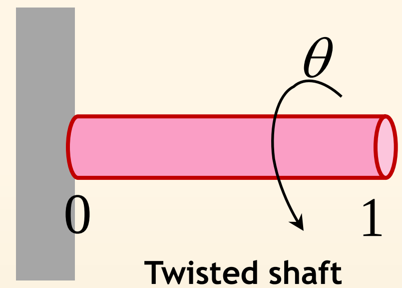

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$$X(x) = c_2 \sin\left(\frac{2n-1}{2}\right)\pi x,\ n = 1, 2, 3, \ldots.$$

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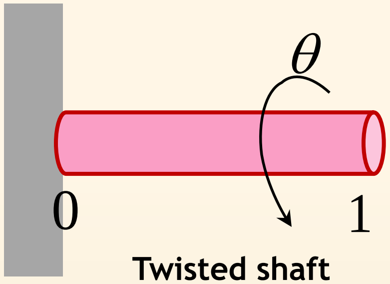

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$$T''+a^2\lambda T=0,\quad \left.\frac{\partial\theta}{\partial t}\right|_{t=0}=0$$

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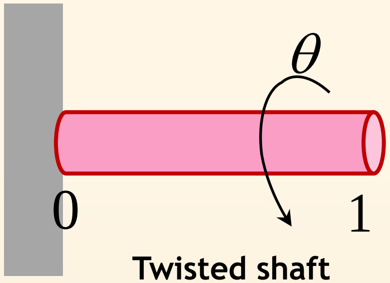

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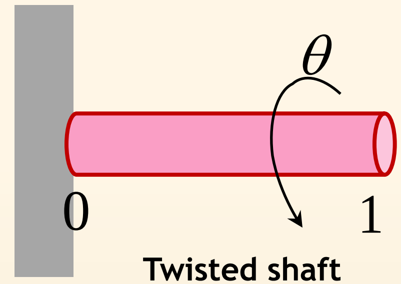

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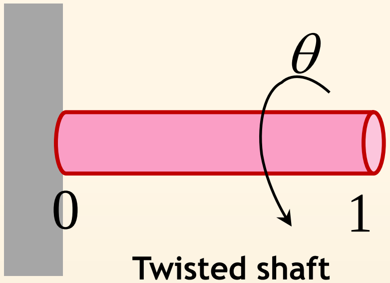

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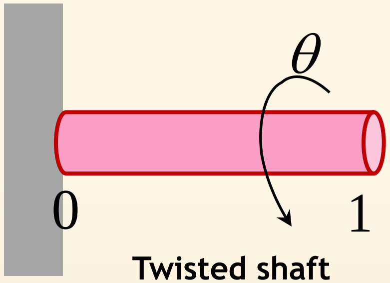

Twisted shaft

Separation variables

$$\theta(x,t) = X(x)T(t)$$

Then, $X'' + \lambda X = 0 \quad , T'' + a^2\lambda T = 0$

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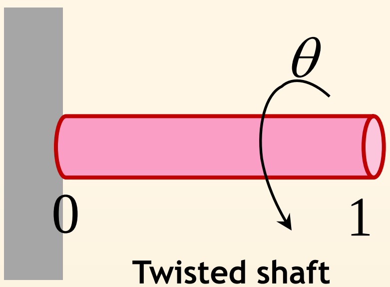

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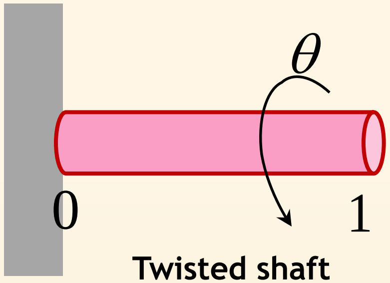

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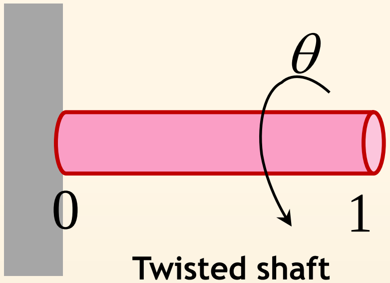

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$,\lambda_n=\alpha_n^2=(2n-1)^2\pi^2/4$

$$\therefore\theta_n(x,t)=X(x)T(t)=A_n\cos a\left(\frac{2n-1}{2}\right)\pi t\sin\left(\frac{2n-1}{2}\right)\pi x$$

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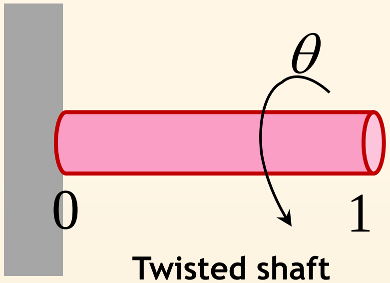

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$$\theta_n(x,t)=A_n\cos a\left(\frac{2n-1}{2}\right)\pi t\sin\left(\frac{2n-1}{2}\right)\pi x$$

Self-adjoint form of $X''+\lambda X=0$

$$\frac{d}{dx}[r(x)X']+[q(x)+\lambda p(x)]X=0$$

$$r(x)=1, q(x)=0, p(x)=1$$

$$\{\sin\alpha_n\},\ \ n=1,2,3,\cdots$$

is an orthogonal set with respect to the weight function $p(x)=1$ **on the interval [0,1].**

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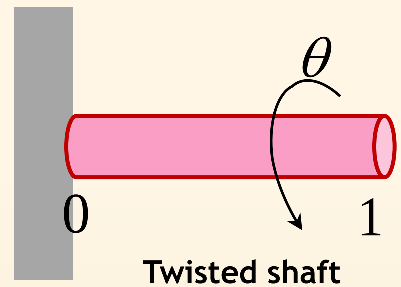

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$$\theta_n(x,t)=A_n\cos a\left(\frac{2n-1}{2}\right)\pi t\sin\left(\frac{2n-1}{2}\right)\pi x$$

$$\theta(x,t)=\sum_{n=1}^{\infty}A_n\cos a\left(\frac{2n-1}{2}\right)\pi t\sin\left(\frac{2n-1}{2}\right)\pi x$$

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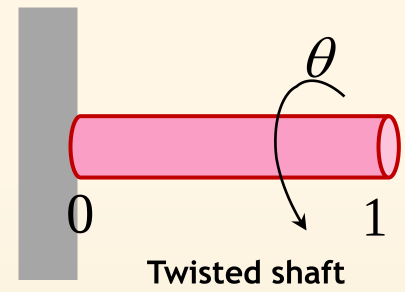

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$$\theta_n(x,t)=A_n\cos a\left(\frac{2n-1}{2}\right)\pi t\sin\left(\frac{2n-1}{2}\right)\pi x$$

$$\theta(x,t)=\sum_{n=1}^{\infty}A_n\cos a\left(\frac{2n-1}{2}\right)\pi t\sin\left(\frac{2n-1}{2}\right)\pi x$$

By initial condition,

$$\theta(x,0)=x=\sum_{n=1}^{\infty}A_n\sin\left(\frac{2n-1}{2}\right)\pi x \text{ , where } A_n=\frac{\int_0^1 1\cdot x\sin\left(\frac{2n-1}{2}\right)\pi x\,dx}{\int_0^1 1\cdot\sin^2\left(\frac{2n-1}{2}\right)\pi x\,dx}$$

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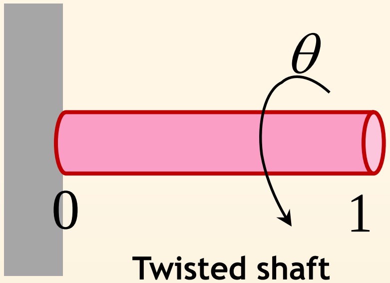

