Naval Architecture & Ocean Engineering

[2008][13-1]

Engineering Mathematics 2

December, 2008

Prof. Kyu-Yeul Lee

Department of Naval Architecture and Ocean Engineering,
Seoul National University of College of Engineering

Fourier Transform(2)
: Fourier Transform Analysis

Basic Fourier Transform Analysis

Fourier Transform

Basic Fourier Transform Analysis

Basic Fourier Transform Analysis

Relationship of Conventional and Transform analysis

Basic Fourier Transform Analysis

Relationship of Conventional and Transform analysis

✓conventional analysis

Basic Fourier Transform Analysis

Relationship of Conventional and Transform analysis

✓conventional analysis

Problem statement

$$y = \frac{x}{z}$$

Basic Fourier Transform Analysis

Relationship of Conventional and Transform analysis

✓conventional analysis

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Problem statement

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↓

Complex analysis

Long-hand division

Basic Fourier Transform Analysis

Relationship of Conventional and Transform analysis

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Transform

$$\log y = \log x - \log z$$

Basic Fourier Transform Analysis

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Basic Fourier Transform Analysis

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$$y = \frac{x}{z}$$

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Problem solution

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Transform

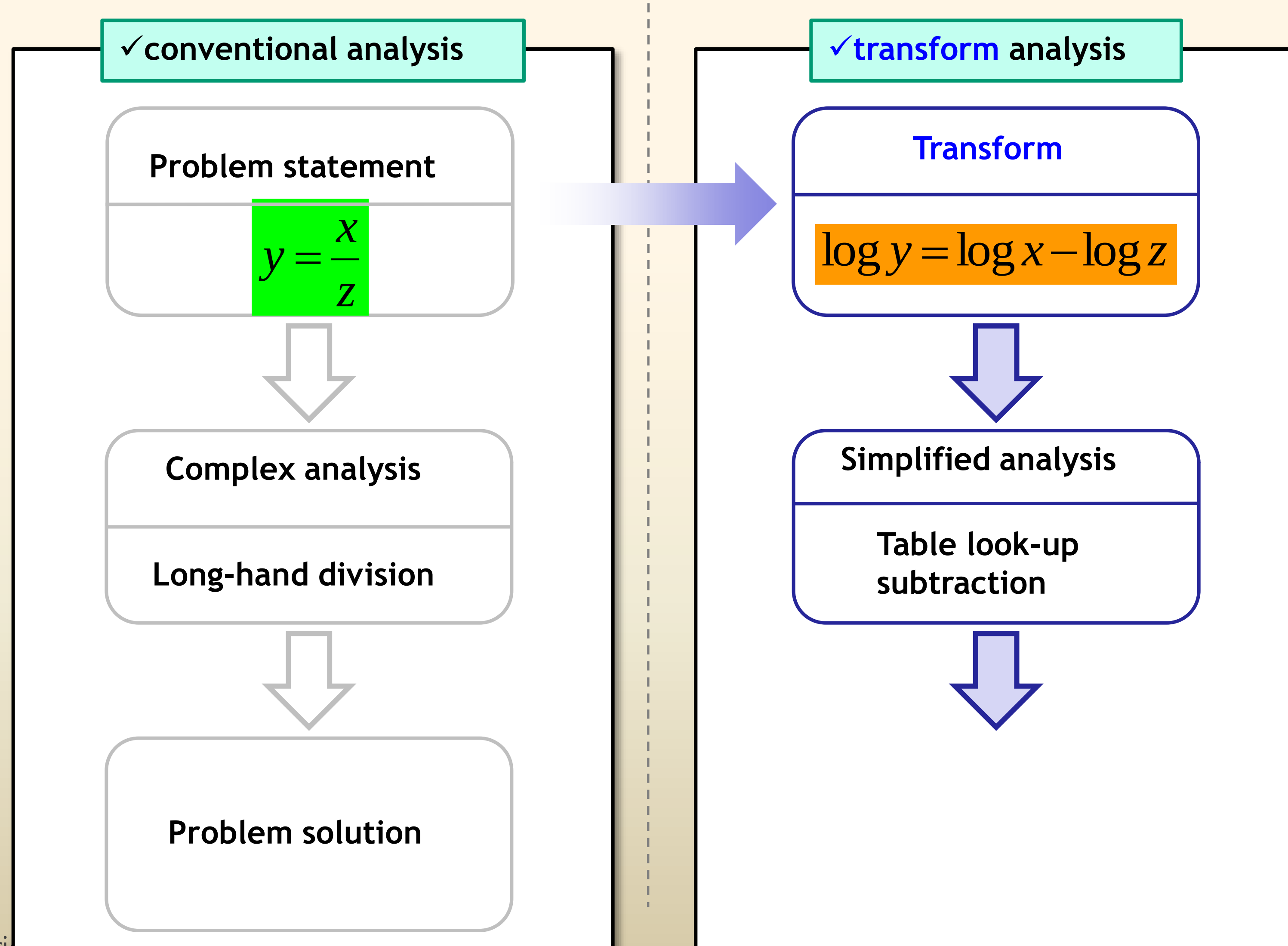
$$\log y = \log x - \log z$$

Simplified analysis

Table look-up
subtraction

Basic Fourier Transform Analysis

Relationship of Conventional and Transform analysis



Basic Fourier Transform Analysis

Relationship of Conventional and Transform analysis

✓conventional analysis

Problem statement

$$y = \frac{x}{z}$$

⇩

Complex analysis

Long-hand division

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Problem solution

⇨

✓transform analysis

Transform

$$\log y = \log x - \log z$$

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Simplified analysis

Table look-up subtraction

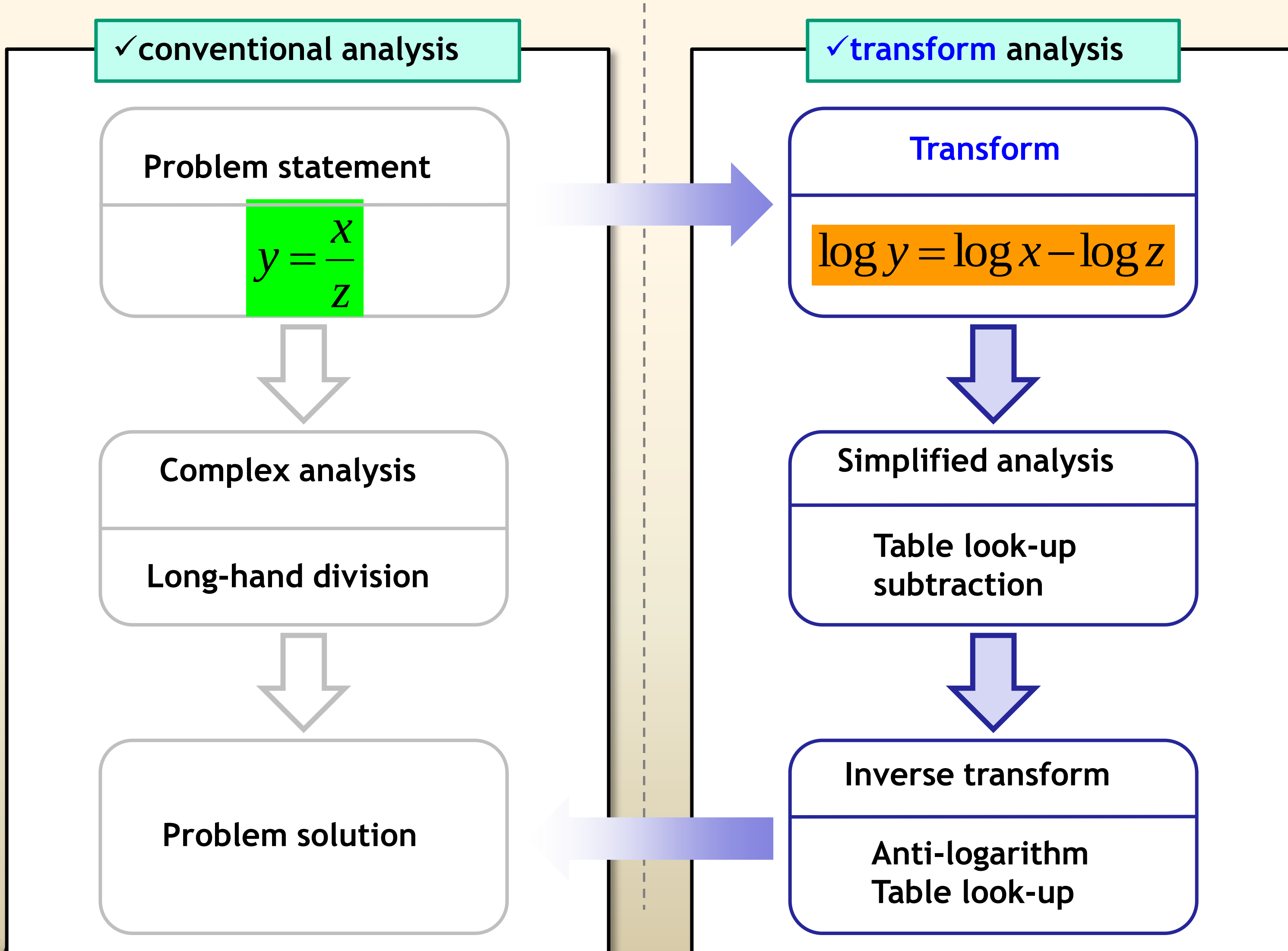
⇩

Inverse transform

Anti-logarithm Table look-up

Basic Fourier Transform Analysis

Relationship of Conventional and Transform analysis



Basic Fourier Transform Analysis

Relationship of Conventional and Transform analysis

complexity reduced

✓conventional analysis

Problem statement

$$y = \frac{x}{z}$$

Complex analysis

Long-hand division

Problem solution

✓transform analysis

Transform

$$\log y = \log x - \log z$$

Simplified analysis

Table look-up subtraction

Inverse transform

Anti-logarithm Table look-up

What kind of Transformation ?

Basic Fourier Transform Analysis

What kind of Transformation ?

$$y = \frac{x}{z} \xrightarrow{\log} \log y = \log x - \log z$$

$$y = x^2 \xrightarrow{\frac{d}{dx}} \frac{d}{dx} y = 2x$$

$$y = x^2 \xrightarrow{\int dx} \int y dx = \frac{1}{3} x^3 + c$$

Basic Fourier Transform Analysis

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Laplace Transform

$$f(t) \xrightarrow{\int_0^\infty e^{-st} \quad dt} \int_0^\infty e^{-st} f(t)dt = \mathscr{L}\{f(t)\}$$

Basic Fourier Transform Analysis

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Laplace Transform

$$f(t) \quad \xrightarrow{\int_0^\infty e^{-st} \quad dt} \quad \int_0^\infty e^{-st} f(t) dt = \mathscr{L}\{ f(t) \}$$

Fourier Transform

$$s(t) \quad \xrightarrow{\int_{-\infty}^\infty e^{-i\omega t} \quad dt} \quad \int_0^\infty e^{-i\omega t} f(t) dt = \hat{f}(\omega)$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

Basic Fourier Transform Analysis

Interpretation of the Fourier Series*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{in\pi t/p} \qquad c_n = \frac{1}{2p}\int_{-p}^{p} f(t) e^{-in\pi t/p} dt$$

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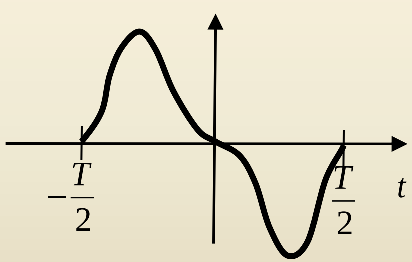
Waveform defined
From -∞ to+ ∞



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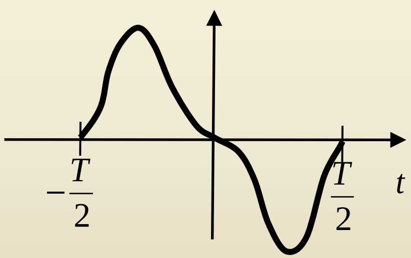

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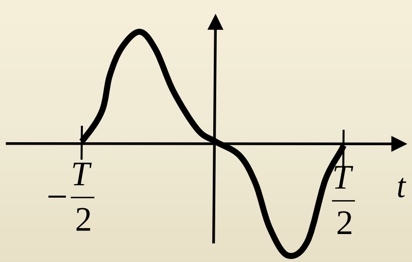

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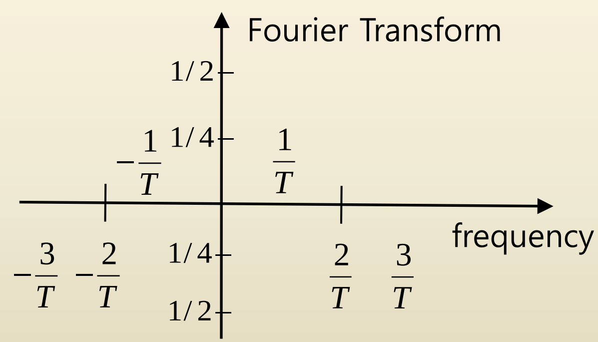

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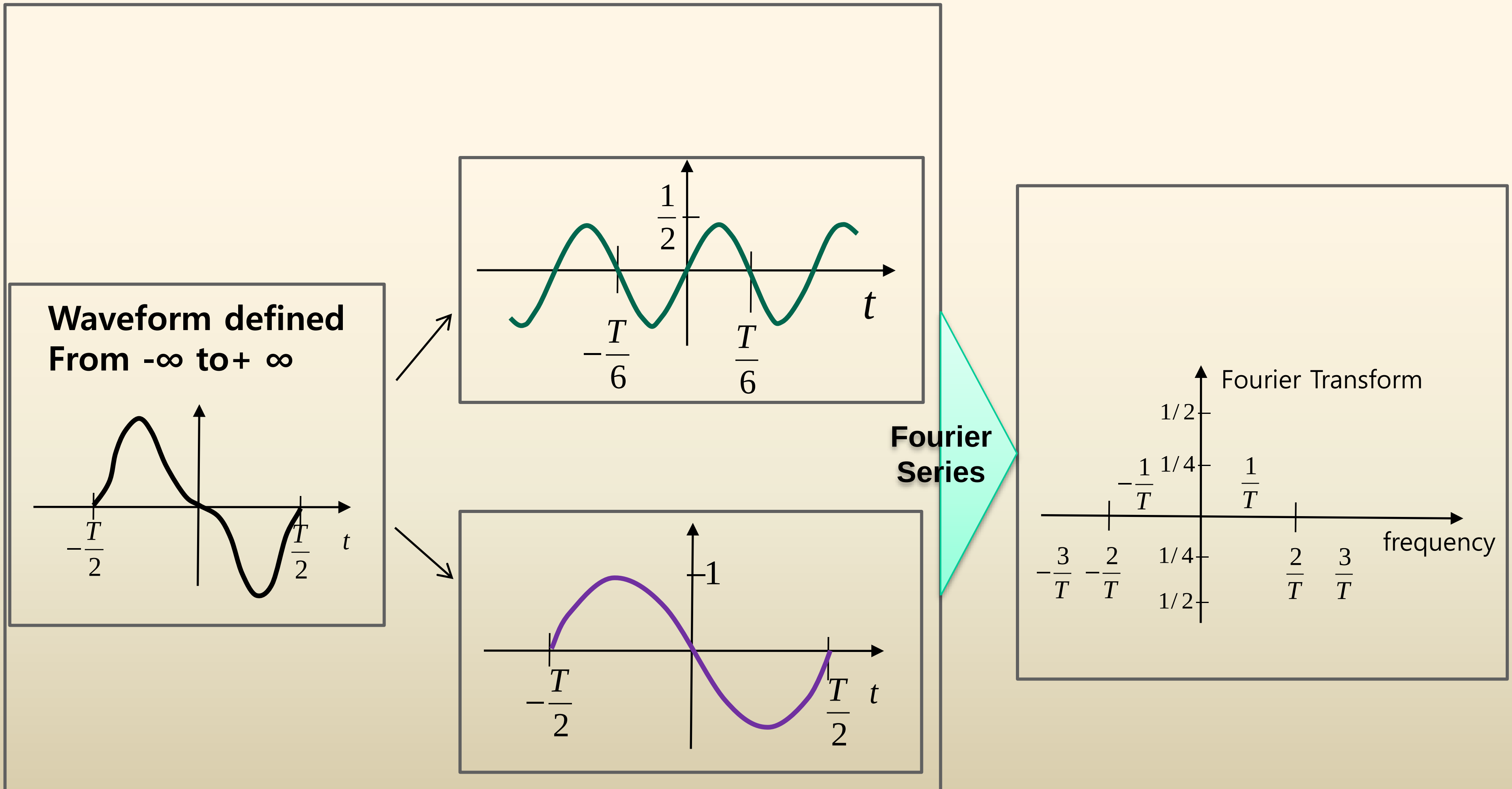

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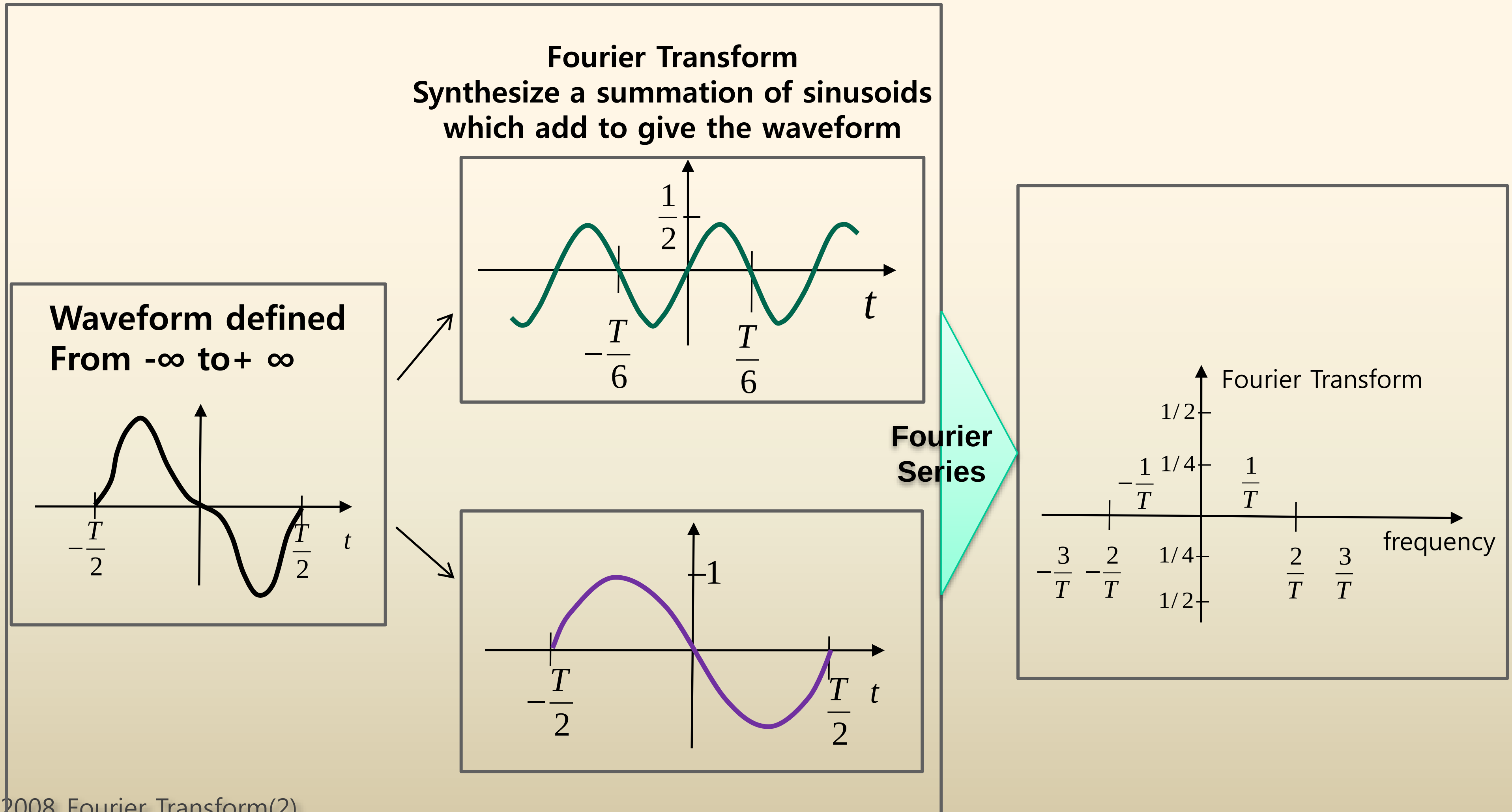

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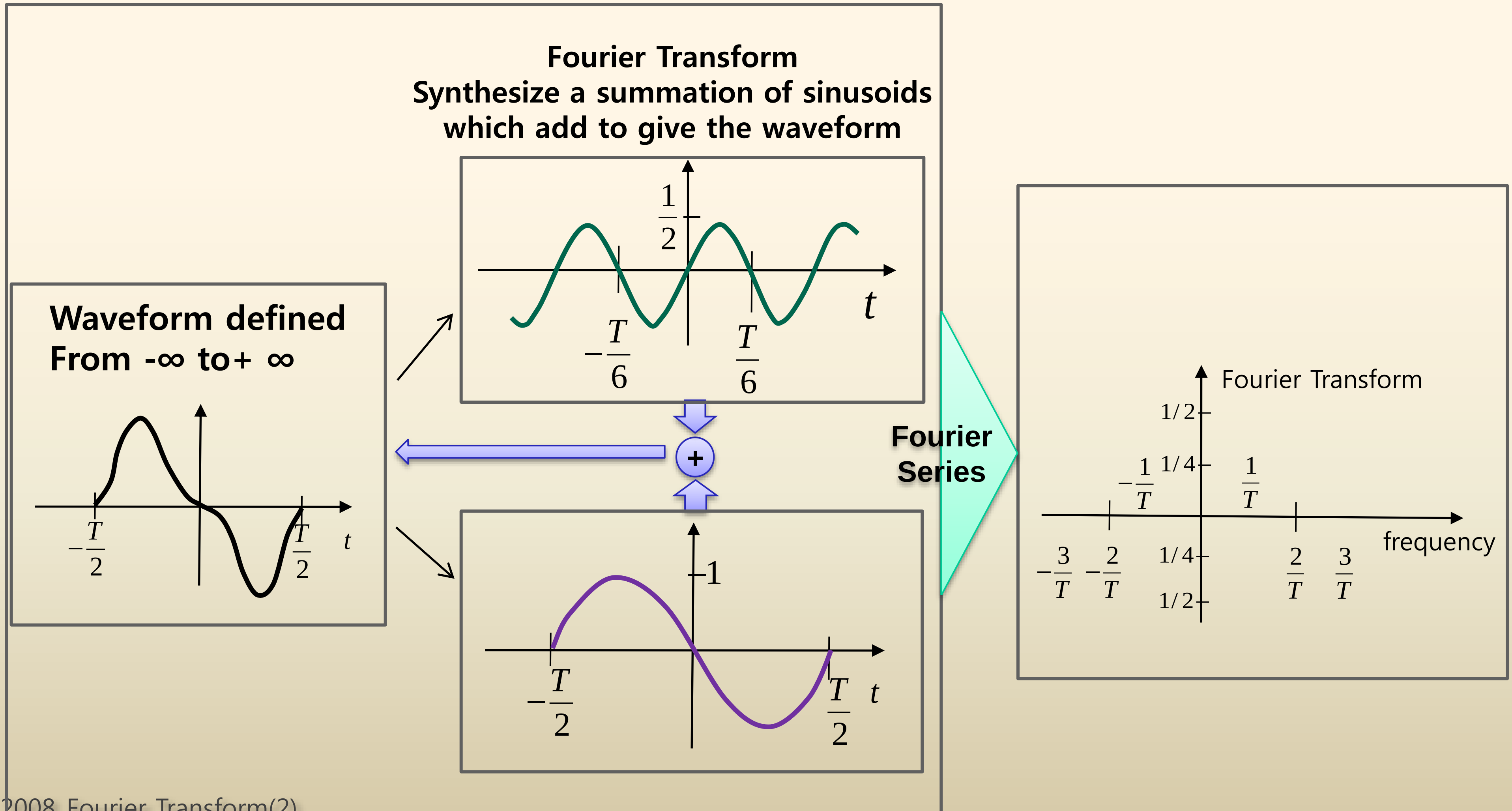

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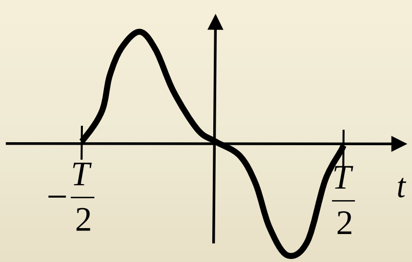
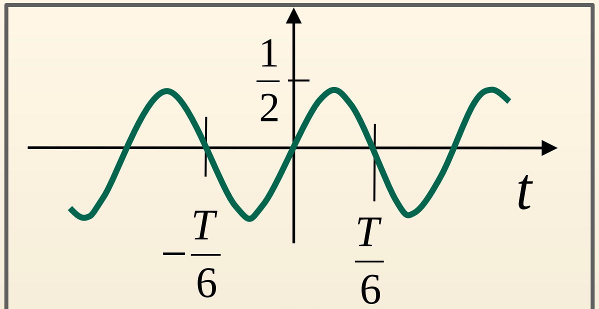
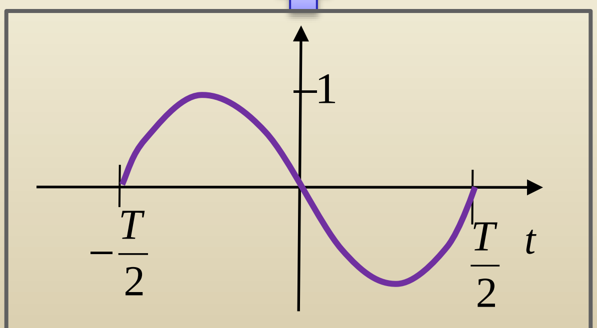


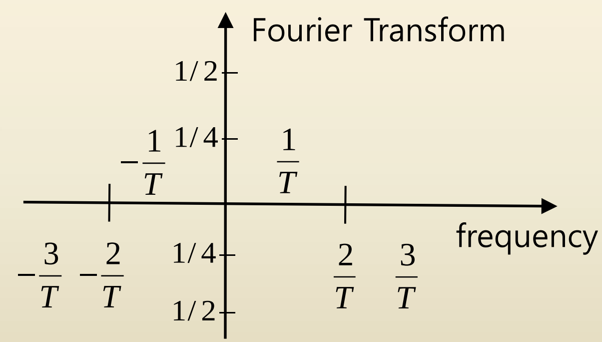
Fourier Transform Synthesize a summation of sinusoids which add to give the waveform





Fourier Series

Construct a diagram which displays amplitude and frequency of each sinusoid



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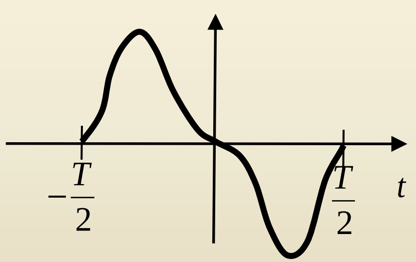

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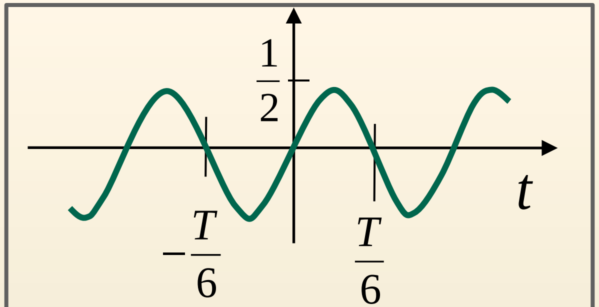

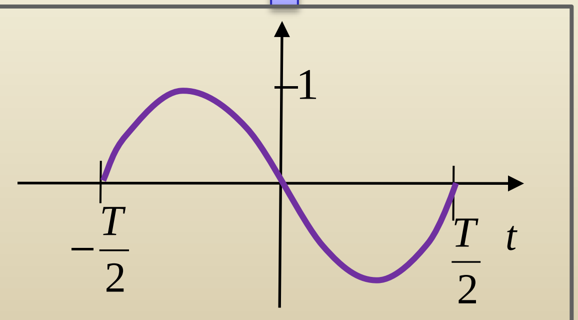

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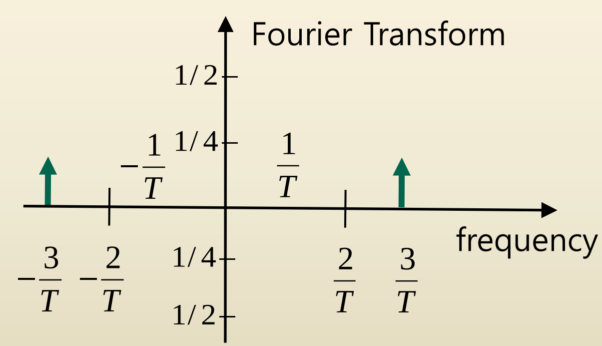

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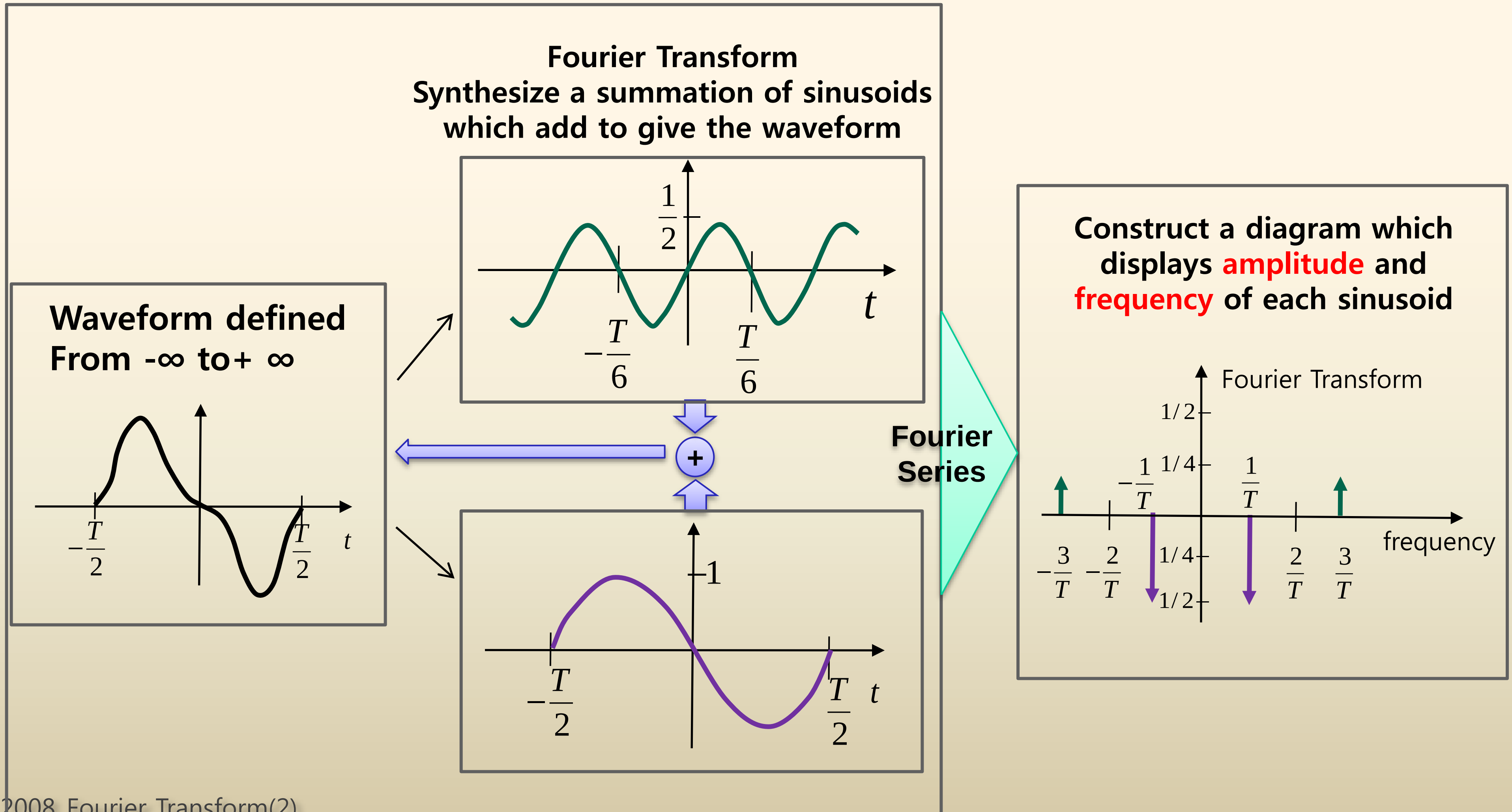

2008 Fourier Transform(2)

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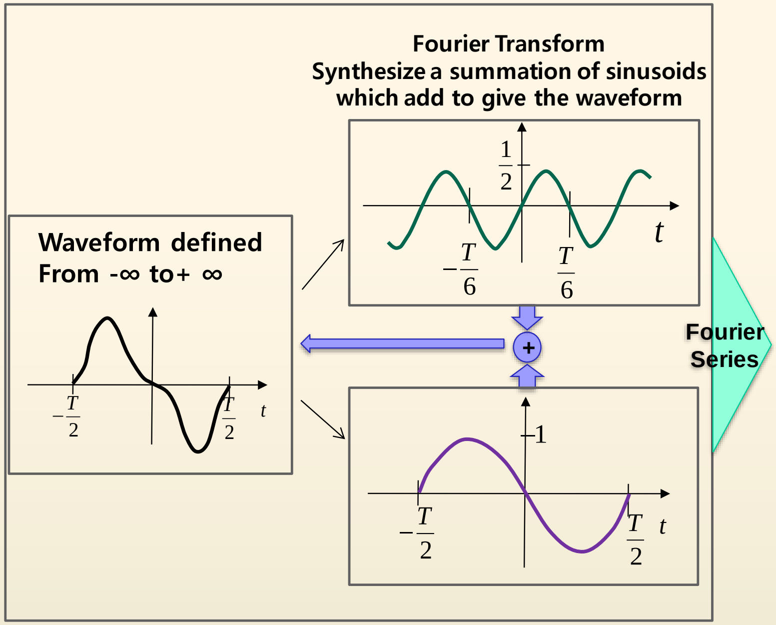

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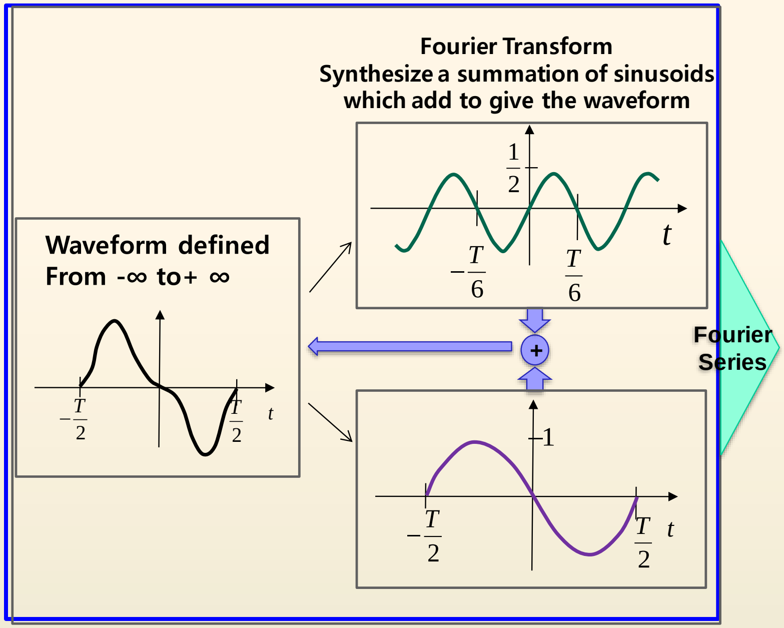

The essence of the Fourier transform of a waveform is

to decompose** or separate the waveform **into a sum of sinusoids of different frequencies

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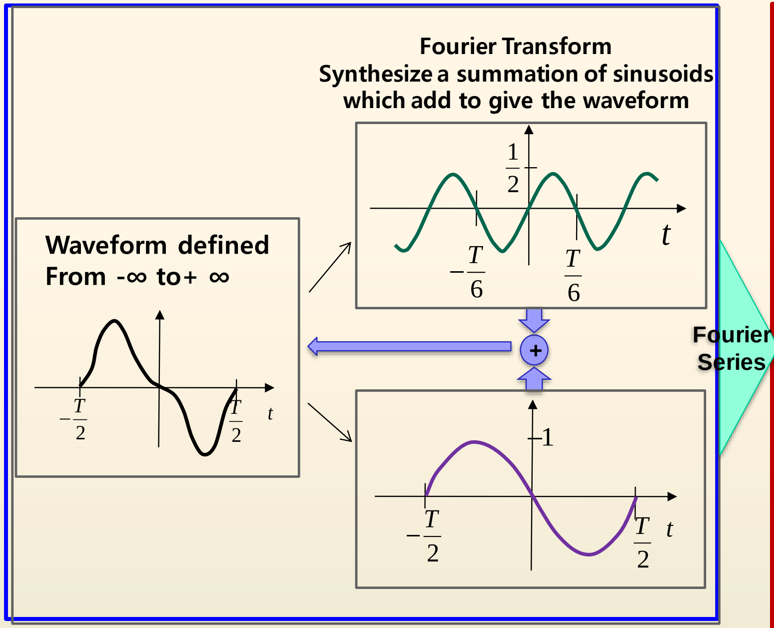
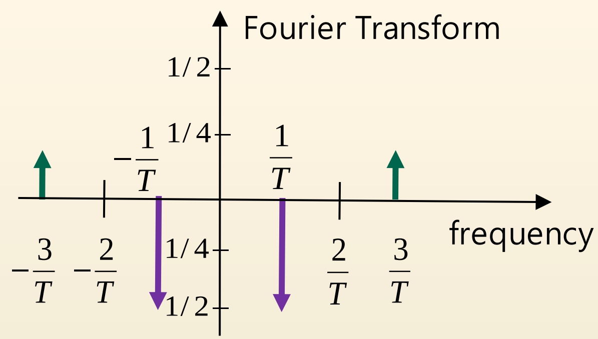


Construct a diagram which displays amplitude and frequency of each sinusoid



Usual convention : displaying both positive and negative frequency sinusoids for each frequencies; the amplitude has been halved accordingly

The essence of the Fourier transform of a waveform is

to decompose or separate the waveform into a sum of sinusoids of different frequencies

The pictorial representation of the Fourier transform is a diagram which displays

the amplitude and frequencies of each of the determined sinusoids

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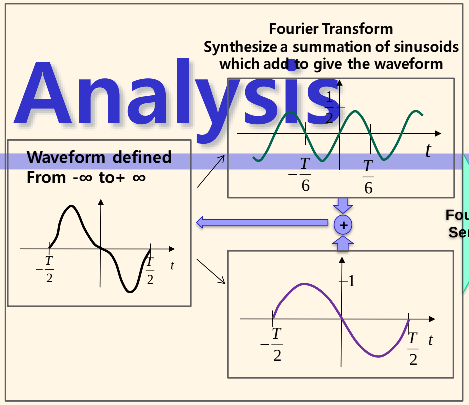
Basic Fourier Transform Analysis



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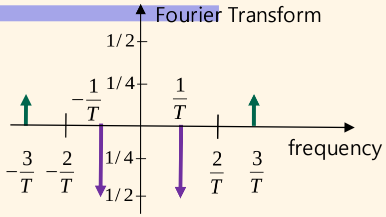
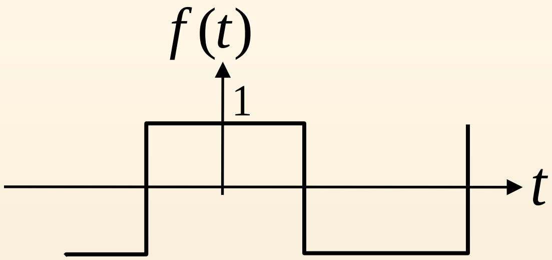
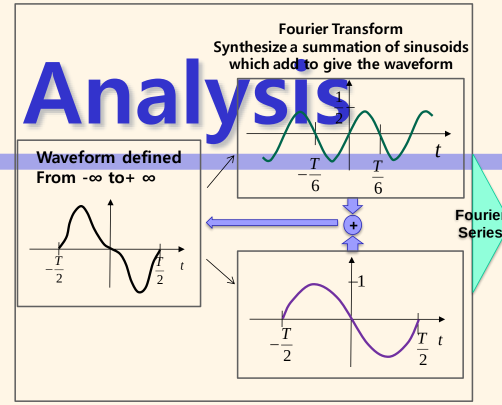


Ex.) Fourier Transform of square wave function



Basic Fourier Transform Analysis



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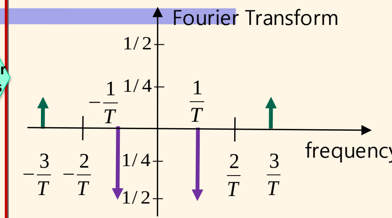

Ex.) Fourier Transform of square wave function

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p}$$

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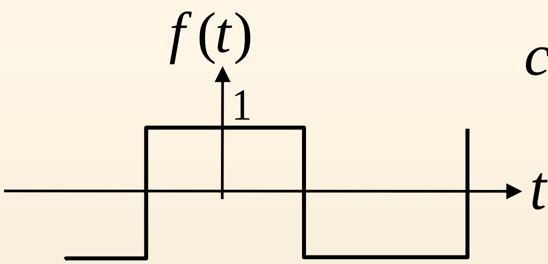
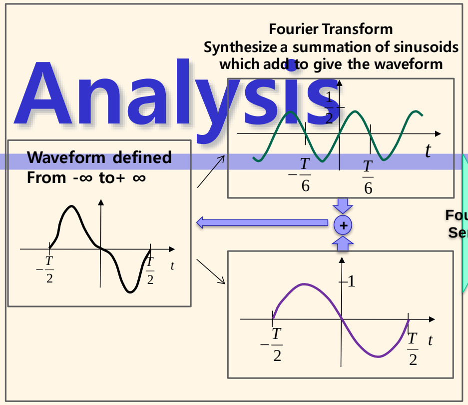
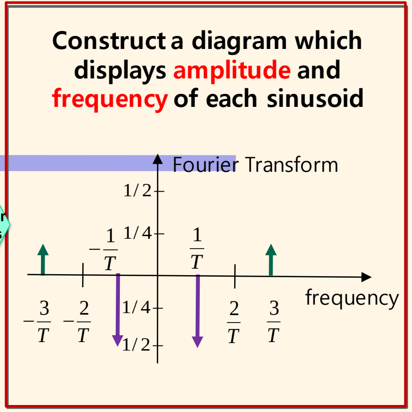


Fourier Series

Basic Fourier Transform Analysis





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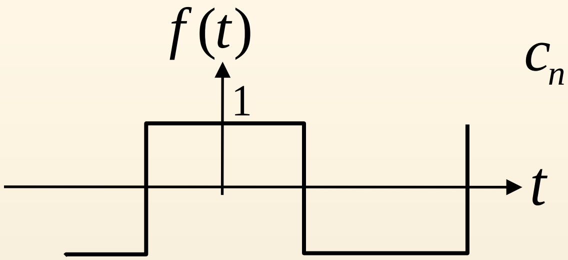
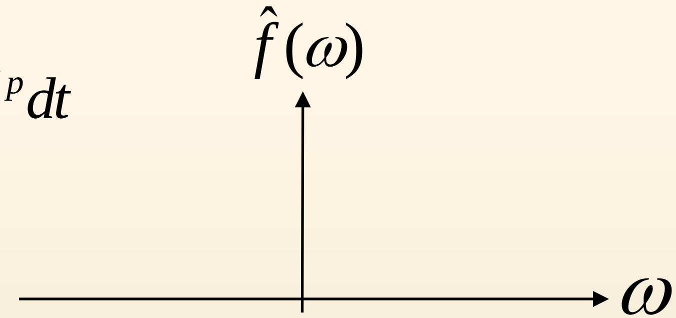


Fourier Series



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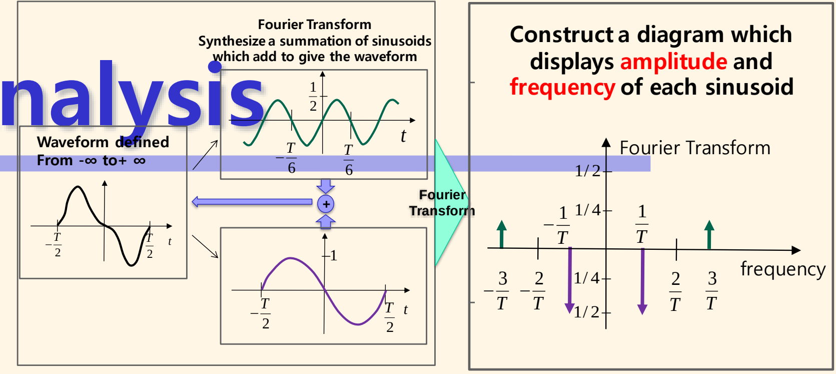

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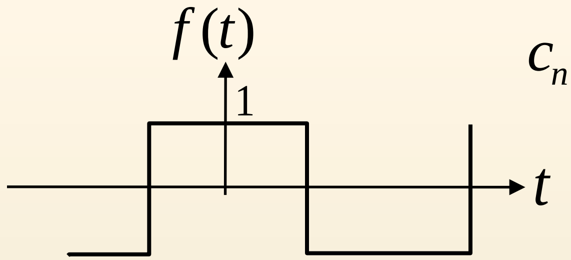