Twisted shaft

By initial condition,

$$\theta(x,0) = x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\right)\pi x$$

$$A_n = \frac{\int_0^1 1 \cdot x \sin\left(\frac{2n-1}{2}\right)\pi x dx}{\int_0^1 1 \cdot \sin^2\left(\frac{2n-1}{2}\right)\pi x dx}$$

Orthogonal Series Expansions

For a certain types of boundary conditions, the method of separation of variables and the superposition principle lead to an expansion of a function in an infinite series that is not a Fourier series

☑ Example 2 Using Orthogonal Series Expansions

The twist angle $\theta(x,t)$ of a torsionally vibrating shaft of unit length is determined from

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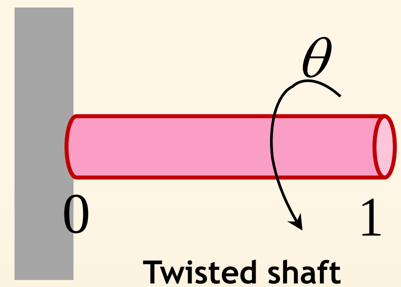

Twisted shaft

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$$\theta(x,0)=x=\sum_{n=1}^{\infty}A_n\sin\left(\frac{2n-1}{2}\right)\pi x$$

$$A_n=\frac{\int_0^1 1\cdot x\sin\left(\frac{2n-1}{2}\right)\pi x\,dx}{\int_0^1 1\cdot\sin^2\left(\frac{2n-1}{2}\right)\pi x\,dx}$$

$$\int_0^1 1\cdot x\sin\left(\frac{2n-1}{2}\right)\pi x\,dx$$

$$=\left[x\left(-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\right)\pi x\right)\right]_0^1-\int_0^1\left(-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\right)\pi x\right)dx$$

$$=\left(-\frac{2}{(2n-1)\pi}\cos\left(\frac{2n-1}{2}\right)\pi\right)+\left[\left(\frac{2}{(2n-1)\pi}\right)^2\sin\left(\frac{2n-1}{2}\right)\pi x\right]_0^1$$

$$=(0)+\left(\left(\frac{2}{(2n-1)\pi}\right)^2\sin\left(\frac{2n-1}{2}\right)\pi\right)$$

$$=\left(\frac{2}{(2n-1)\pi}\right)^2(-1)^{n+1}$$

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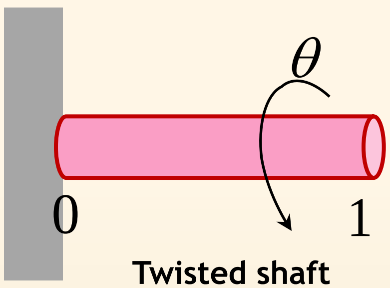

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$$A_n=\frac{\int_0^1 1\cdot x\sin\left(\frac{2n-1}{2}\right)\pi x dx}{\int_0^1 1\cdot \sin^2\left(\frac{2n-1}{2}\right)\pi x dx}$$

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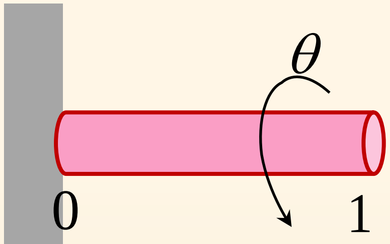

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$$\int_0^1 1\cdot x\sin\left(\frac{2n-1}{2}\right)\pi x\,dx=\left(\frac{2}{(2n-1)\pi}\right)^2(-1)^{n+1}$$

$$\int_0^1\sin^2\left(\frac{2n-1}{2}\right)\pi x\,dx=\frac{1}{2}\int_0^1\left(1-\cos 2\left(\frac{2n-1}{2}\right)\pi x\right)dx$$

$$=\frac{1}{2}\left[x-\frac{1}{(2n-1)\pi}\sin(2n-1)\pi x\right]_0^1$$

$$=\frac{1}{2}$$

Orthogonal Series Expansions

For a certain types of boundary conditions, the method of separation of variables and the superposition principle lead to an expansion of a function in an infinite series that is not a Fourier series

☑ Example 2 Using Orthogonal Series Expansions

The twist angle $\theta(x,t)$ of a torsionally vibrating shaft of unit length is determined from

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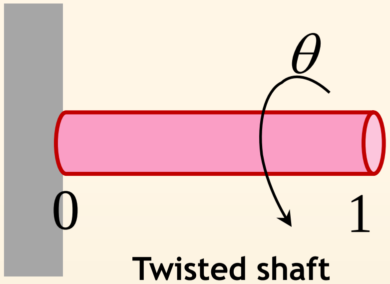


Twisted shaft

$$\theta(x,t)=\sum_{n=1}^{\infty}A_n\cos a\left(\frac{2n-1}{2}\right)\pi t\sin\left(\frac{2n-1}{2}\right)\pi x\ ,\ A_n=\frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}$$

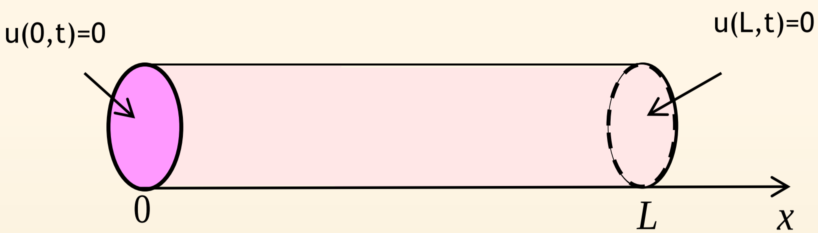
$$\theta(x,t)=\sum_{n=1}^{\infty}\frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}\cos a\left(\frac{2n-1}{2}\right)\pi t\sin\left(\frac{2n-1}{2}\right)\pi x$$

Fourier Series in Two Variables

Fourier Series in Two Variables

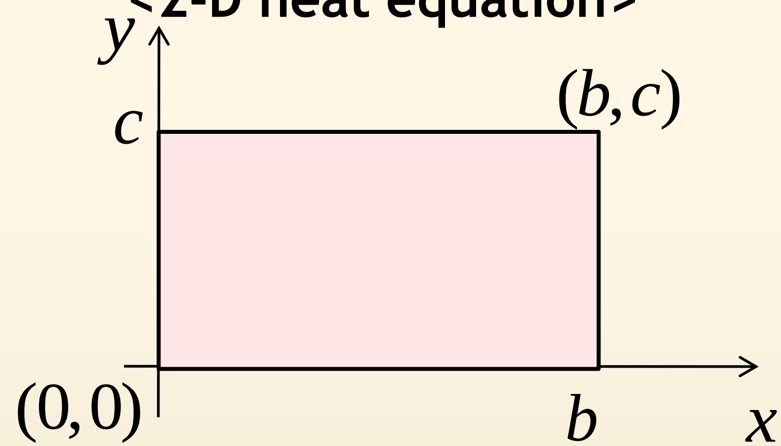
Heat equation in Two dimensions

<1-D heat equation>



$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\ 0<x<L,\ t>0$$

<2-D heat equation>

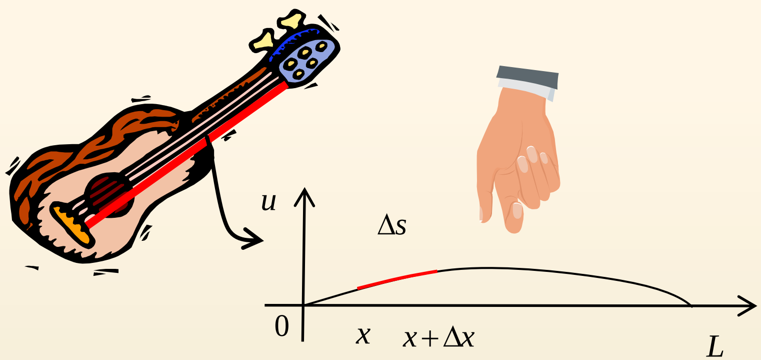


$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}$$

Fourier Series in Two Variables

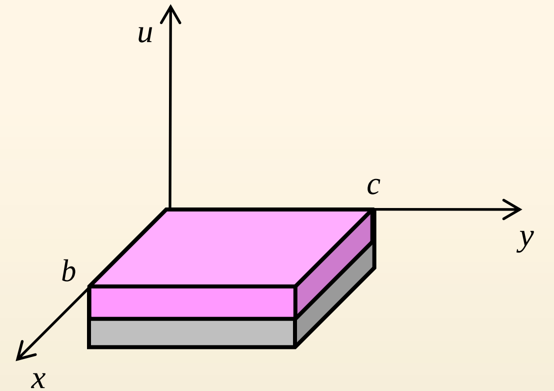
Wave equation in Two dimensions

<1-D wave equation>



$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

<2-D wave equation>



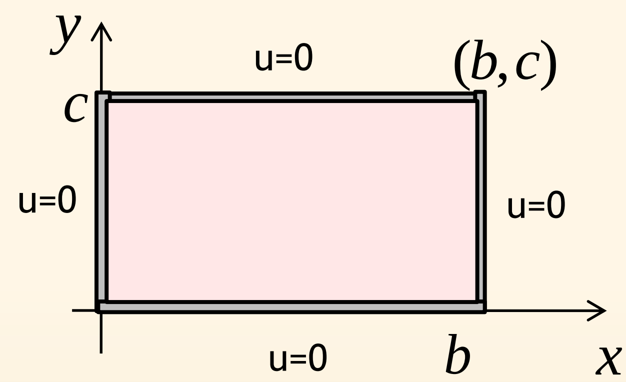
$$a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}$$

Fourier Series in Two Variables

☑ Example 1 Temperatures in a Plate

Solve equation

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t},\ \ 0<x<b,\ 0<y<c,t>0$$



subject to

$$u(0,y,t)=0,\ u(b,y,t)=0,\ 0<y<c,\ t>0$$

$$u(x,0,t)=0,\ u(x,c,t)=0,\ 0<x<b,\ t>0$$

$$u(x,y,0)=f(x,y),\ \ 0<x<b,\ \ 0<y<c$$

Product Method

$$u(x,y,t)=X(x)Y(y)T(t)\longrightarrow k\left(u_{xx}+u_{yy}\right)=u_t$$

Fourier Series in Two Variables

☑ **Example 1** **Temperatures in a Plate**

Solve equation

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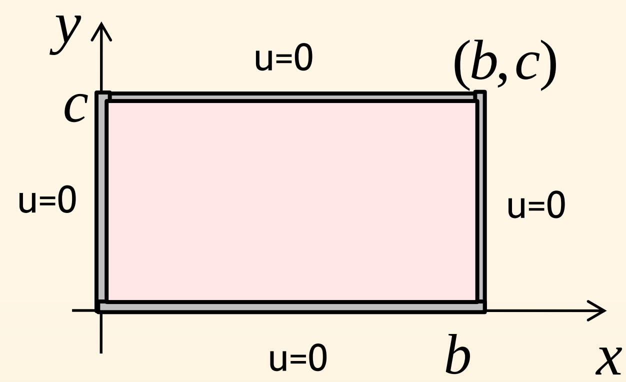

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Fourier Series in Two Variables

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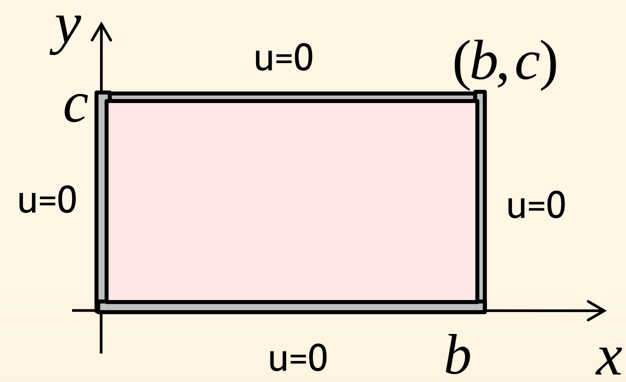

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dividing by $kXYT$

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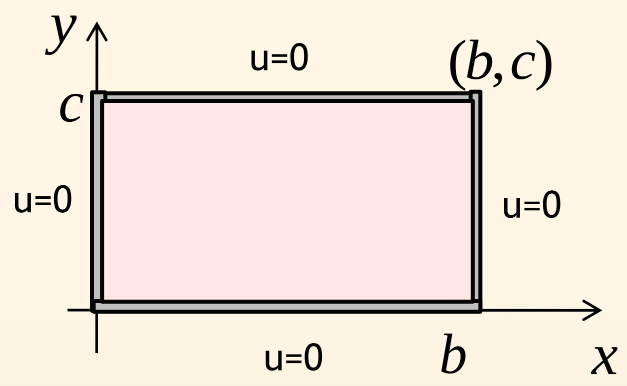

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$$\frac{X''}{X}=-\frac{Y''}{Y}+\frac{T'}{kT}=-\lambda$$

Fourier Series in Two Variables

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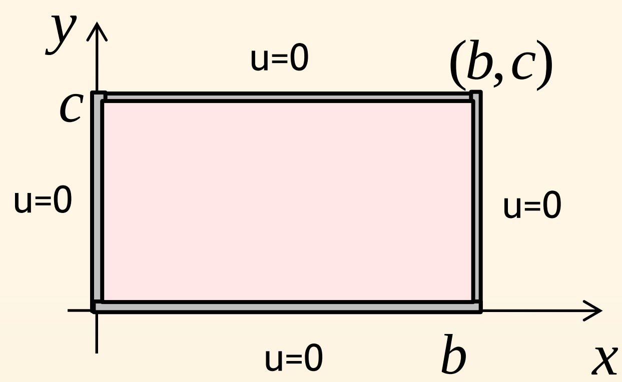

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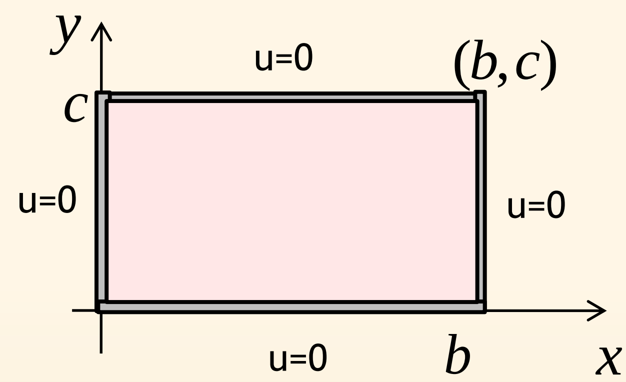

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dividing by $kXYT$

$$\frac{X''}{X}=-\frac{Y''}{Y}+\frac{T'}{kT}=-\lambda \qquad \Rightarrow \qquad X''+\lambda X=0$$

Fourier Series in Two Variables

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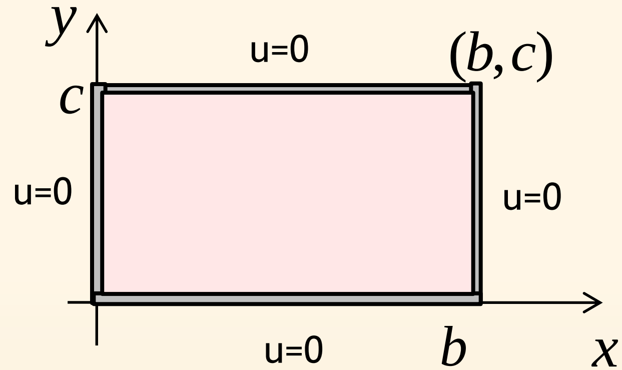