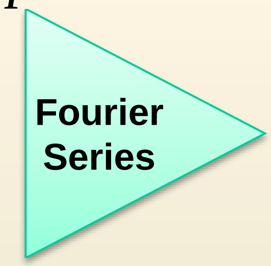

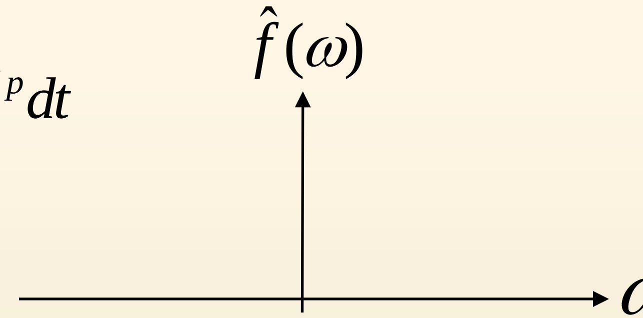

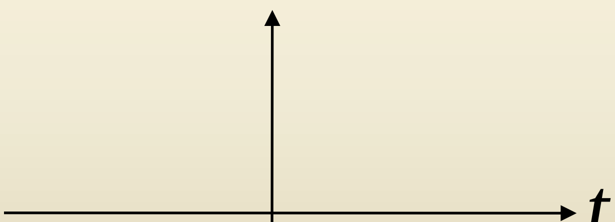

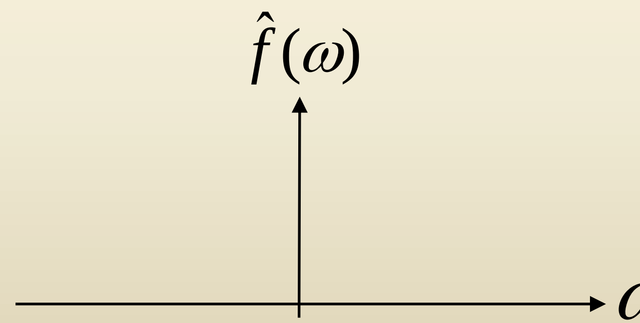

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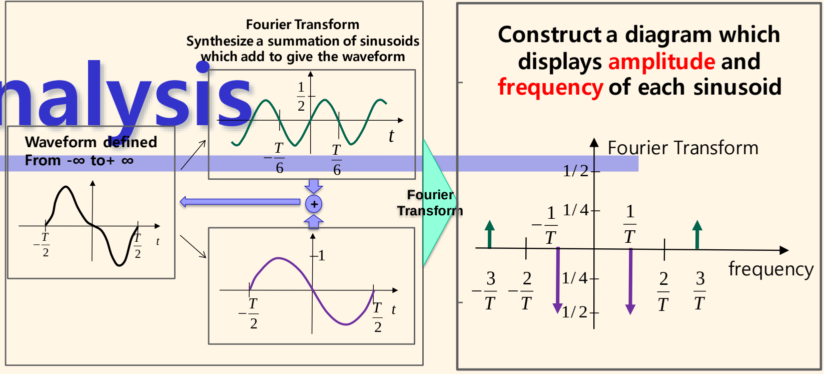

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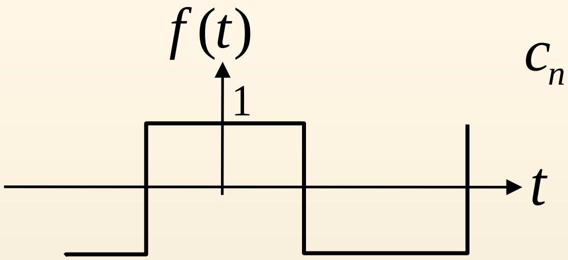
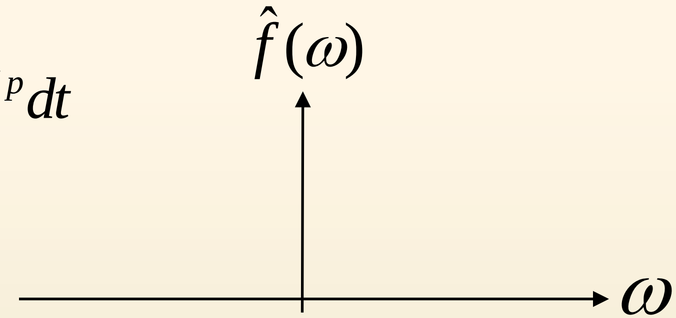
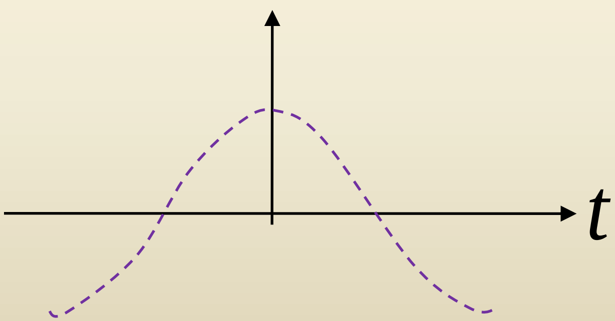
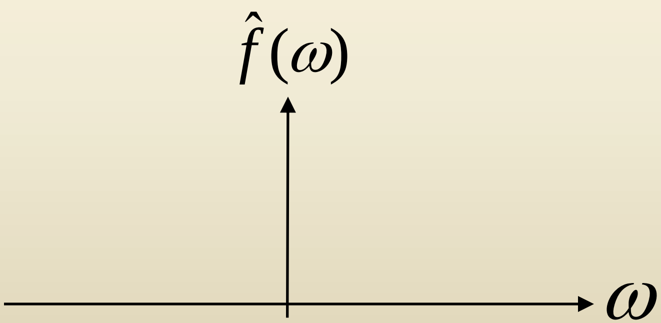


Fourier Series

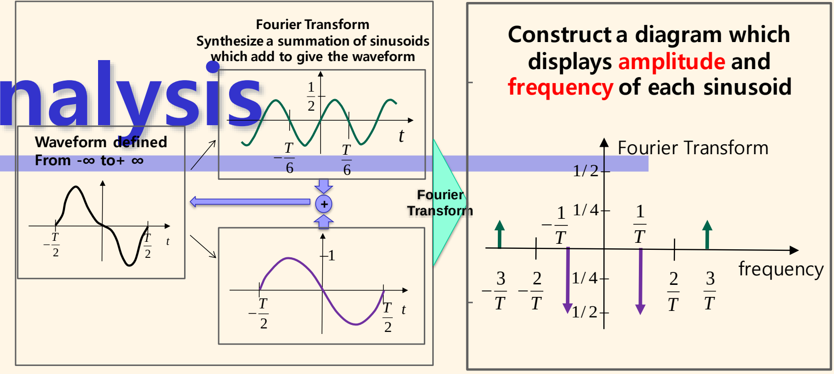






$\cos(2\pi f_0 t)$

Basic Fourier Transform Analysis



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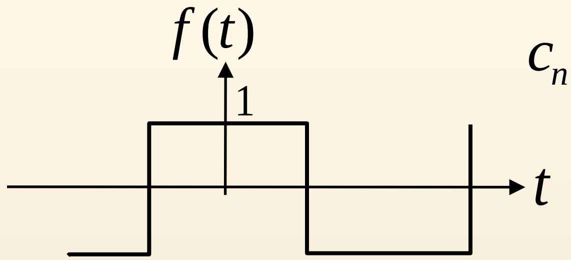
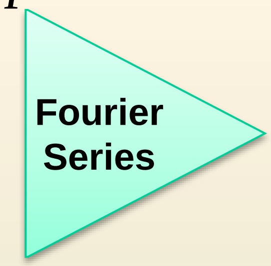
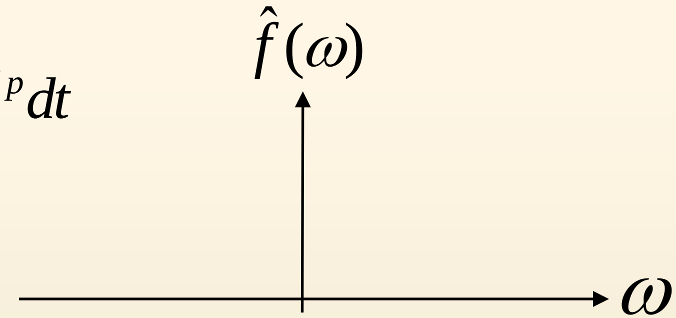
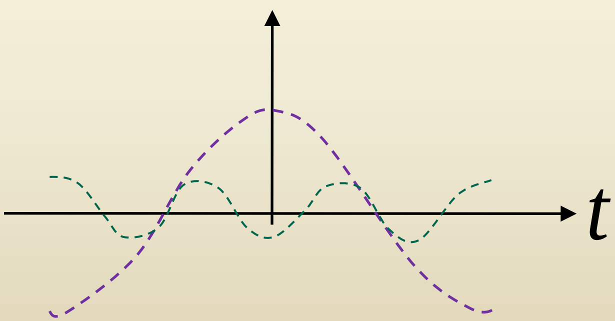
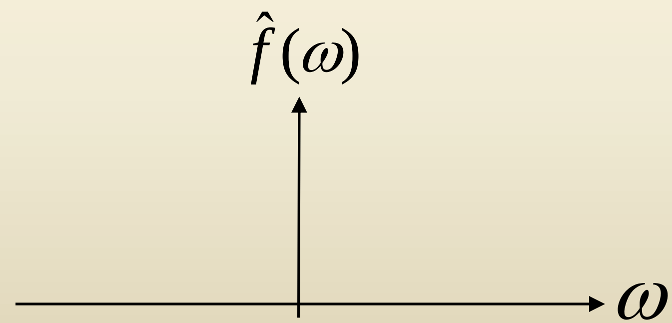




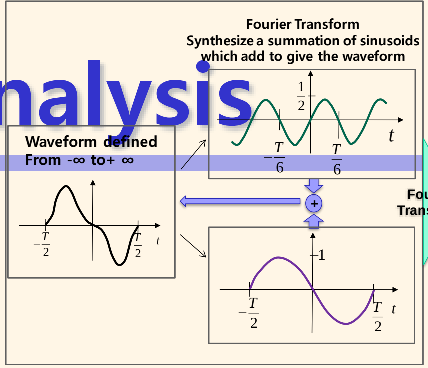


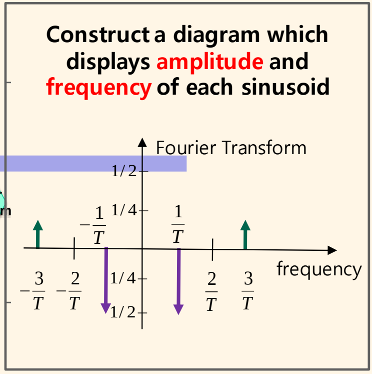




$$\cos(2\pi f_0 t) - \frac{1}{3}\cos(6\pi f_0 t)$$

Basic Fourier Transform Analysis





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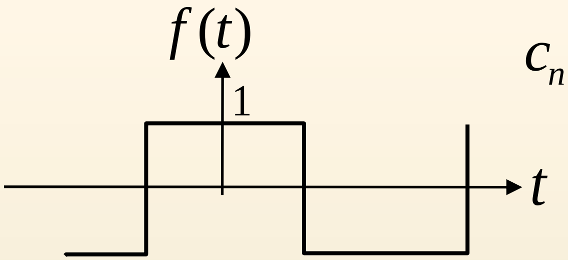
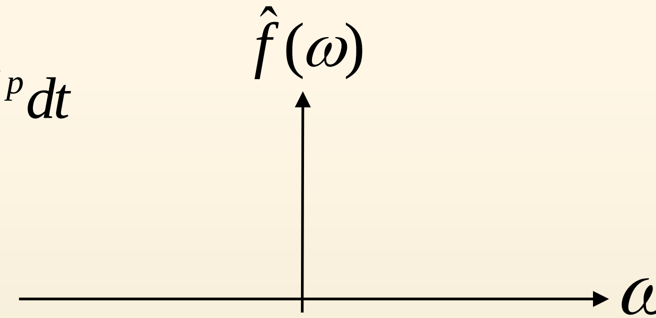
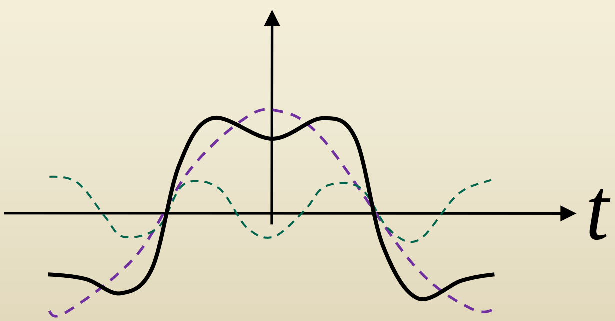
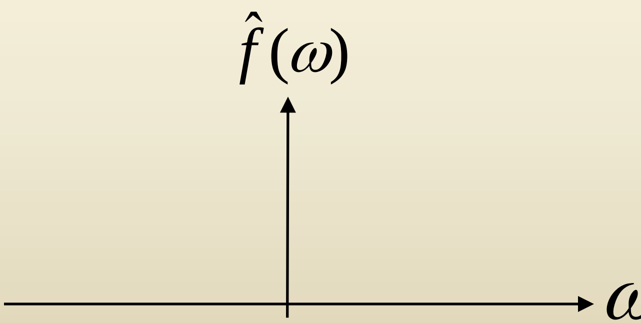


Fourier Series







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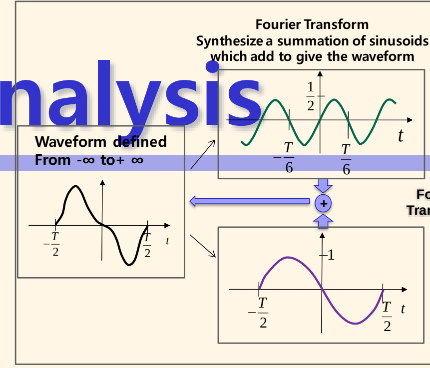
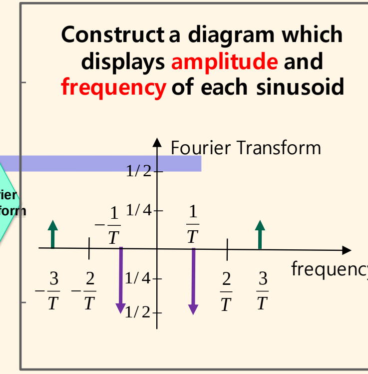
Basic Fourier Transform Analysis





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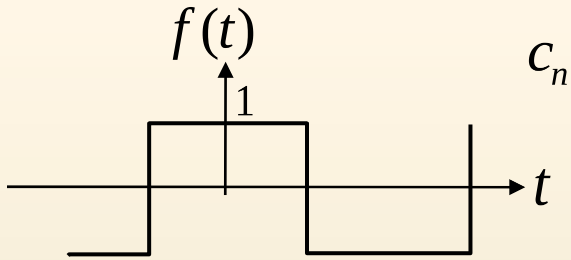
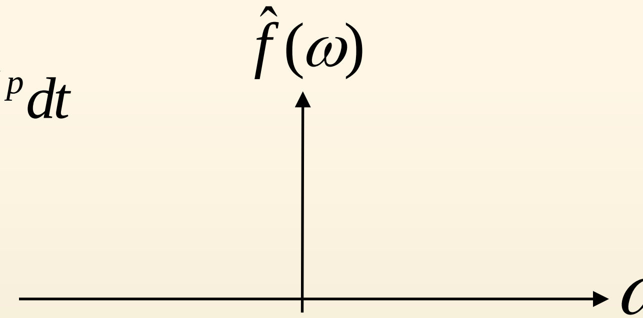
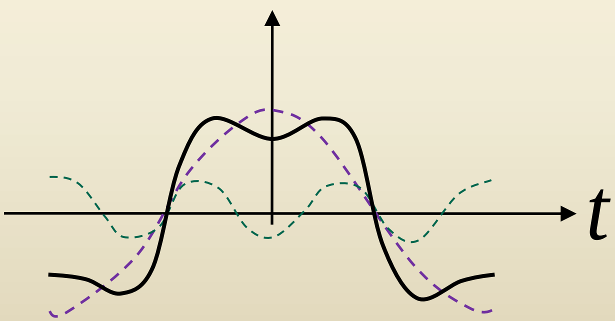
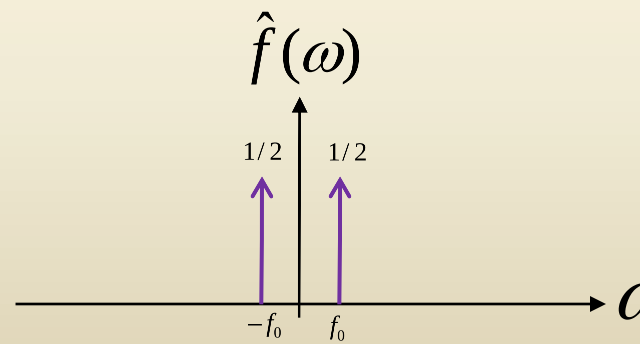


Fourier Series







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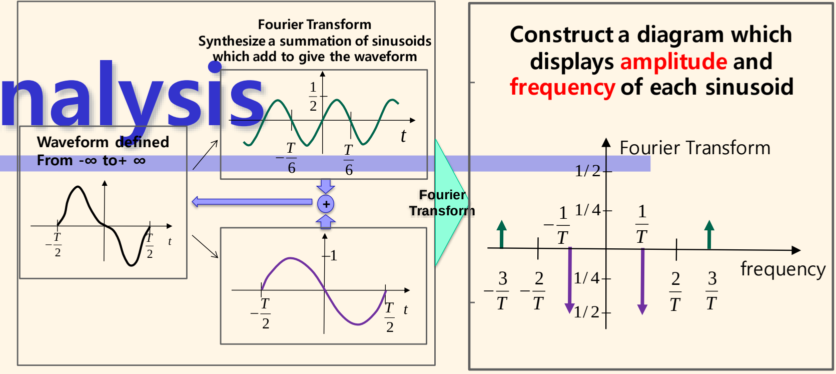
Basic Fourier Transform Analysis



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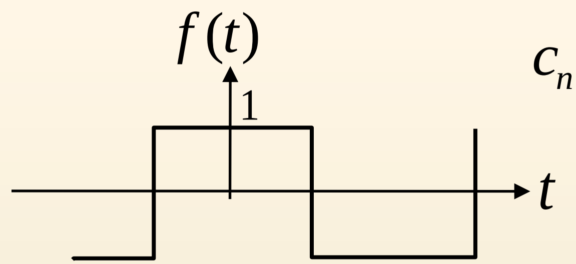

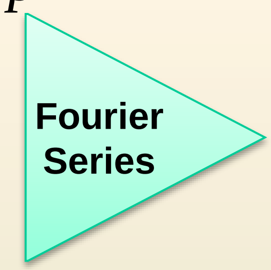

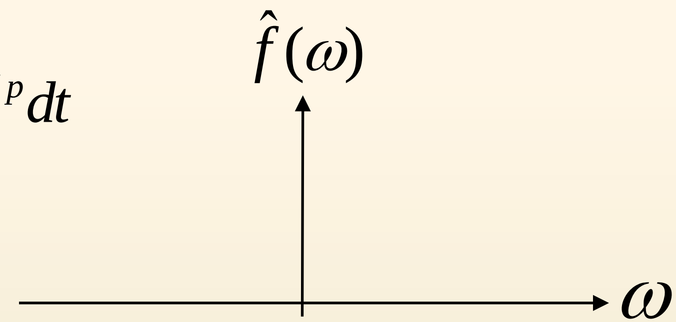

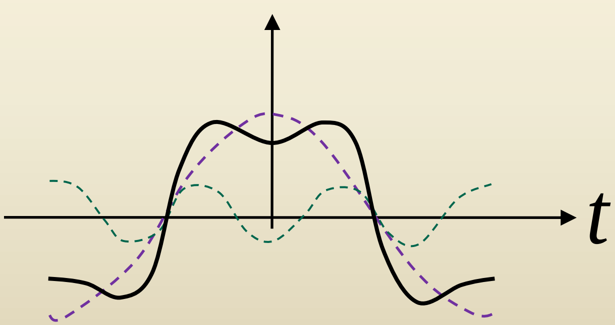

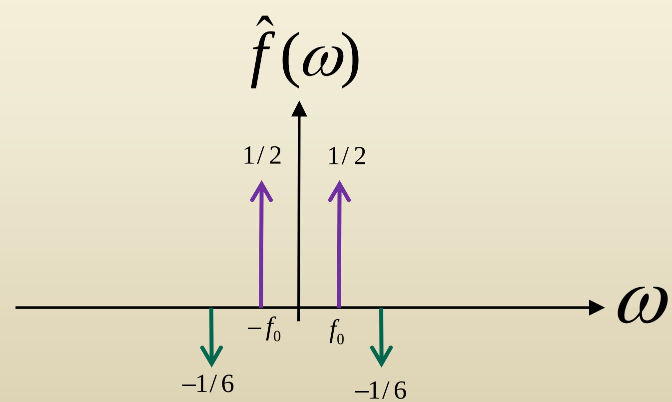

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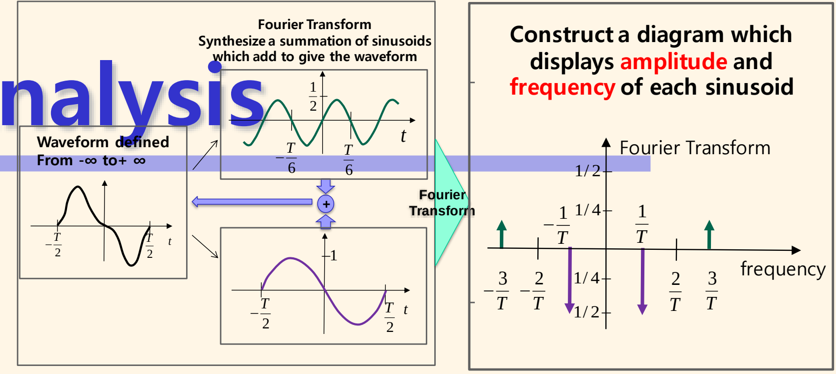
Basic Fourier Transform Analysis



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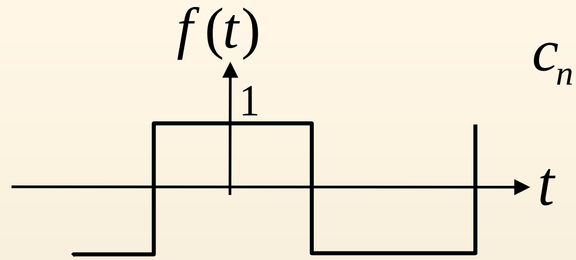

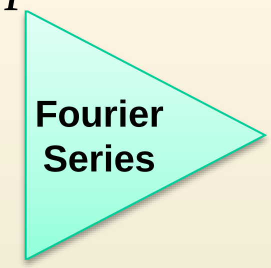

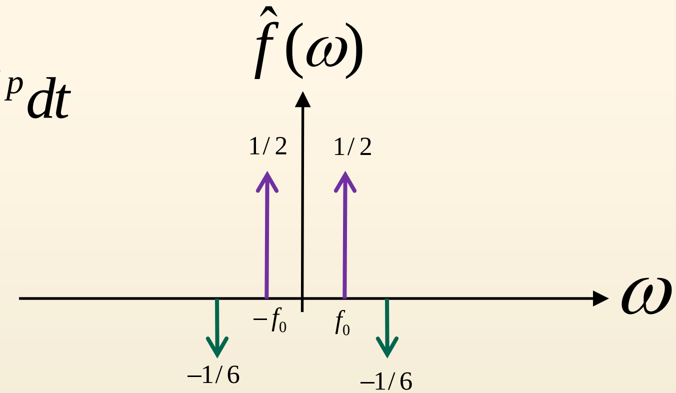

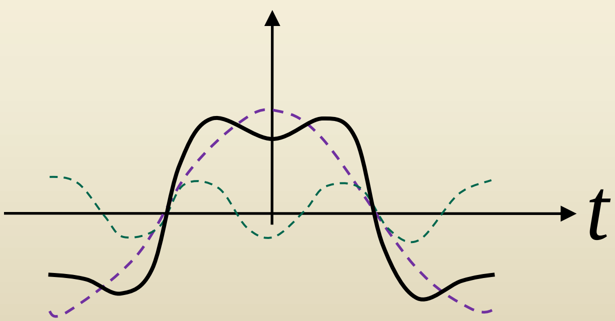

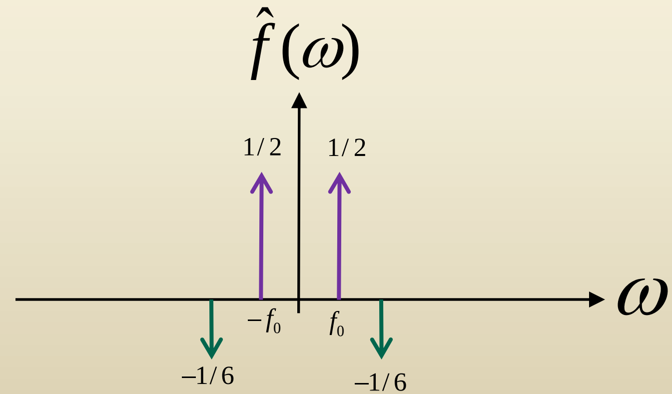

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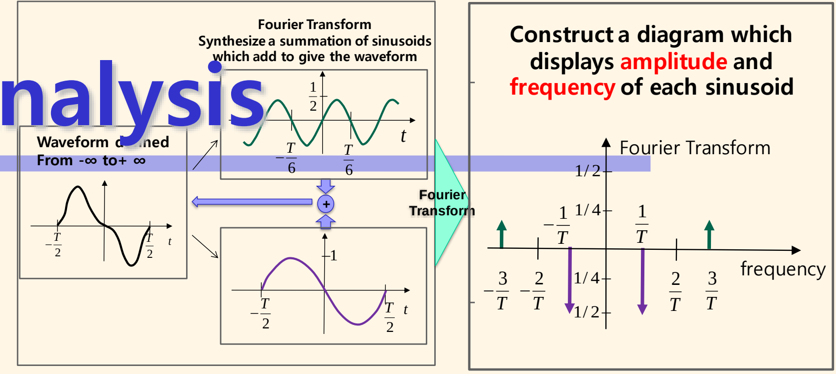
Basic Fourier Transform Analysis



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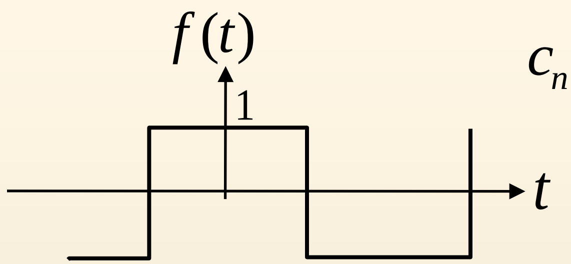
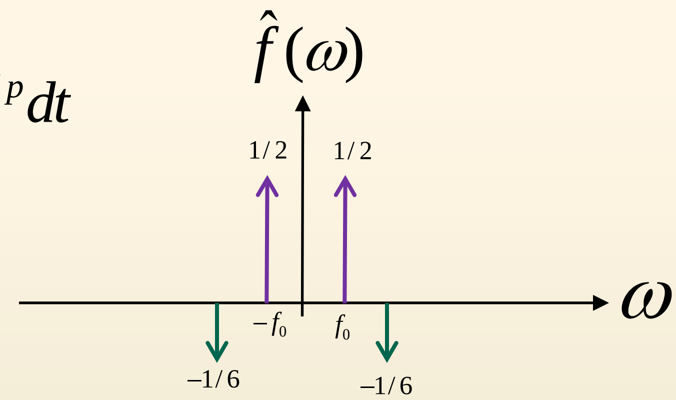
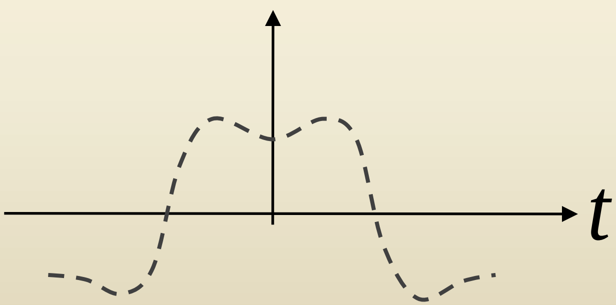


Fourier Series





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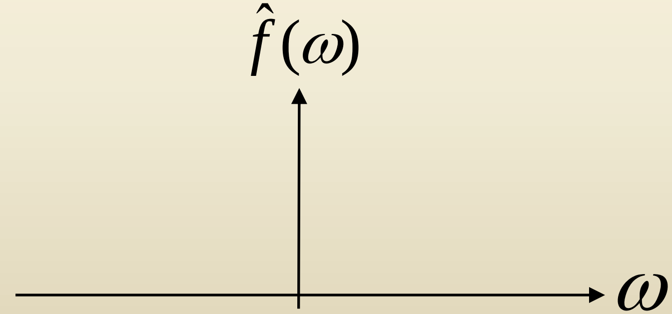
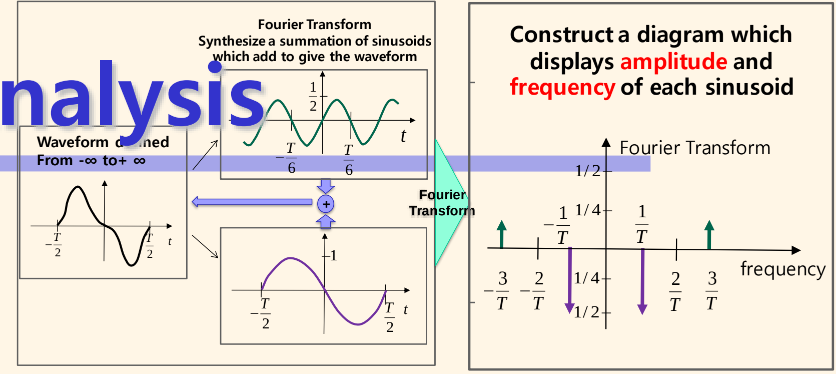


Basic Fourier Transform Analysis



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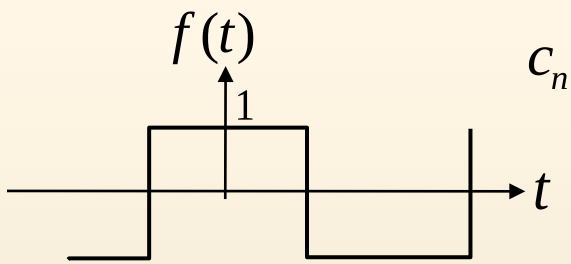
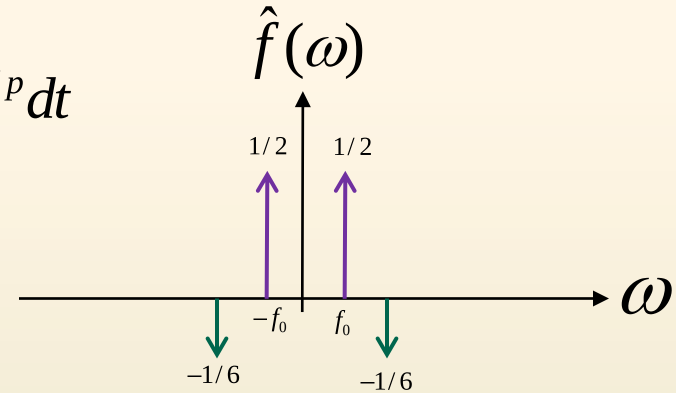
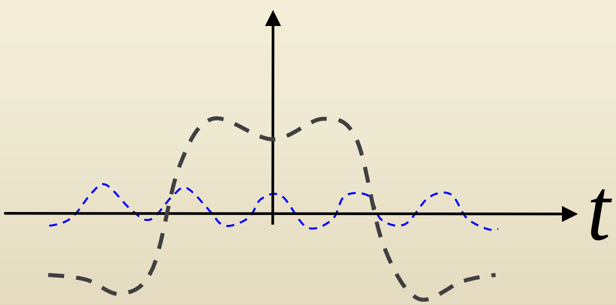
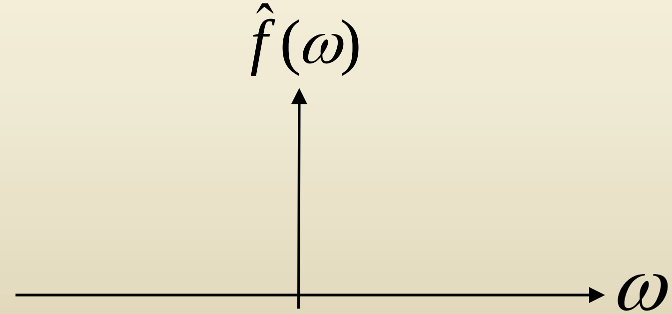


Fourier Series







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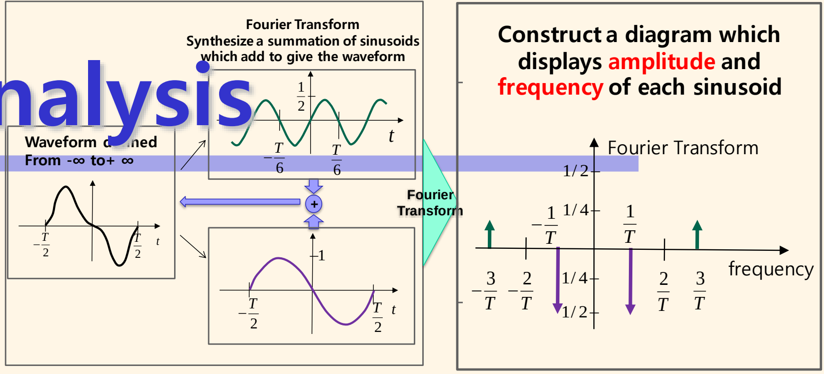
Basic Fourier Transform Analysis



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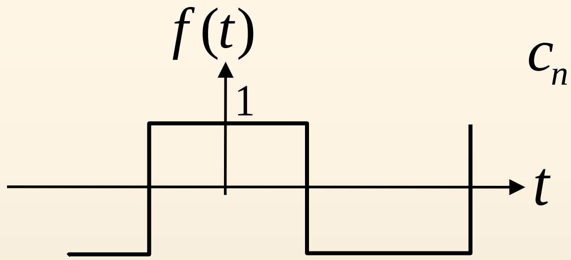

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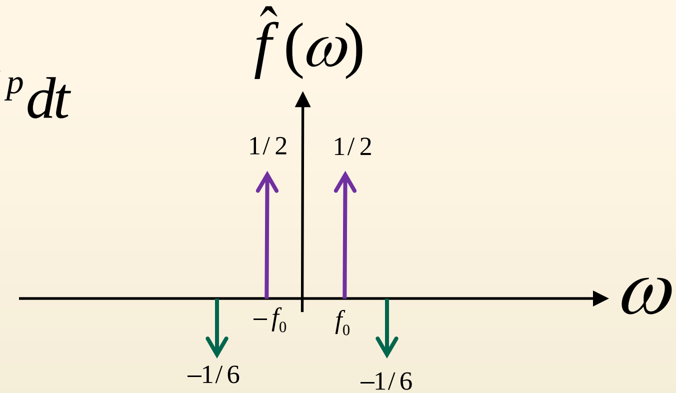
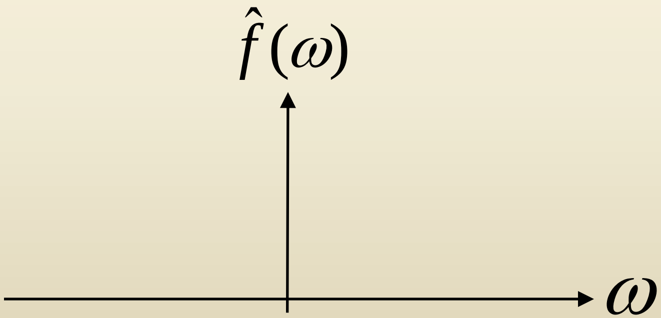
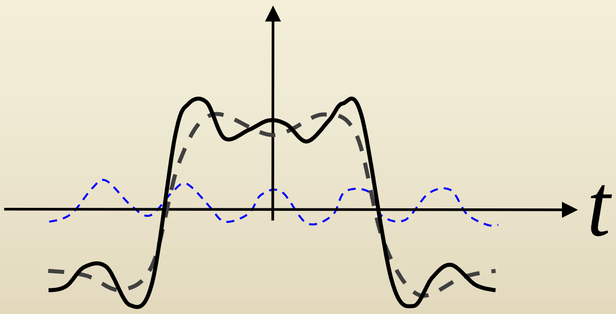
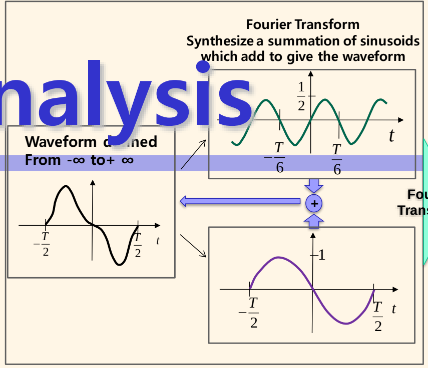


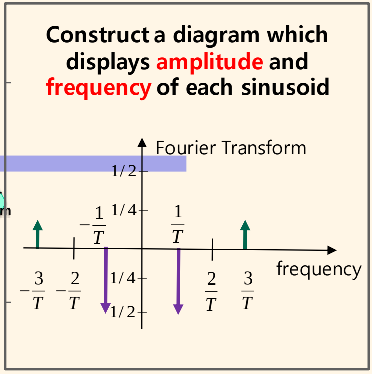




$$s_2(t)=\cos(2\pi f_0 t)-\frac{1}{3}\cos(6\pi f_0 t)+\frac{1}{5}\cos(10\pi f_0 t)$$

Basic Fourier Transform Analysis





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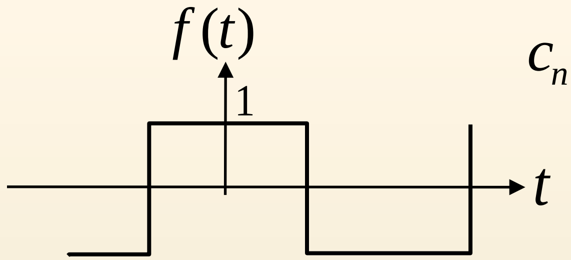

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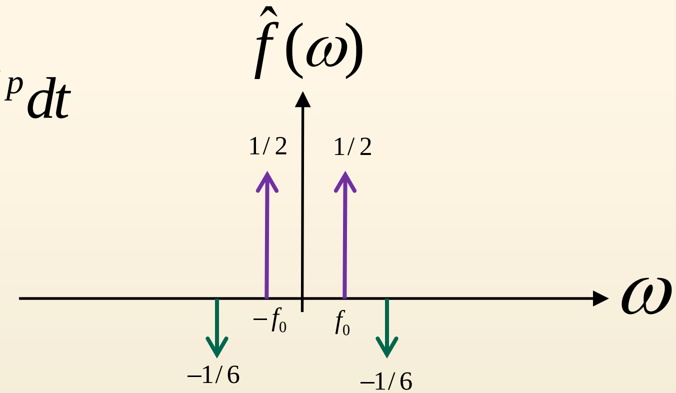

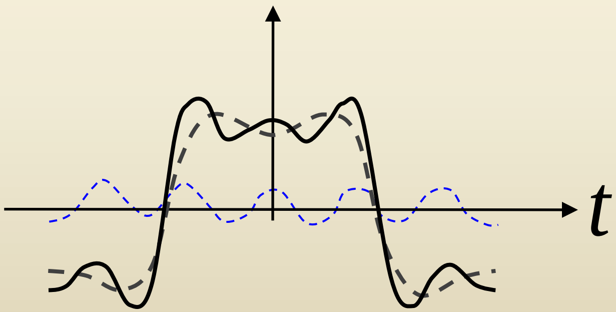

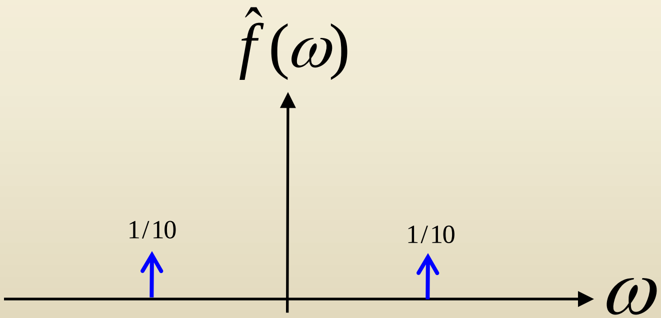

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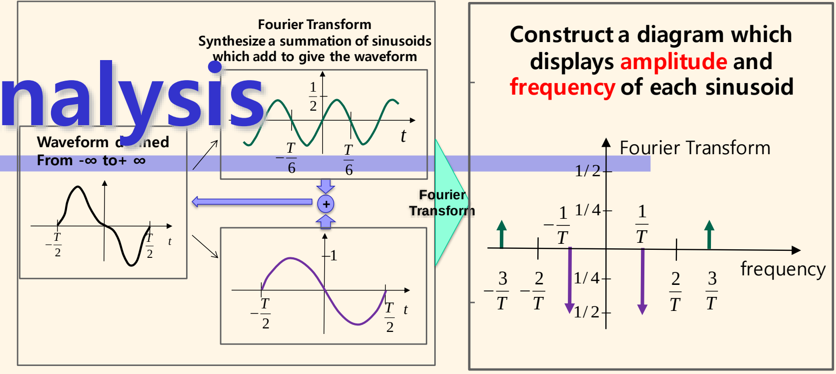
Basic Fourier Transform Analysis



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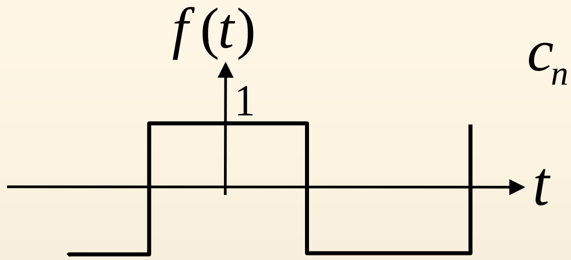

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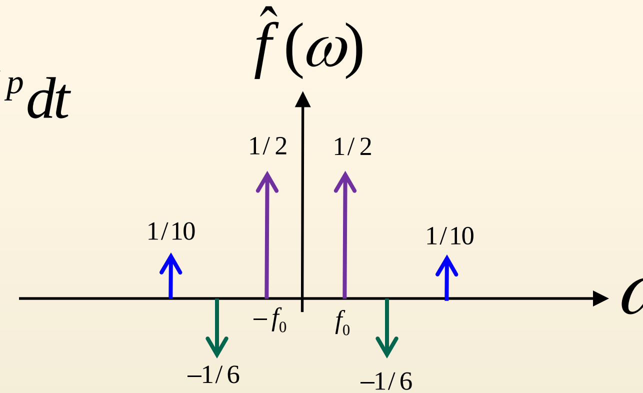

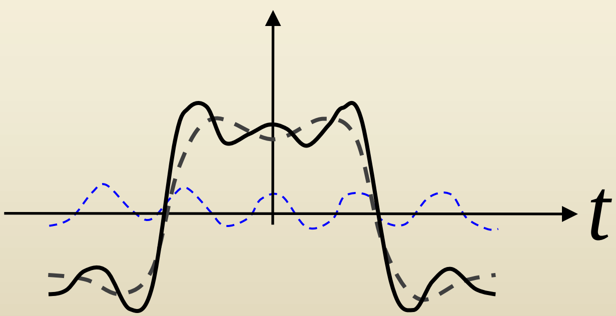

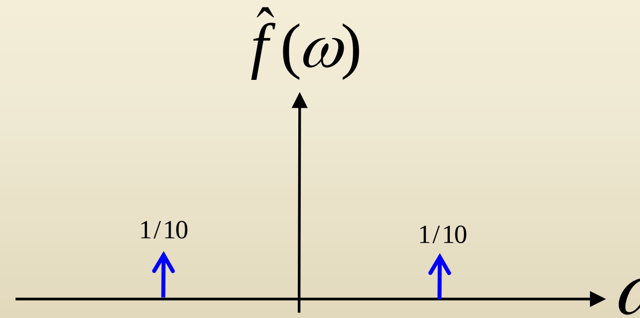

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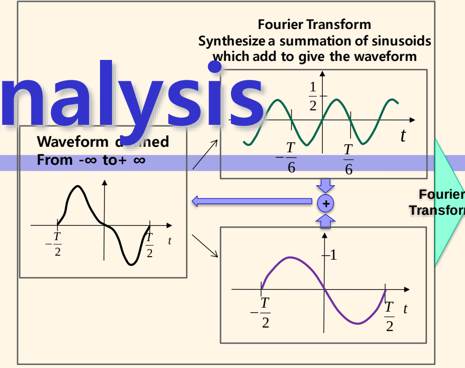
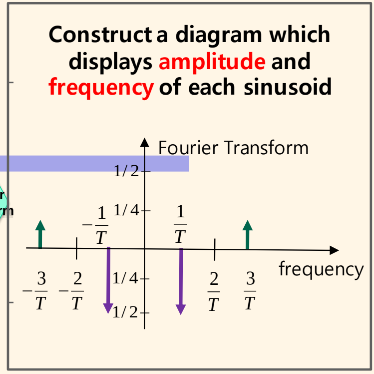
Basic Fourier Transform Analysis





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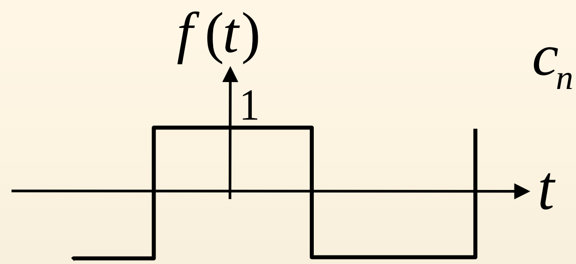