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$$u(x,y,0)=f(x,y),\ \ 0<x<b,\ \ 0<y<c$$

$$\frac{Y''}{Y}=\frac{T'}{kT}+\lambda$$

$$X''+\lambda X=0$$

Fourier Series in Two Variables

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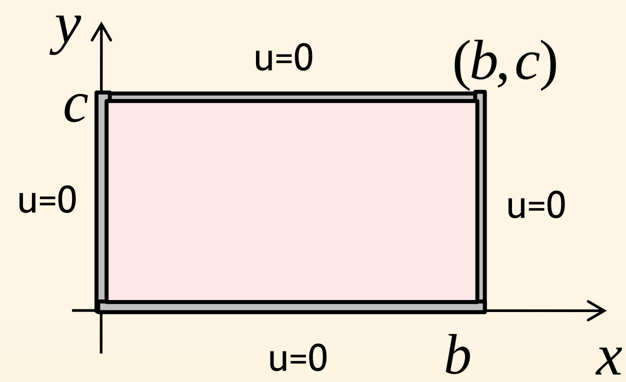

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$$X''+\lambda X=0$$

Introduce another separation constant,

Fourier Series in Two Variables

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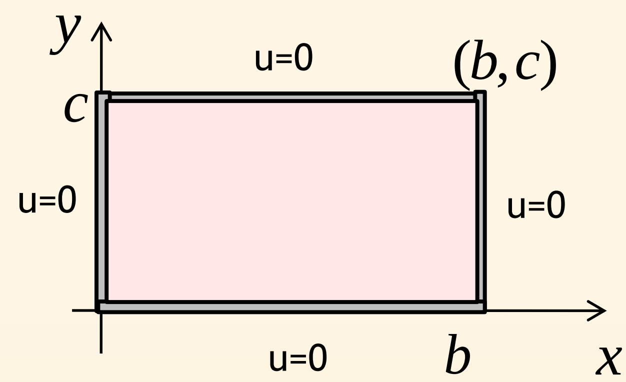

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$$X''+\lambda X=0$$

Introduce another separation constant,

$$\frac{Y''}{Y}=-\mu,\quad \text{then}\quad \frac{T'}{kT}+\lambda=-\mu$$

Fourier Series in Two Variables

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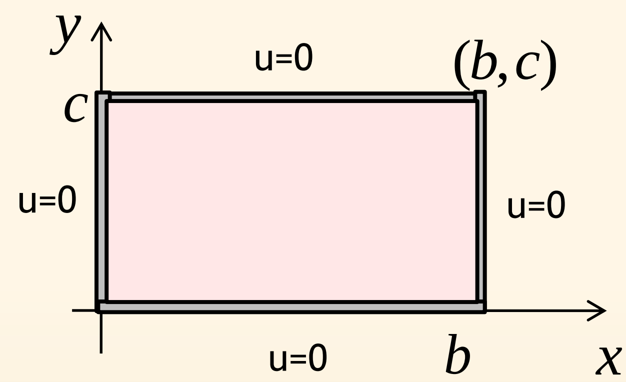

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Fourier Series in Two Variables

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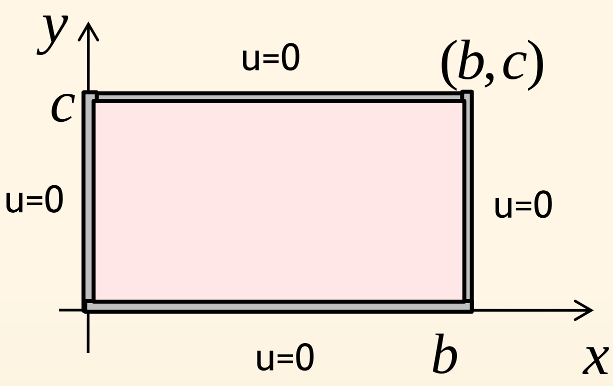

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$$X''+\lambda X=0$$

Introduce another separation constant,

$$\frac{Y''}{Y}=-\mu,\quad \text{then}\ \ \frac{T'}{kT}+\lambda=-\mu \quad \Rightarrow \quad Y''+\mu Y=0,\ \ T'+k(\lambda+\mu)T=0$$

Fourier Series in Two Variables

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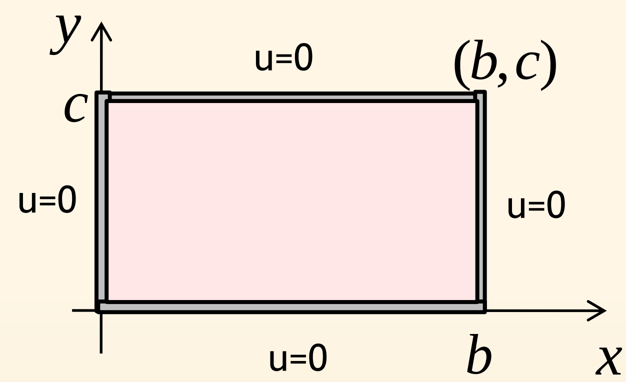

subject to

$$u(0,y,t)=0,\ u(b,y,t)=0,\ 0<y<c,\ t>0$$

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Separation variables

$$u(x,y,t)=X(x)Y(y)T(t)$$

$$\Downarrow$$

$$k\left(u_{xx}+u_{yy}\right)=u_t$$

$$\begin{cases} X''+\lambda X=0 \\ Y''+\mu Y=0 \\ T'+k(\lambda+\mu)T=0 \end{cases}$$

Fourier Series in Two Variables

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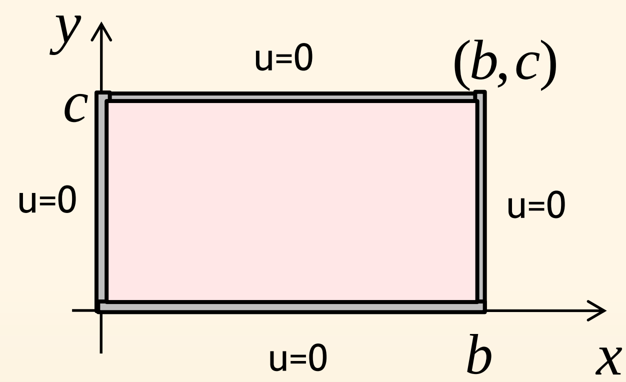
Solve equation

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t},\ \ 0<x<b,\ 0<y<c, t>0$$



subject to

$$u(0,y,t)=0,\ u(b,y,t)=0,\ 0<y<c,\ t>0$$

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Separation variables

$$u(x,y,t)=X(x)Y(y)T(t)$$

$$\Downarrow$$

$$k\left(u_{xx}+u_{yy}\right)=u_t$$

$$\begin{cases} X''+\lambda X=0 \\ Y''+\mu Y=0 \\ T'+k(\lambda+\mu)T=0 \end{cases}$$

Boundary conditions

$$u(0,y,t)=X(0)Y(y)T(t)=0$$

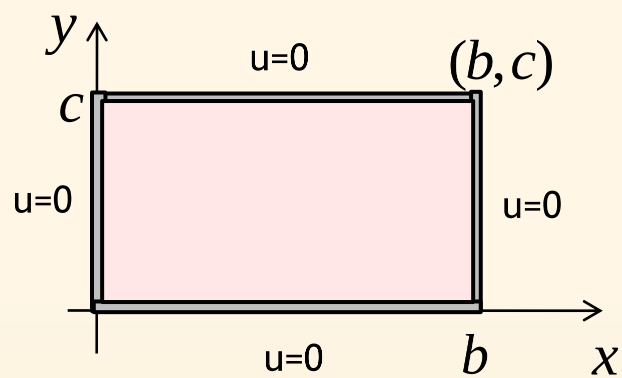
$$u(b,y,t)=X(0)Y(y)T(t)=0$$

Fourier Series in Two Variables

☑ Example 1 Temperatures in a Plate

Solve equation

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t},\ \ 0<x<b,\ 0<y<c, t>0$$



subject to

$$u(0,y,t)=0,\ u(b,y,t)=0,\ 0<y<c,\ t>0$$

$$u(x,0,t)=0,\ u(x,c,t)=0,\ 0<x<b,\ t>0$$

$$u(x,y,0)=f(x,y),\ \ 0<x<b,\ \ 0<y<c$$

Separation variables

$$u(x,y,t)=X(x)Y(y)T(t)$$

$$k\left(u_{xx}+u_{yy}\right)=u_t$$

$$\begin{cases} X''+\lambda X=0 \\ Y''+\mu Y=0 \\ T'+k(\lambda+\mu)T=0 \end{cases}$$