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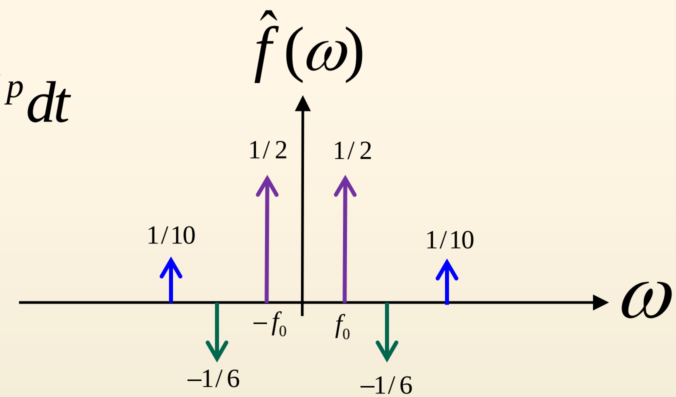

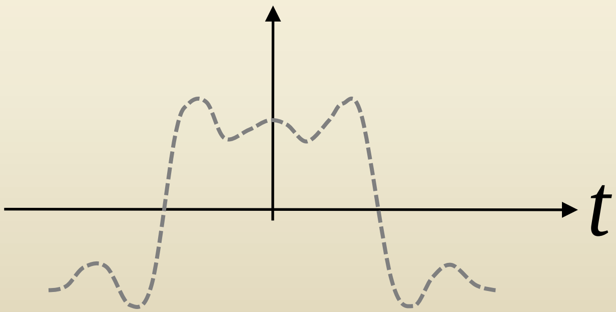

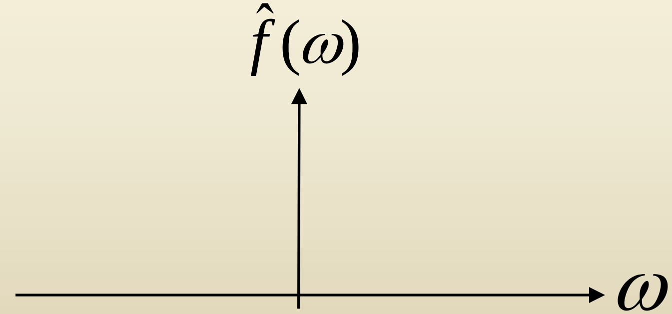

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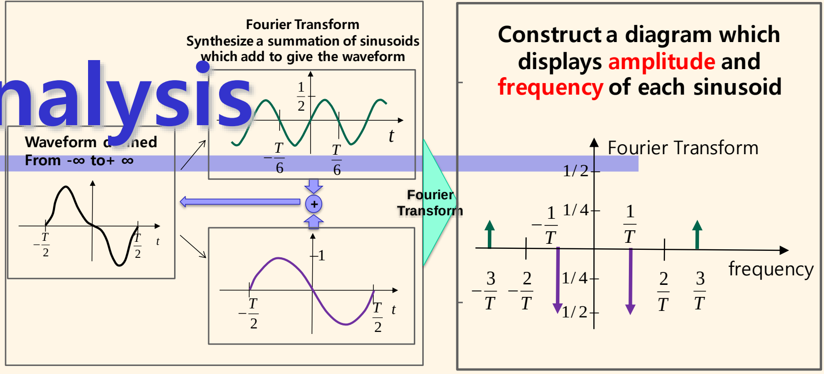
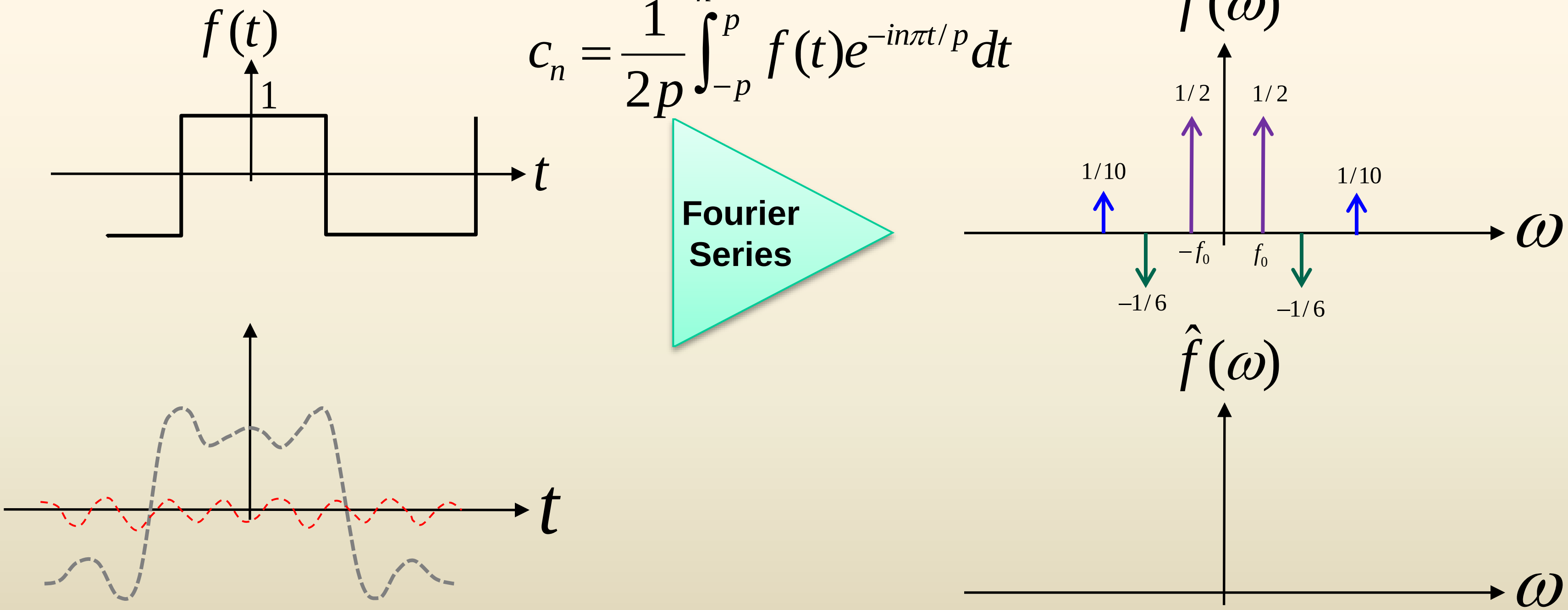
Basic Fourier Transform Analysis



Ex.) **Fourier Transform of square wave function**

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p}$$

$$c_n=\frac{1}{2p}\int_{-p}^{p} f(t)e^{-in\pi t/p}dt$$



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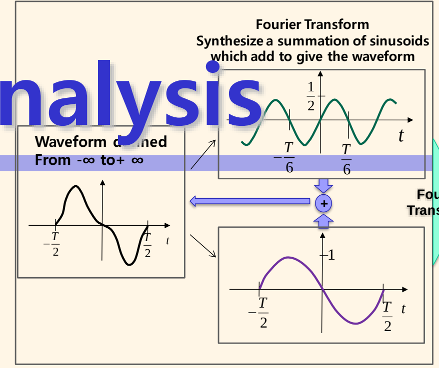
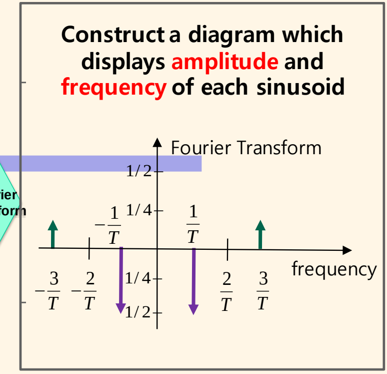
Basic Fourier Transform Analysis





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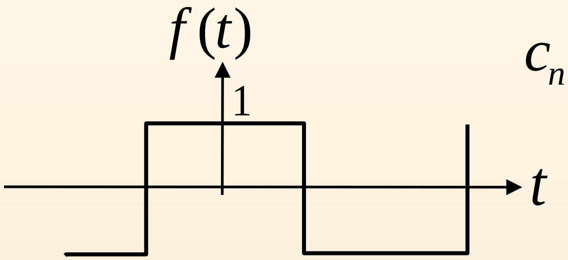
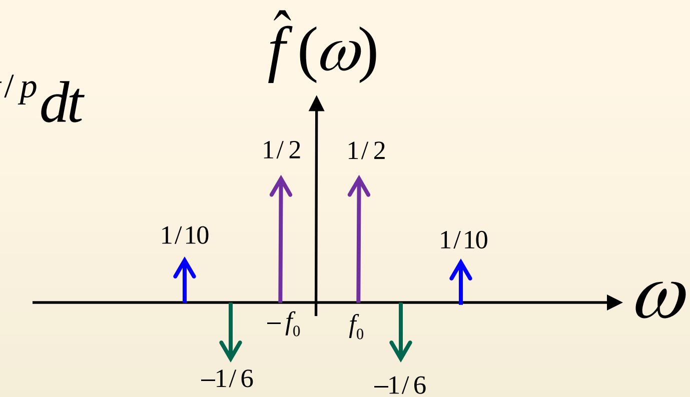
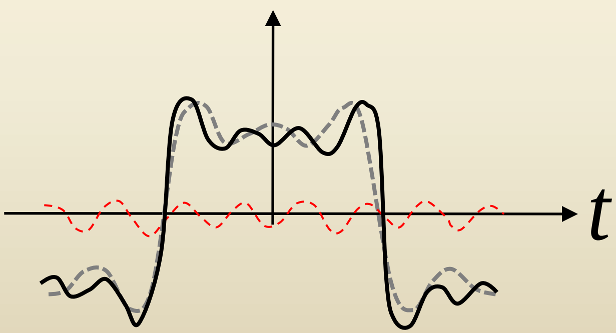
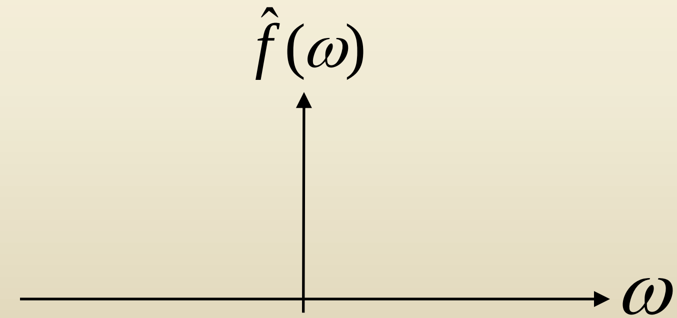


Fourier Series







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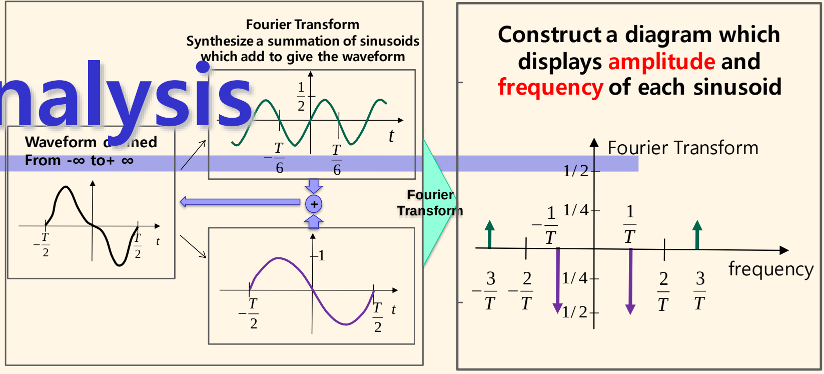
Basic Fourier Transform Analysis



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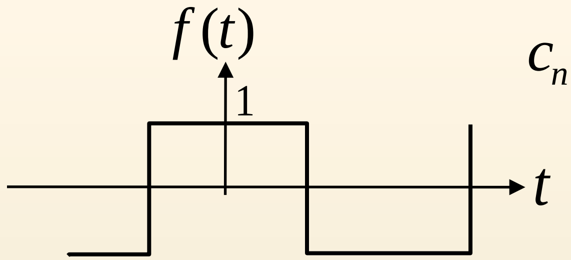

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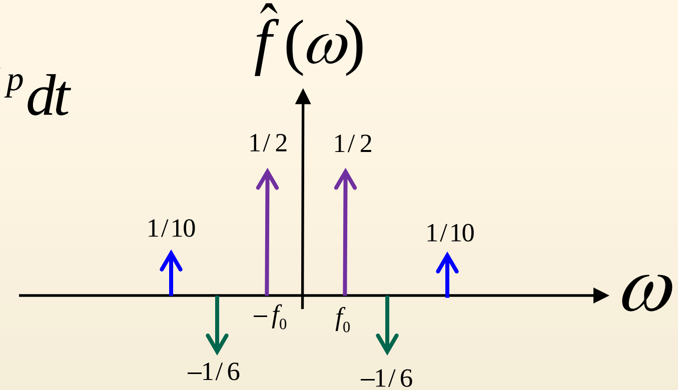

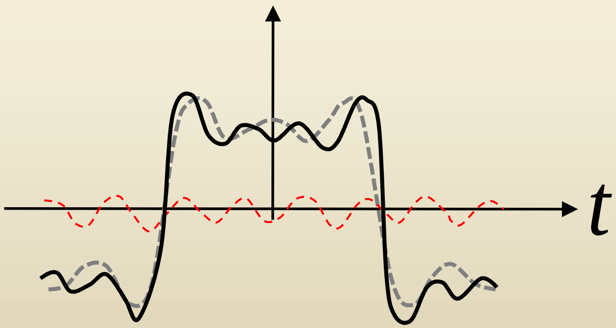

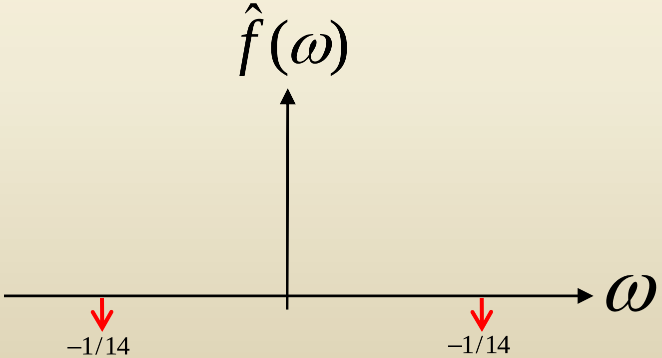

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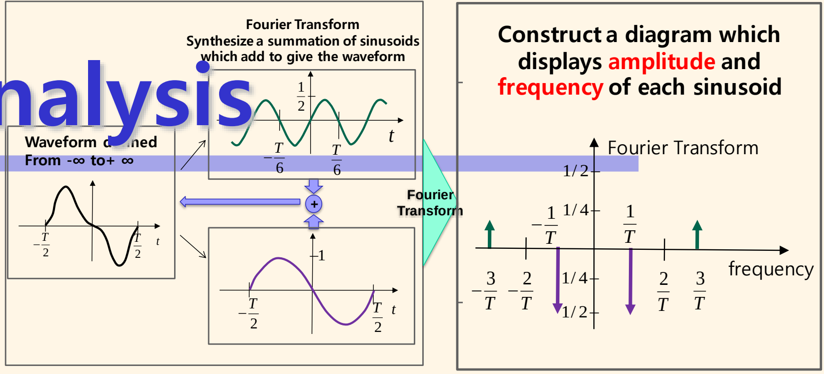
Basic Fourier Transform Analysis



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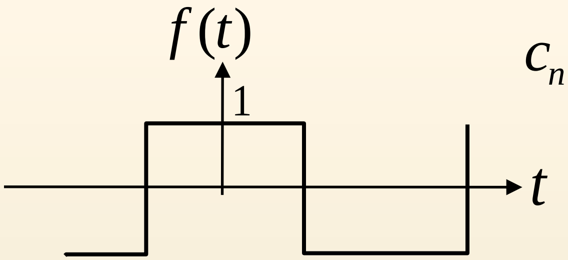

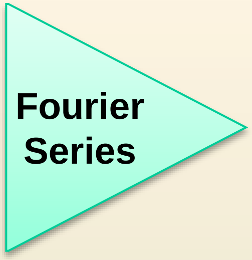

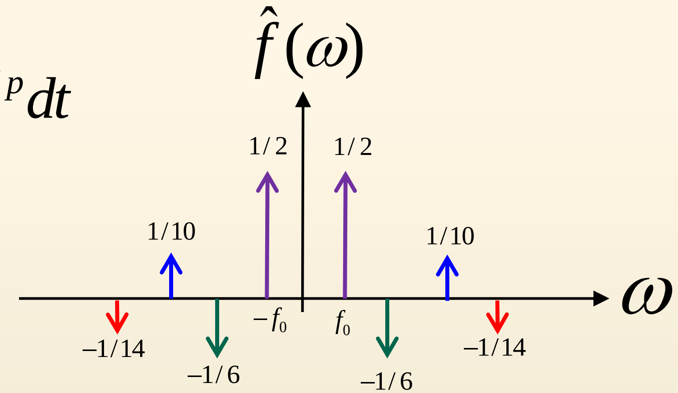

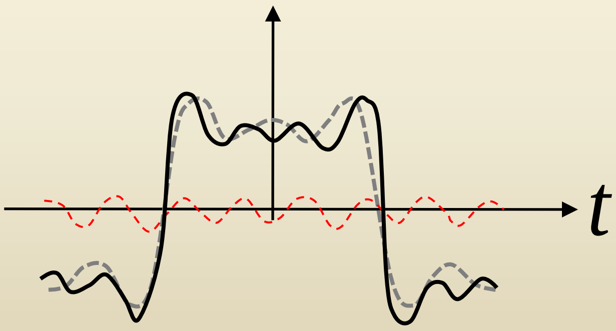

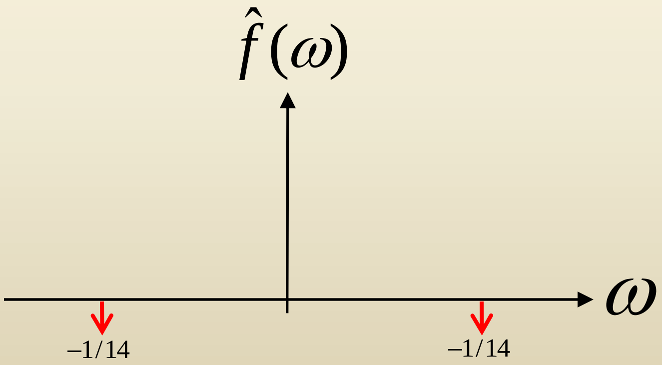

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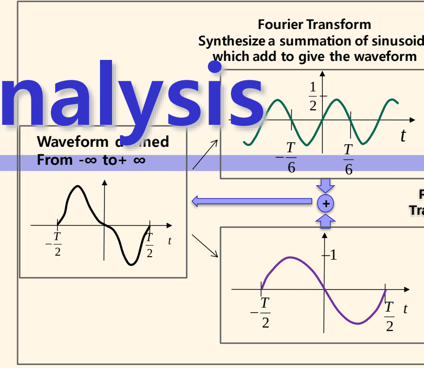
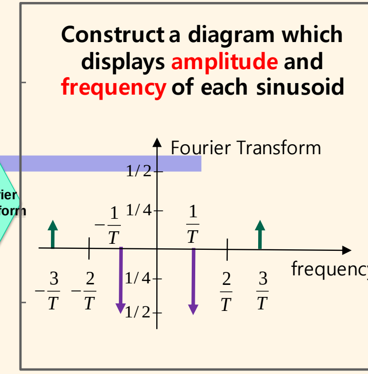
Basic Fourier Transform Analysis





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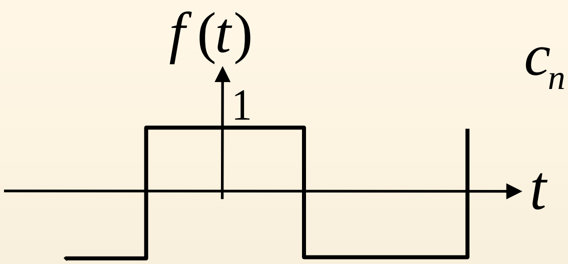

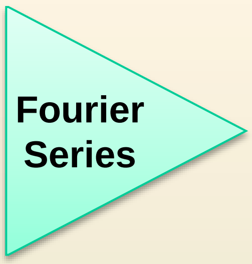

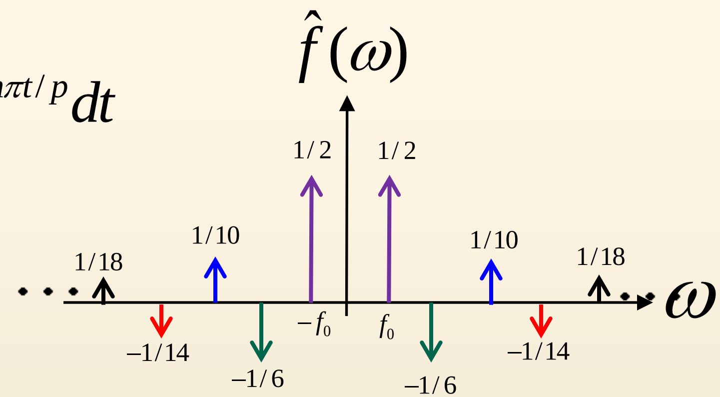

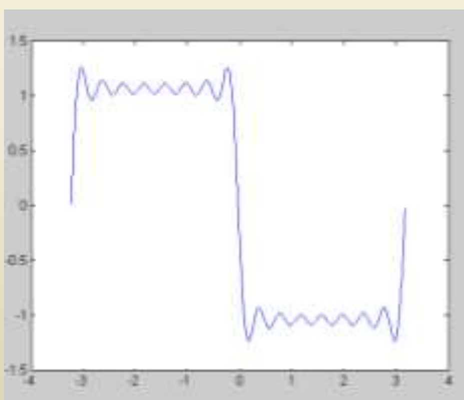

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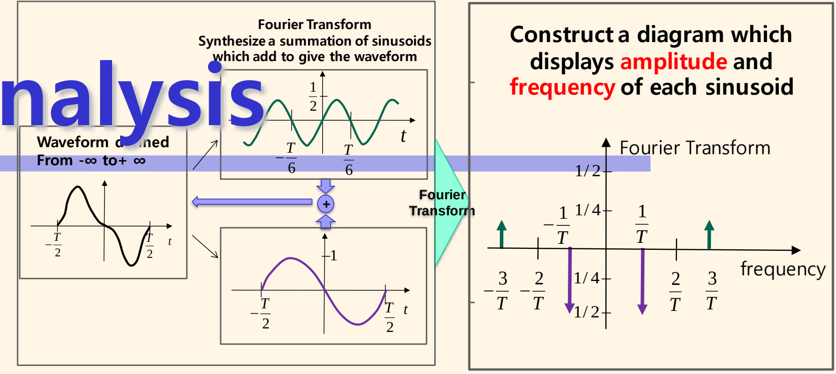
Basic Fourier Transform Analysis



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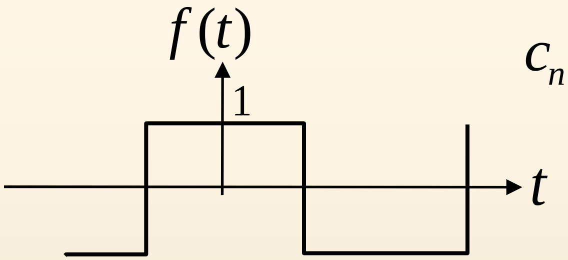
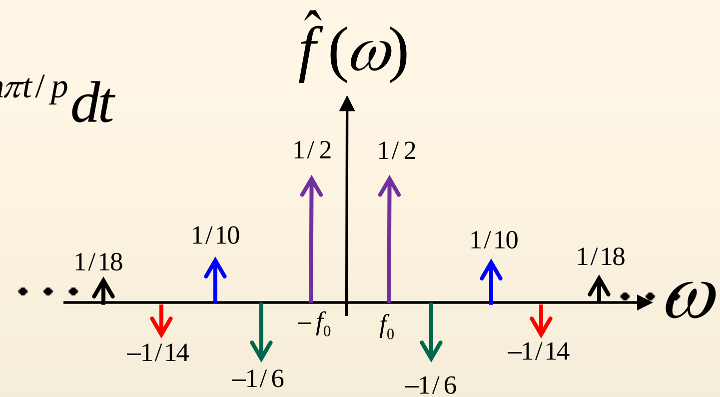
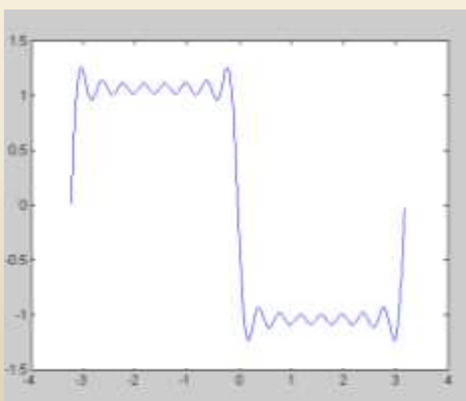


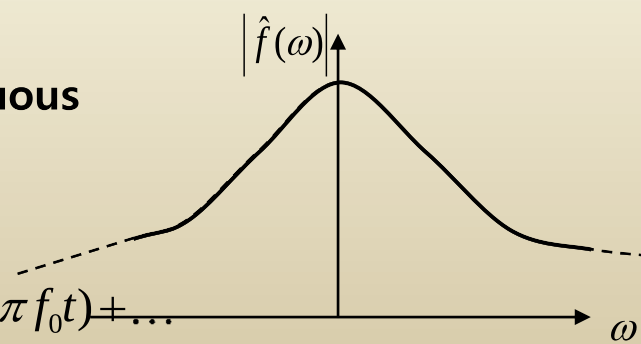
Fourier Series





$$\sum_{n=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$$

Discrete **Continuous**



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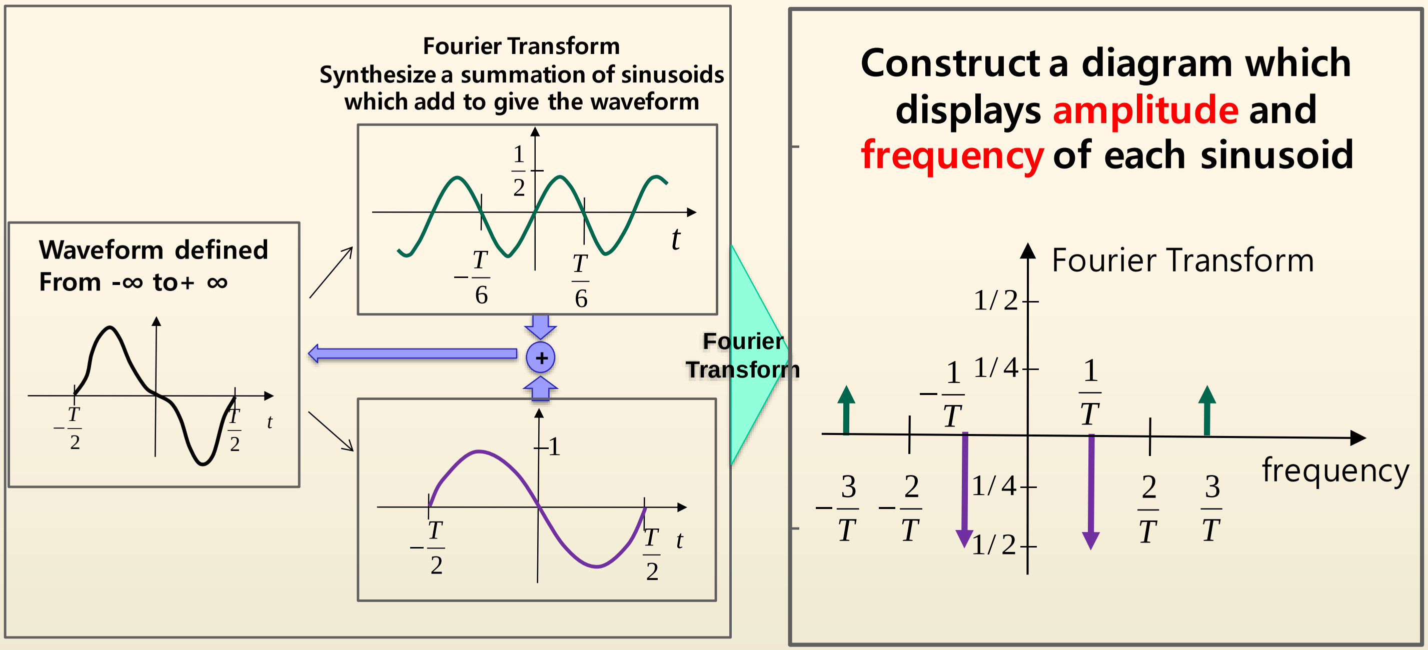
Basic Fourier Transform Analysis

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

Interpretation of the Fourier Transform*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/p} \qquad c_n = \frac{1}{2p}\int_{-p}^{p} f(t) e^{-in\pi t/p} dt$$



*Brigham E.O., The Fast Fourier Transform Prentice-Hall, Inc., 1974

Basic Fourier Transform Analysis

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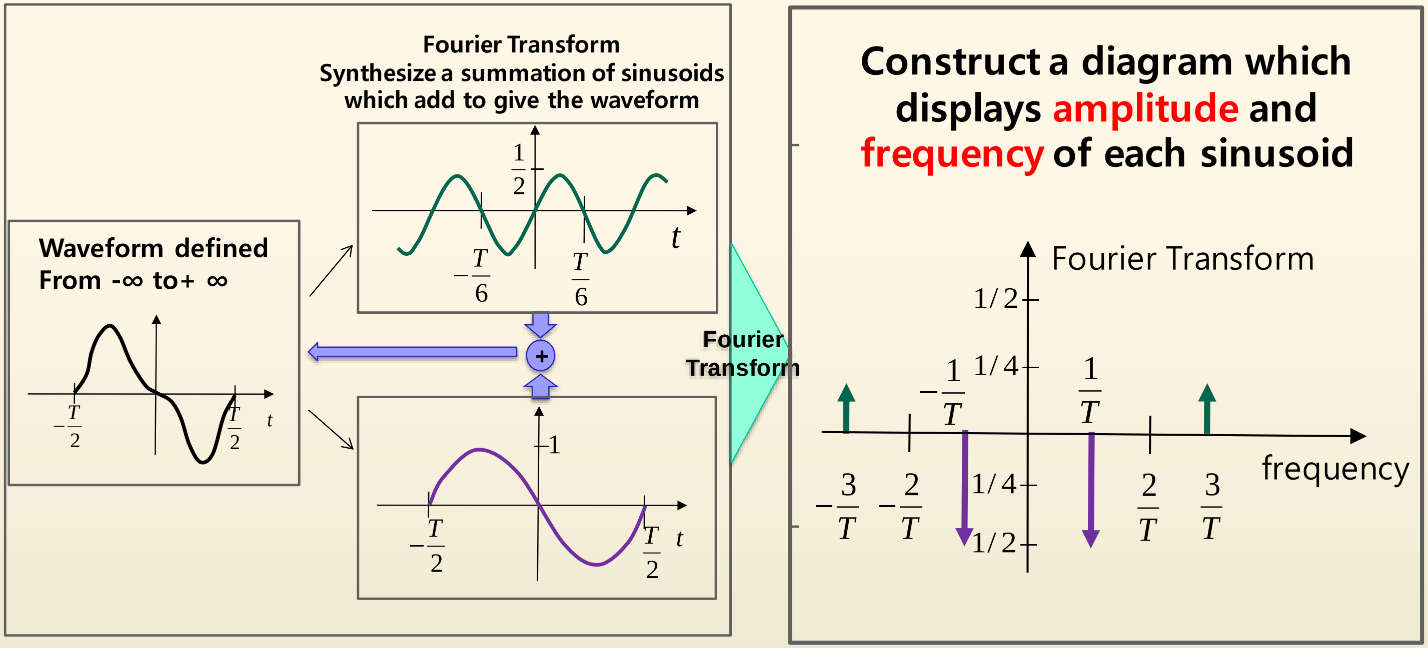

The Fourier transform is , then, a frequency domain representation of a function

Fourier transform frequency domain contains exactly the same information as that of the original function ; they differ only in the manner of presentation of the information

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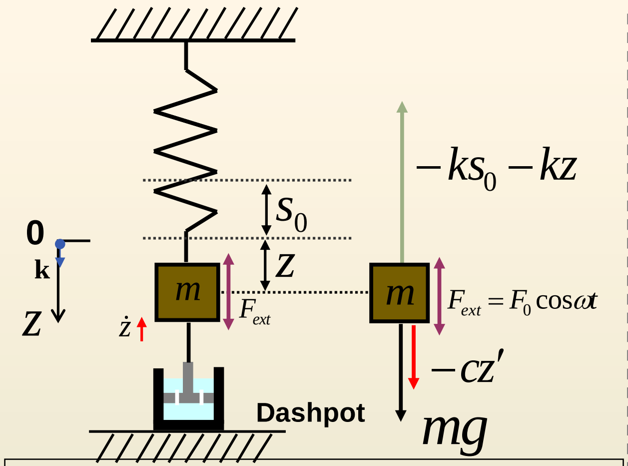
Application of Fourier Series

Application of Fourier Series

✓ Application 1) Express a function which is periodic but not a pure sine or cosine function to a linear combination of sine or cosine functions.



$$m\mathbf{z}'' = \mathbf{F}$$
$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F}_0 \cos \omega t$$
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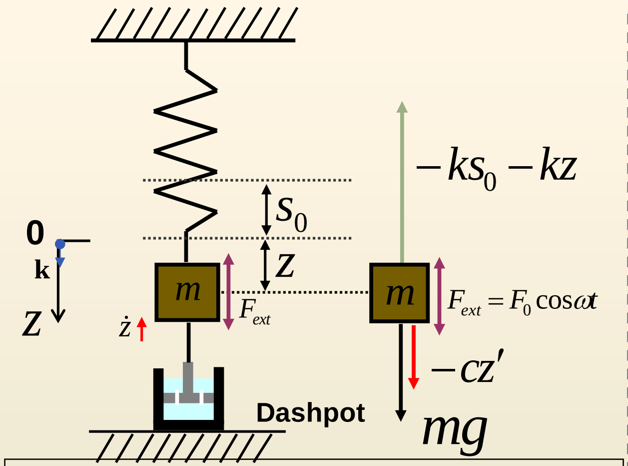
ex) Forced damped mass-spring system



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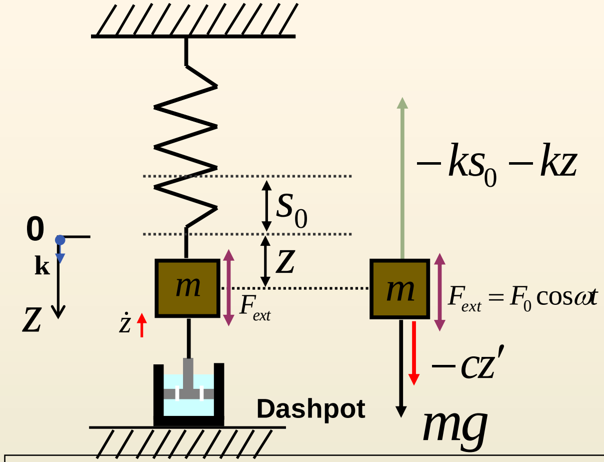
$$m=1, c=0.05, k=25$$



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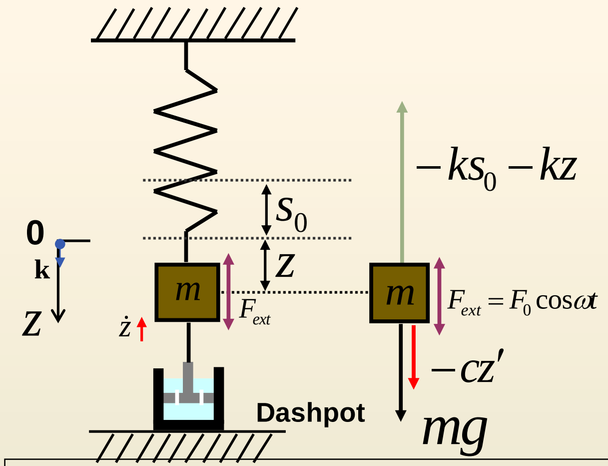

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$$\Downarrow$$

$$y'' + 0.05y' + 25y = r(t)$$

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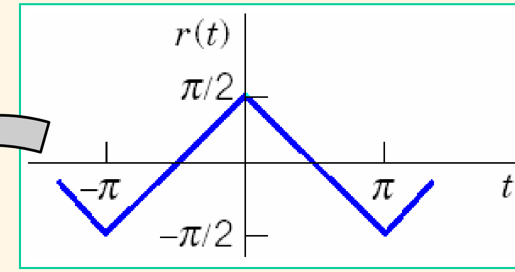

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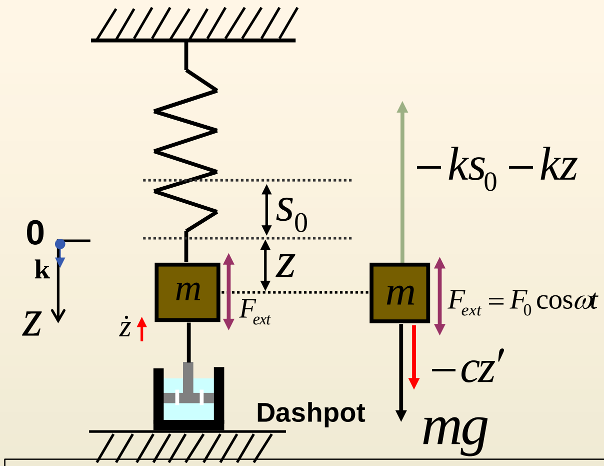

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periodic but not a sine or cosine

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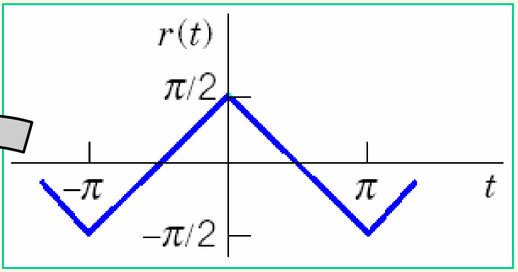

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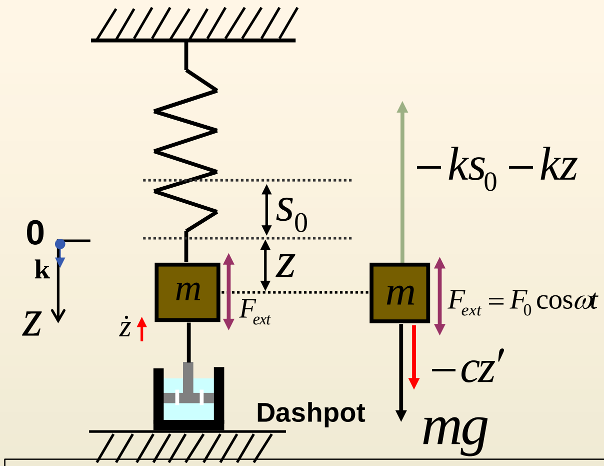

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periodic but not a sine or cosine

Fourier Series

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

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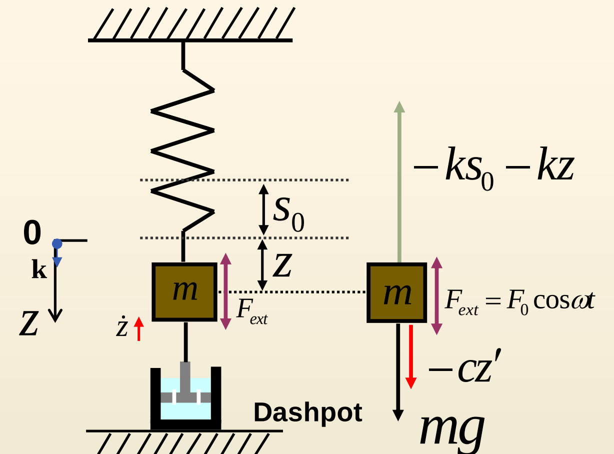

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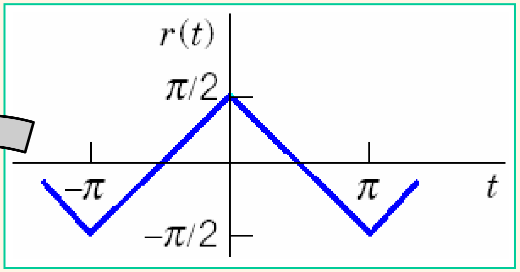

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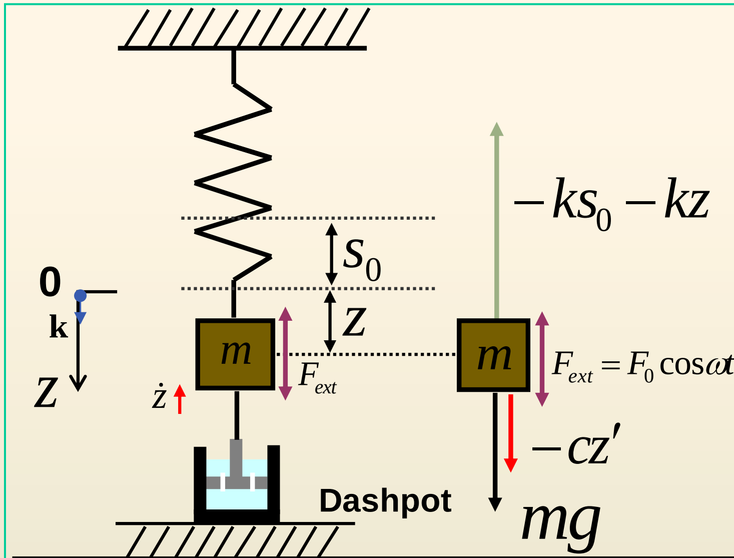
미정계수법(Method of undetermined coefficient)을 사용하여 y_p 를 구함

Forced Oscillations

- **Mass-Spring system**

$$mz'' + cz' + kz = F_0 \cos \omega t$$



$$m\mathbf{z}'' = \mathbf{F}$$

$$= mg\mathbf{k} - ks_0\mathbf{k} - kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0} \cos \omega t$$

$$= -kz\mathbf{k} - cz'\mathbf{k} + \mathbf{F_0} \cos \omega t$$

$$m\mathbf{z}'' + c\mathbf{z}' + k\mathbf{z} = \mathbf{F}_0 \cos \omega t$$

$$mz'' + cz' + kz = F_0 \cos \omega t$$ (z component)

Forced Oscillations

the method of undetermined coefficient (Sec. 2.7)

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

$$mz'' + cz' + kz = F_0 \cos \omega t$$

- **Mass-Spring system**

$$my'' + cy' + ky = r(t)$$

- **If $r(t)$ is a sine or cosine function and if there is damping ($c > 0$), then the steady-state solution is a harmonic oscillation with frequency equal to that of $r(t)$.**

- **If $r(t)$ is not a pure sine or cosine function but is any other periodic function, then the steady-state solution will be a superposition of harmonic oscillations with frequencies equal to that of $r(t)$ and integer multiples of the latter.**

Forced Oscillations

Superposition Principle — Nonhomogeneous Equations*

Let $y_1, y_2, \ldots, y_k$ be k particular solutions of the nonhomogeneous linear n-th differential equation on an interval I corresponding, in turn, to k distinct functions $r_1, r_2, \ldots, r_k$. That is, suppose y_i denotes a particular solution of the corresponding differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r_i(x)$$

where $i = 1,2,\ldots,k$. Then

$$y(x) = y_1(x) + y_2(x) + \cdots + y_k(x)$$

is a particular solution of

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r_1(x) + r_2(x) + \cdots + r_k(x).$$

* Zill, Cullen, Advanced Engineering Mathematics 3rd edition, Jones and Bartlett, pp113

Forced Oscillations

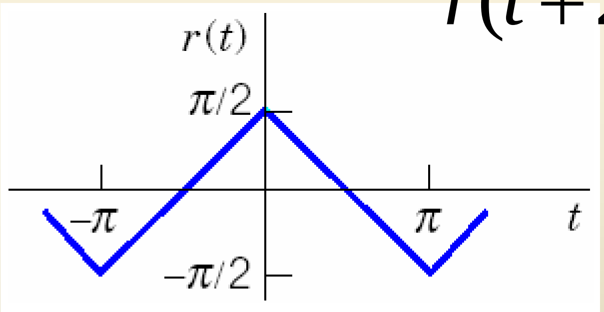
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx,\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

$$y'' + 0.05y' + 25y = r(t)$$

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if } -\pi < t < 0, \\ -t + \dfrac{\pi}{2} & \text{if } \quad 0 < t < \pi, \end{cases}$$

$$r(t + 2\pi) = r(t)$$



Find the steady-state solution $y(t)$

Forced Oscillations

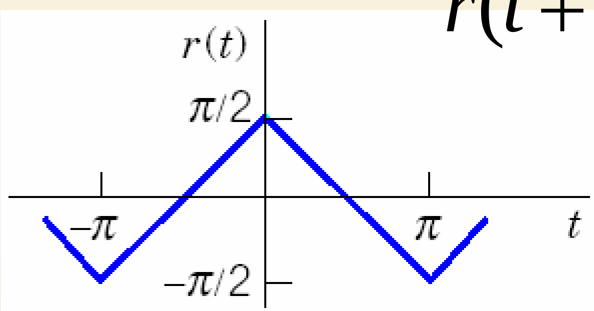
$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

$$y''+0.05y'+25y=r(t)$$

$$r(t)=\begin{cases} t+\dfrac{\pi}{2} & \text{if } -\pi<t<0, \\ -t+\dfrac{\pi}{2} & \text{if } \quad 0<t<\pi, \end{cases}$$

$$r(t+2\pi)=r(t)$$



Find the steady-state solution *y*(*t*)

1. represent *r*(*t*) by a Fourier series

Forced Oscillations

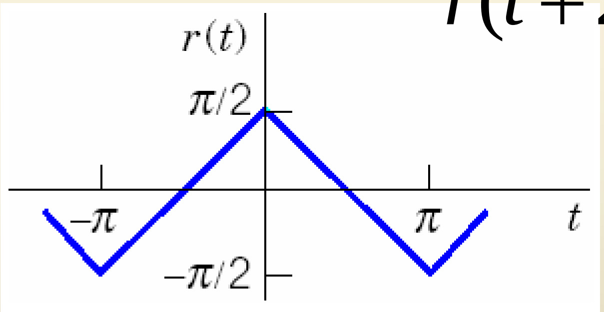
$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

$$y''+0.05y'+25y=r(t)$$

$$r(t)=\begin{cases} t+\dfrac{\pi}{2} & \text{if } -\pi<t<0, \\ -t+\dfrac{\pi}{2} & \text{if } \quad 0<t<\pi, \end{cases}$$

$$r(t+2\pi)=r(t)$$



Find the **steady-state solution $y(t)$**

1. represent $r(t)$ by a Fourier series

$$a_0=\frac{1}{\pi}\int_0^{\pi}(-t+\frac{\pi}{2})dt$$

Forced Oscillations

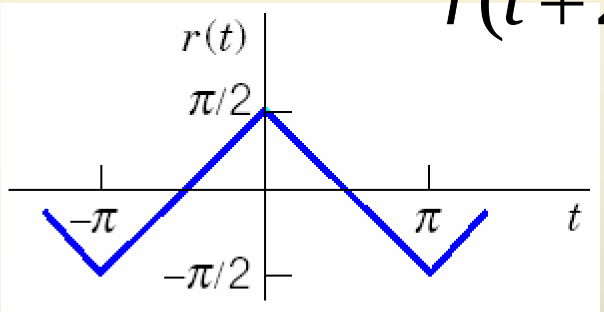
$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

$$y''+0.05y'+25y=r(t)$$

$$r(t)=\begin{cases} t+\dfrac{\pi}{2} & \text{if } -\pi<t<0, \\ -t+\dfrac{\pi}{2} & \text{if } \quad 0<t<\pi, \end{cases}$$

$$r(t+2\pi)=r(t)$$



Find the **steady-state solution** $y(t)$

1. represent $r(t)$ by a Fourier series

$$a_0=\frac{1}{\pi}\int_0^{\pi}\left(-t+\frac{\pi}{2}\right)dt$$

$$=\frac{1}{\pi}\left[-\frac{t^2}{2}+\frac{\pi}{2}t\right]_0^{\pi}$$

Forced Oscillations

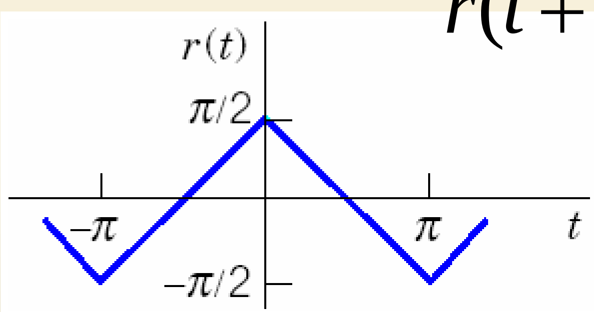
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx,\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

$$y'' + 0.05y' + 25y = r(t)$$

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if } -\pi < t < 0, \\ -t + \dfrac{\pi}{2} & \text{if } \quad 0 < t < \pi, \end{cases}$$

$$r(t + 2\pi) = r(t)$$



Find the **steady-state solution $y(t)$**

1. represent $r(t)$ by a Fourier series

$$a_0 = \frac{1}{\pi}\int_0^{\pi}(-t + \frac{\pi}{2})dt$$

$$= \frac{1}{\pi}\left[-\frac{t^2}{2} + \frac{\pi}{2}t\right]_0^{\pi}$$

$$= 0$$

Forced Oscillations

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx,\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

1. represent *r*(*t*) by a Fourier series

$$y'' + 0.05y' + 25y = r(t)$$

$$= \begin{cases} t + \frac{\pi}{2} & \text{if } -\pi < t < 0 \\ -t + \frac{\pi}{2} & \text{if } \quad 0 < t < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} r(t)dt = 0$$

Forced Oscillations

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

1. represent $r(t)$ by a Fourier series

$$y''+0.05y'+25y=r(t)=\begin{cases} t+\frac{\pi}{2} & \text{if } -\pi<t<0 \\ -t+\frac{\pi}{2} & \text{if } \quad 0<t<\pi \end{cases}$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}r(t)dt=0$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(-t+\frac{\pi}{2})\cos nt\,dt$$

Forced Oscillations

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

1. represent *r*(*t*) by a Fourier series

$$y''+0.05y'+25y=r(t)=\begin{cases}t+\frac{\pi}{2} & \text{if } -\pi<t<0\\ -t+\frac{\pi}{2} & \text{if } \quad 0<t<\pi\end{cases}$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}r(t)dt=0$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(-t+\frac{\pi}{2})\cos nt\,dt=\frac{2}{\pi}\left\{\frac{1}{n}\left(-t+\frac{\pi}{2}\right)\sin nt\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\sin nt\,dt\right\}$$

Forced Oscillations

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

1. represent *r*(*t*) by a Fourier series

$$y''+0.05y'+25y=r(t)=\begin{cases} t+\dfrac{\pi}{2} & \text{if } -\pi<t<0 \\ -t+\dfrac{\pi}{2} & \text{if } \quad 0<t<\pi \end{cases}$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}r(t)dt=0$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(-t+\frac{\pi}{2})\cos nt\,dt\ =\frac{2}{\pi}\left\{\frac{1}{n}\left(-t+\frac{\pi}{2}\right)\sin nt\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\sin nt\,dt\right\}$$

$$=\frac{2}{\pi}\left\{-\frac{1}{n^2}\cos nt\Big|_0^{\pi}\right\}$$

Forced Oscillations

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

1. represent *r*(*t*) by a Fourier series

$$y''+0.05y'+25y=r(t)=\begin{cases}t+\dfrac{\pi}{2} & \text{if } -\pi<t<0\\ -t+\dfrac{\pi}{2} & \text{if } \quad 0<t<\pi\end{cases}$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}r(t)dt=0$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(-t+\frac{\pi}{2})\cos nt\,dt=\frac{2}{\pi}\left\{\frac{1}{n}\left(-t+\frac{\pi}{2}\right)\sin nt\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\sin nt\,dt\right\}$$

$$=\frac{2}{\pi}\left\{-\frac{1}{n^2}\cos nt\Big|_0^{\pi}\right\}=\frac{2}{\pi}\left\{-\frac{1}{n^2}(\cos n\pi-1)\right\}$$

Forced Oscillations

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

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$$a_0=\frac{1}{\pi}\int_0^{\pi}r(t)dt=0$$

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$$=\frac{2}{\pi}\left\{-\frac{1}{n^2}\cos nt\Big|_0^{\pi}\right\}=\frac{2}{\pi}\left\{-\frac{1}{n^2}(\cos n\pi-1)\right\}$$

$$=\begin{cases}\dfrac{4}{\pi}\dfrac{1}{n^2} & \text{if } n=odd\\ 0 & \text{if } n=even\end{cases}$$

Forced Oscillations

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx,\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

$$y'' + 0.05y' + 25y = r(t) = \begin{cases} t + \frac{\pi}{2} & \text{if } -\pi < t < 0 \\ -t + \frac{\pi}{2} & \text{if } 0 < t < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} r(t)dt = 0, \qquad a_n = \begin{cases} \frac{4}{\pi}\frac{1}{n^2} & \text{if } n = odd \\ 0 & \text{if } n = even \end{cases}$$

Forced Oscillations

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx,\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

$$y'' + 0.05y' + 25y = r(t)$$

$$= \begin{cases} t + \dfrac{\pi}{2} & \text{if } -\pi < t < 0 \\ -t + \dfrac{\pi}{2} & \text{if } \quad 0 < t < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} r(t)dt = 0, \qquad a_n = \begin{cases} \dfrac{4}{\pi}\dfrac{1}{n^2} & \text{if } \; n = odd \\ 0 & \text{if } \; n = even \end{cases}$$

$$\therefore r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

Forced Oscillations

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx,\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

$$y'' + 0.05y' + 25y = r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if } -\pi < t < 0 \\ -t + \dfrac{\pi}{2} & \text{if } \quad 0 < t < \pi \end{cases}$$

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$$\therefore r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$\therefore y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

Forced Oscillations

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

$$y''+0.05y'+25y=r(t)=\begin{cases}t+\frac{\pi}{2} & \text{if } -\pi<t<0\\ -t+\frac{\pi}{2} & \text{if } \quad 0<t<\pi\end{cases}$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}r(t)dt=0,\qquad a_n=\begin{cases}\frac{4}{\pi}\frac{1}{n^2} & \text{if } n=odd\\ 0 & \text{if } n=even\end{cases}$$

$$\therefore r(t)=\frac{4}{\pi}\left(\cos t+\frac{1}{3^2}\cos 3t+\frac{1}{5^2}\cos 5t+\cdots\right)$$

$$\therefore y''+0.05y'+25y=\frac{4}{\pi}\left(\cos t+\frac{1}{3^2}\cos 3t+\frac{1}{5^2}\cos 5t+\cdots\right)$$

Let y_n is a solution of following ODE

Forced Oscillations

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,\ \ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx,\ \ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

$$y'' + 0.05y' + 25y = r(t) = \begin{cases} t + \dfrac{\pi}{2} & \text{if } -\pi < t < 0 \\ -t + \dfrac{\pi}{2} & \text{if } \quad 0 < t < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} r(t)dt = 0, \qquad a_n = \begin{cases} \dfrac{4}{\pi}\dfrac{1}{n^2} & \text{if } \ n = odd \\ 0 & \text{if } \ n = even \end{cases}$$

$$\therefore r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$\therefore y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

Let y_n is a solution of following ODE

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \ , \ (n = 1, 3, 5, \cdots)$$

Forced Oscillations

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

$$y''+0.05y'+25y=r(t)$$

$$=\begin{cases} t+\dfrac{\pi}{2} & \text{if } -\pi<t<0 \\ -t+\dfrac{\pi}{2} & \text{if } \quad 0<t<\pi \end{cases}$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}r(t)dt=0,\qquad a_n=\begin{cases}\dfrac{4}{\pi}\dfrac{1}{n^2} & \text{if } n=odd \\ 0 & \text{if } n=even\end{cases}$$

$$\therefore r(t)=\frac{4}{\pi}\left(\cos t+\frac{1}{3^2}\cos 3t+\frac{1}{5^2}\cos 5t+\cdots\right)$$

$$\therefore y''+0.05y'+25y=\frac{4}{\pi}\left(\cos t+\frac{1}{3^2}\cos 3t+\frac{1}{5^2}\cos 5t+\cdots\right)$$

Let y_n is a solution of following ODE

$$y''+0.05y'+25y=\frac{4}{n^2\pi}\cos nt\ ,\ (n=1,3,5,\cdots)$$

then, the solution of the given ODE is

Forced Oscillations

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right),$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi}{L}xdx$$

$$y''+0.05y'+25y=r(t)=\begin{cases} t+\frac{\pi}{2} & \text{if } -\pi<t<0 \\ -t+\frac{\pi}{2} & \text{if } \quad 0<t<\pi \end{cases}$$

$$a_0=\frac{1}{\pi}\int_0^{\pi}r(t)dt=0,\qquad a_n=\begin{cases}\frac{4}{\pi}\frac{1}{n^2} & \text{if } n=odd \\ 0 & \text{if } n=even\end{cases}$$

$$\therefore r(t)=\frac{4}{\pi}\left(\cos t+\frac{1}{3^2}\cos 3t+\frac{1}{5^2}\cos 5t+\cdots\right)$$

$$\therefore y''+0.05y'+25y=\frac{4}{\pi}\left(\cos t+\frac{1}{3^2}\cos 3t+\frac{1}{5^2}\cos 5t+\cdots\right)$$

Let y_n is a solution of following ODE

$$y''+0.05y'+25y=\frac{4}{n^2\pi}\cos nt\ ,\ (n=1,3,5,\cdots)$$

then, the solution of the given ODE is

$$y=y_1+y_3+y_5+\cdots \qquad \cdots(7)$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \ , \ (n = 1,3,5,\cdots)$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \ , \ (n = 1, 3, 5, \cdots)$$

- **solution of above ODE** y_n

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

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- **solution of above ODE y_n**

Let $y_n = A_n \cos nt + B_n \sin nt$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt\ ,\ (n = 1, 3, 5, \cdots)$$

- **solution of above ODE y_n**

Let $y_n = A_n \cos nt + B_n \sin nt$

then $y_n' = nB_n \cos nt - nA_n \sin nt$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

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- **solution of above ODE y_n**

Let $y_n = A_n \cos nt + B_n \sin nt$

then $y_n' = nB_n \cos nt - nA_n \sin nt$

$$y_n'' = -n^2 A_n \cos nt - n^2 B_n \sin nt$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

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- **solution of above ODE y_n**

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then $y_n' = nB_n \cos nt - nA_n \sin nt$

$$y_n'' = -n^2 A_n \cos nt - n^2 B_n \sin nt$$

Substituting y_n, y'_n, y''_n

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \ , \ (n = 1, 3, 5, \cdots)$$

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Substituting y_n, y'_n, y''_n

$$y'' + 0.05y' + 25y =$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

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then $y_n' = nB_n \cos nt - nA_n \sin nt$

$$y_n'' = -n^2 A_n \cos nt - n^2 B_n \sin nt$$

Substituting y_n, y'_n, y''_n

$$y'' + 0.05y' + 25y =$$

$$-n^2 A_n \cos nt - n^2 B_n \sin nt + 0.05(nB_n \cos nt - nA_n \sin nt)$$

$$+25(A_n \cos nt + B_n \sin nt) = \frac{4}{n^2\pi}\cos nt$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \ , \ (n = 1,3,5,\cdots)$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \;,\; (n = 1,3,5,\cdots)$$

$$-n^2 A_n \cos nt - n^2 B_n \sin nt$$
$$+ 0.05(nB_n \cos nt - nA_n \sin nt)$$
$$+ 25(A_n \cos nt + B_n \sin nt) = \frac{4}{n^2\pi}\cos nt$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

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$$+25(A_n \cos nt + B_n \sin nt) = \frac{4}{n^2\pi}\cos nt$$

$$(-n^2 A_n - 25A_n + 0.05nB_n)\cos nt - (0.05nA_n + n^2 B_n - 25B_n)\sin nt = \frac{4}{n^2\pi}\cos nt + 0 \cdot \sin nt$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \ , \ (n = 1, 3, 5, \cdots)$$

$$-n^2 A_n \cos nt - n^2 B_n \sin nt$$
$$+0.05(nB_n \cos nt - nA_n \sin nt)$$
$$+25(A_n \cos nt + B_n \sin nt) = \frac{4}{n^2\pi}\cos nt$$

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$$\Downarrow$$

$$-(n^2 + 25)A_n + 0.05nB_n = \frac{4}{n^2\pi}$$
$$0.05nA_n + (n^2 - 25)B_n = 0$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \ , \ (n = 1,3,5,\cdots)$$

$$-(n^2 + 25)A_n + 0.05nB_n = \frac{4}{n^2\pi}$$

$$0.05nA_n + (n^2 - 25)B_n = 0$$

$$\therefore A_n = \frac{4(25-n^2)}{n^2\pi D_n},\ \ B_n = \frac{0.2}{n\pi D_n},\ \ \left(\text{where } D_n = (25-n^2)^2 + (0.05n)^2\right)$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

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$$\therefore y_n = A_n \cos nt + B_n \sin nt$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \;, \; (n = 1,3,5,\cdots)$$

$$-(n^2+25)A_n + 0.05nB_n = \frac{4}{n^2\pi}$$

$$0.05nA_n + (n^2-25)B_n = 0$$

$$\therefore A_n = \frac{4(25-n^2)}{n^2\pi D_n}, \;\; B_n = \frac{0.2}{n\pi D_n}, \;\; \left(\text{where } D_n = (25-n^2)^2 + (0.05n)^2\right)$$

$$\therefore y_n = A_n \cos nt + B_n \sin nt$$

$$= \frac{4(25-n^2)}{n^2\pi D_n}\cos nt + \frac{0.2}{n\pi D_n}\sin nt$$

$$\left(\text{where } D_n = (25-n^2)^2 + (0.05n)^2\right)$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

$$y = y_1 + y_3 + y_5 + \cdots$$

$$y_n = A_n \cos nt + B_n \sin nt,$$

$$A_n = \frac{4(25-n^2)}{n^2\pi D_n}, \quad B_n = \frac{0.2}{n\pi D_n},$$

$$\left(\text{where } D_n = (25-n^2)^2 + (0.05n)^2\right)$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

$$y = y_1 + y_3 + y_5 + \cdots$$

$$y_n = A_n \cos nt + B_n \sin nt,$$

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$$\left(\text{where } D_n = (25-n^2)^2 + (0.05n)^2\right)$$

$$\therefore A_1 = 0.0531, \quad B_1 = 0.0001105$$

$$A_3 = 0.0088, \quad B_3 = 0.0001964$$

$$A_5 = 0, \quad B_5 = 0.2037$$

$$A_7 = -0.0011, \quad B_7 = 0.0000$$

$$A_9 = -0.0033, \quad B_9 = 0.0000$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

$$y = y_1 + y_3 + y_5 + \cdots$$

$$y_n = A_n \cos nt + B_n \sin nt,$$

$$A_n = \frac{4(25-n^2)}{n^2\pi D_n},\quad B_n = \frac{0.2}{n\pi D_n},$$

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- **amplitude of solution y_n**

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

$$y = y_1 + y_3 + y_5 + \cdots$$

$$y_n = A_n \cos nt + B_n \sin nt,$$

$$A_n = \frac{4(25-n^2)}{n^2\pi D_n}, \quad B_n = \frac{0.2}{n\pi D_n},$$

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$$A_9 = -0.0033, \quad B_9 = 0.0000$$

- **amplitude of solution y_n**

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{D_n}}$$

$$C_1 = 0.0531, \quad C_3 = 0.0088$$

$$C_5 = 0.2037, \quad C_7 = 0.0011$$

$$C_9 = 0.0003$$

Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

$$y = y_1 + y_3 + y_5 + \cdots$$

$$y_n = A_n \cos nt + B_n \sin nt,$$

$$A_n = \frac{4(25-n^2)}{n^2\pi D_n}, \quad B_n = \frac{0.2}{n\pi D_n},$$

$$\left(\text{where } D_n = (25-n^2)^2 + (0.05n)^2\right)$$

$$\therefore A_1 = 0.0531, \quad B_1 = 0.0001105$$

$$A_3 = 0.0088, \quad B_3 = 0.0001964$$

$$A_5 = 0, \quad B_5 = 0.2037$$

$$A_7 = -0.0011, \quad B_7 = 0.0000$$

$$A_9 = -0.0033, \quad B_9 = 0.0000$$

- **amplitude of solution y_n**

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{D_n}}$$

$$C_1 = 0.0531, \quad C_3 = 0.0088$$

$$C_5 = 0.2037, \quad C_7 = 0.0011$$

$$C_9 = 0.0003$$

- **C_5 is so large that y_5 is dominating term among the solutions**

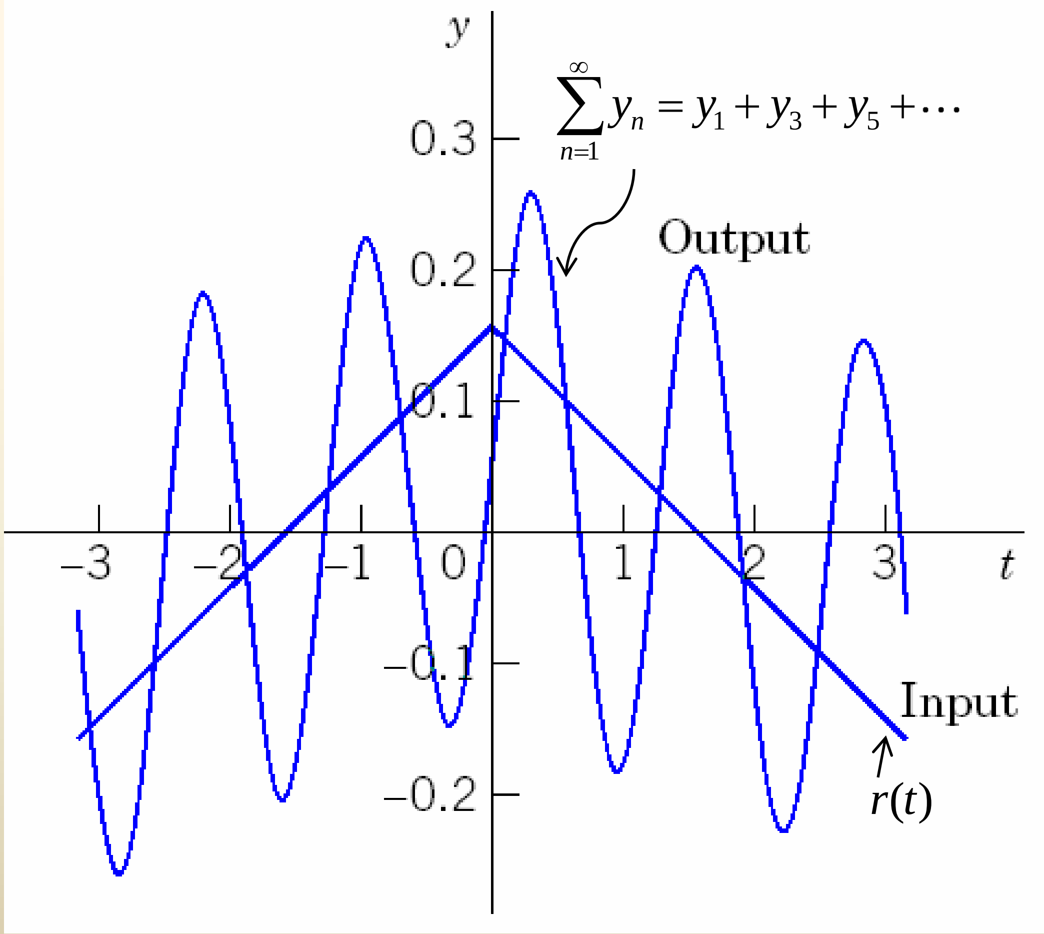
Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

$$y = y_1 + y_3 + y_5 + \cdots$$

$$y_n = A_n \cos nt + B_n \sin nt,$$



Forced Oscillations

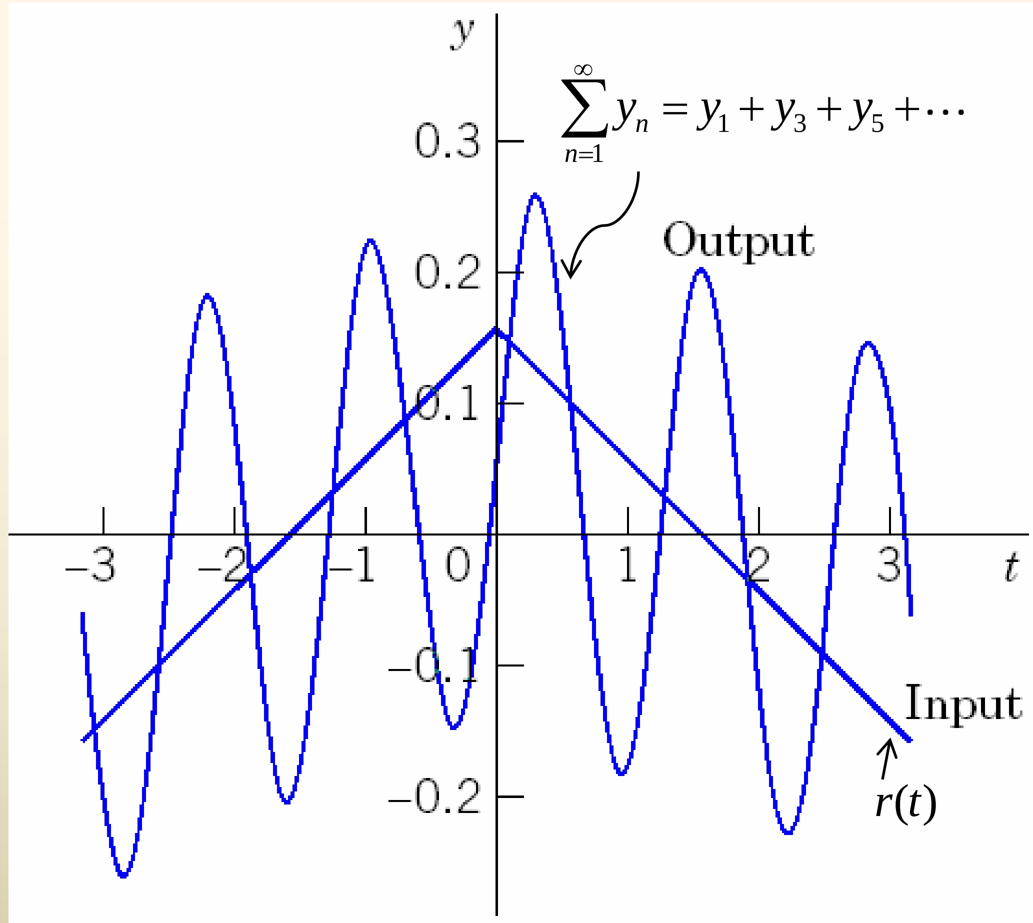
$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

$$y = y_1 + y_3 + y_5 + \cdots$$

$$y_n = A_n \cos nt + B_n \sin nt,$$

- **solution of the given ODE**



Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

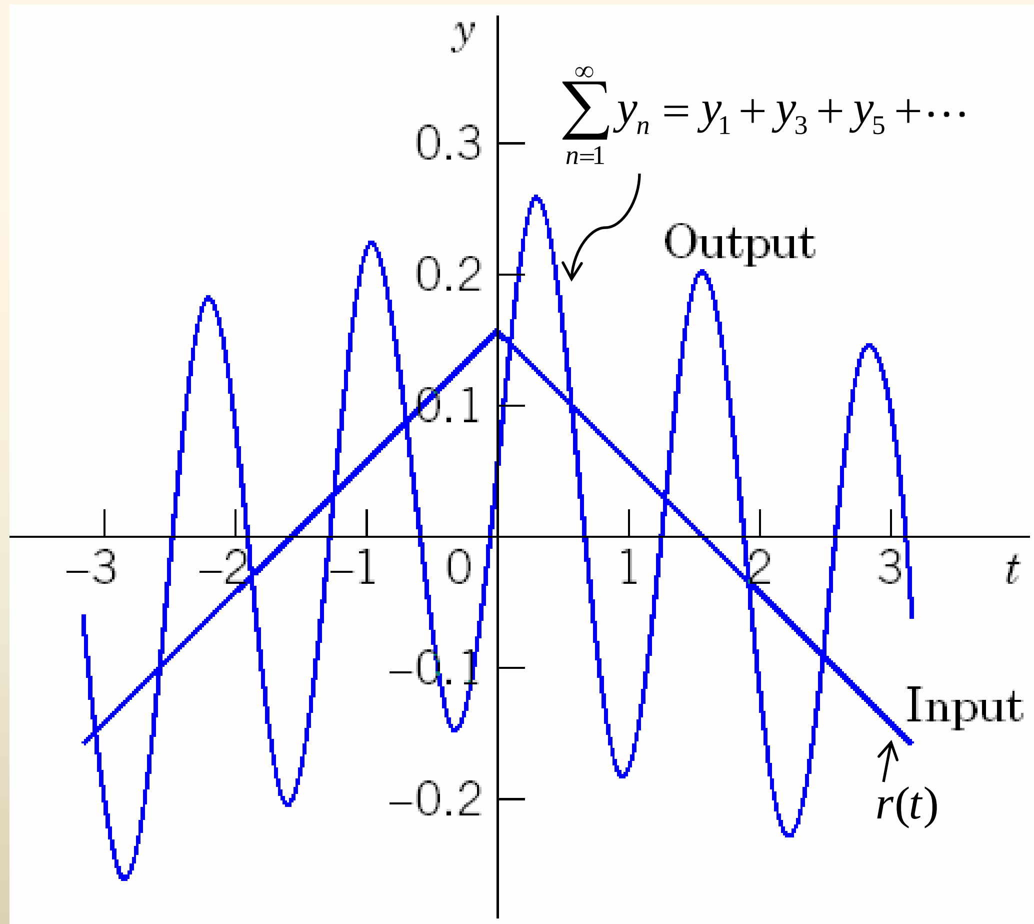
$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

$$y = y_1 + y_3 + y_5 + \cdots$$

$$y_n = A_n \cos nt + B_n \sin nt,$$

- **solution of the given ODE**

$$y = y_1 + y_3 + y_5 + y_7 + \cdots$$



Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

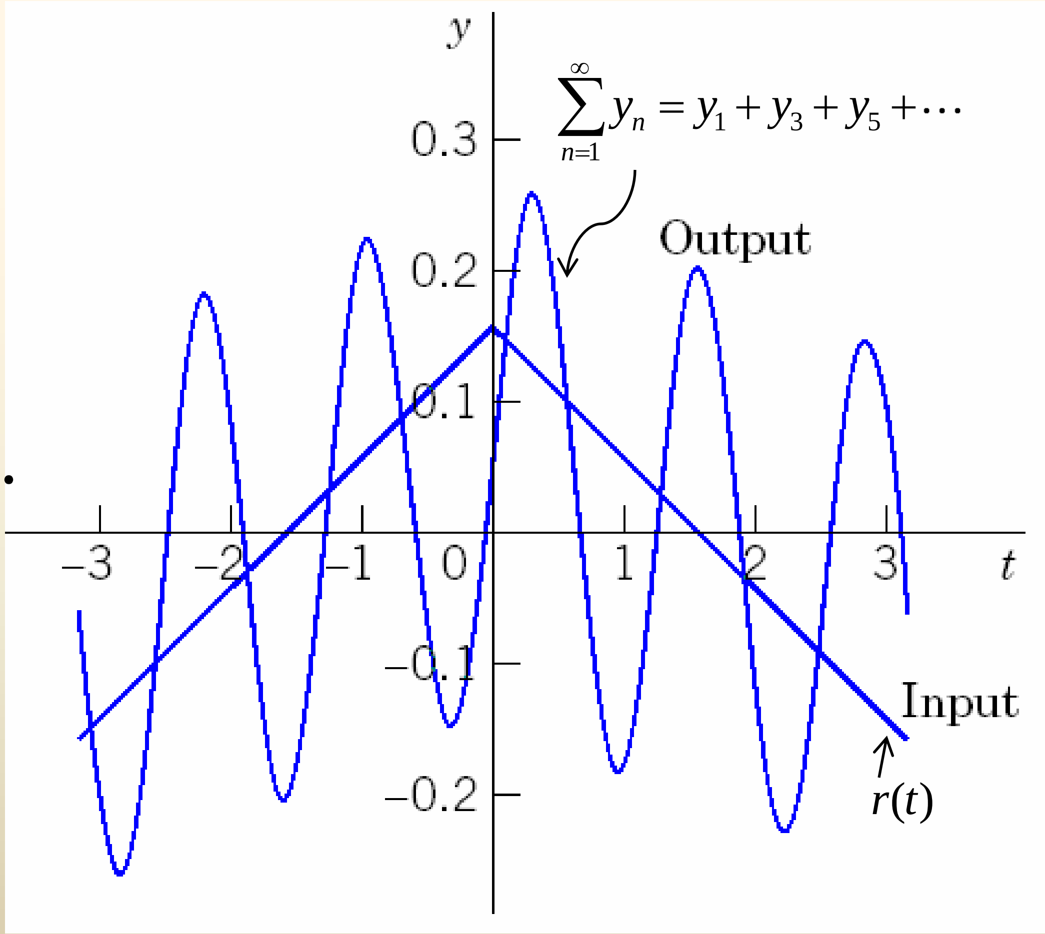
$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

$$y = y_1 + y_3 + y_5 + \cdots$$

$$y_n = A_n \cos nt + B_n \sin nt,$$

- **solution of the given ODE**

$$\begin{aligned} y &= y_1 + y_3 + y_5 + y_7 + \cdots \\ &= A_1 \cos t + B_1 \sin t \\ &\quad + A_3 \cos 3t + B_3 \sin 3t \\ &\quad + A_5 \cos 5t + B_5 \sin 5t + \cdots \end{aligned}$$



Forced Oscillations

$$y'' + 0.05y' + 25y = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt$$

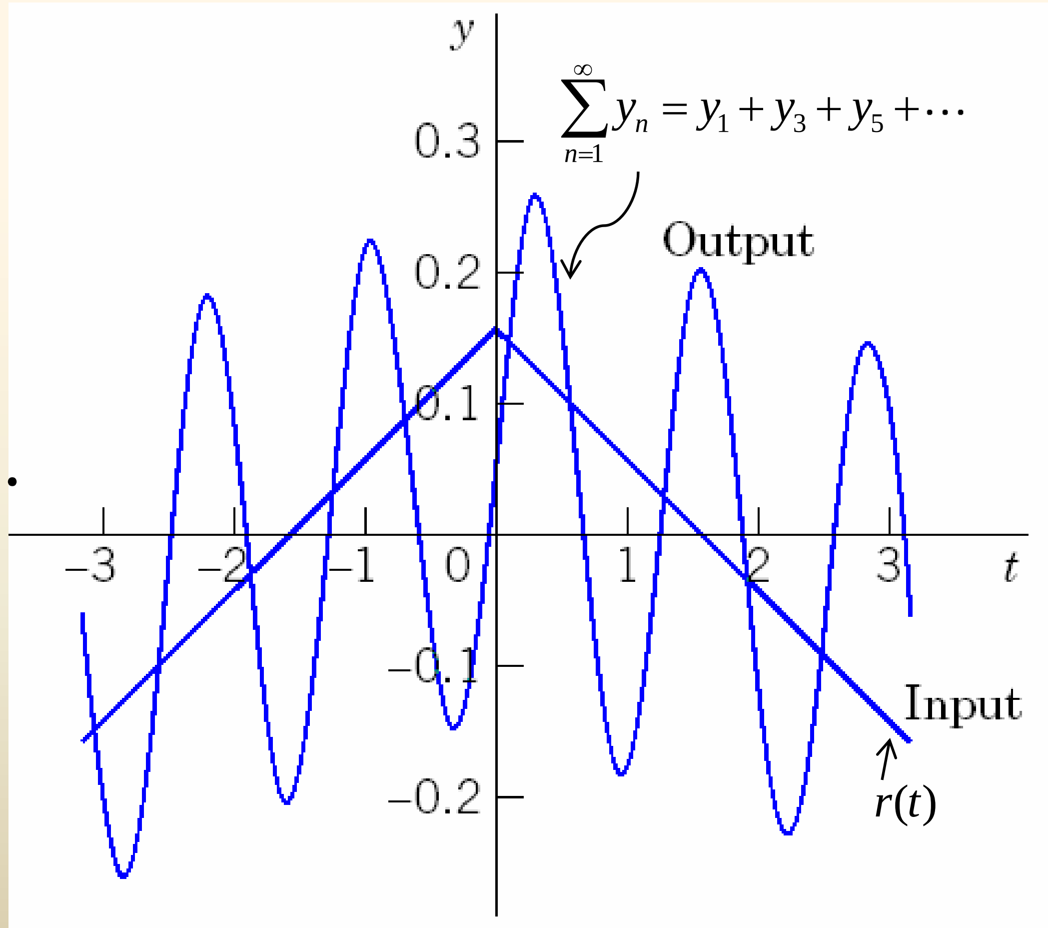
$$y_n = A_n \cos nt + B_n \sin nt,$$

$$y = y_1 + y_3 + y_5 + \cdots$$

- **solution of the given ODE**

$$\begin{aligned} y &= y_1 + y_3 + y_5 + y_7 + \cdots \\ &= A_1 \cos t + B_1 \sin t \\ &\quad + A_3 \cos 3t + B_3 \sin 3t \\ &\quad + A_5 \cos 5t + B_5 \sin 5t + \cdots \end{aligned}$$

$$\begin{aligned} \therefore A_1 &= 0.0531, & B_1 &= 0.0001105 \\ A_3 &= 0.0088, & B_3 &= 0.0001964 \\ A_5 &= 0, & B_5 &= 0.2037 \\ A_7 &= -0.0011, & B_7 &= 0.0000 \\ A_9 &= -0.0033, & B_9 &= 0.0000 \end{aligned}$$



Application of Fourier Series

✓ Application 2) Fourier transform

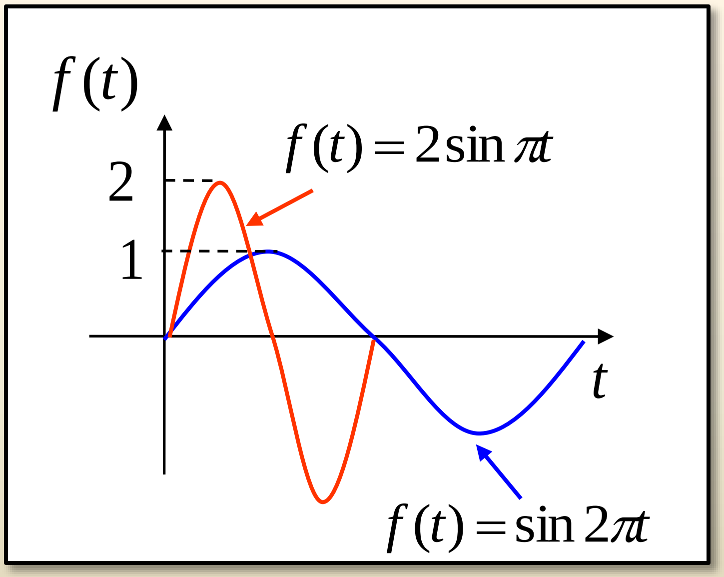
: Transform between time domain and frequency domain.

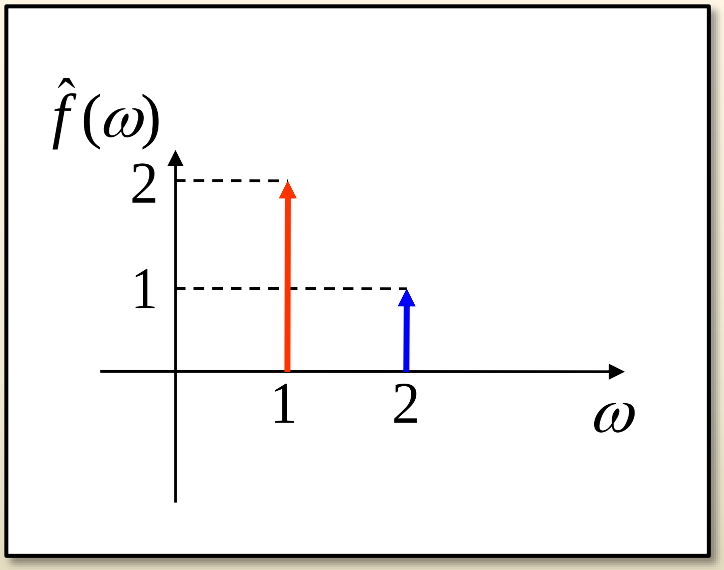
ex) Interpretation of the Fourier transform

Inverse Fourier Transform

Time Domain ← → Frequency Domain

Fourier Transform





Frequency와 Amplitude로 표현됨

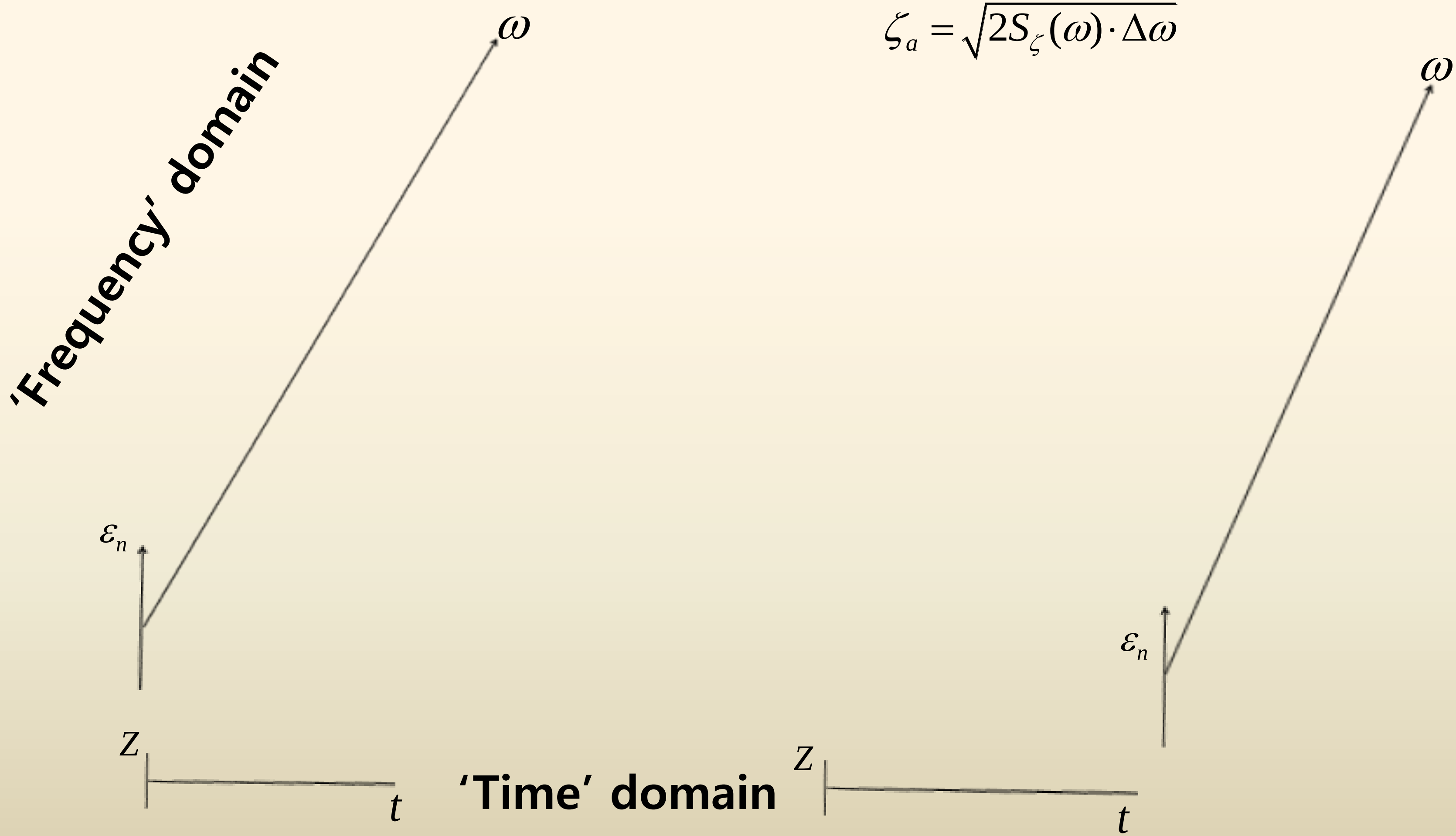
=> 시계열의 운동 복원 가능

Application of Fourier Series

✓ Application 2) Fourier transform

: Transform between time domain and frequency domain. ▷wave spectrum

$$\zeta_a = \sqrt{2S_\zeta(\omega)\cdot\Delta\omega}$$



Measured Wave Record **Generated Wave Record**

ζ_a:wave amplitude

*Journee J. M. J., Massie W. W., Offshore Hydrodynamics, First Edition, Delft University of Technology, 2001, Figure 5.38

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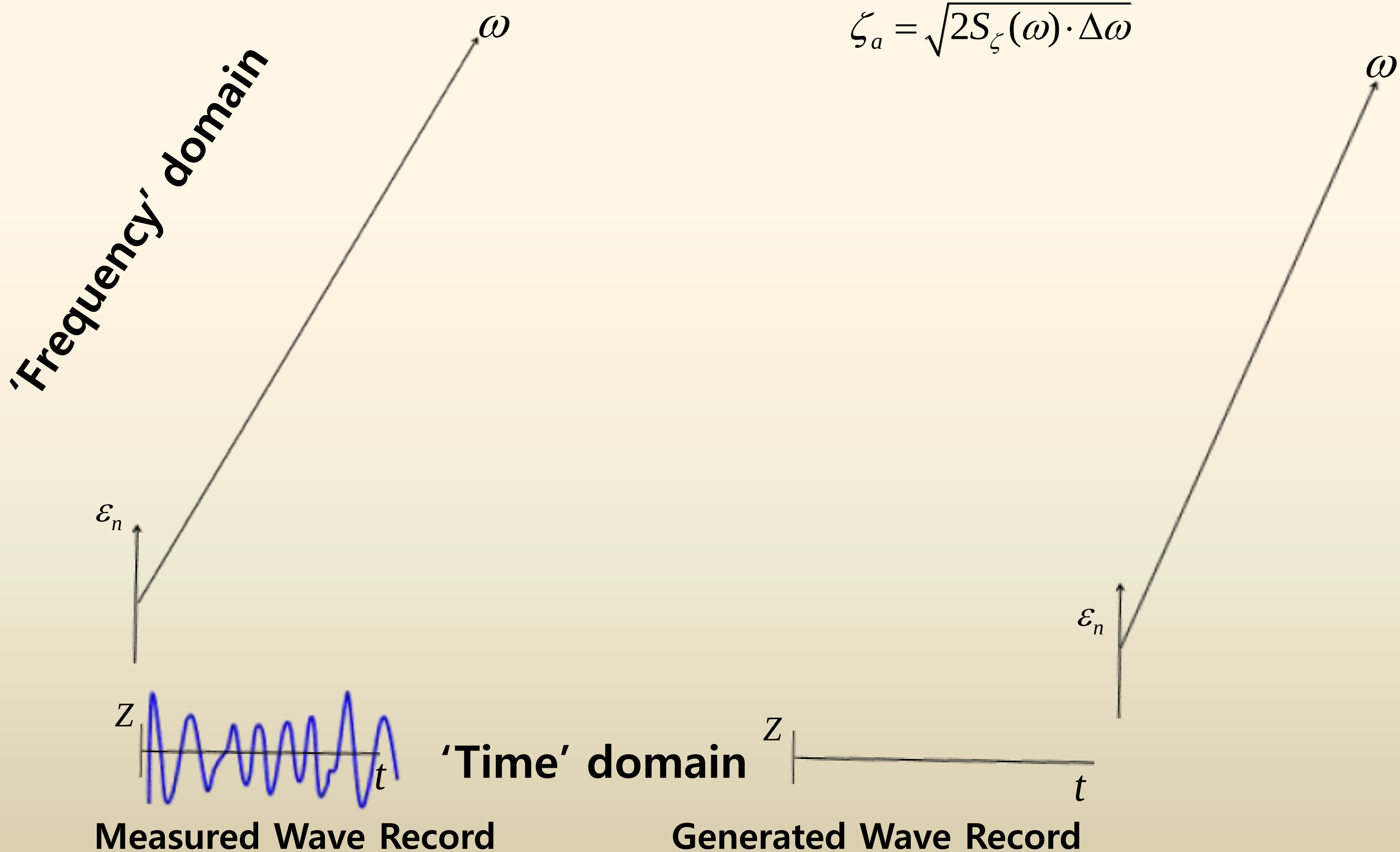