Boundary conditions

$$u(0,y,t)=X(0)Y(y)T(t)=0$$

$$u(b,y,t)=X(0)Y(y)T(t)=0$$

imply $X(0)=0,\ X(b)=0$

Fourier Series in Two Variables

☑ Example 1 Temperatures in a Plate

Solve equation

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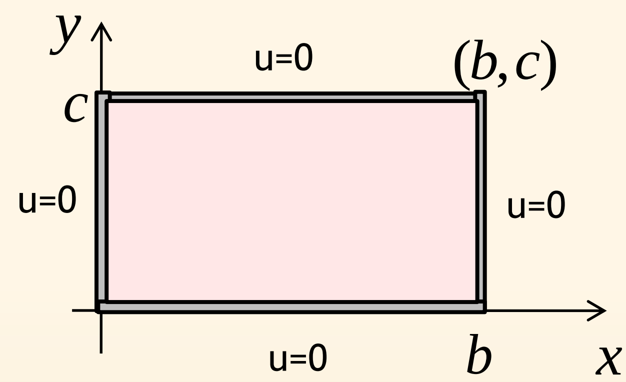

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Otherwise $\left(Y(y)=0\ \ or\ \ T(t)=0\right)$, u(x,y,t) : trivial solution

Fourier Series in Two Variables

☑ **Example 1** **Temperatures in a Plate**

Solve equation

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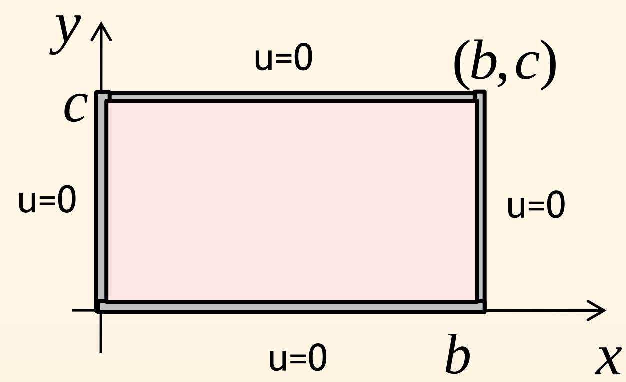

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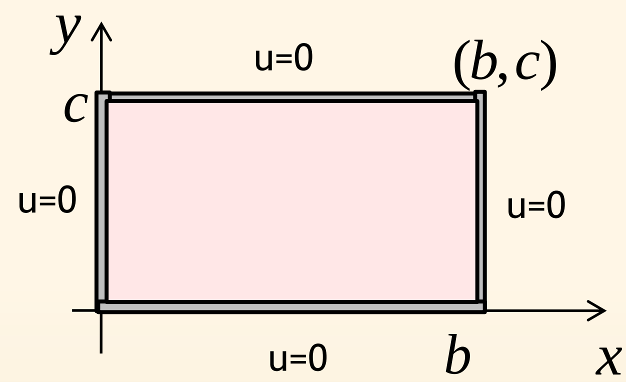

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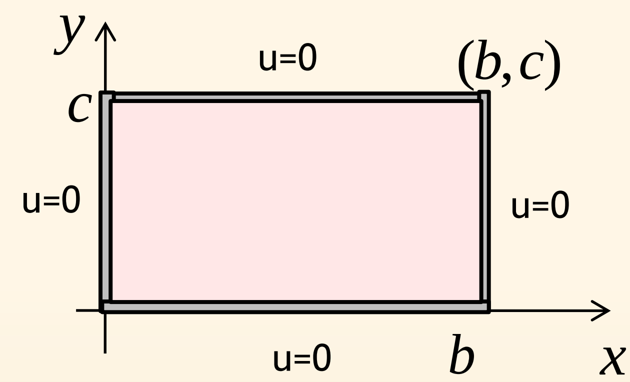

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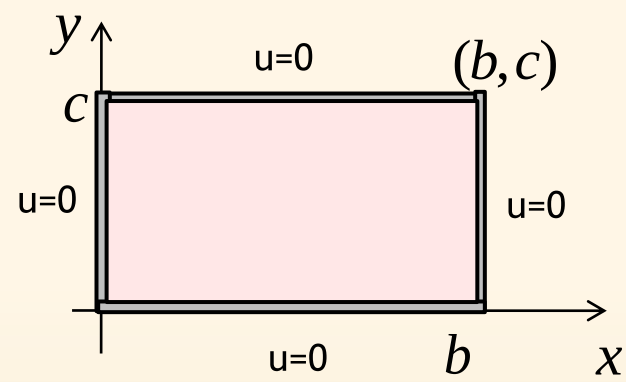

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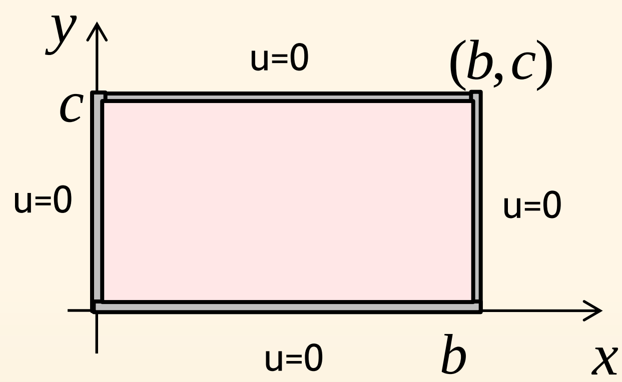

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So, We have two Sturm-Liouville problems,

Fourier Series in Two Variables

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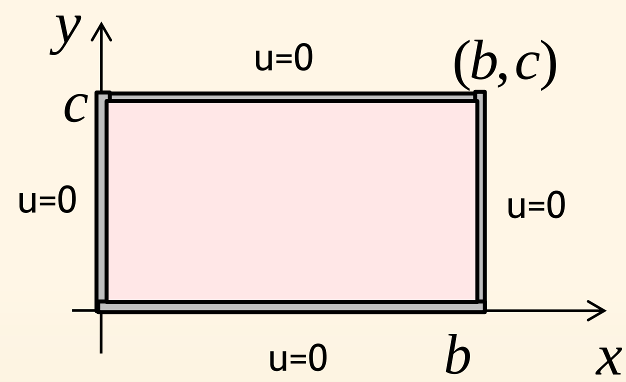

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$$\Downarrow$$

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$$\Rightarrow\quad\left[\begin{array}{l} X''+\lambda X=0 \\ Y''+\mu Y=0 \\ T'+k(\lambda+\mu)T=0 \end{array}\right.$$

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$$\begin{cases} X'' + \lambda X = 0 \\ Y'' + \mu Y = 0 \\ T' + k(\lambda + \mu)T = 0 \end{cases}$$