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2008_Fourier Transform(2)

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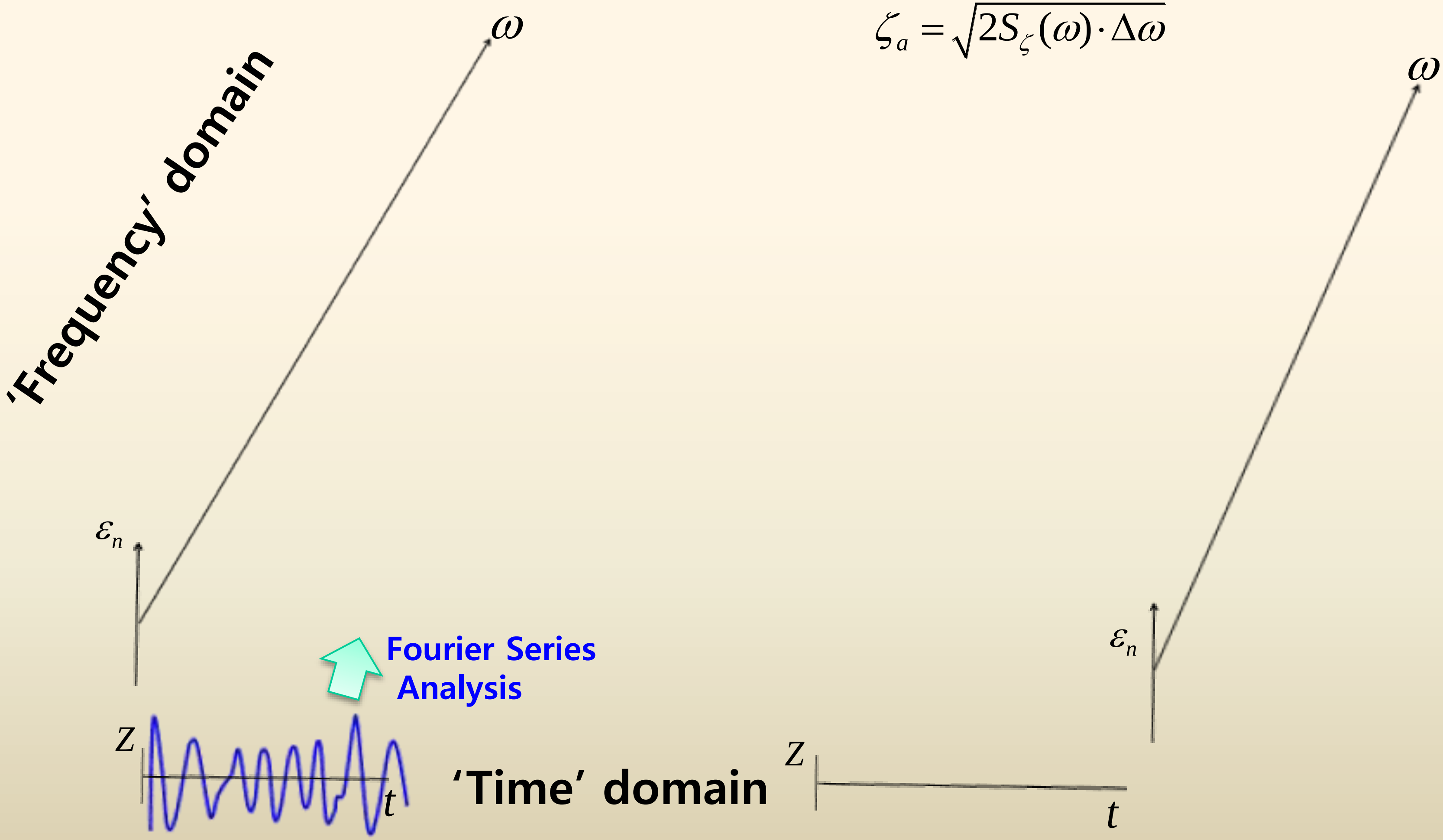

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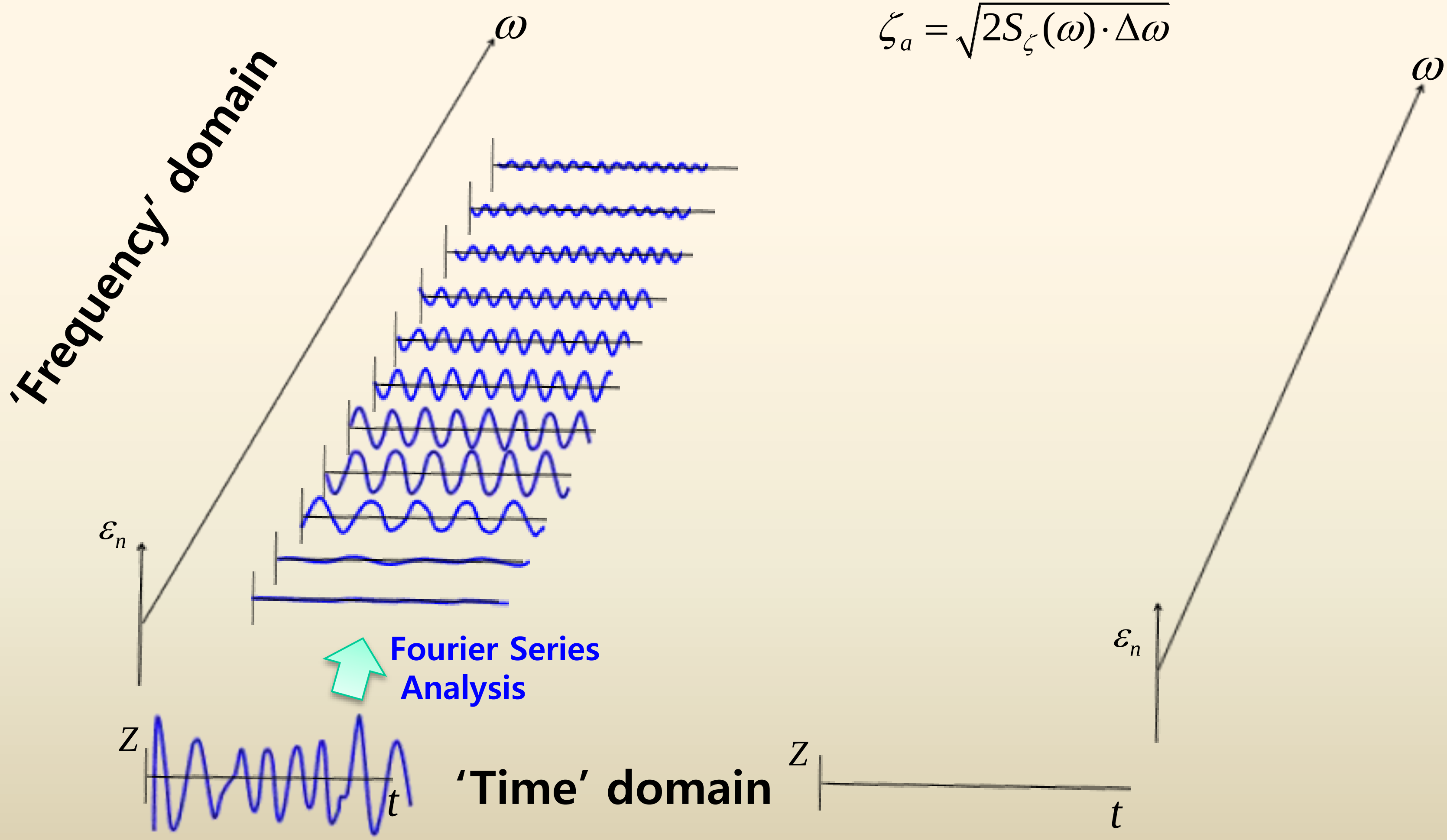

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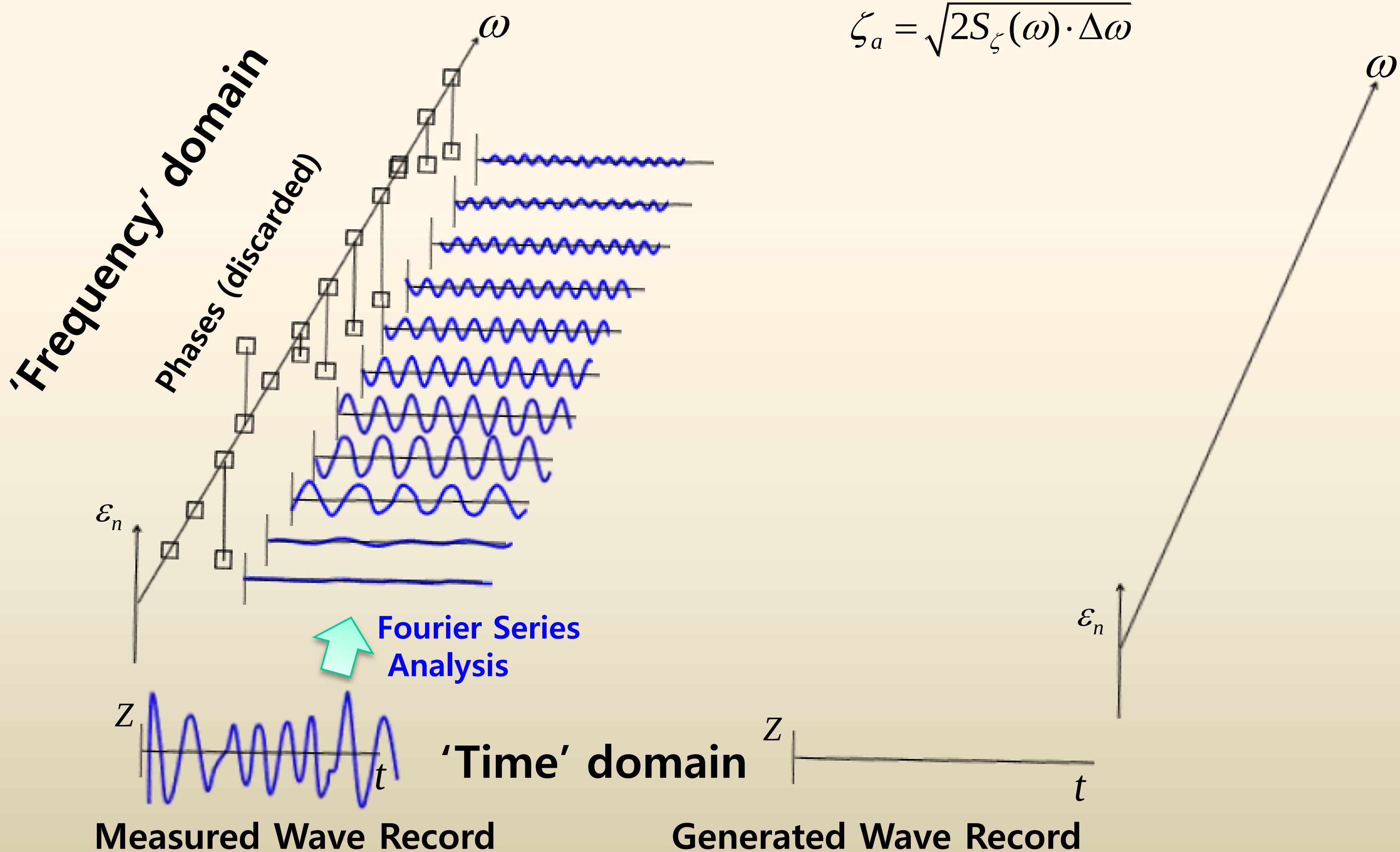

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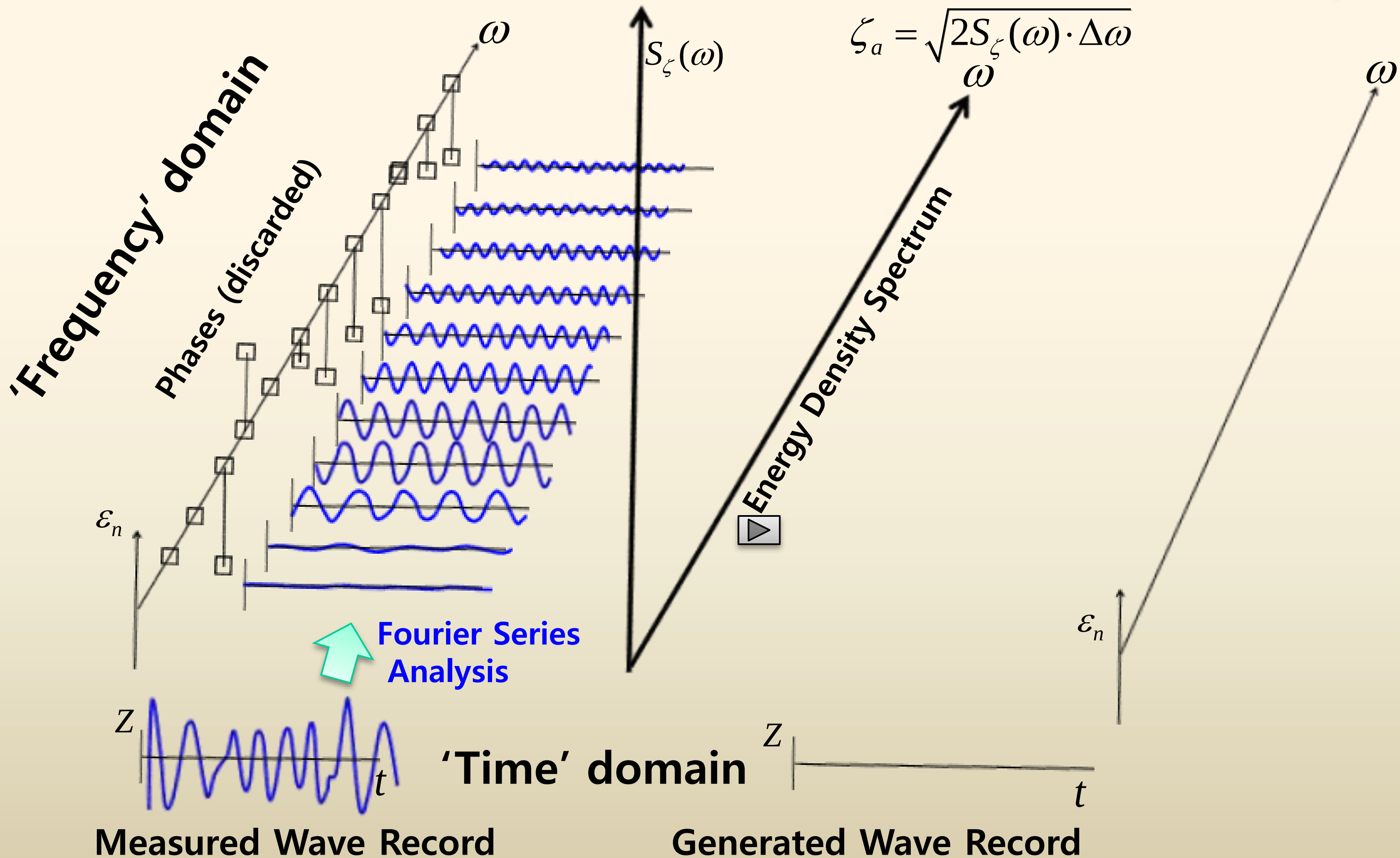

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2008_Fourier Transform(2)

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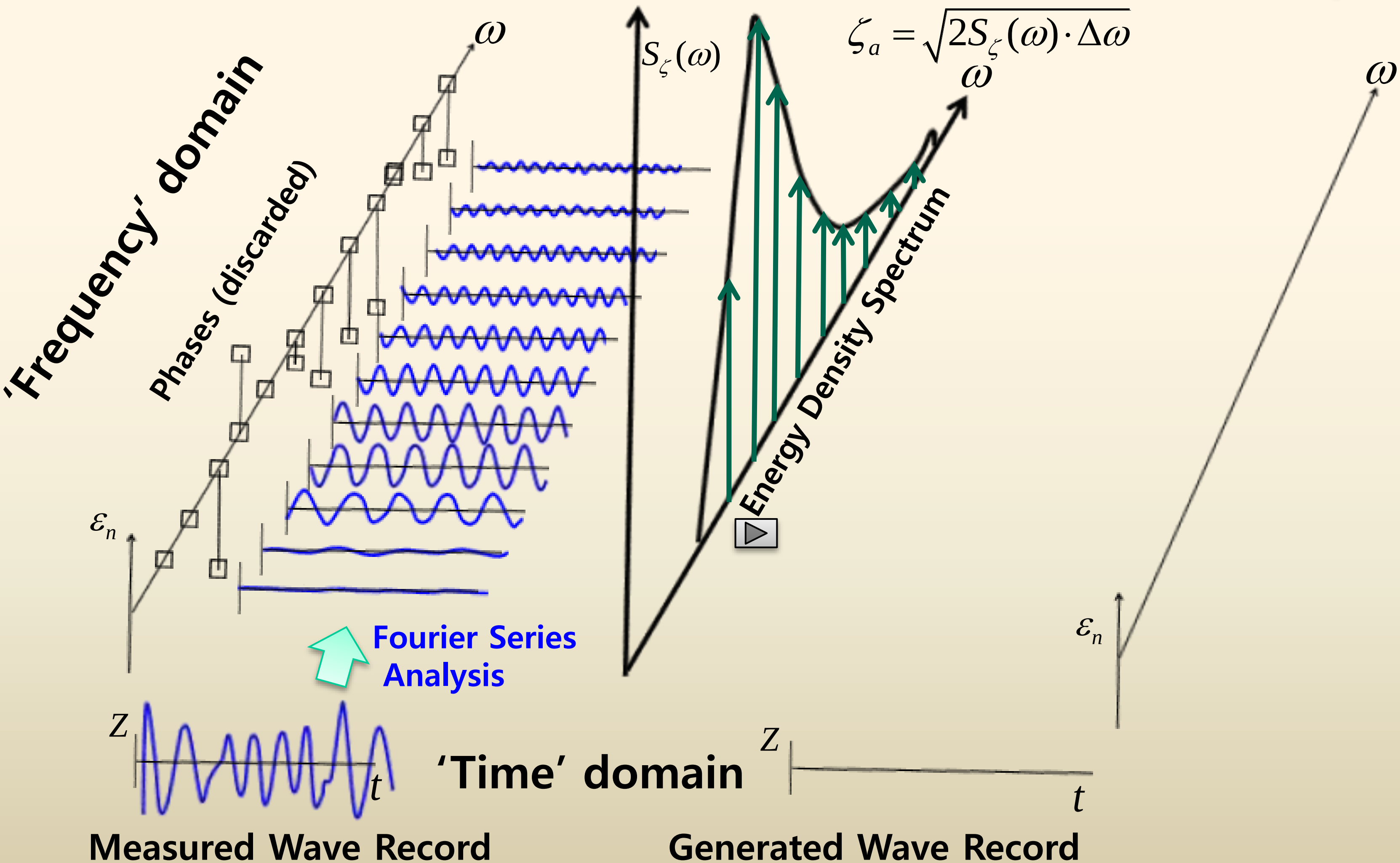

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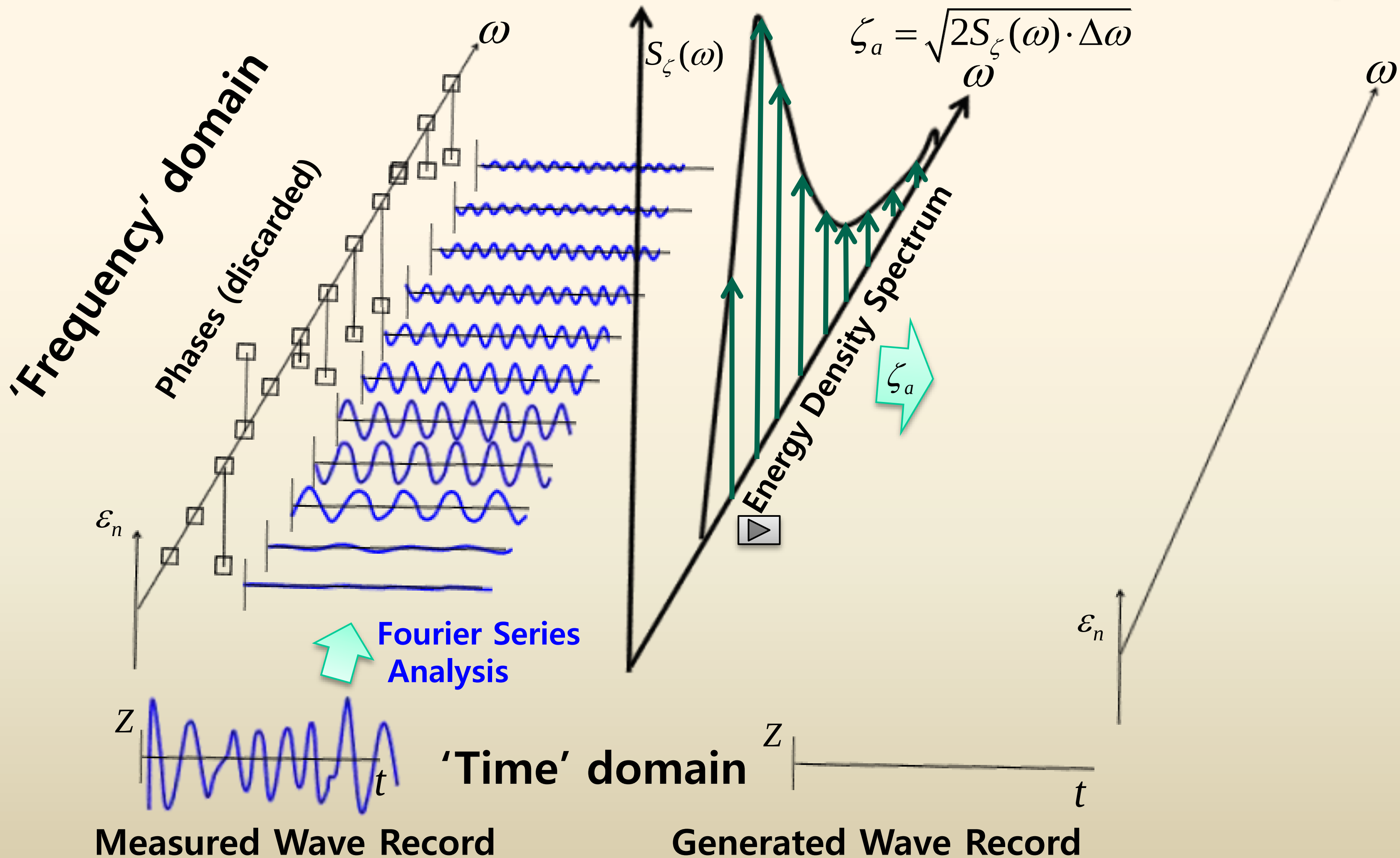

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Application of Fourier Series

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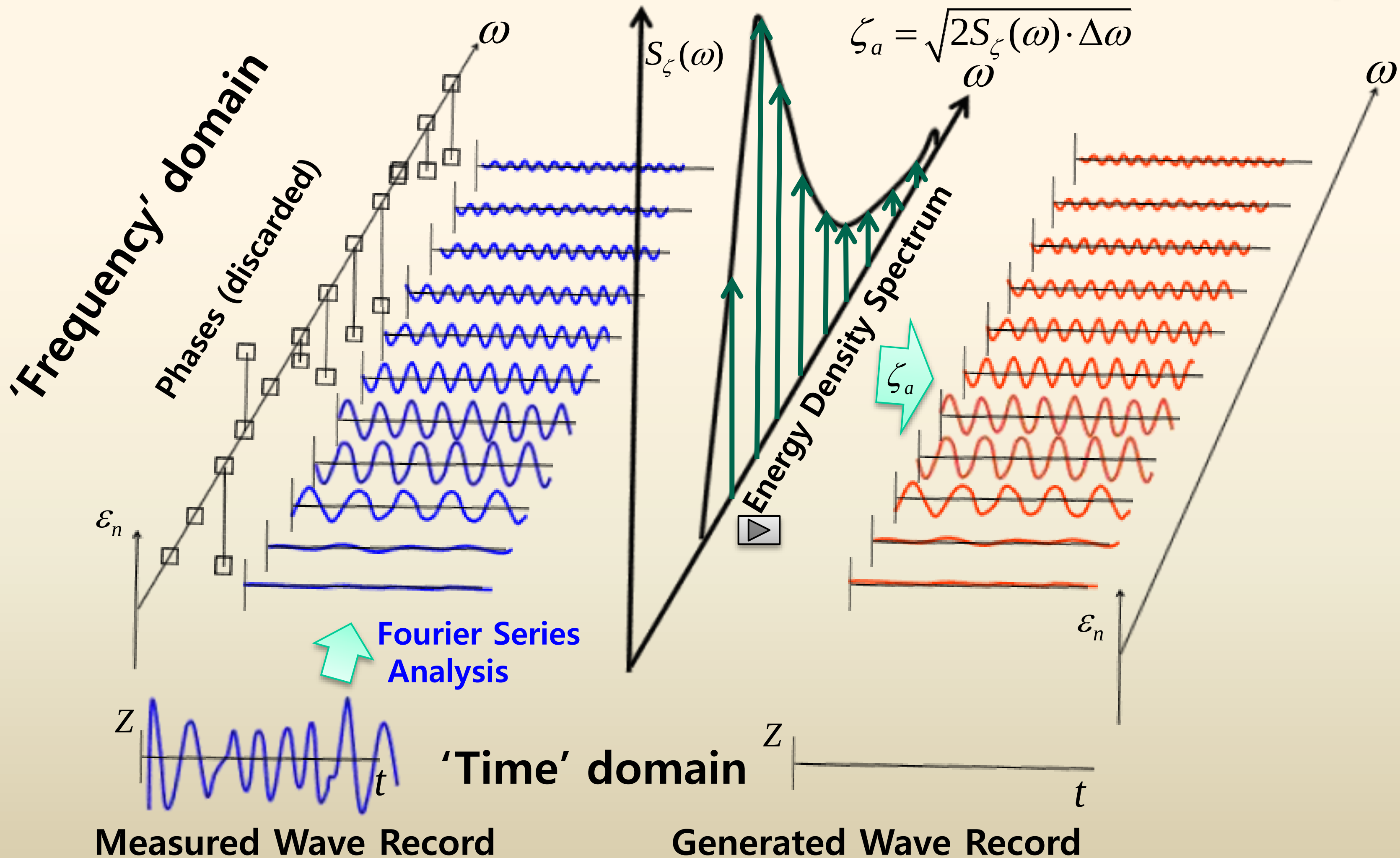
: Transform between time domain and frequency domain. ▷ wave spectrum



ζ_a:wave amplitude

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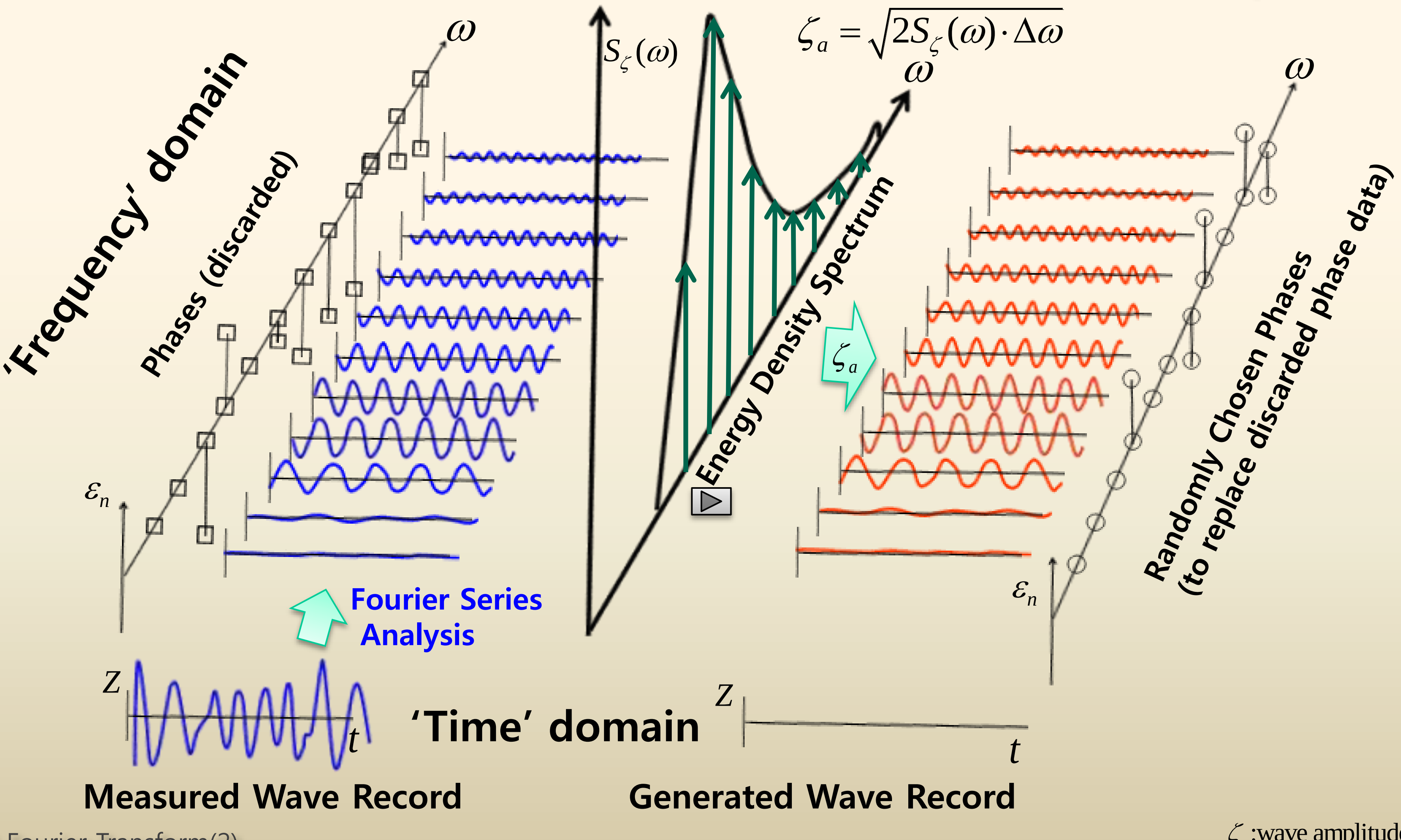

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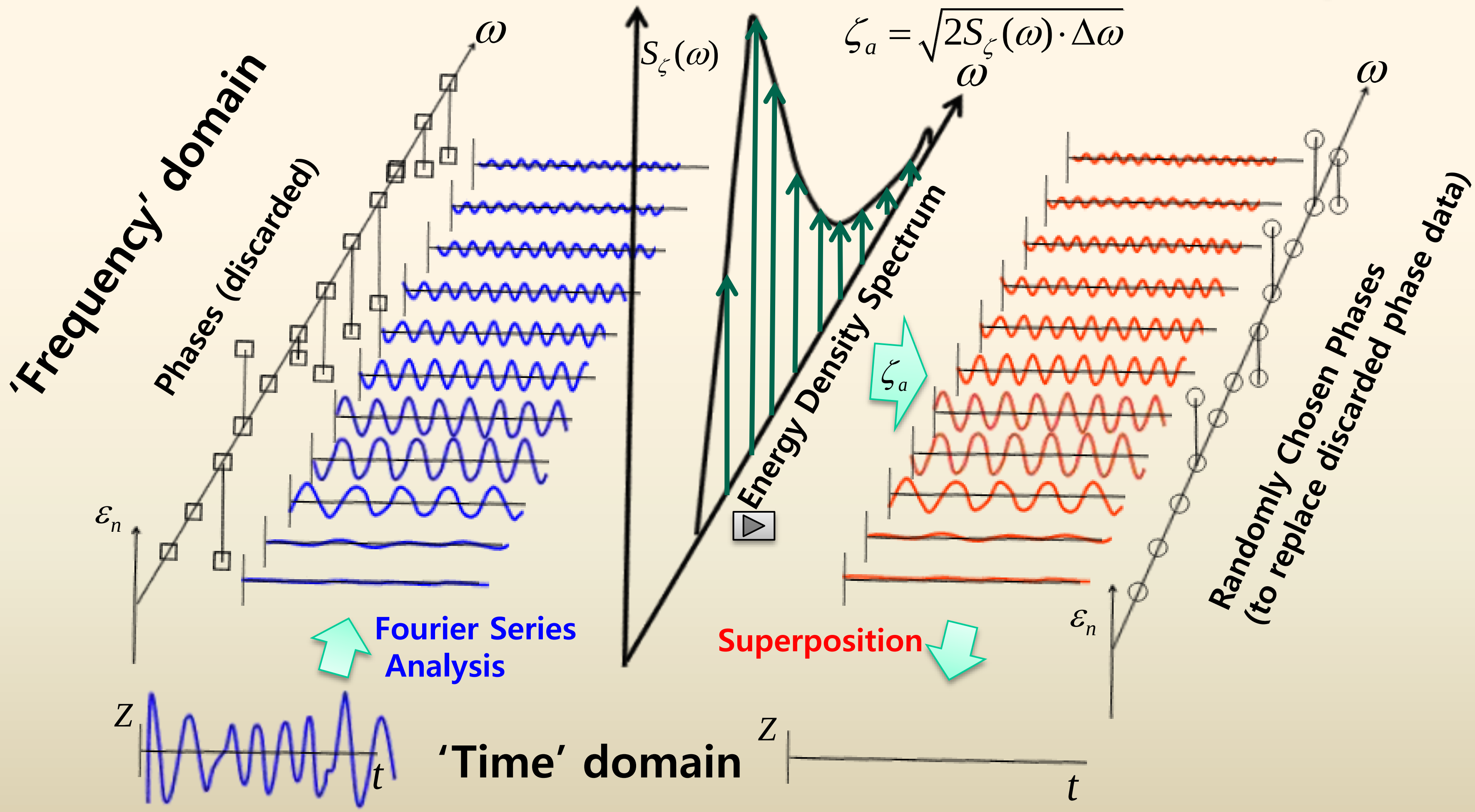

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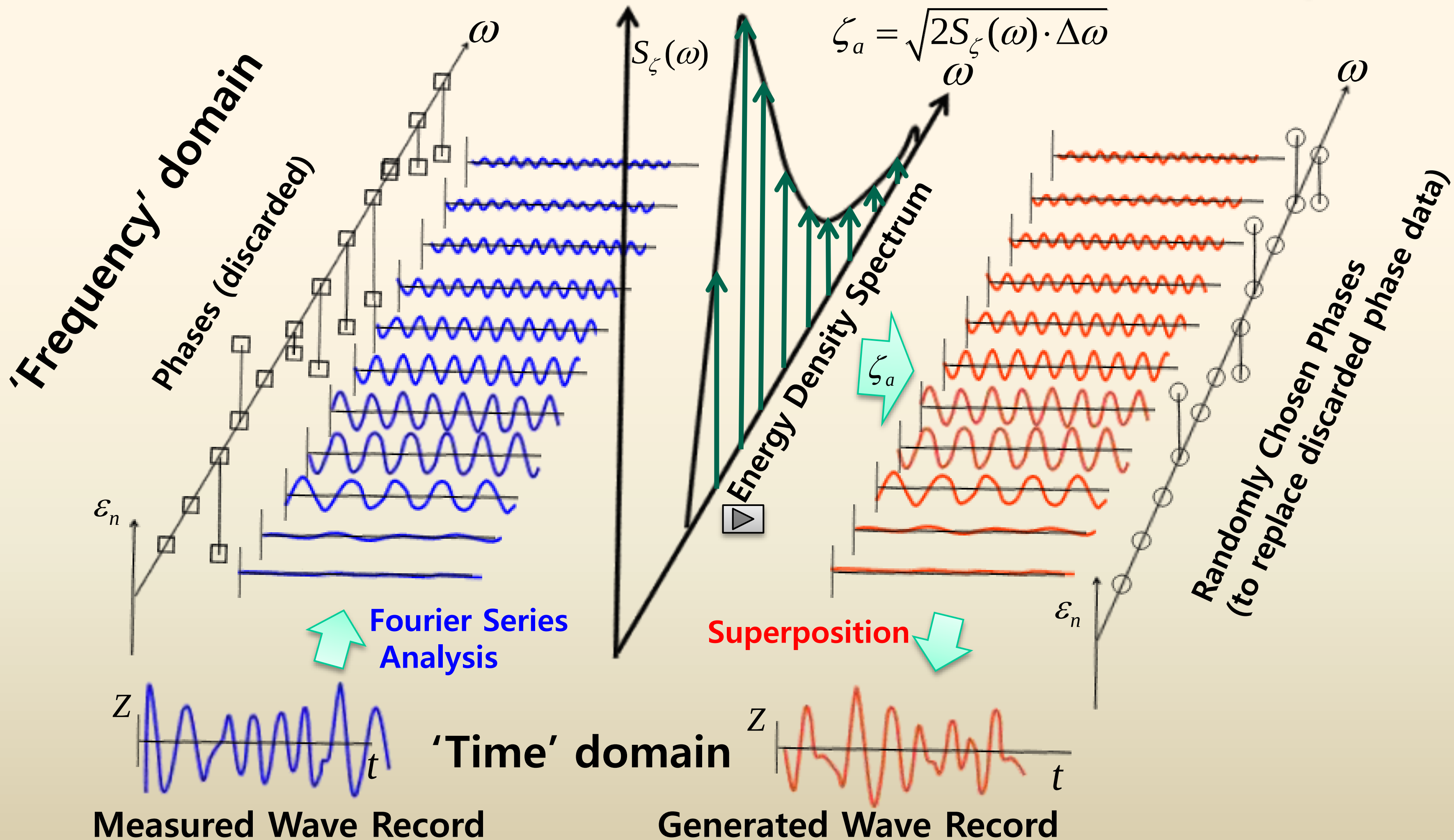

2008_Fourier Transform(2)

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Wave Spectrum

Wave Spectrum → 3학년 해양파 역학

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2p} = \frac{\pi}{p}$$

•Linear wave

-Sum of many simple sine waves makes an irregular sea*

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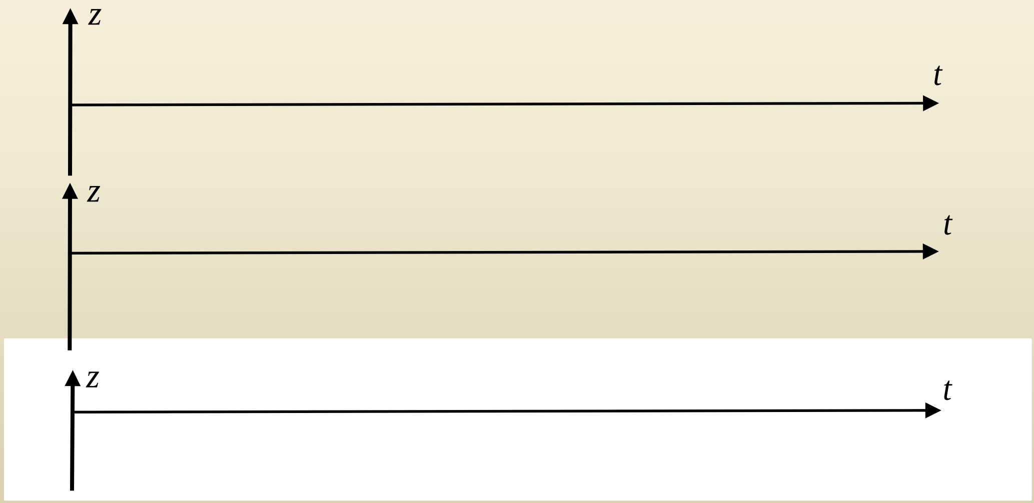
-Superposition of two uni-directional harmonic waves***



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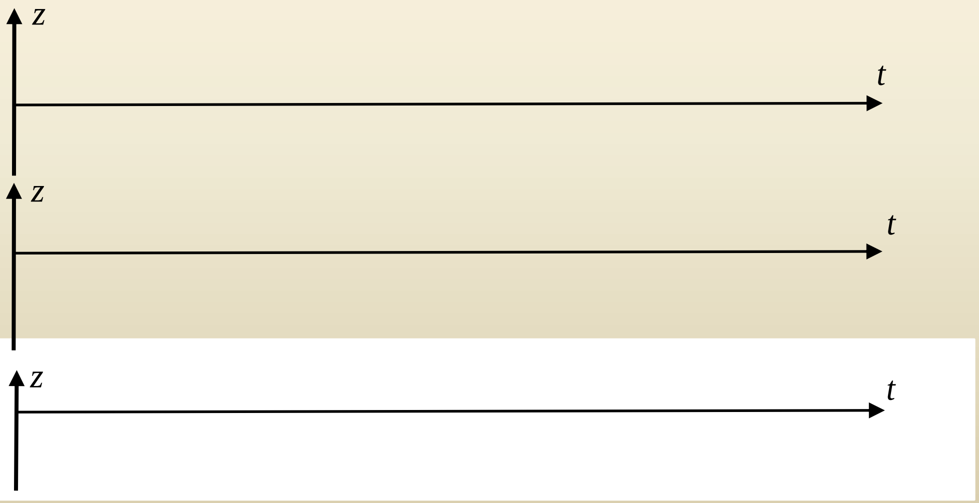

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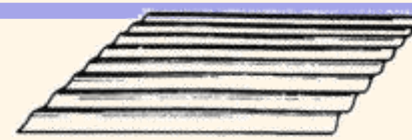

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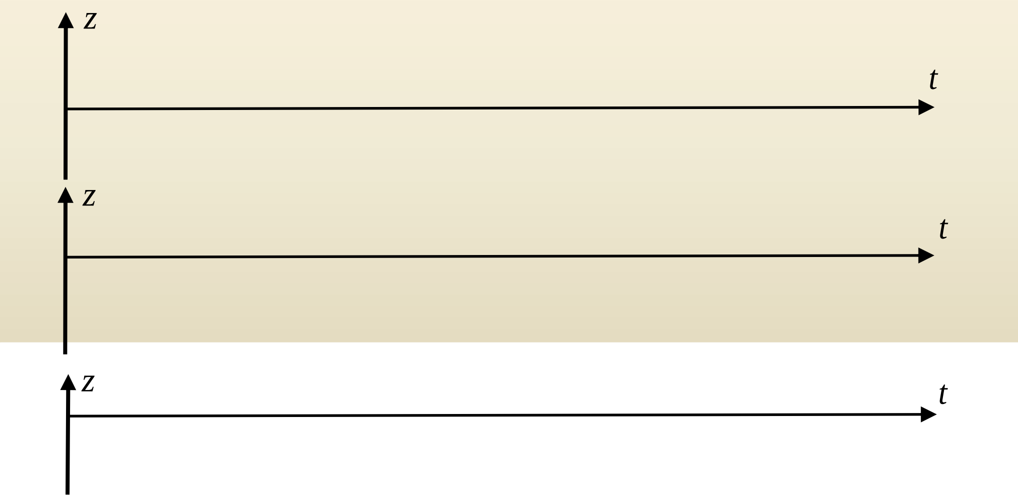

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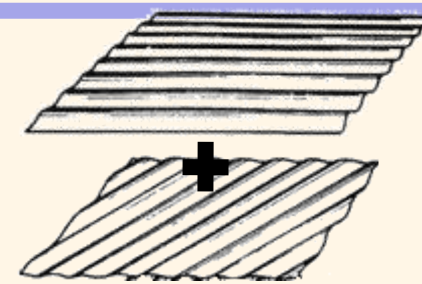

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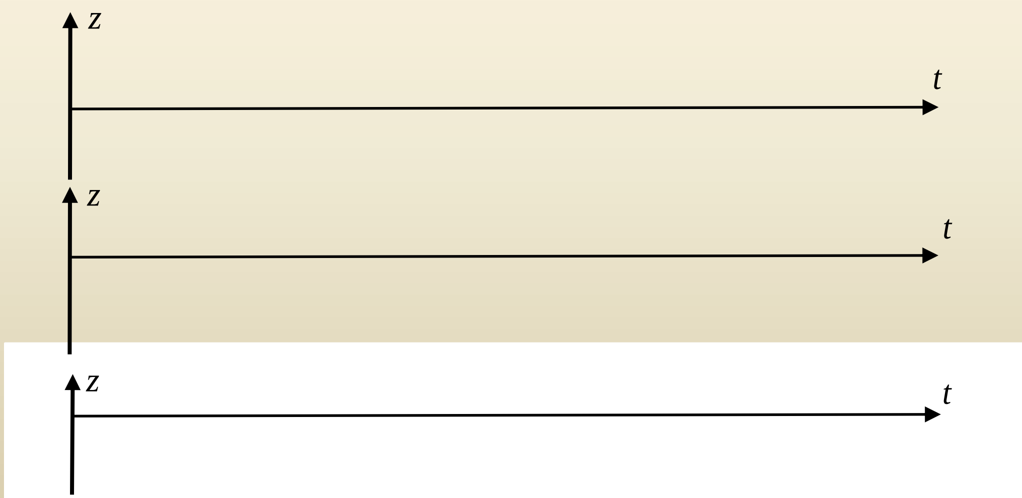