Fourier Series in Two Variables

$u(x, y, t) = X(x)Y(y)T(t)$

☑ Example 1 Temperatures in a Plate

Solve equation

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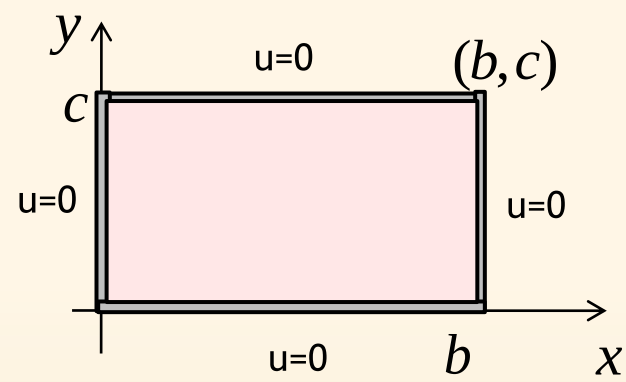

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$$u(0, y, t) = 0, \ u(b, y, t) = 0, \ 0 < y < c, \ t > 0$$

$$u(x, 0, t) = 0, \ u(x, c, t) = 0, \ 0 < x < b, \ t > 0$$

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Fourier Series in Two Variables

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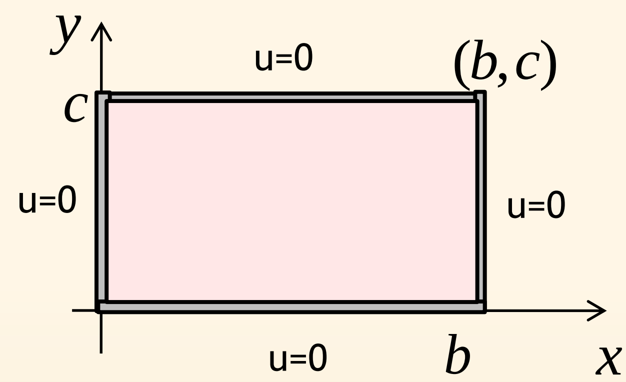

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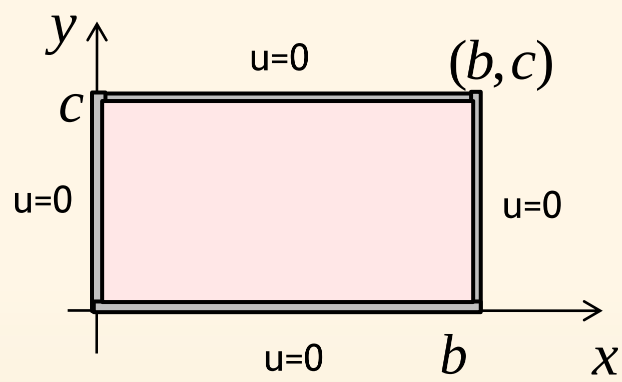

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Fourier Series in Two Variables

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' + \mu Y = 0 \\ T' + k(\lambda + \mu)T = 0 \end{cases}$$

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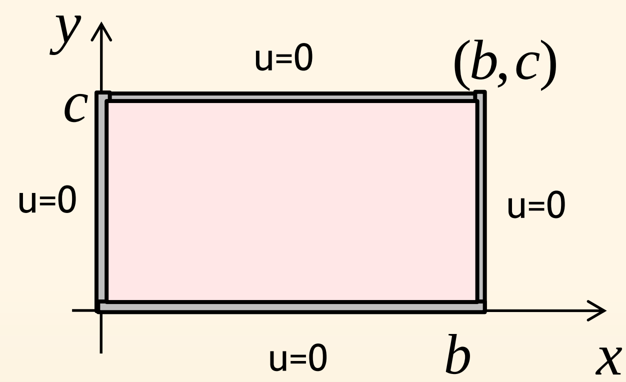

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$$Y'' + \mu Y = 0,\ Y(0) = 0,\ Y(c) = 0$$

⇩

$$\mu_n = \frac{n^2\pi^2}{c^2} \quad \therefore Y(y) = c_4 \sin\frac{n\pi}{c} y,\ n = 1,2,3,\ldots$$

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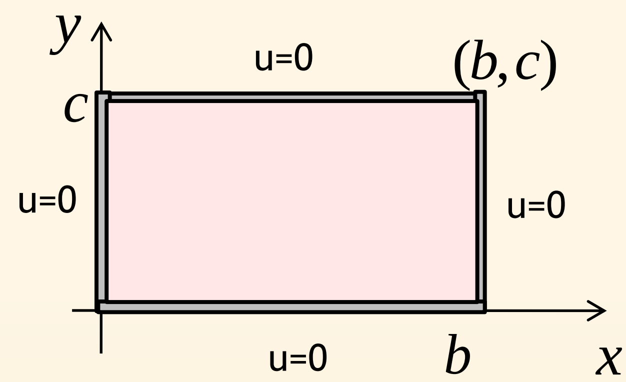

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We have two Sturm-Liouville problems,

$$X'' + \lambda X = 0,\ \ X(0) = 0,\ \ X(b) = 0 \qquad \lambda_m = \frac{m^2\pi^2}{b^2} \qquad X(x) = c_2 \sin\frac{m\pi}{b}x,\ m = 1, 2, 3, \ldots$$

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Fourier Series in Two Variables

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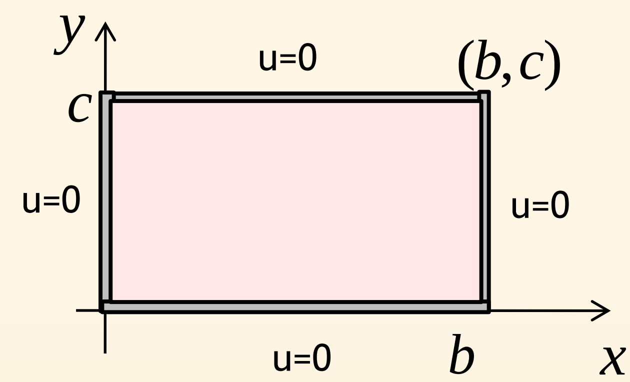

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Solving

$$T' + k(\lambda+\mu)T = 0 \ \Rightarrow\ T(t) = c_5 e^{-k[\lambda+\mu]t} = c_5 e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t}$$

Fourier Series in Two Variables

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' + \mu Y = 0 \\ T' + k(\lambda + \mu)T = 0 \end{cases}$$

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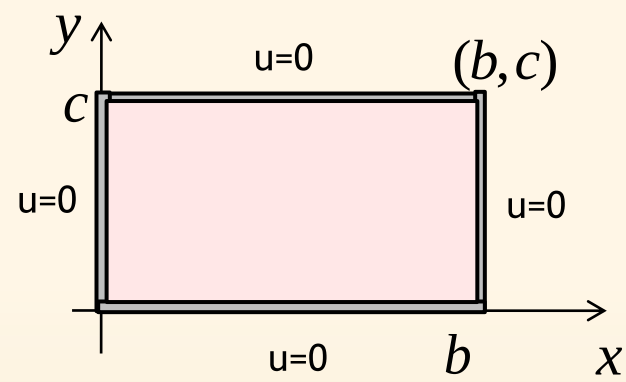

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$$u(x,y,0) = f(x,y),\ \ 0 < x < b,\ \ 0 < y < c$$

$$u_{mn}(x,y,t) = X(x)Y(y)T(t)$$

$$= A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b} x \sin\frac{n\pi}{c} y$$

$$X(x) = c_2 \sin\frac{m\pi}{b} x,\ m = 1, 2, 3, \ldots$$

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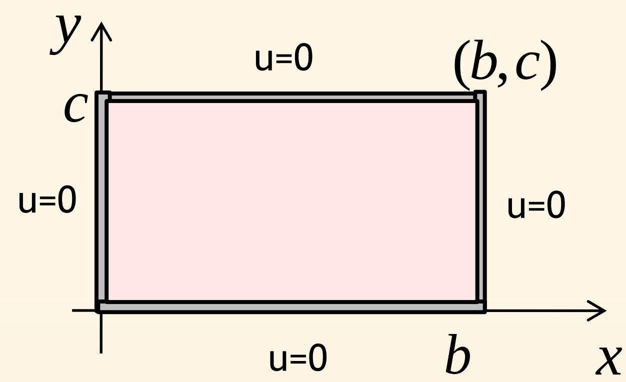