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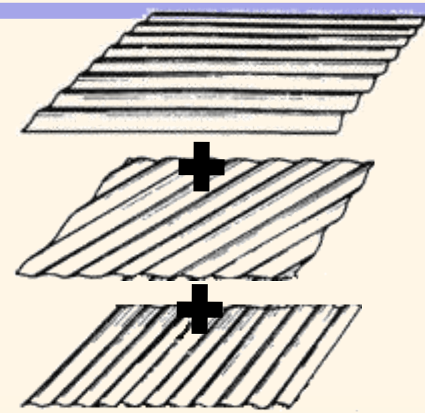

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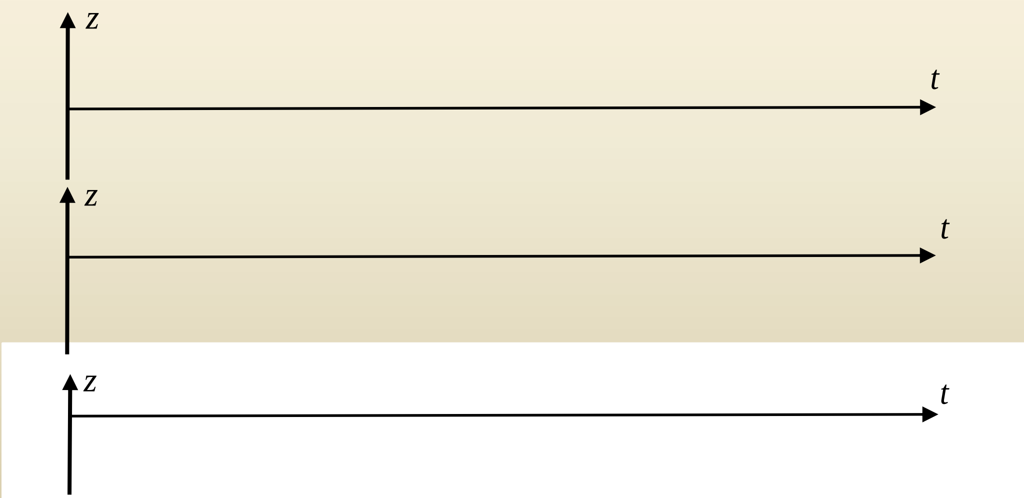

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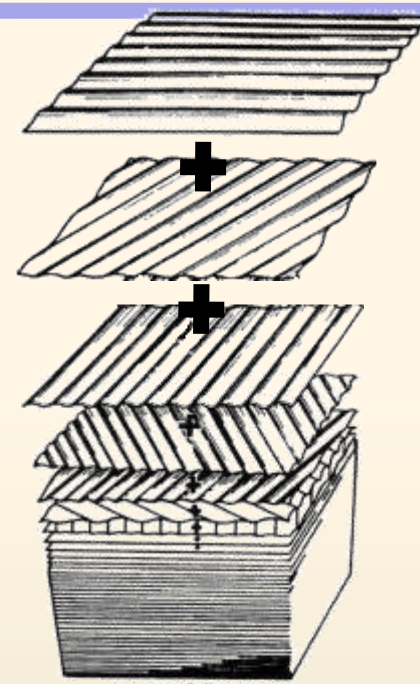

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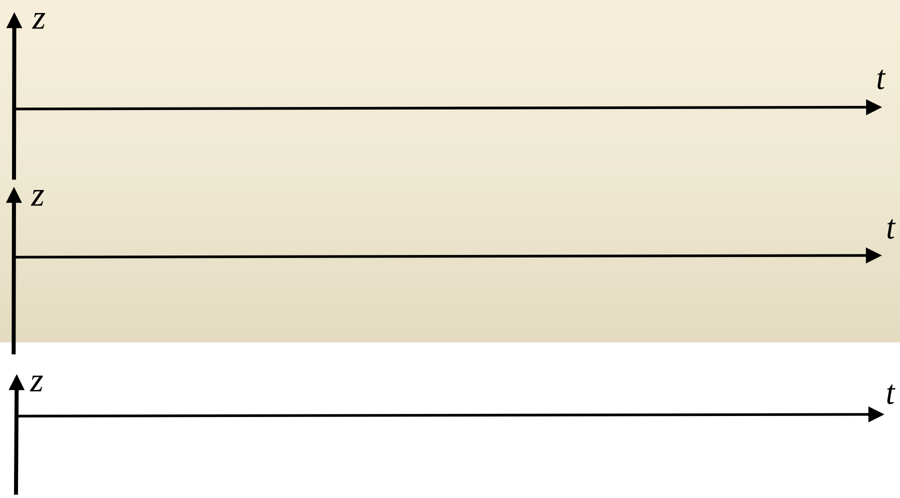

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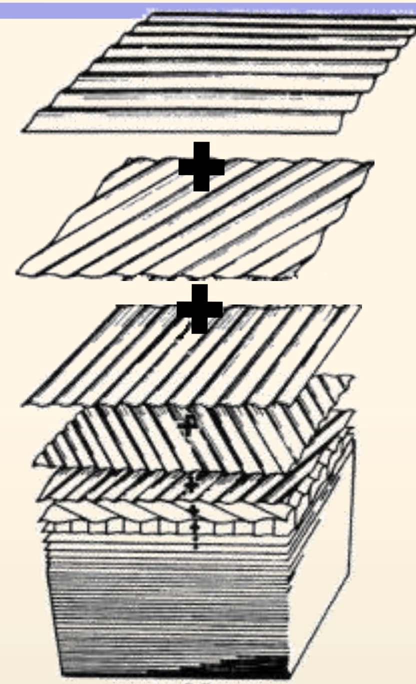

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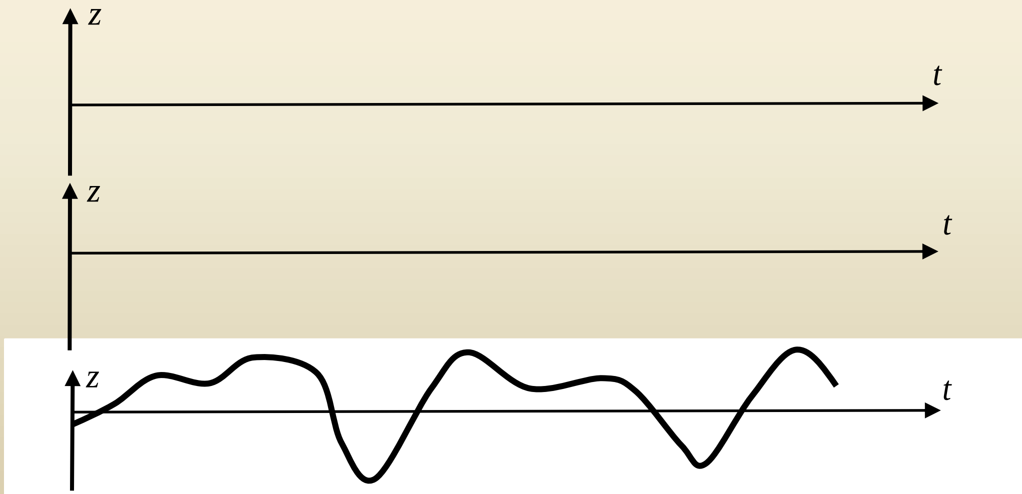

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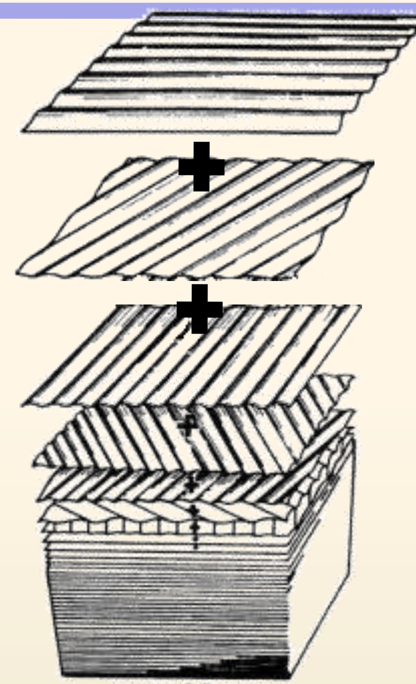

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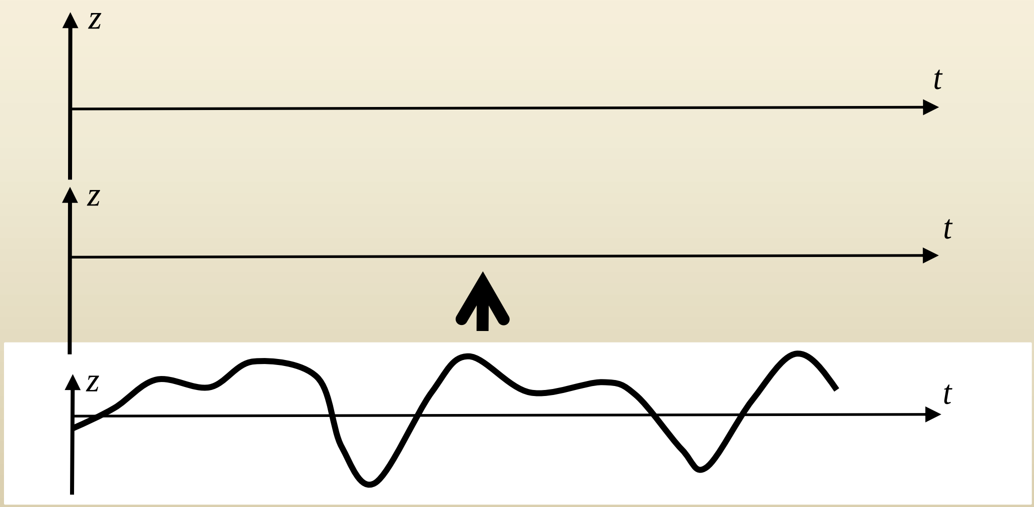

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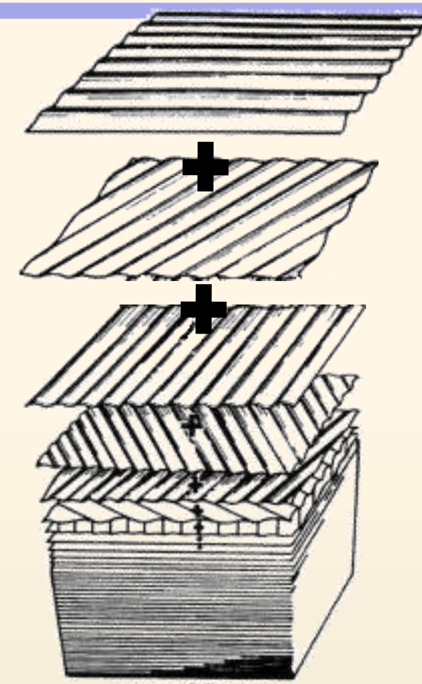

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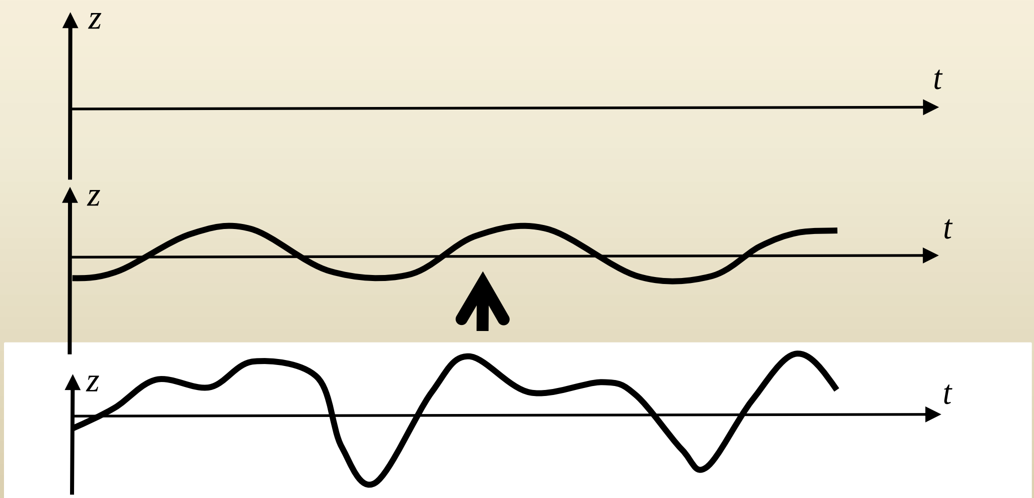

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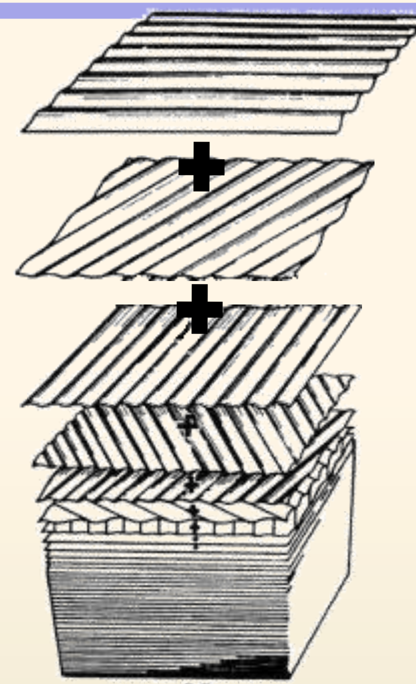

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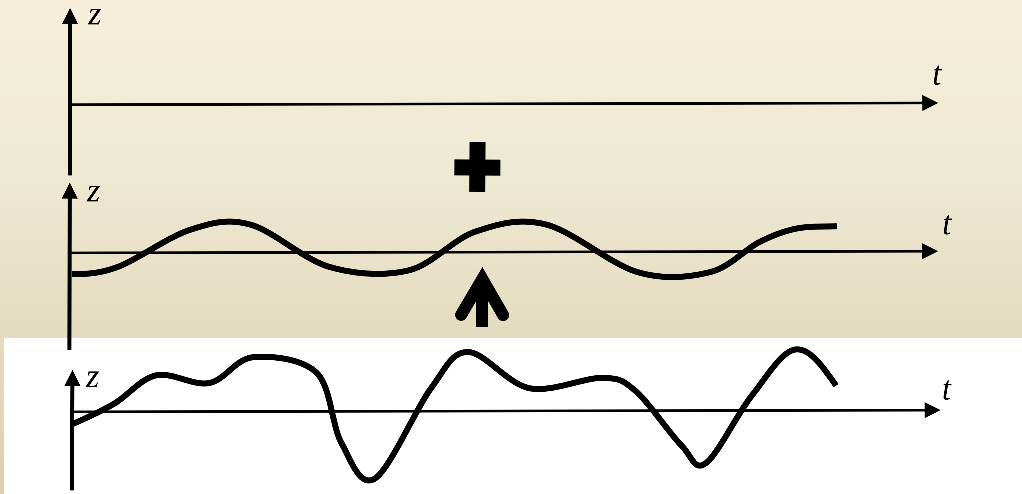

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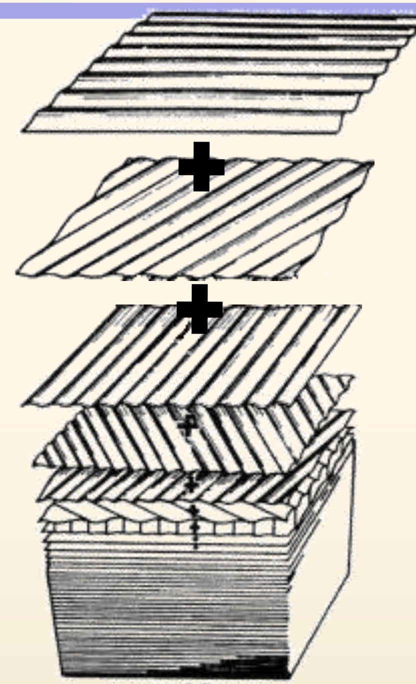

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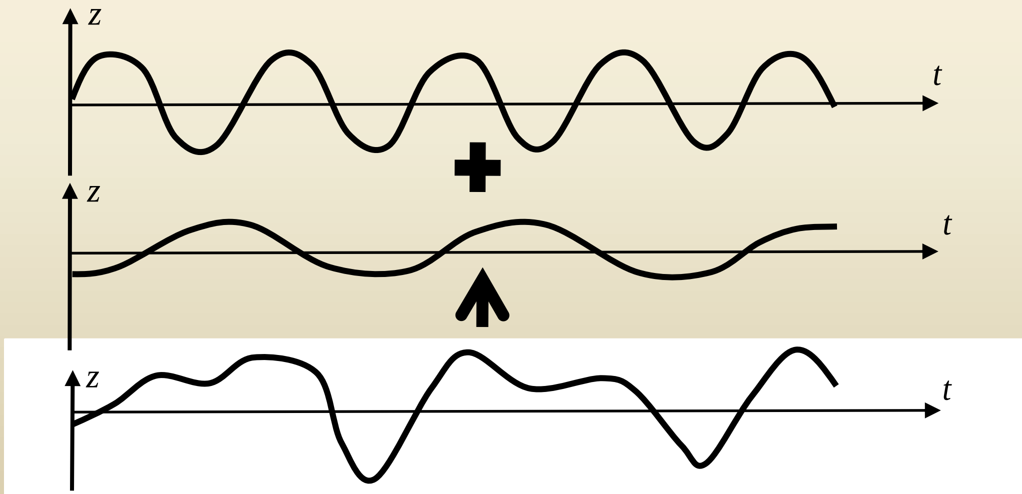

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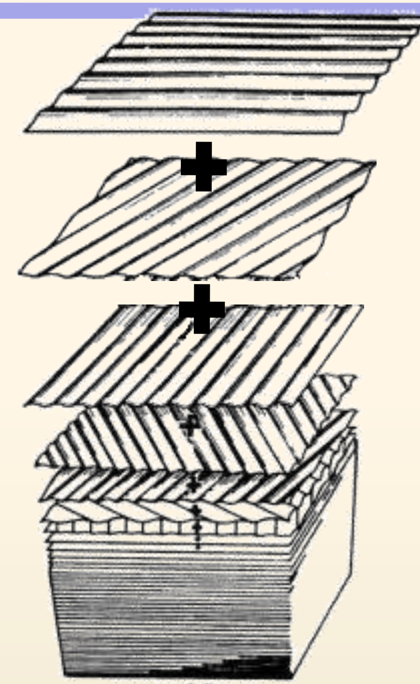

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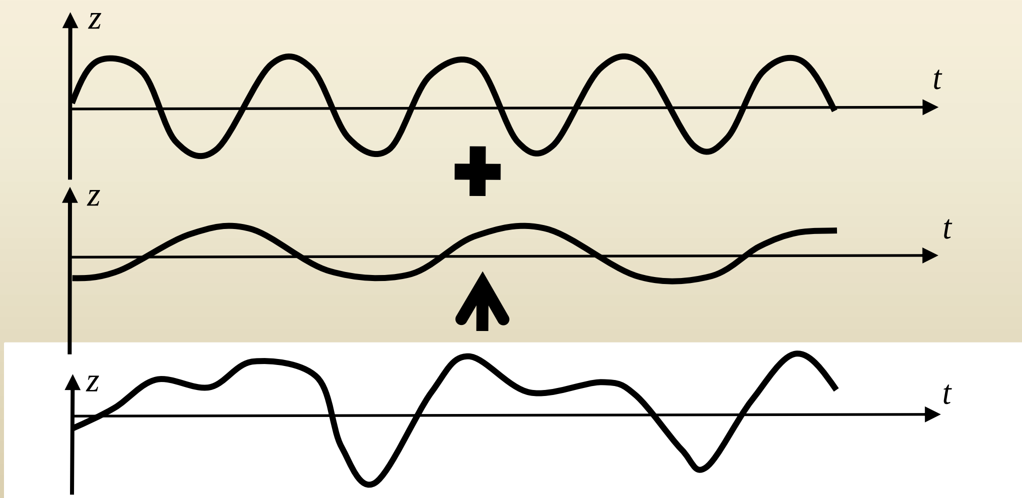

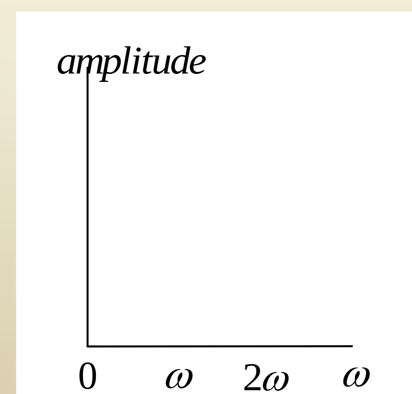

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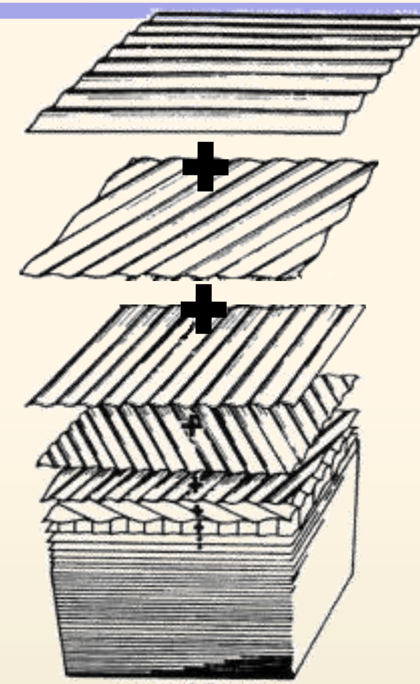

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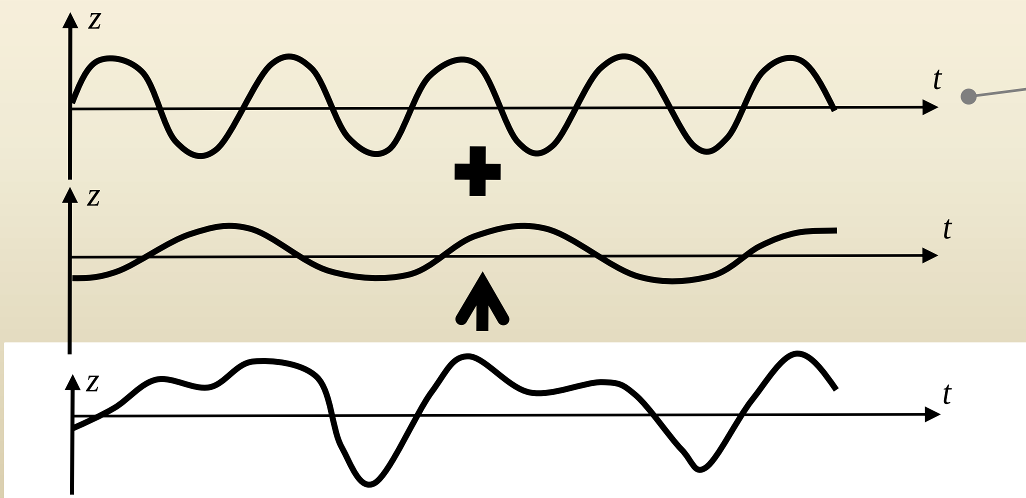

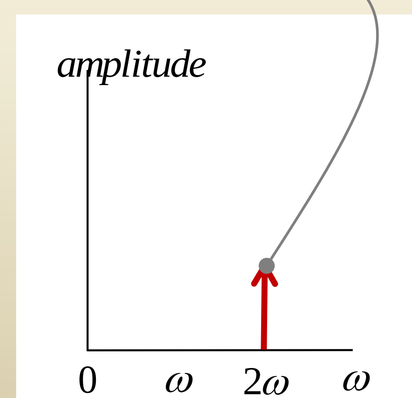

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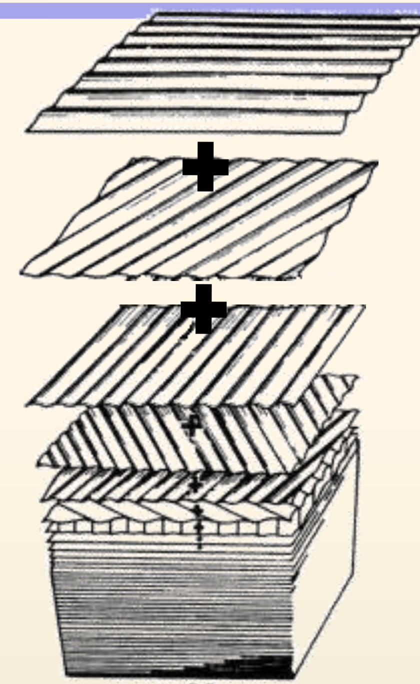

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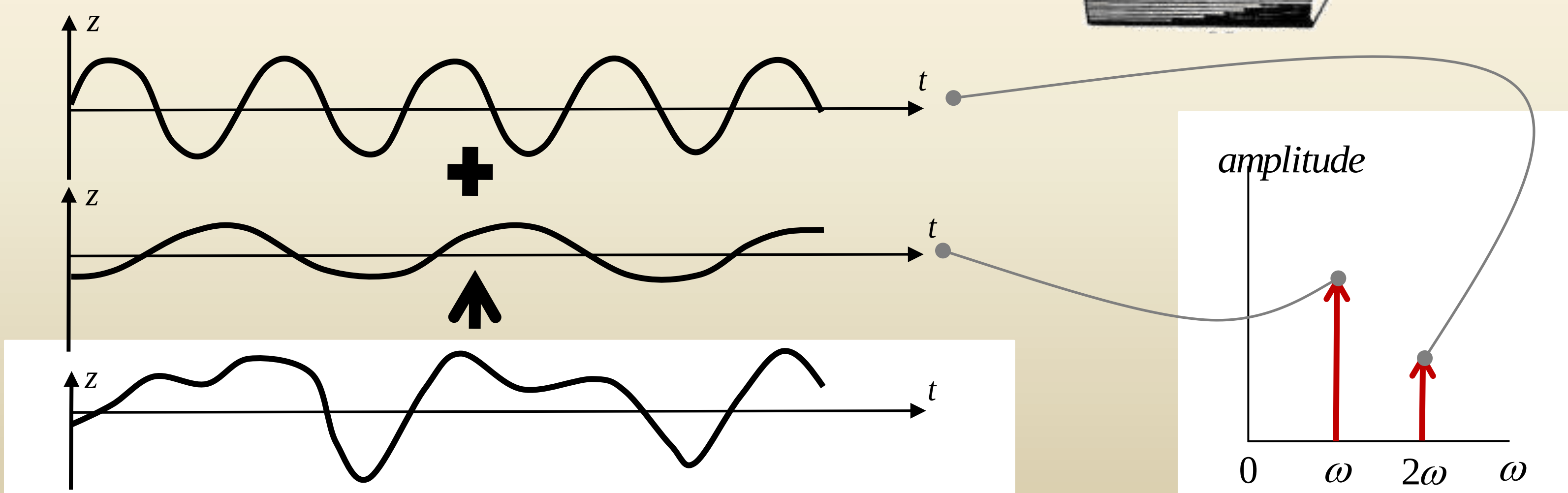

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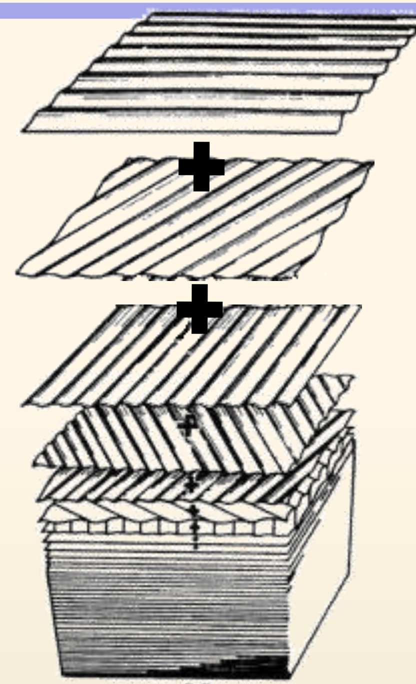

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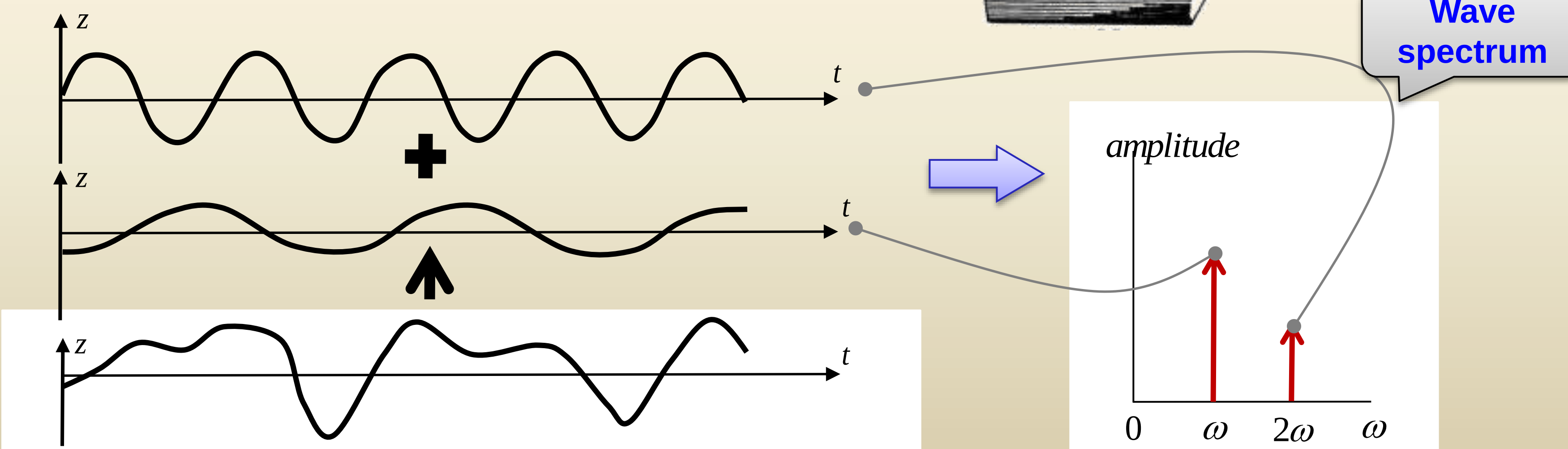

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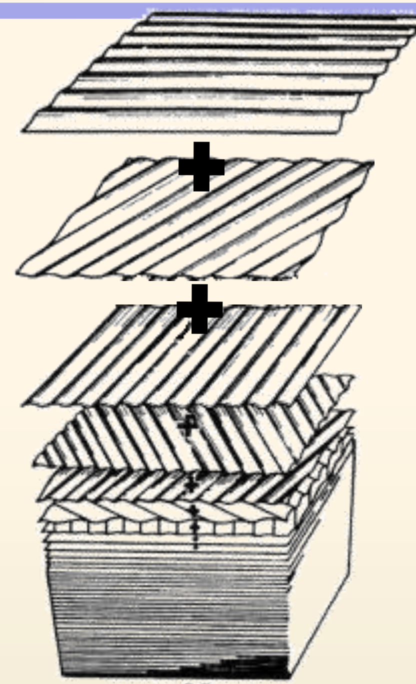

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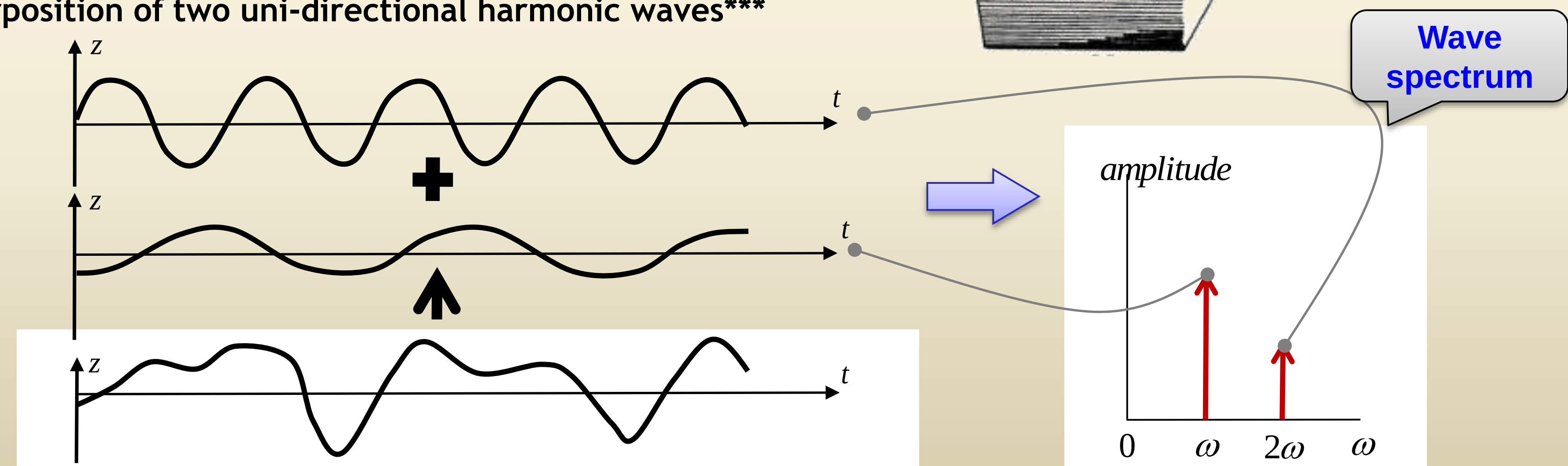

frequency domain contains exactly the same information as that of the time domain

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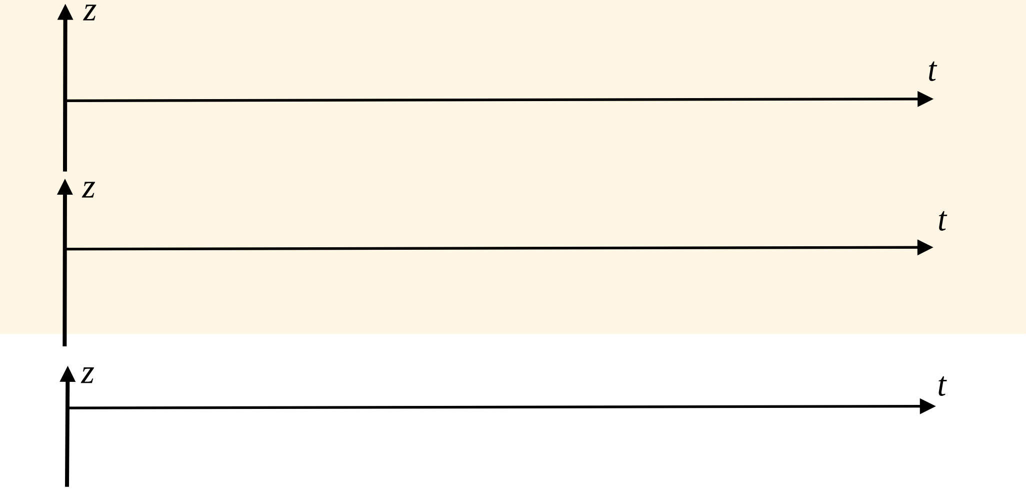
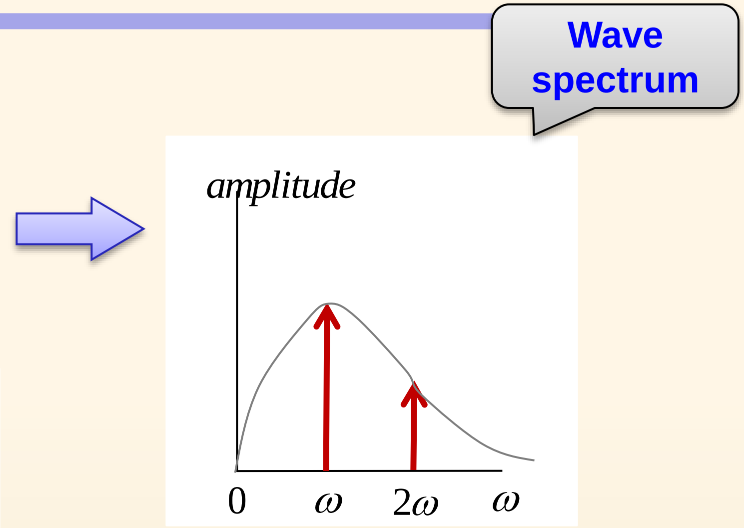
Wave Spectrum





If you know wave spectrum, can 're-construct' the original wave

ITTC : International Towing Tank Conference/ISSC : International Ship Structures Congress
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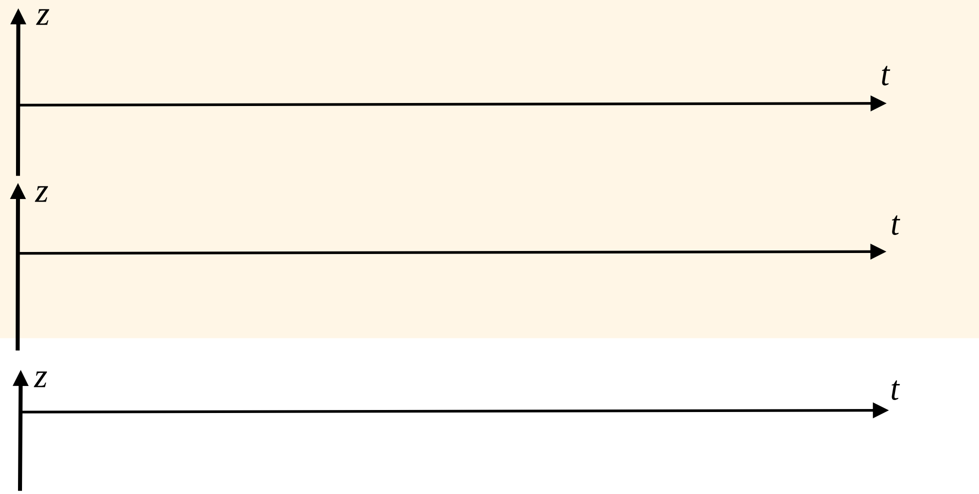

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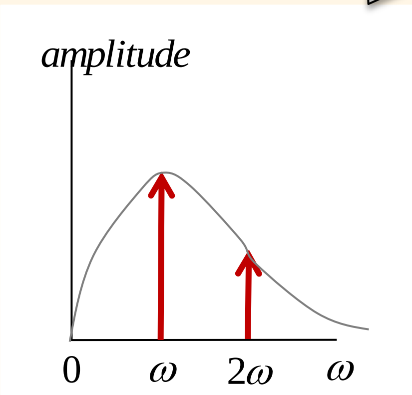

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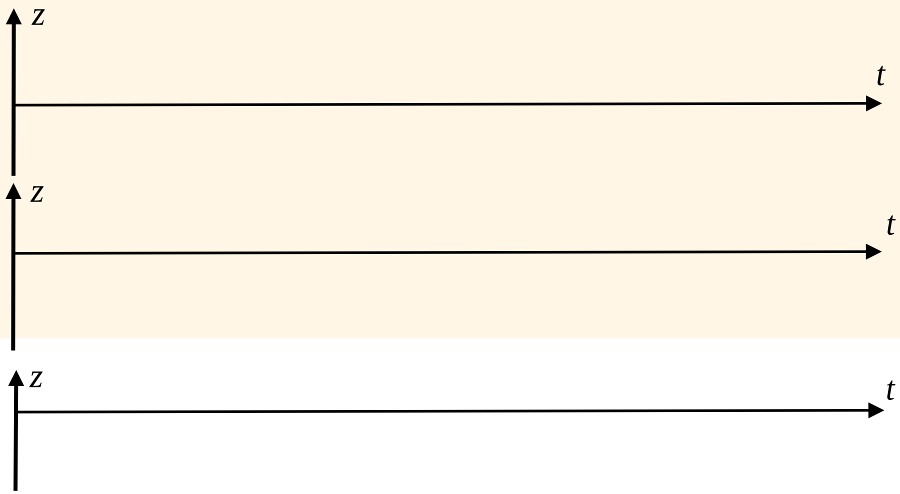
Wave Spectrum



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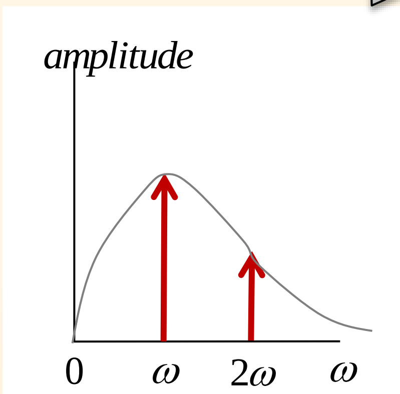

If you know wave spectrum, can 're-construct' the original wave

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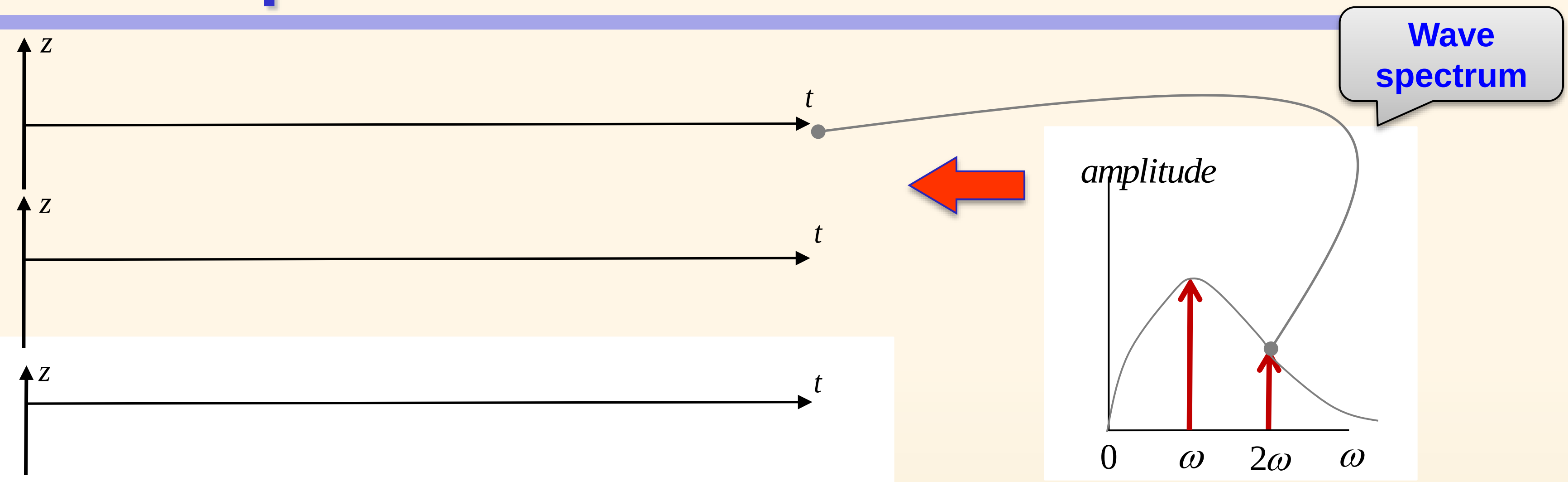

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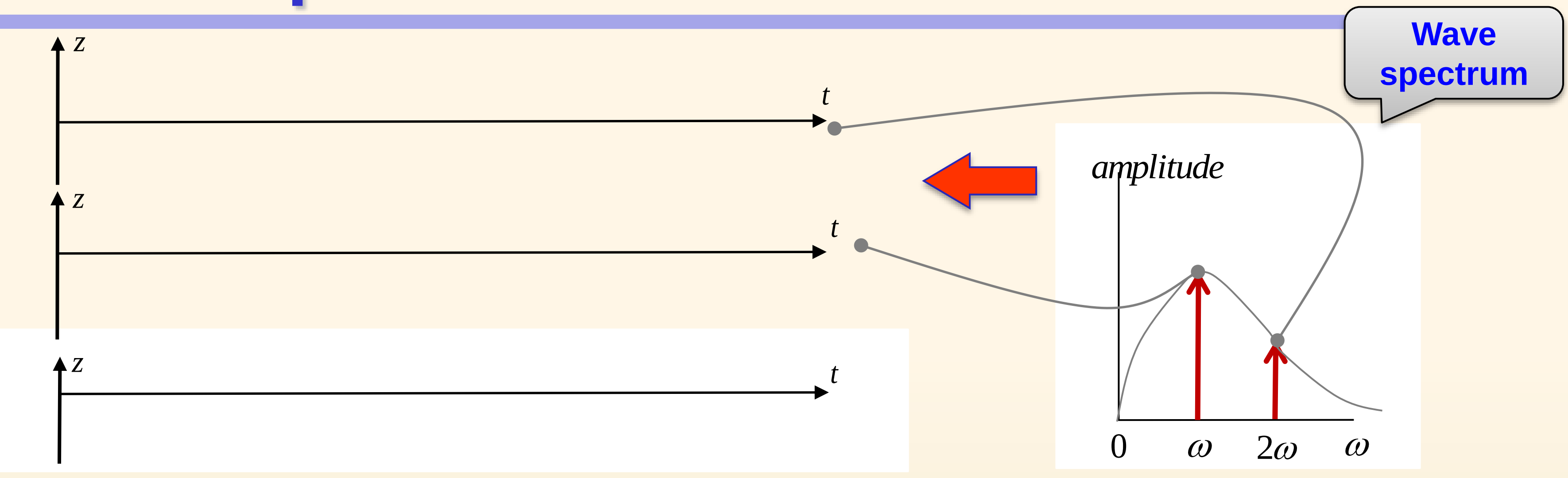