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$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k[(m\pi/b)^2+(n\pi/c)^2]t}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y$$

$$X(x)=c_2\sin\frac{m\pi}{b}x,\ m=1,2,3,\ldots$$

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Fourier Series in Two Variables

☑ **Example 1** **Temperatures in a Plate**

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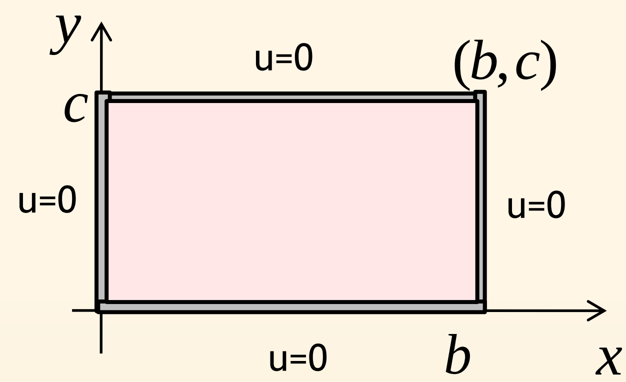

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$$u(x,0,t)=0,\ u(x,c,t)=0,\ 0<x<b,\ t>0$$

$$u(x,y,0)=f(x,y),\ \ 0<x<b,\ \ 0<y<c$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k[(m\pi/b)^2+(n\pi/c)^2]t}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y$$

Fourier Series in Two Variables

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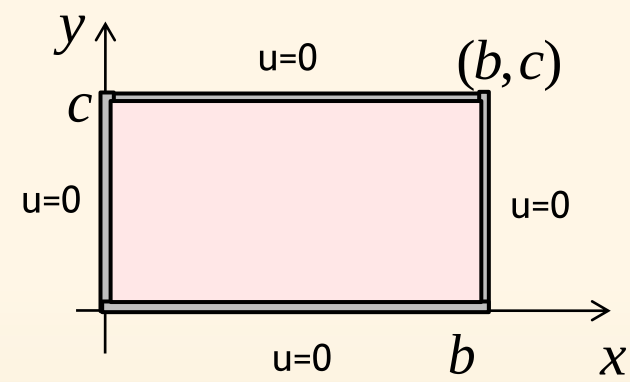
$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t},\ \ 0<x<b,\ 0<y<c, t>0$$



subject to

$$u(0,y,t)=0,\ u(b,y,t)=0,\ 0<y<c,\ t>0$$

$$u(x,0,t)=0,\ u(x,c,t)=0,\ 0<x<b,\ t>0$$

$$u(x,y,0)=f(x,y),\ \ 0<x<b,\ \ 0<y<c$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k[(m\pi/b)^2+(n\pi/c)^2]t}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y$$

Intial condition:

$$u(x,y,0)=f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{c}$$

Fourier Series in Two Variables

☑ **Example 1** **Temperatures in a Plate**

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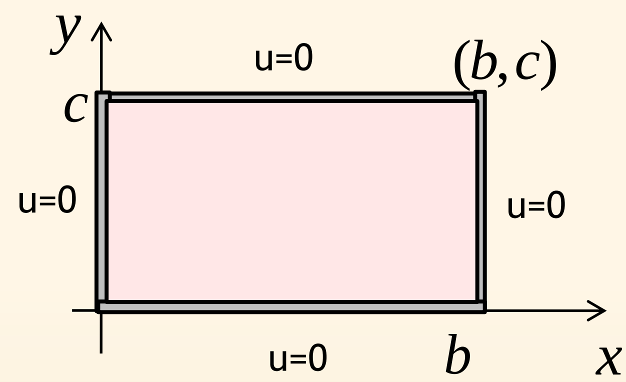

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"Double Sine Series(Sine Series in Two Variables)"

Fourier Series in Two Variables

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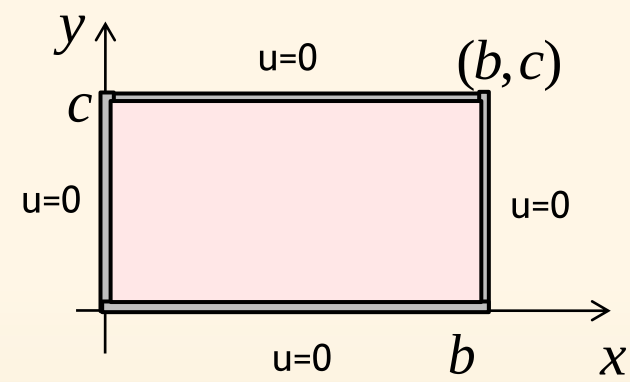

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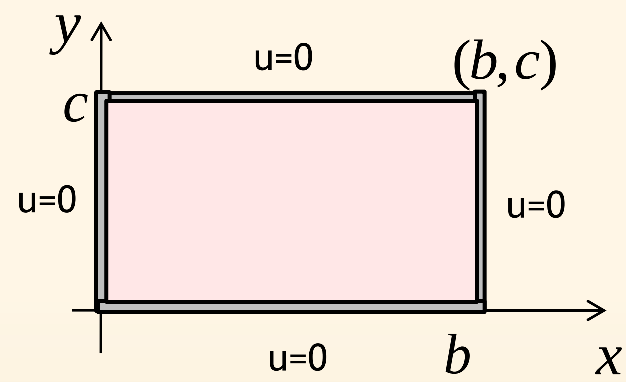

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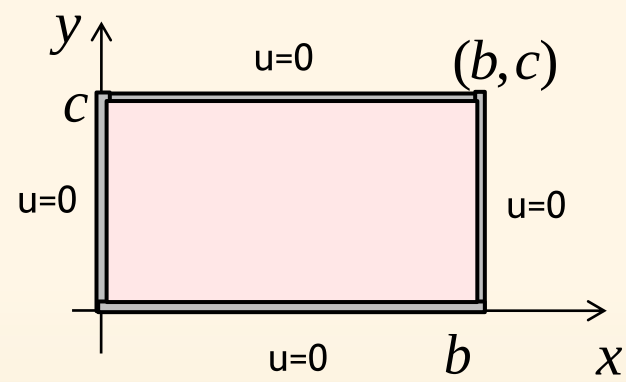

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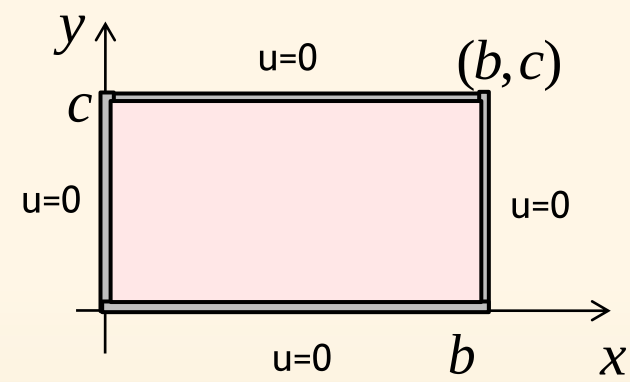

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$$\text{setting } K_m(y)=\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi y}{c},$$

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Fourier sine series

Fourier Series in Two Variables

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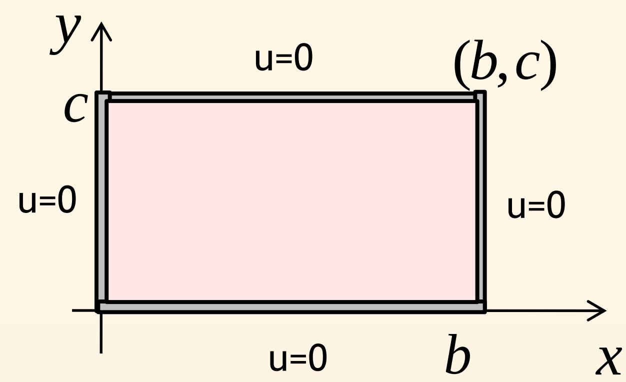