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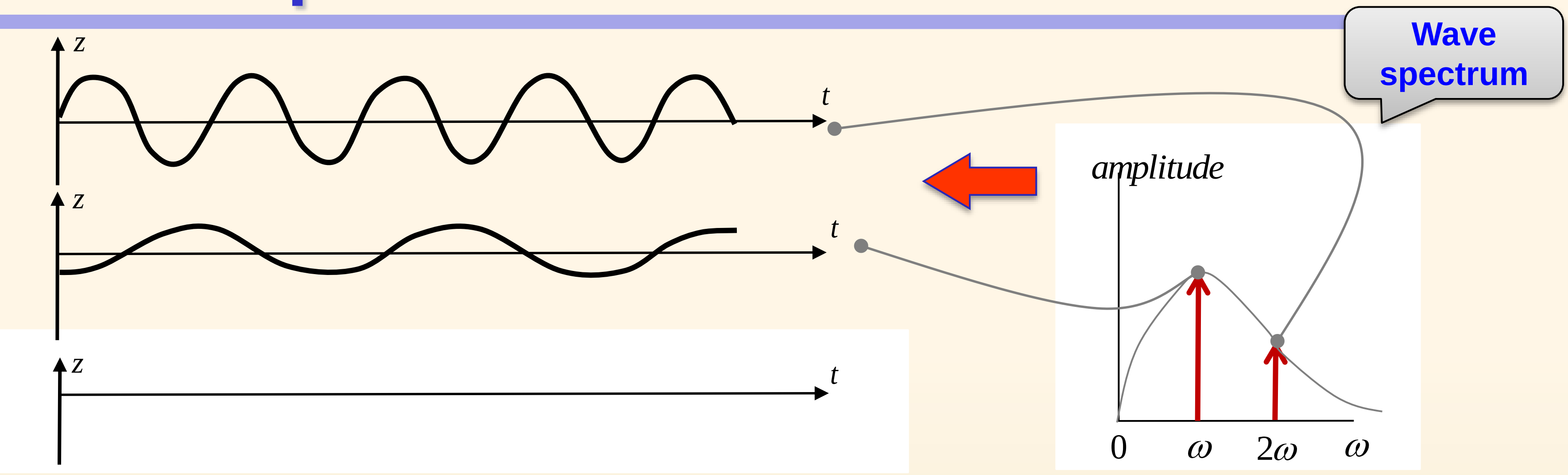

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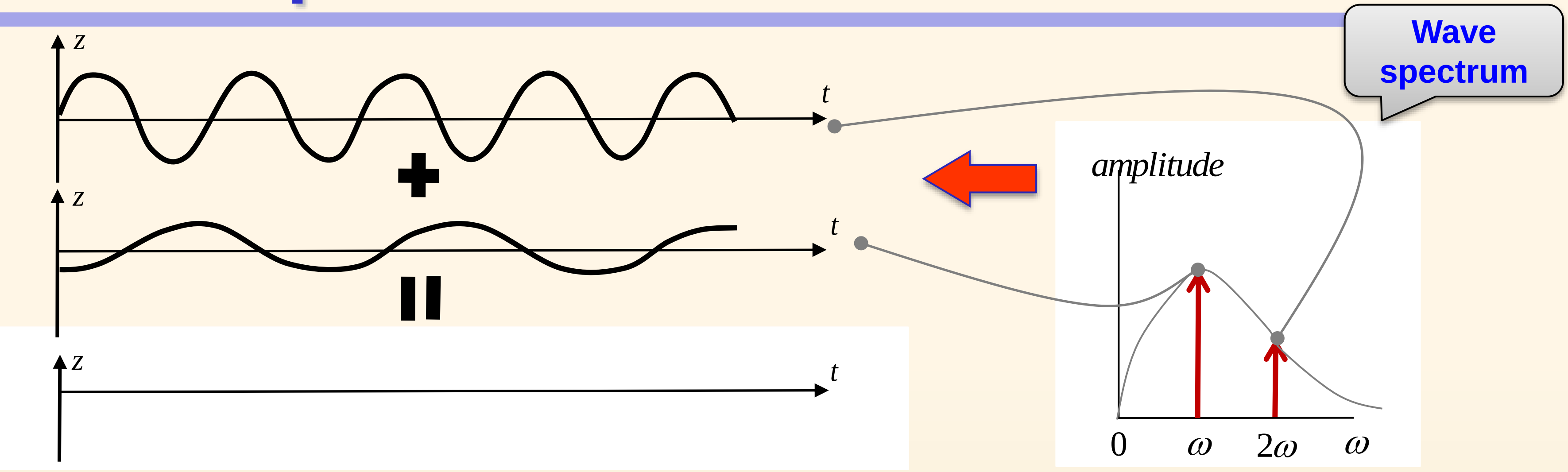

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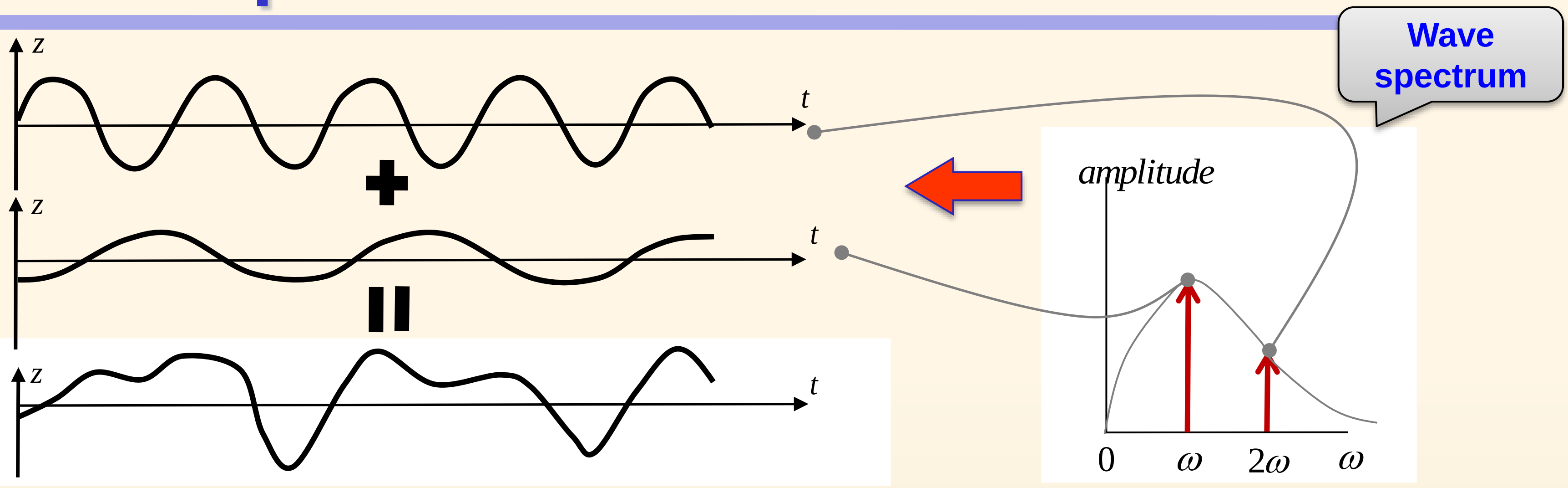

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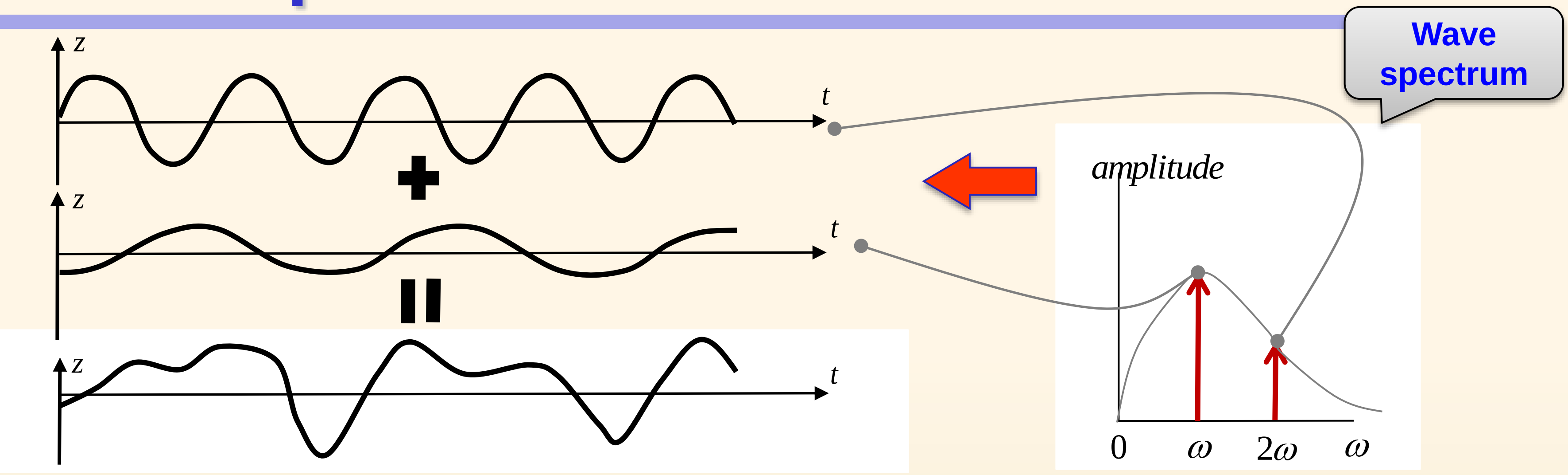

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<u>Standard Wave Spectrum</u>

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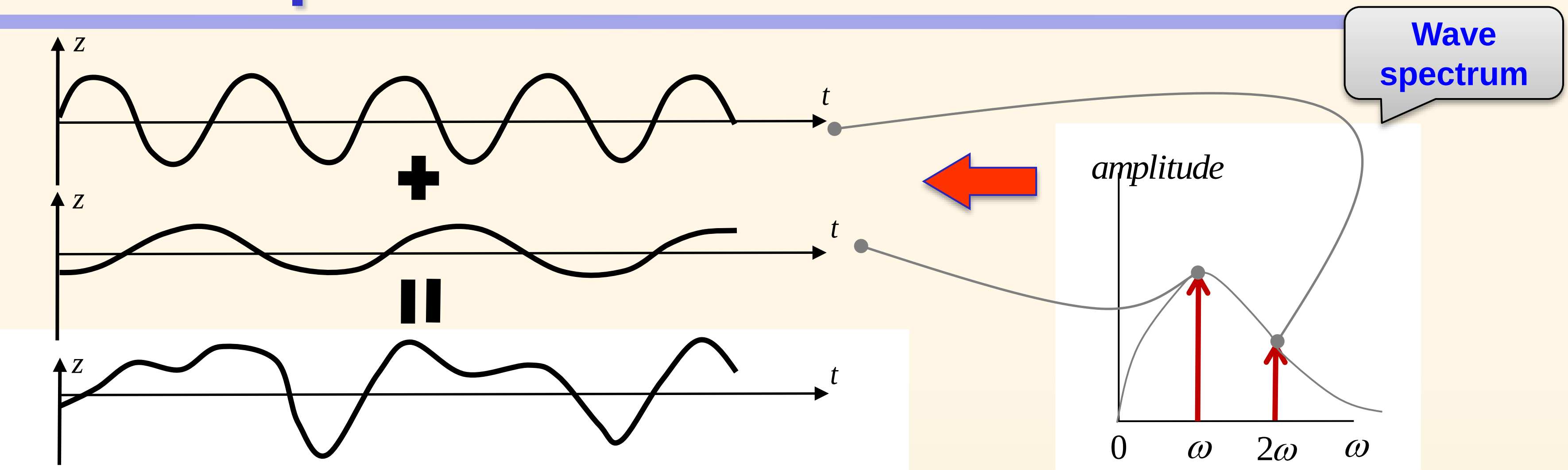

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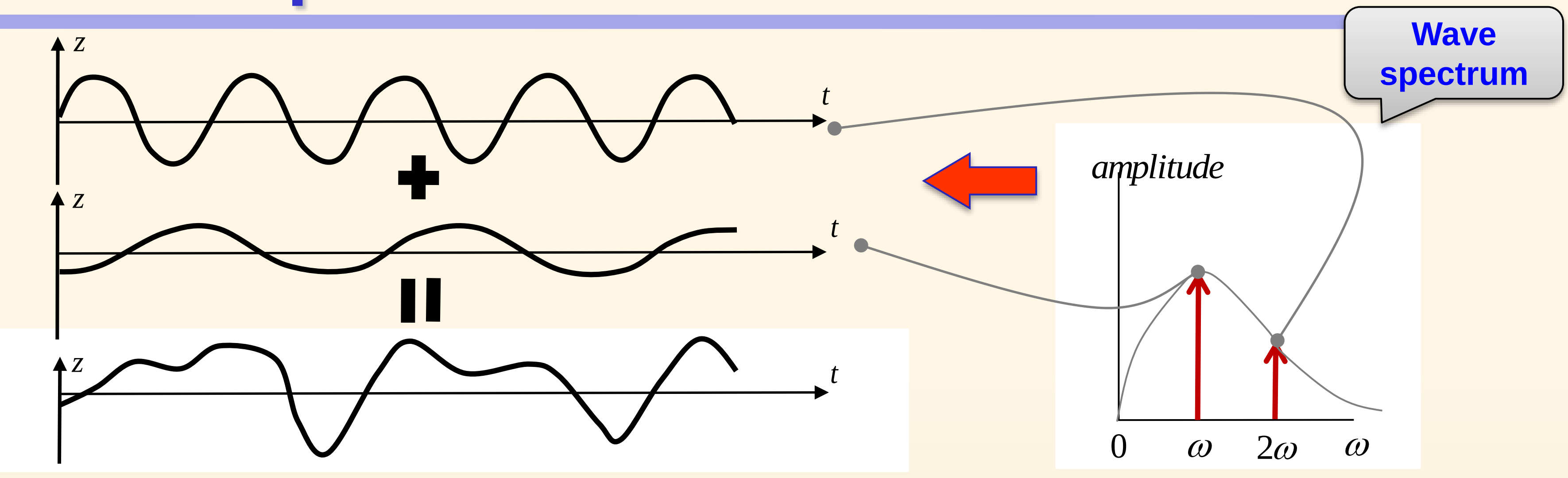

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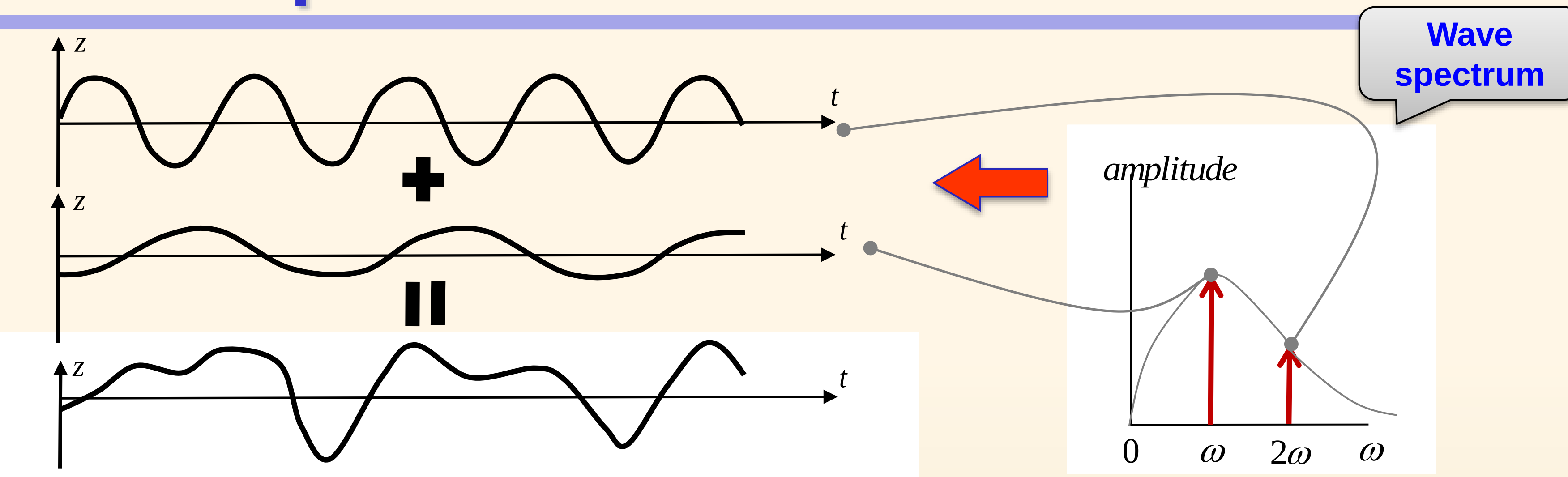

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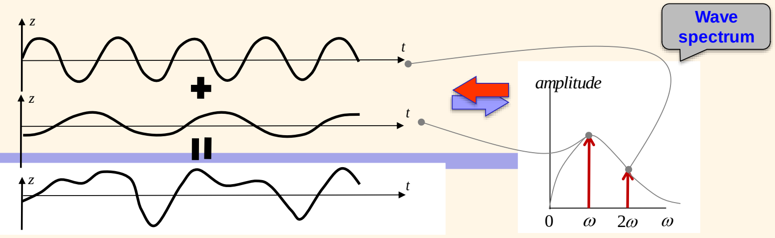

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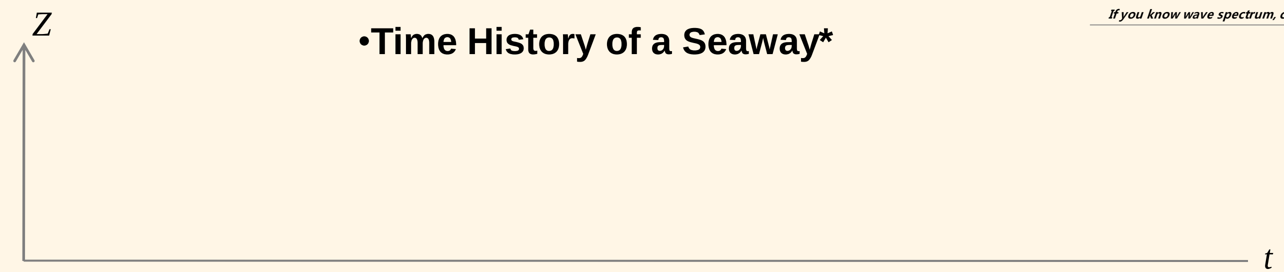
How to use Standard Wave Spectrum

•Time History of a Seaway*



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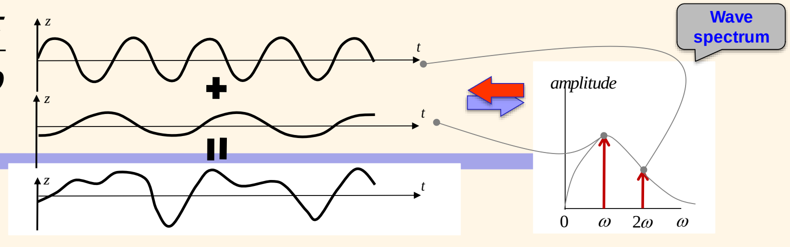

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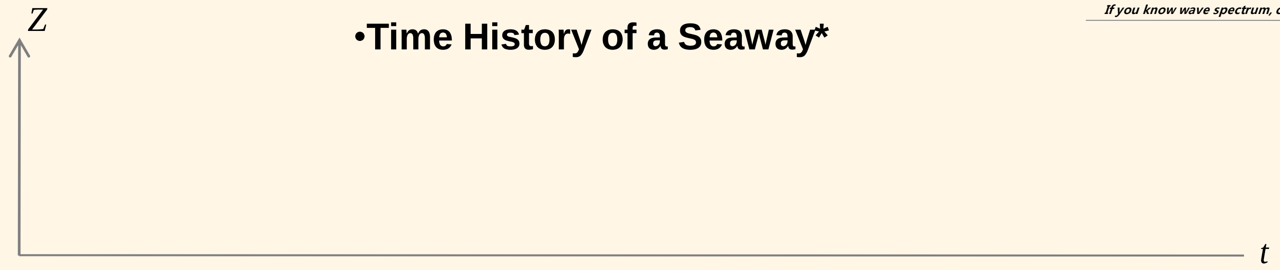

Measuring

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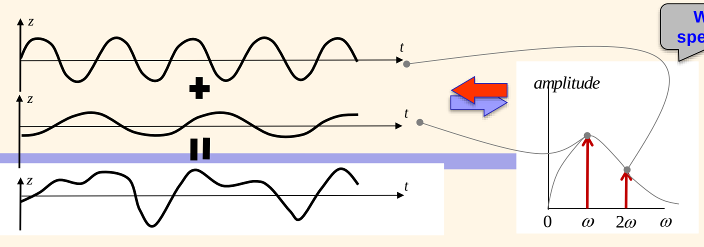

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Wave spectrum

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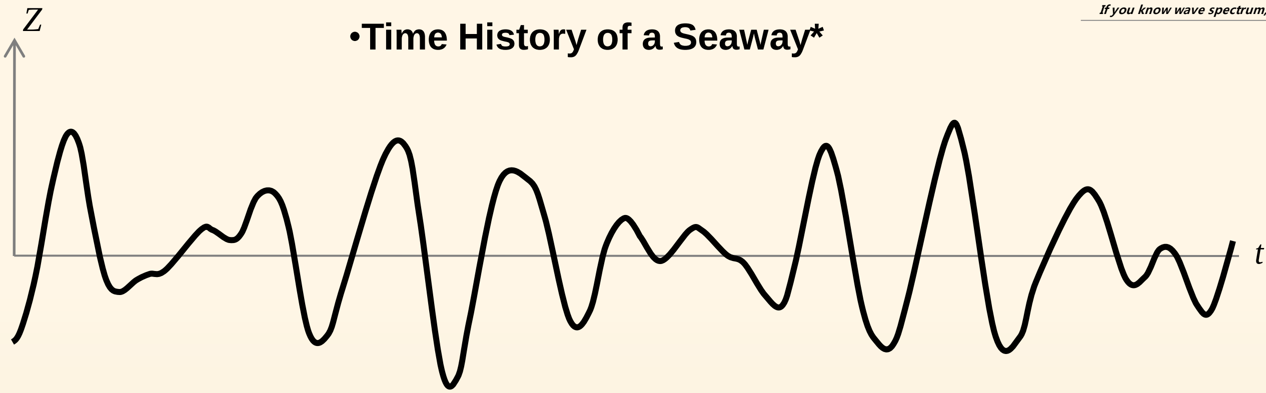

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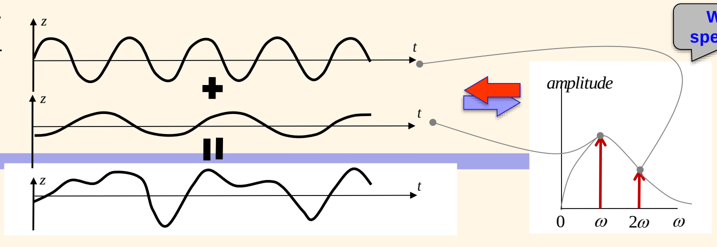

How to use Standard Wave Spectrum

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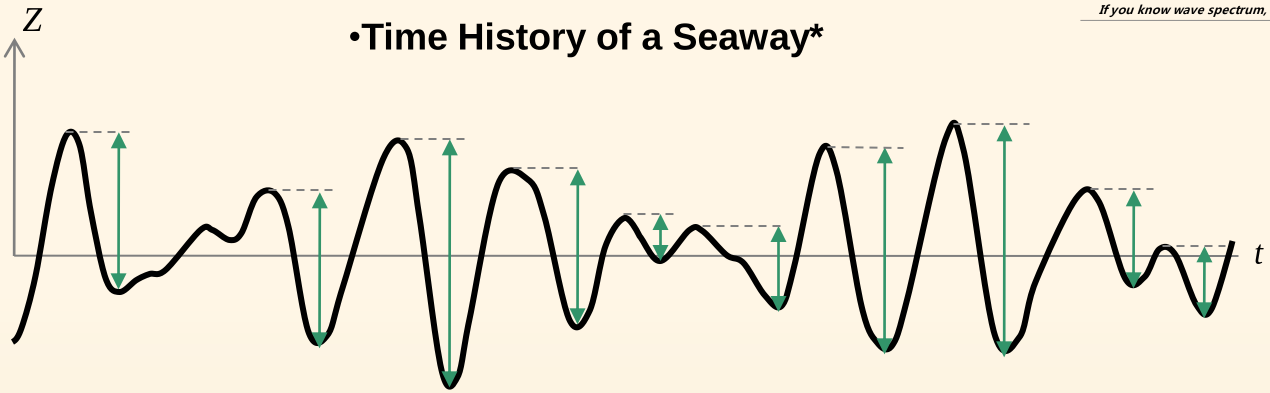

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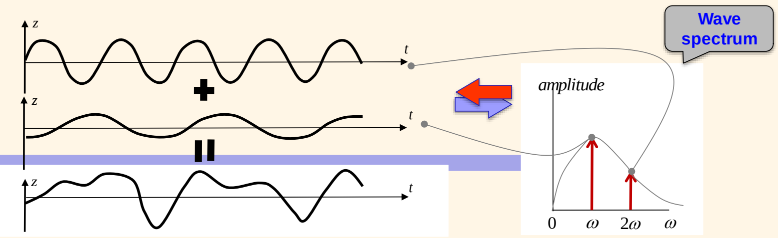

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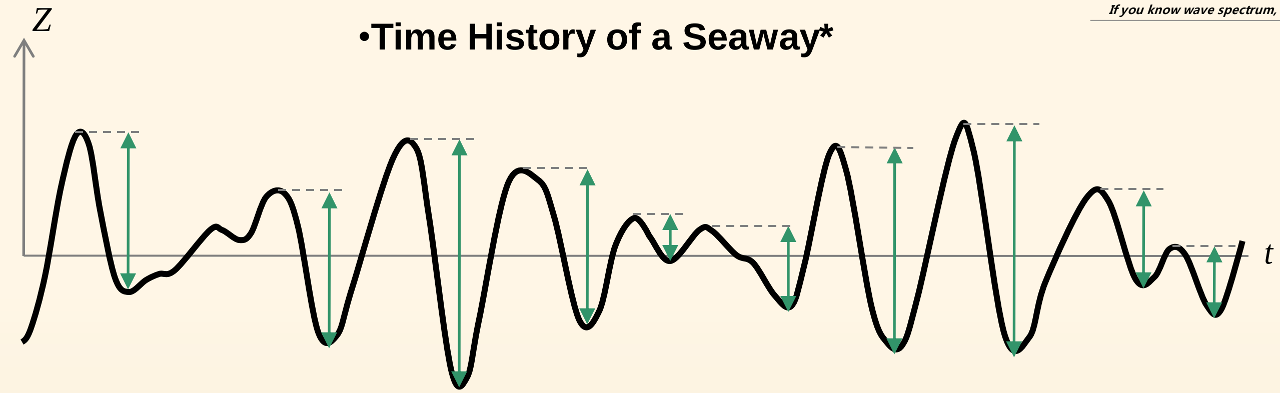

Measuring

Wave Height

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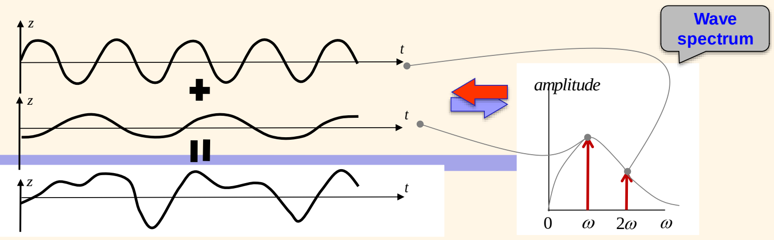

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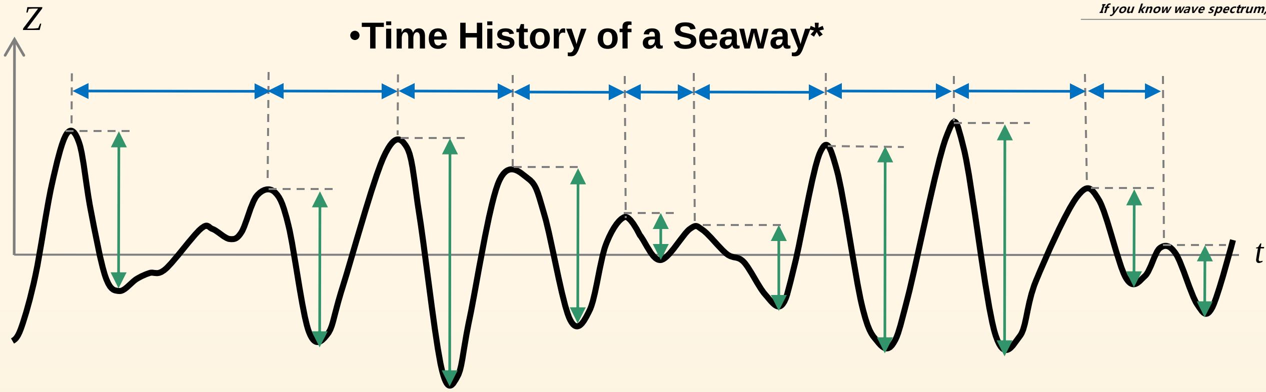

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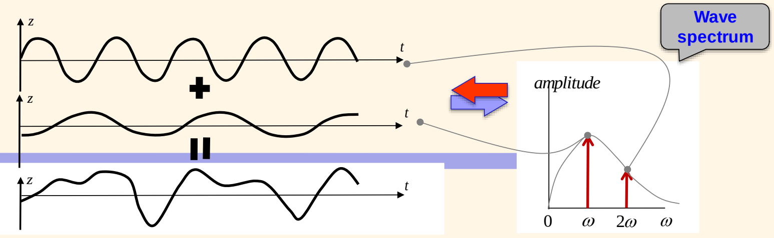

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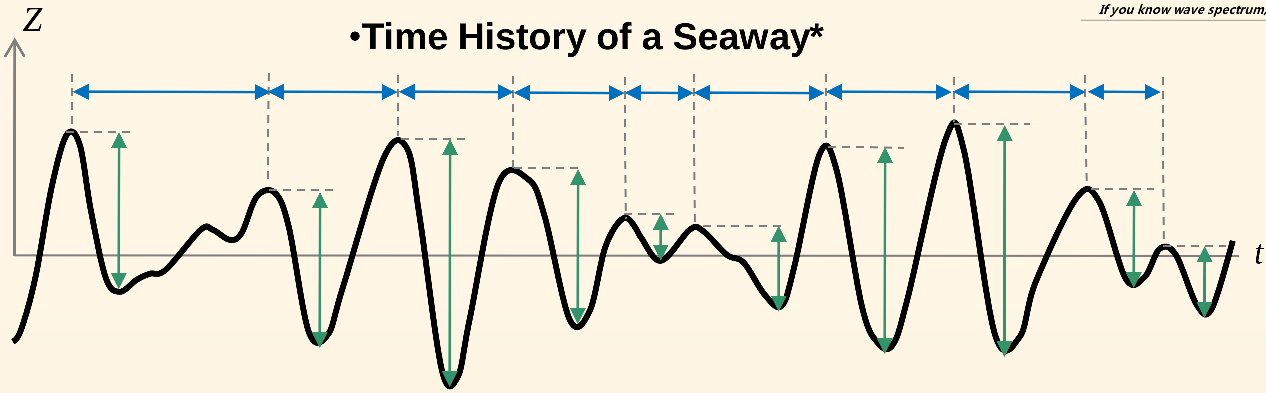

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Peak-to-Peak Wave Period

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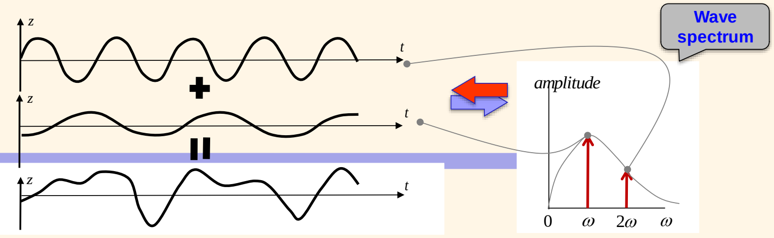

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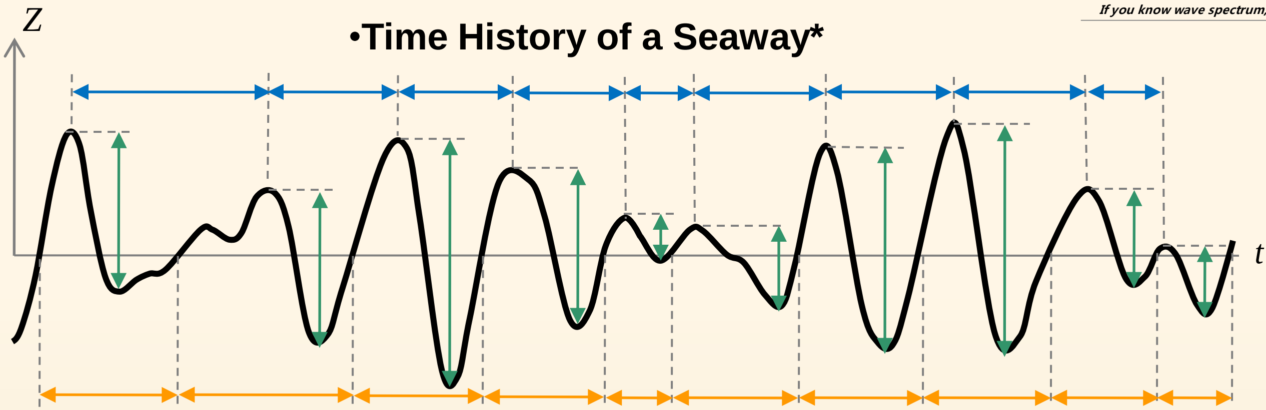

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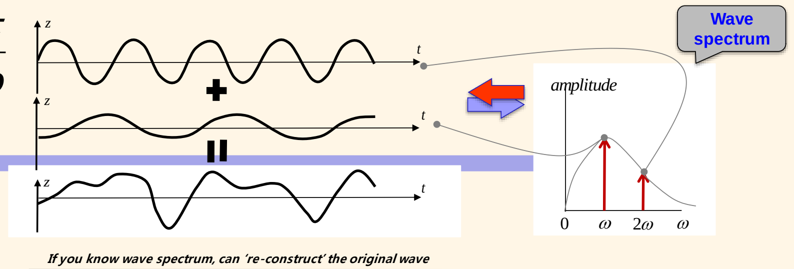

How to use Standard Wave Spectrum

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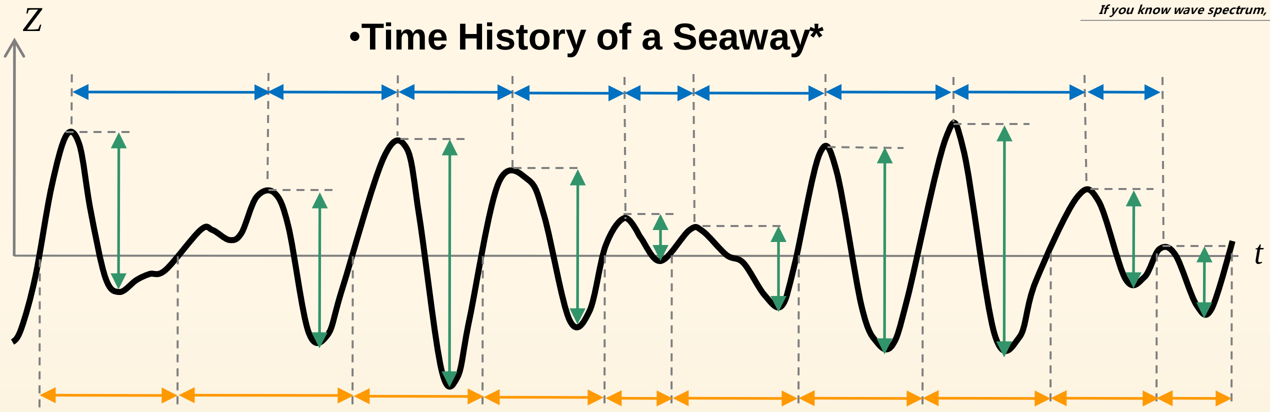

Measuring

- Wave Height
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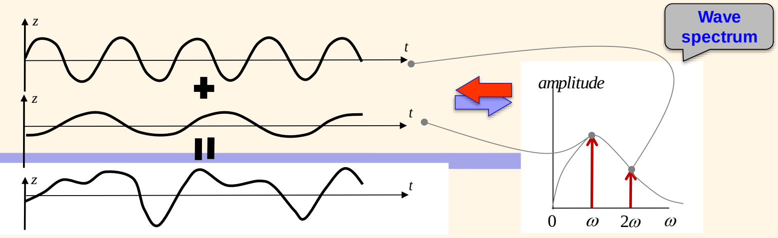

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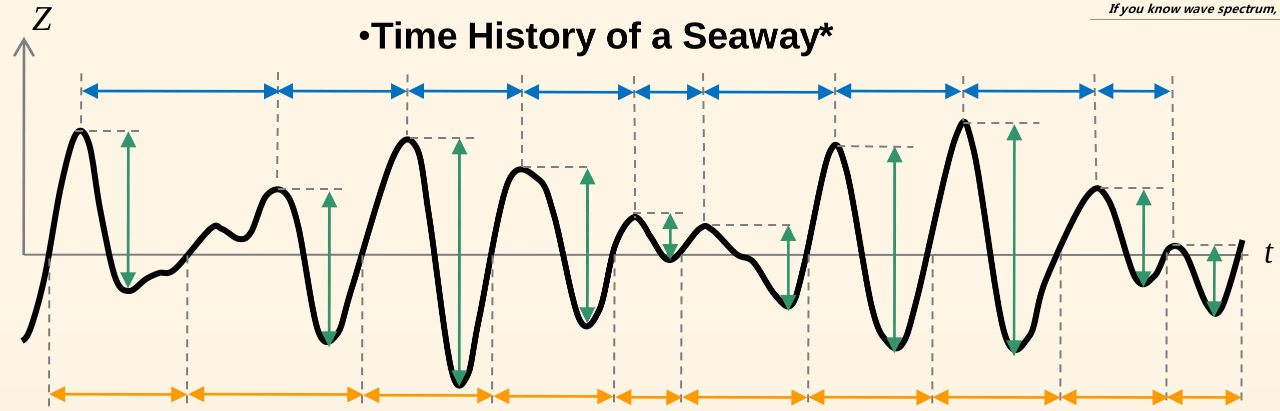
How to use Standard Wave Spectrum

•Time History of a Seaway*



Measuring

- Wave Height
- Peak-to-Peak Wave Period
- Zero-Crossing Wave Period

Parameters

•Bretschneider (Modified Two-Parameter Pierson-Moskowitz/ ITTC/ ISSC) Wave Spectrum*

$$S(\omega) = \frac{173 \cdot H_{1/3}^2}{T_1^4} \cdot \omega^{-5} \cdot \exp\left\{ \frac{-692}{T_1^4} \cdot \omega^{-4} \right\}$$

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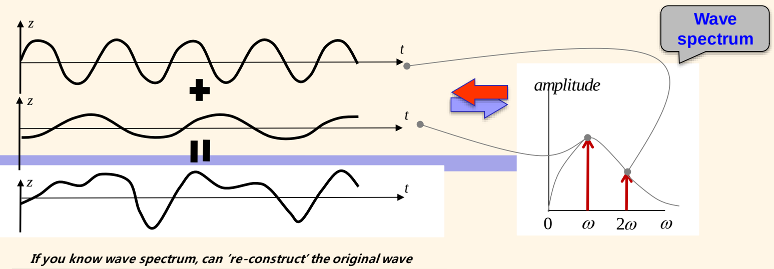

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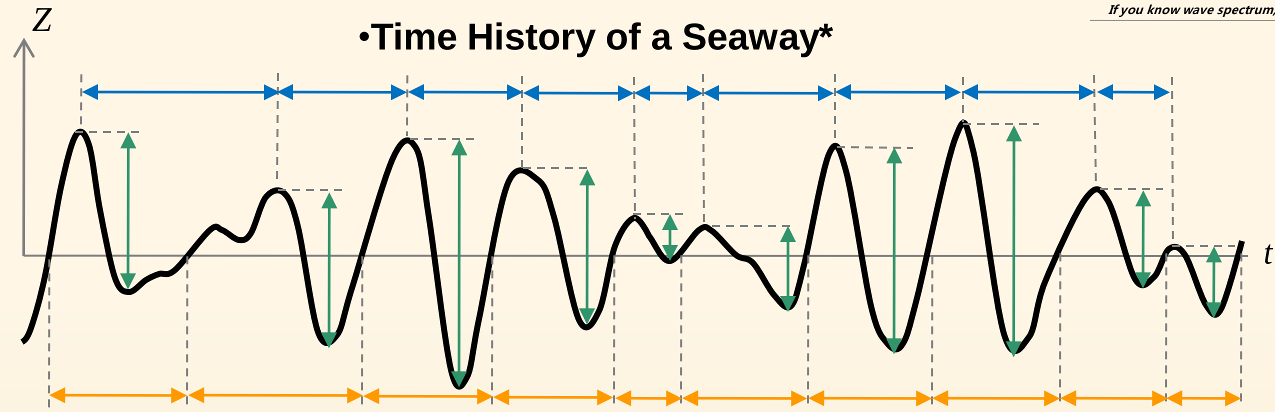

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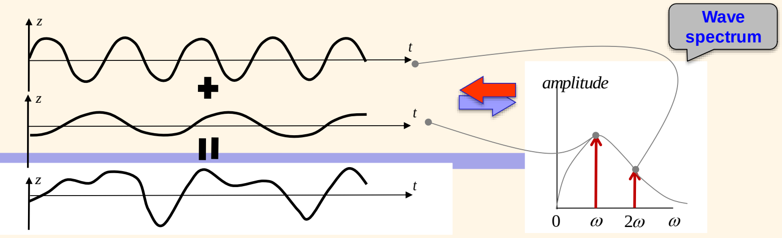

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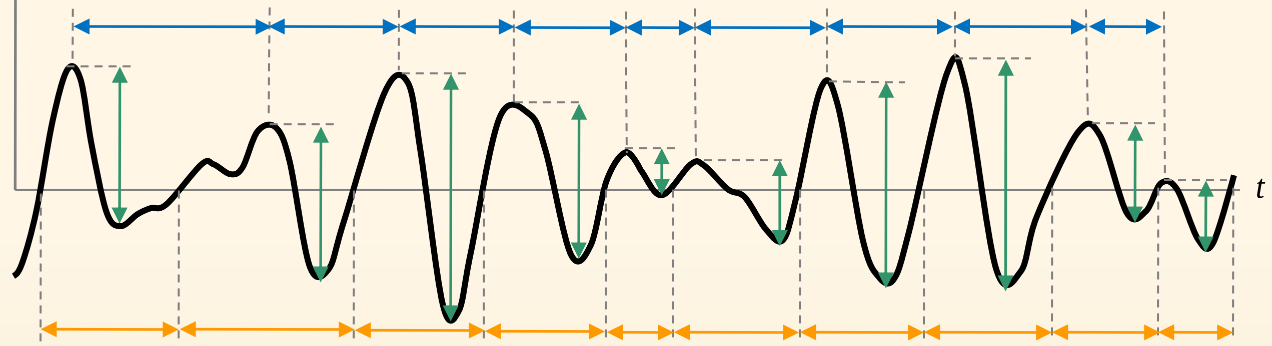

Measuring

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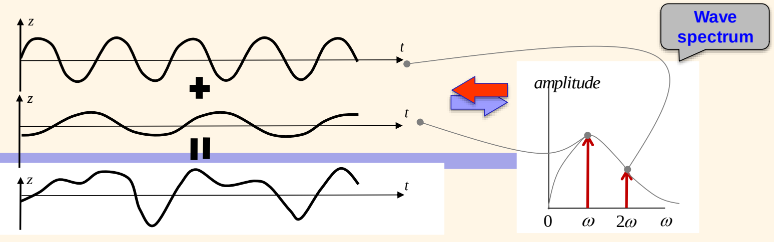

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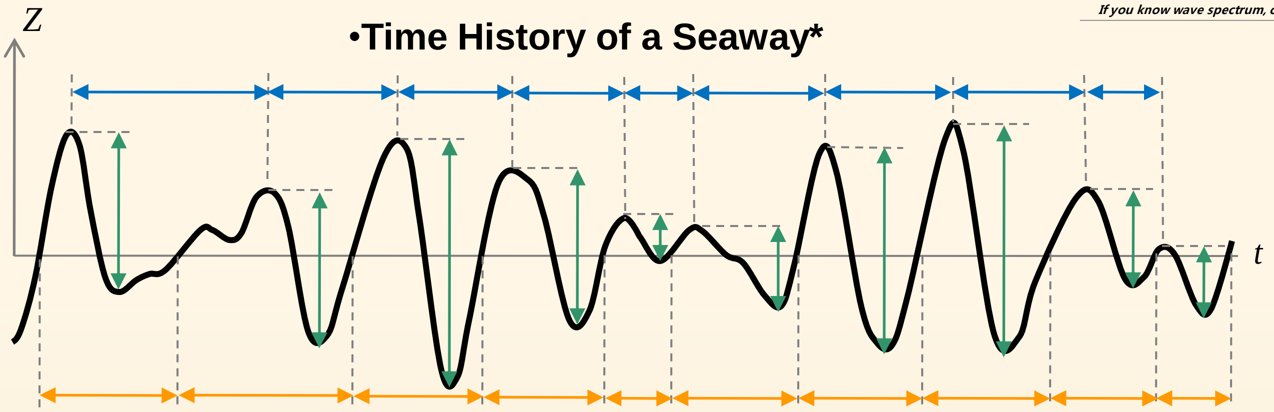

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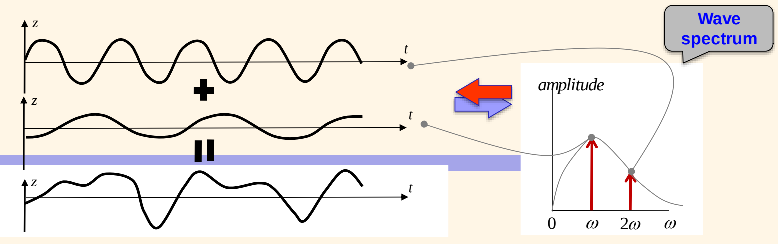