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Fourier sine series

$$\therefore K_m(y)=\frac{2}{b}\int_0^b f(x,y)\sin\frac{m\pi x}{b}dx$$

Fourier sine series Coefficients

Fourier Series in Two Variables

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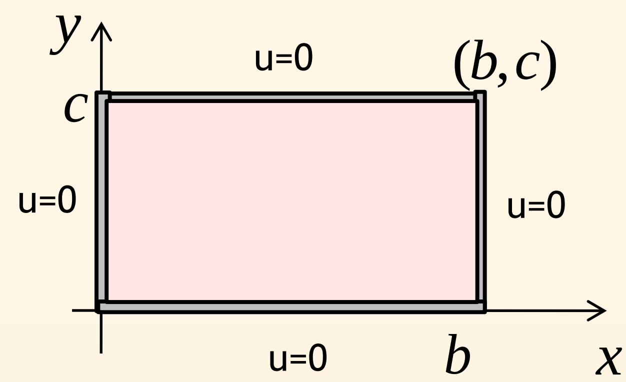

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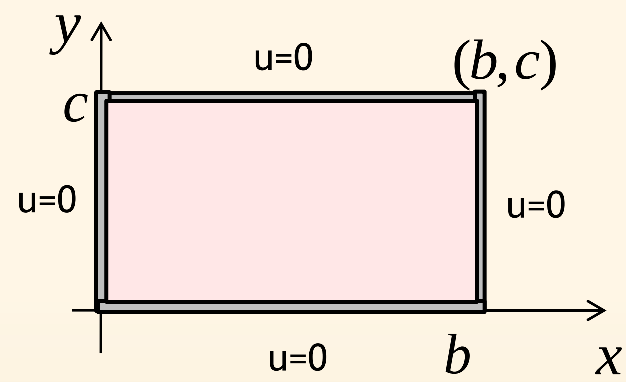

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$$\text{setting}\ \ \underline{K_m(y)=\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi y}{c}},$$

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Fourier Series in Two Variables

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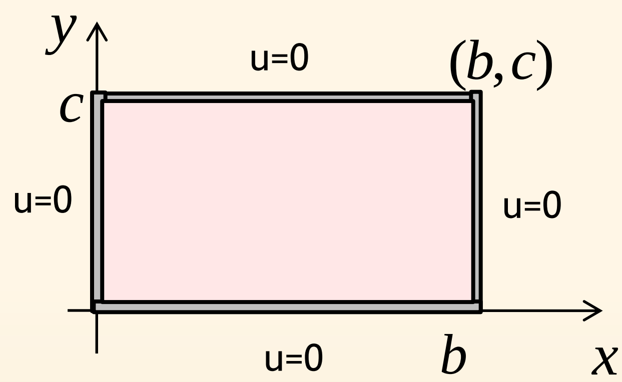

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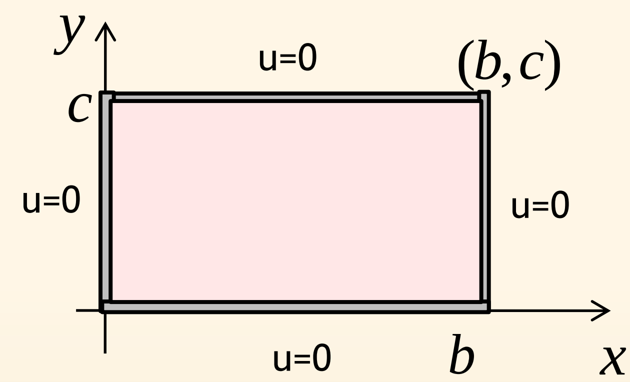

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Fourier Series in Two Variables

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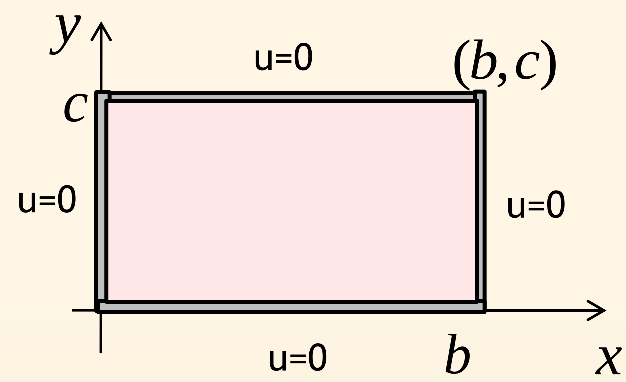

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setting $K_m(y)=\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi y}{c}$,

Fourier sine series

$$\therefore A_{mn}=\frac{2}{c}\int_0^c K_m(y)\sin\frac{n\pi y}{c}dy$$

Fourier sine series Coefficients

then, $f(x,y)=\sum_{m=1}^{\infty}K_m(y)\sin\frac{m\pi x}{b}$ ⇨ $\therefore K_m(y)=\frac{2}{b}\int_0^b f(x,y)\sin\frac{m\pi x}{b}dx$

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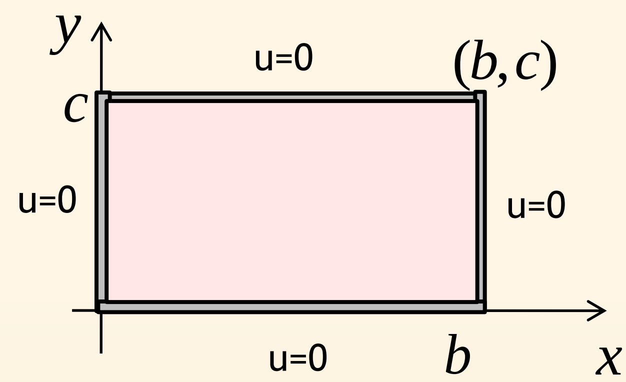

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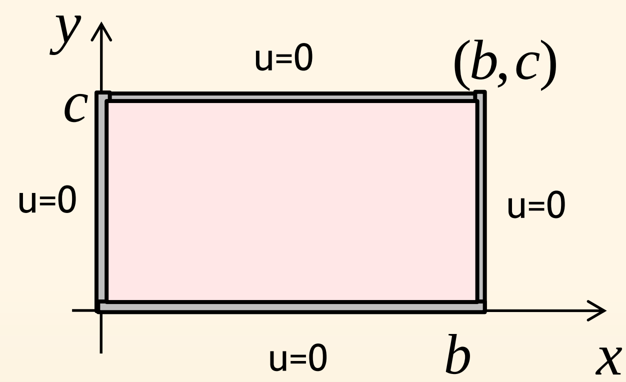

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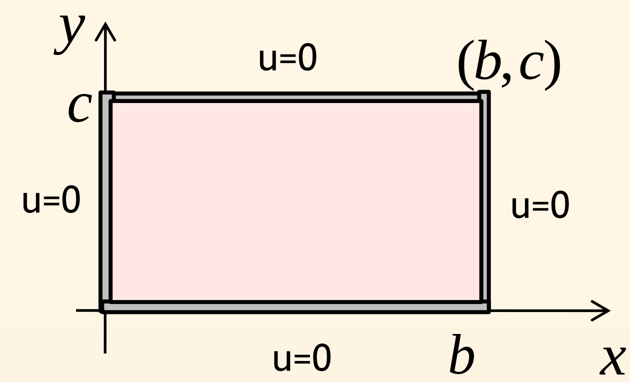

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$$\therefore u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{b}\sin\frac{n\pi y}{c}$$

$$where,\ \ A_{mn}=\frac{4}{bc}\int_0^c\int_0^b f(x,y)\sin\frac{m\pi x}{b}dx\sin\frac{n\pi y}{c}dy$$