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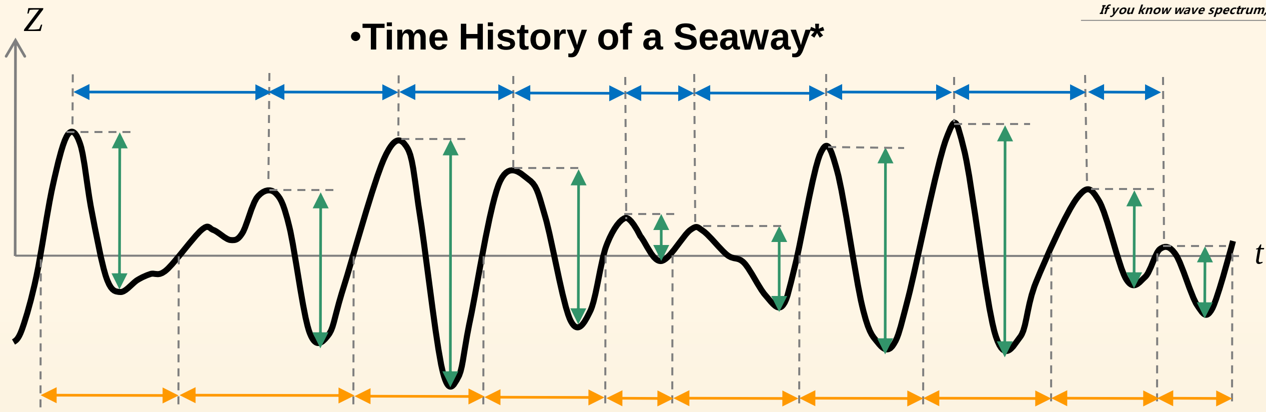

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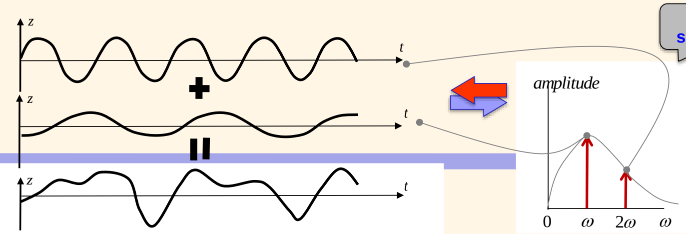

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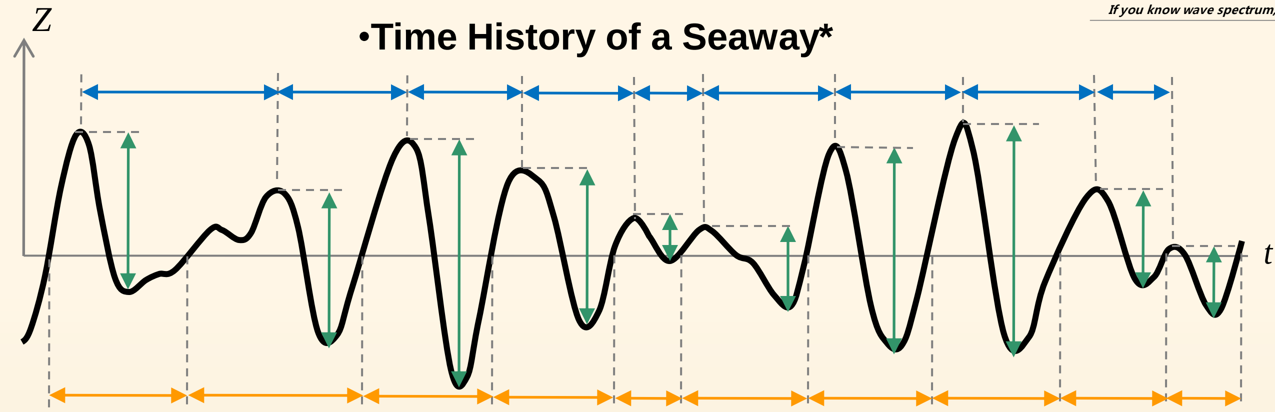
How to use Standard Wave Spectrum



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$$T_1 = 1.086 T_2, \quad T_0 = 1.408 T_2$$

•JONSWAP Wave Spectrum

$$T_1 = 0.834 T_0 = 1.073 T_2$$

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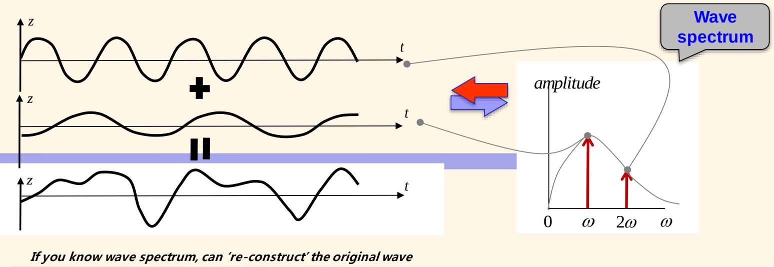
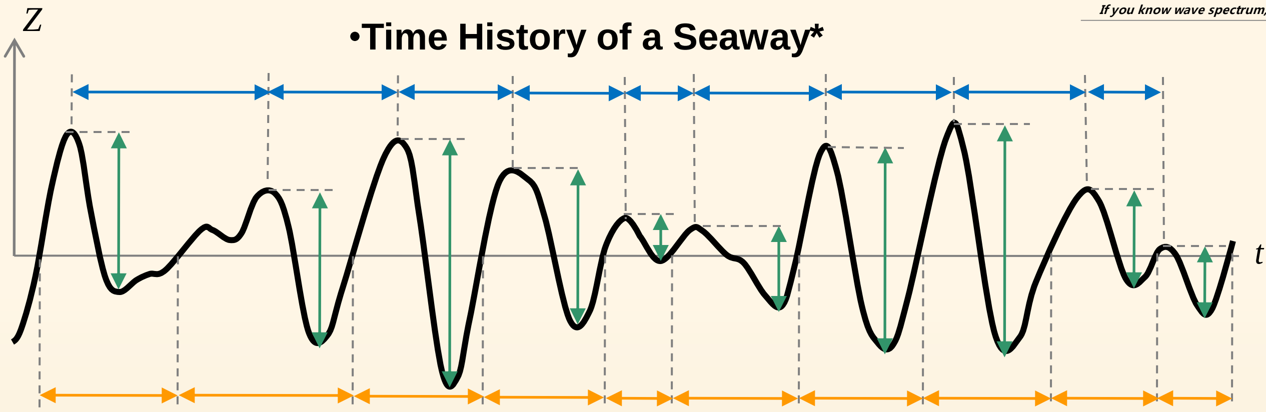


How to use Standard Wave Spectrum

•**Time History of a Seaway***



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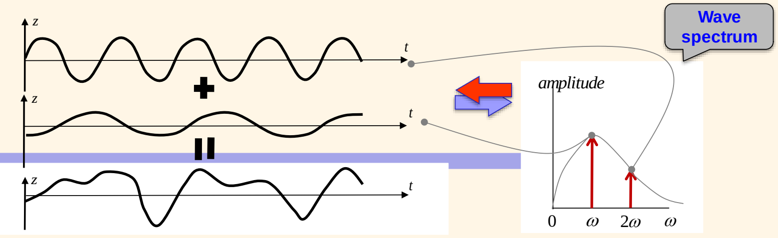

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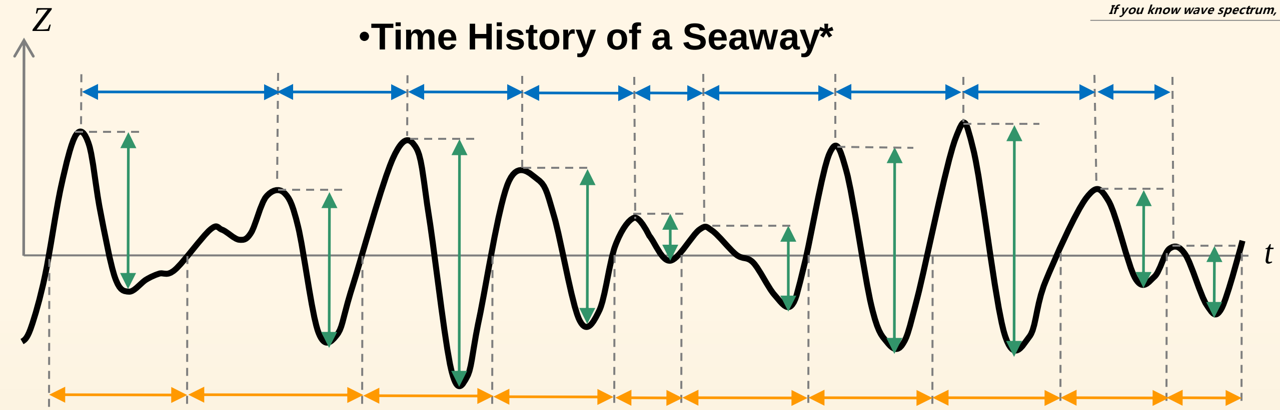

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2008_Fourier Transform(2)

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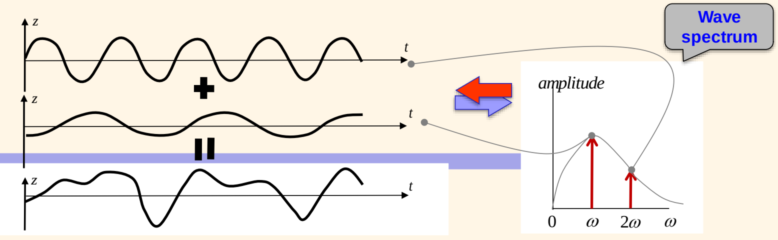

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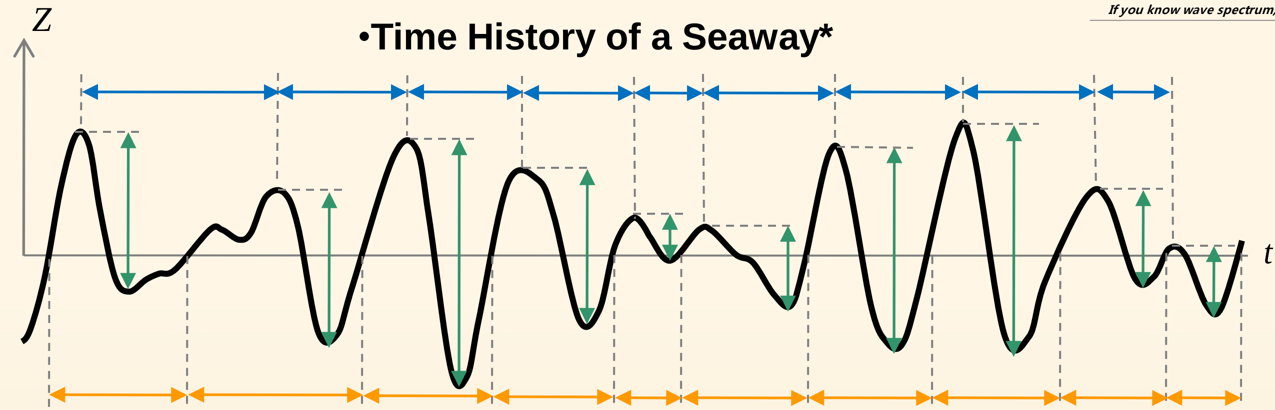

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Wave Spectrum

ζ_a:wave amplitude

$S_\zeta(\omega)$:energy density spectrum

$T_1,\ H_{1/3}$

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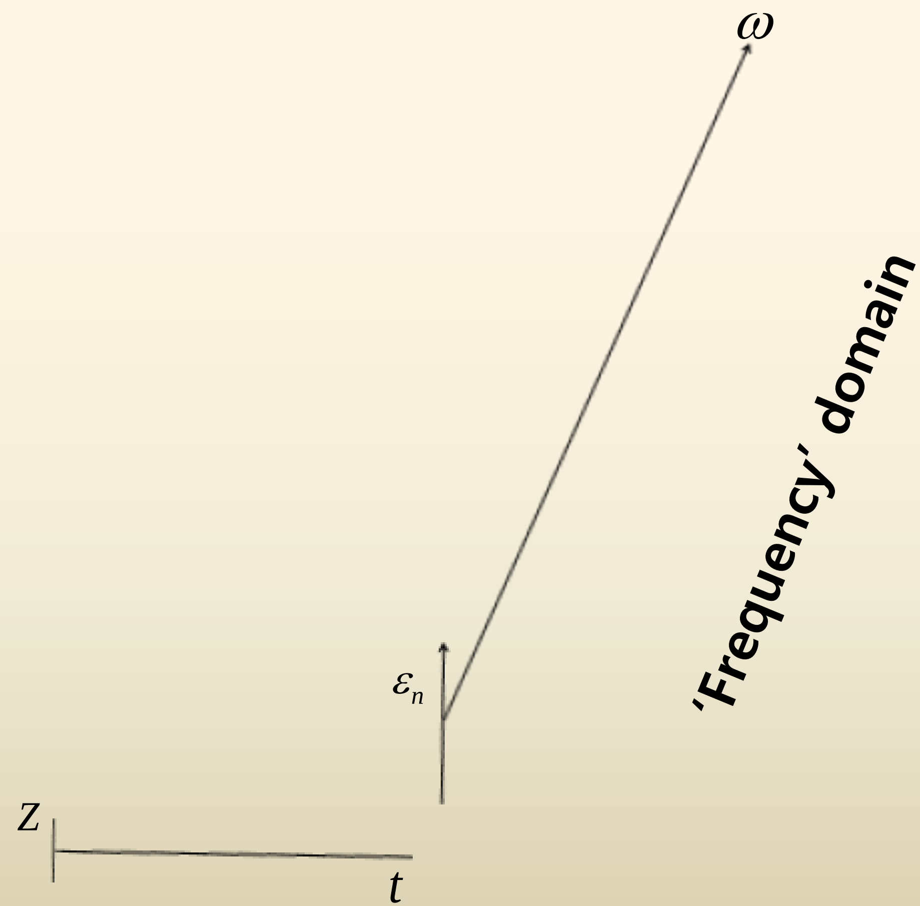

'Time' domain

Generated Wave Record

*Journee J. M. J., Massie W. W., Offshore Hydrodynamics, First Edition, Delft University of Technology, 2001, Figure 5.38

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Wave Spectrum

$T_1,\ H_{1/3}$

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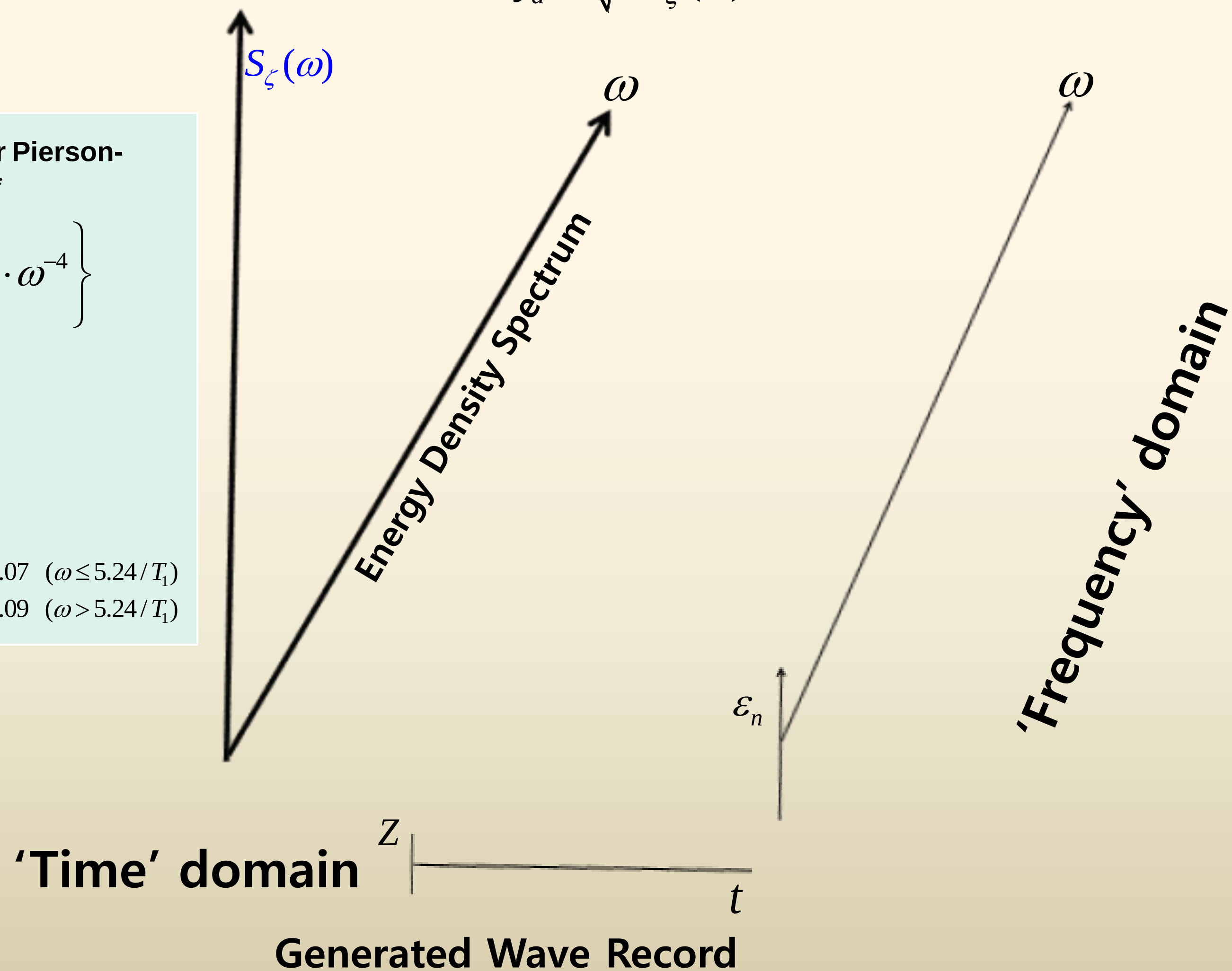

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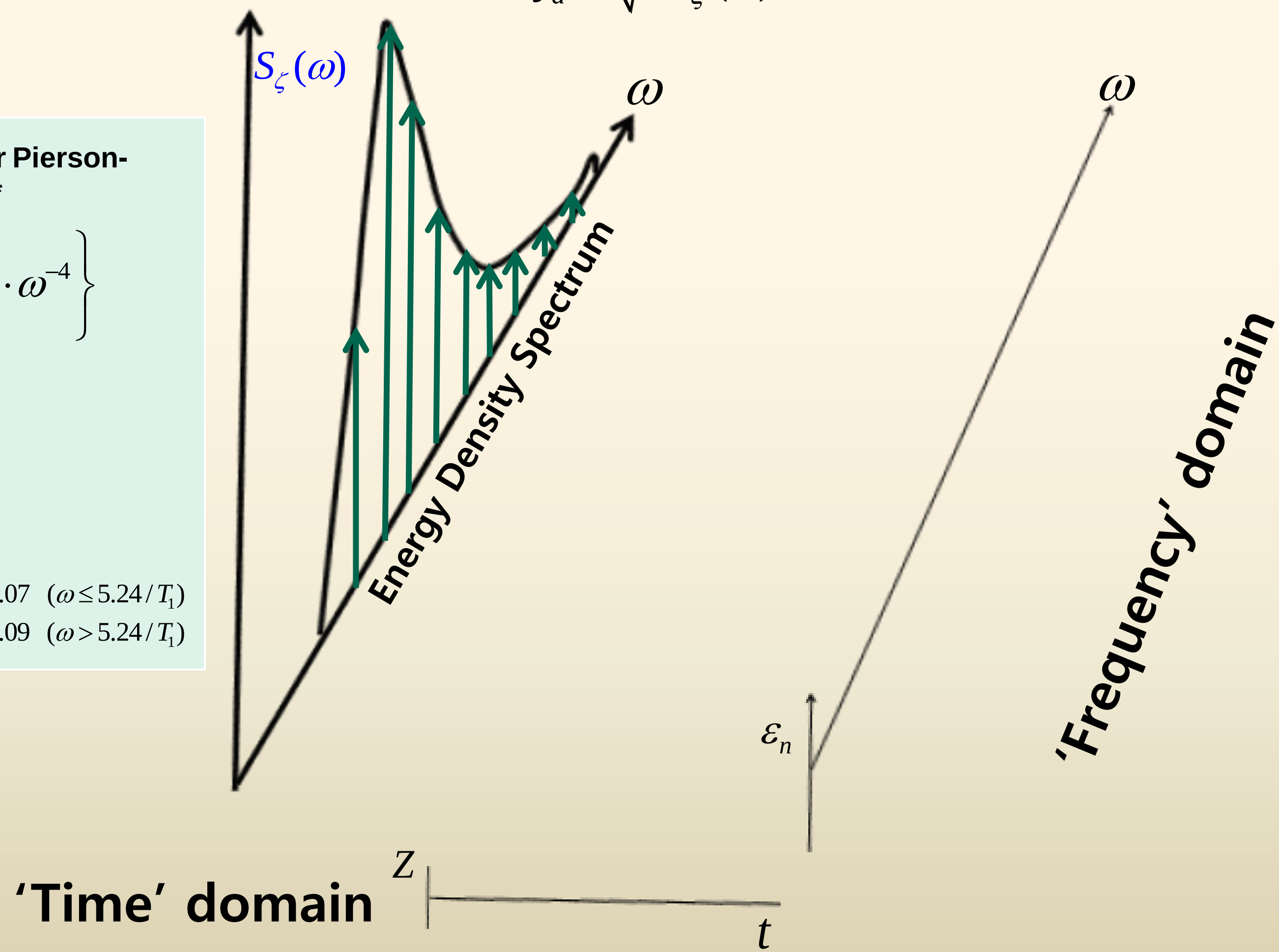

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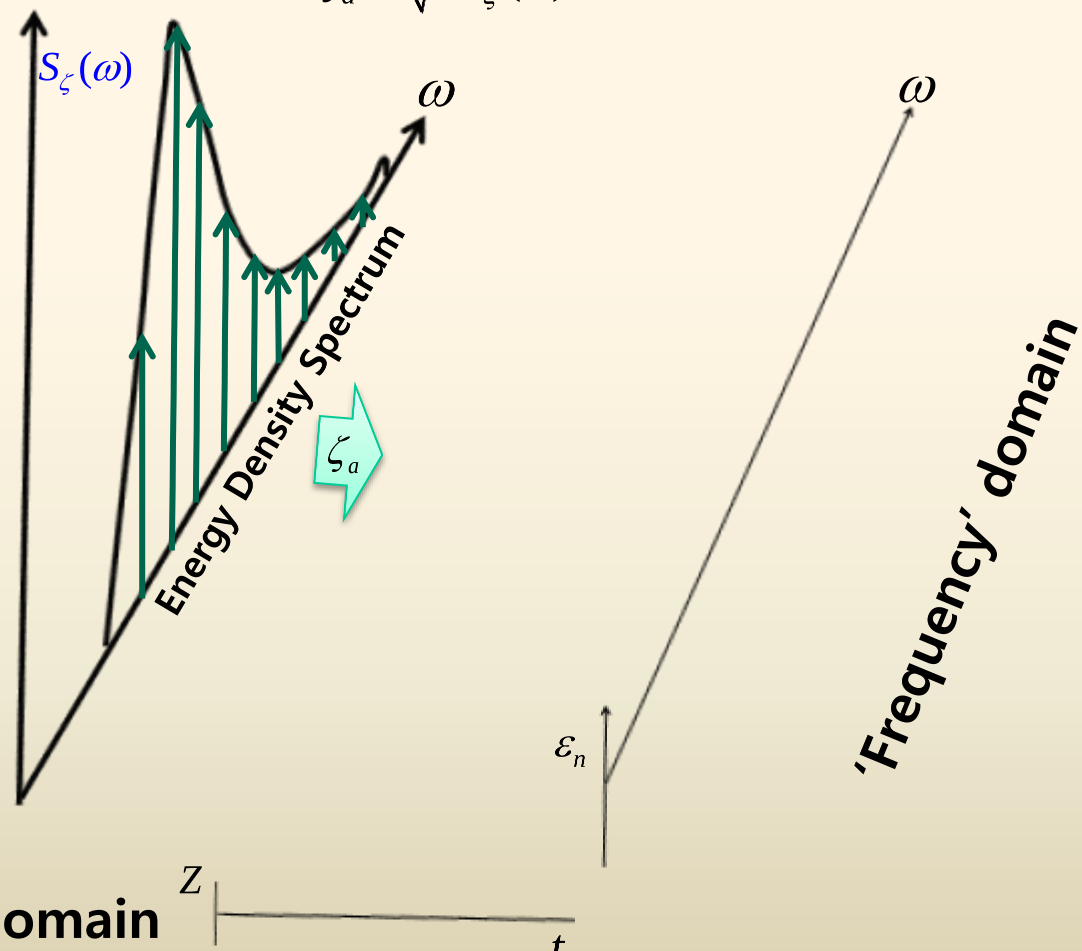

*Journee J. M. J., Massie W. W., Offshore Hydrodynamics, First Edition, Delft University of Technology, 2001, Figure 5.38

Wave Spectrum

ζ_a: wave amplitude

$S_\zeta(\omega)$: energy density spectrum

$T_1,\ H_{1/3}$

•**Bretschneider (Modified Two-Parameter Pierson-Moskowitz/ ITTC/ ISSC) Wave Spectrum***

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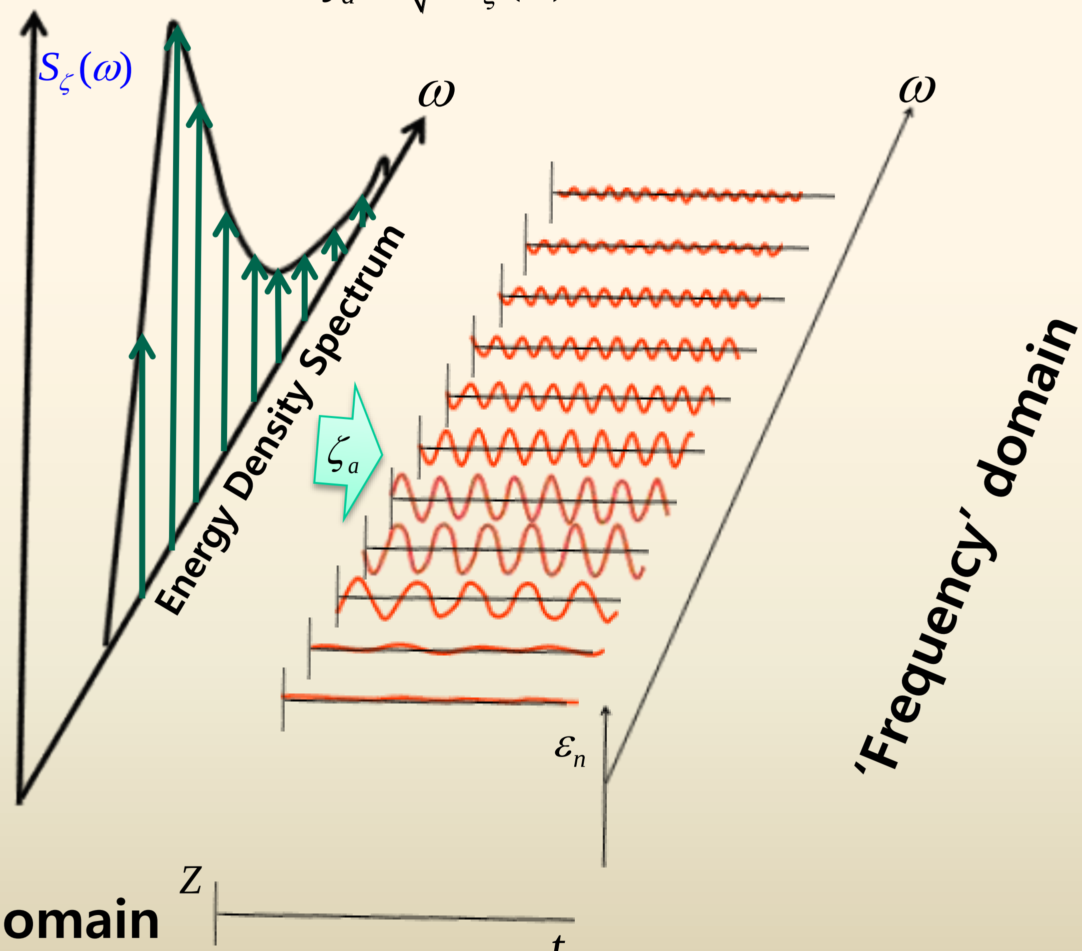

Generated Wave Record

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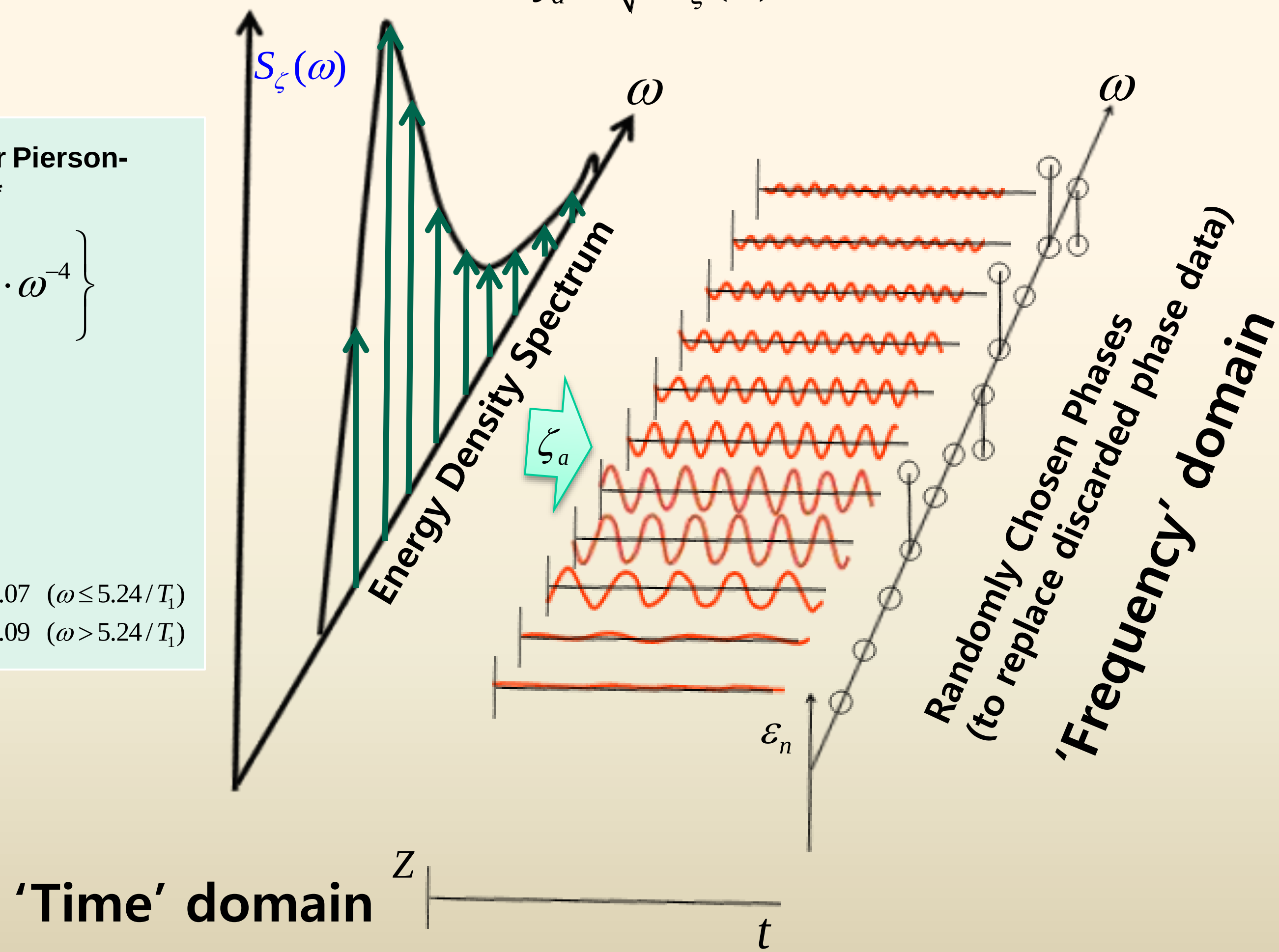

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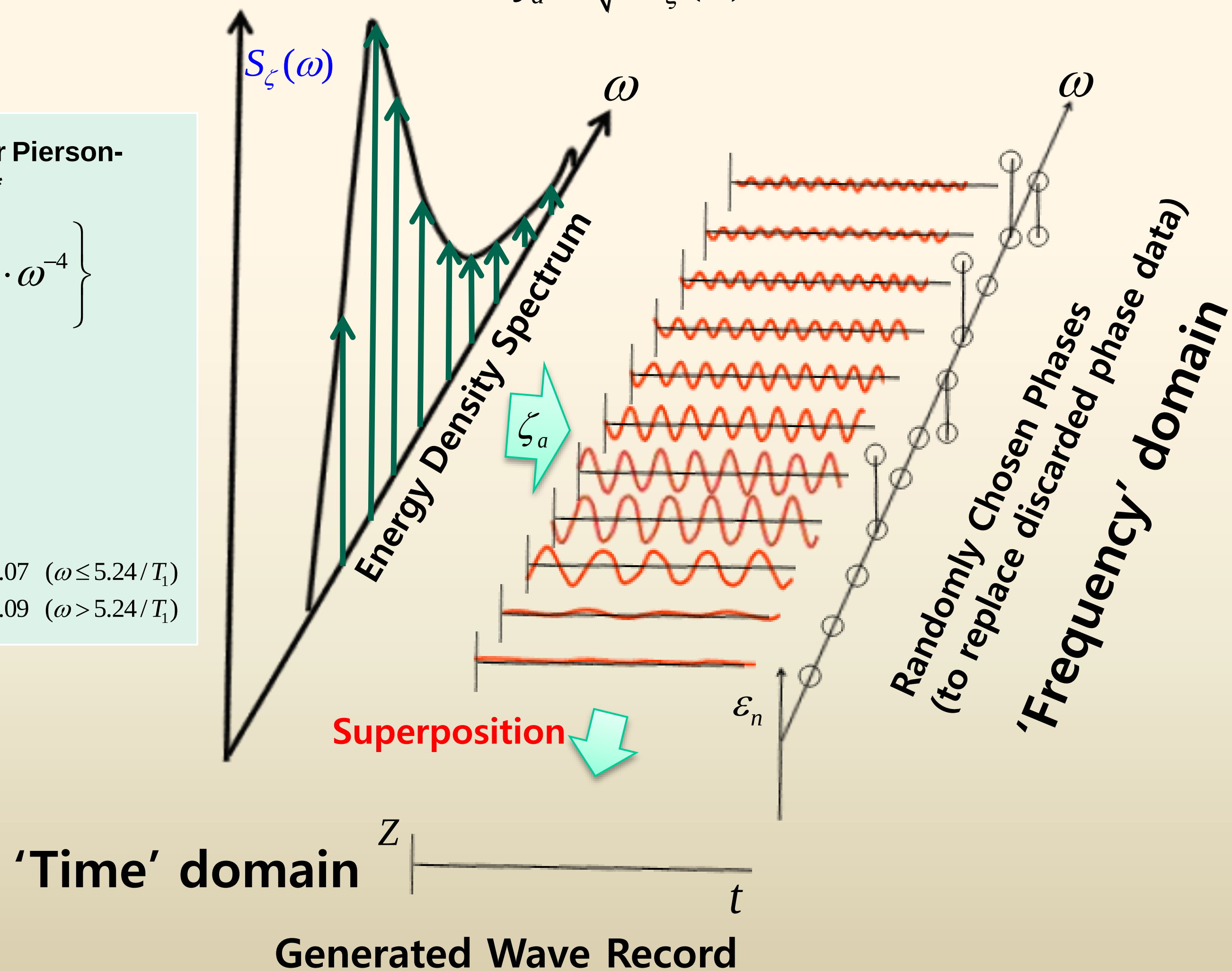

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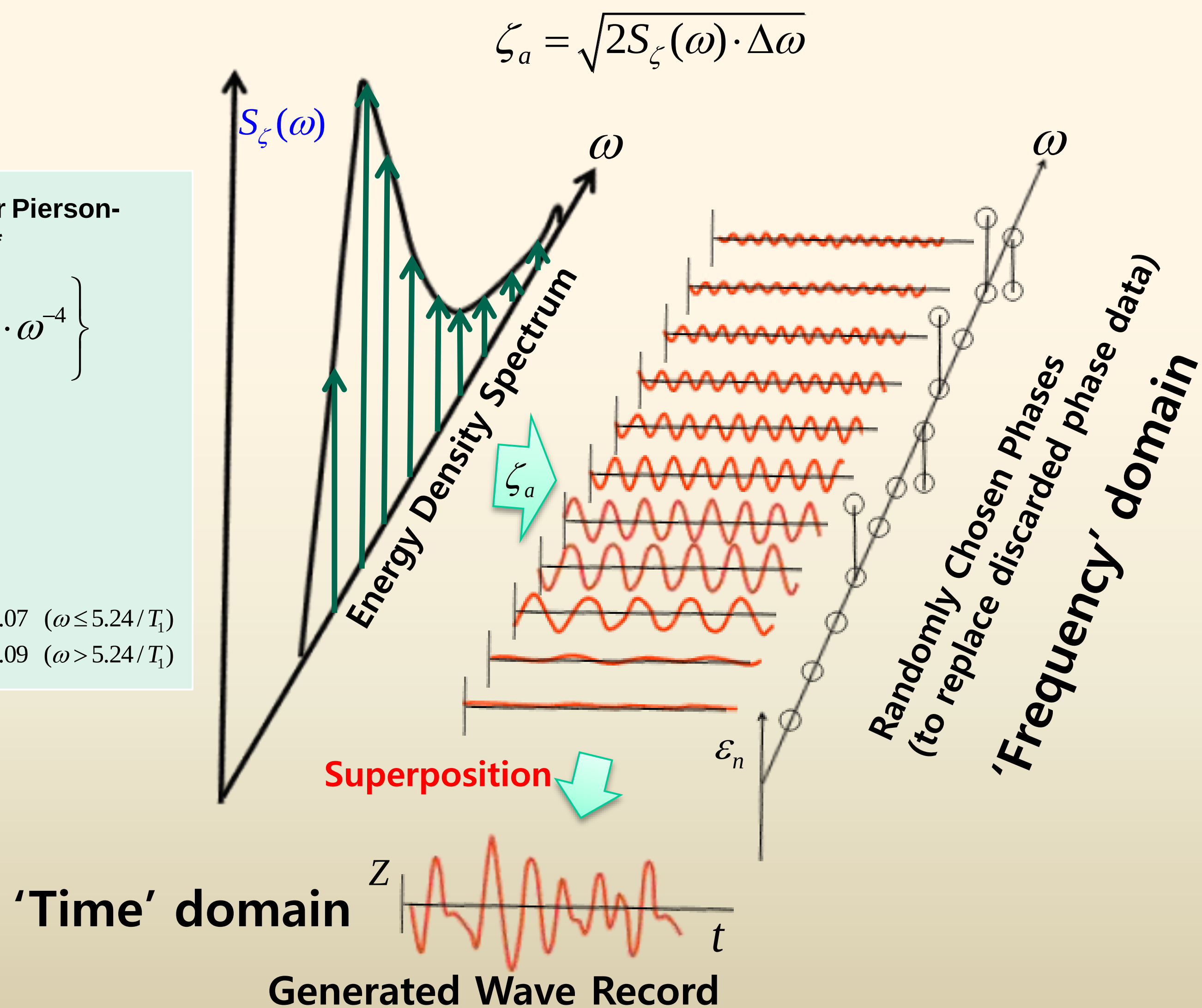

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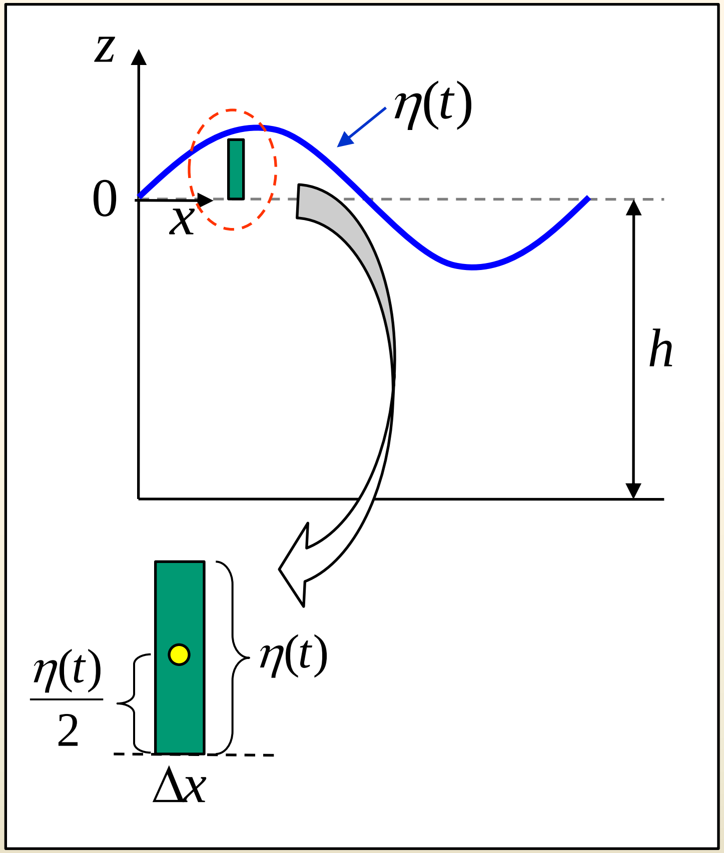
Wave Energy

Wave Energy

$$\eta = \eta_0 \sin(kx - \omega t)$$



(Q) 한 주기 파의 Potential Energy 합은?

(A) 0 이 아니다.

sin 파의 앞쪽 반주기 동안은 물입자를 들어올리는데 일을 하였고, 뒤의 반주기 동안은 내리는데 일을 하였음.

- 미소 부피에 대한 Potential Energy

$$dE = \rho g \eta(t) dx \times \frac{\eta(t)}{2} = \frac{\rho g \{\eta(t)\}^2}{2} dx$$

- 한 주기 파의 평균 Potential Energy

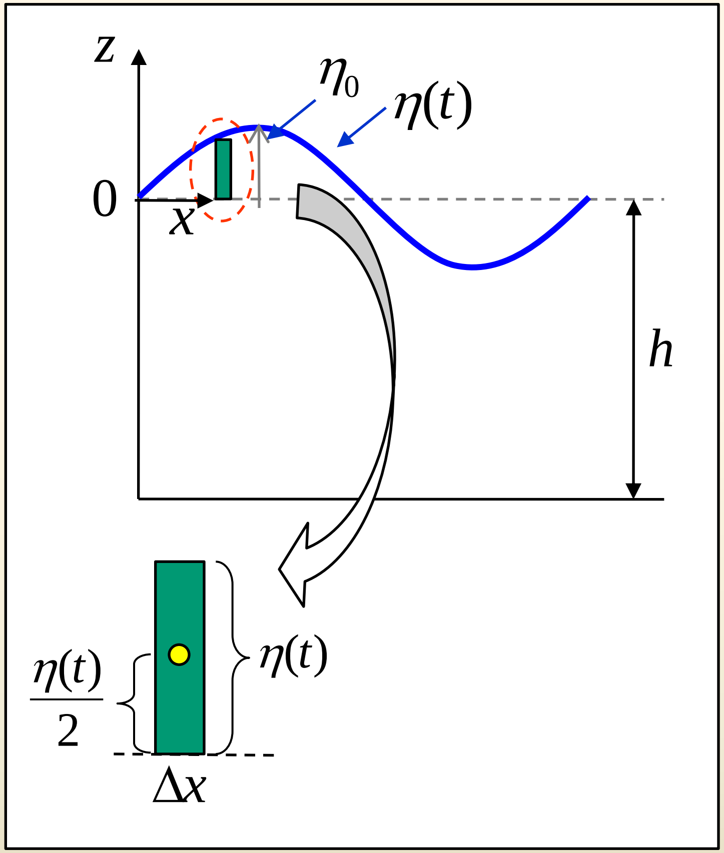
$$PE = \frac{1}{L}\int dE = \frac{1}{L}\int_0^L \frac{\rho g \{\eta(t)\}^2}{2} dx$$

$$= \frac{\rho g}{2L}\int_0^L \eta_0^2 \sin^2(kx - \omega t) dx = \left[\frac{1}{2}x - \frac{2}{k}\sin(kx - wt)\right]_0^L$$

$$= \frac{\rho g}{2L} \cdot \frac{\eta_0^2 L}{2} = \frac{1}{4}\rho g \eta_0^2$$

Wave Energy

Wave velocity potential* $\Phi = \frac{g\eta_0}{\omega} \cdot \frac{\cosh k(z+h)}{\cosh kh} \cdot \sin(kx - \omega t)$



- 미소 부피에 대한 Kinetic Energy

$$dE = \frac{dm}{2} \cdot \left|\nabla\Phi\right|^2 = \frac{\rho}{2}(u^2 + w^2)dzdx$$

$$u = \frac{\partial\Phi}{\partial x} = \frac{g\eta_0}{\omega} \cdot \frac{\cosh k(z+h)}{\cosh kh} \cdot k\cos(kx - \omega t)$$

$$w = \frac{\partial\Phi}{\partial z} = \frac{g\eta_0}{\omega} \cdot k\frac{\sinh k(z+h)}{\cosh kh} \cdot \sin(kx - \omega t)$$

$$dE = \frac{\rho}{2} \cdot \left(\frac{g\eta_0 k}{\omega\cosh kh}\right)^2 \begin{pmatrix} \cosh^2 k(z+h)\cos^2(kx - \omega t) \\ +\sinh^2 k(z+h)\sin^2(kx - \omega t) \end{pmatrix} dzdx$$

*권순홍 외 공역, 해양환경하중, 동명사, 1990, p27 표2.1, (원저: Faltisen O.M., Sea Load on Ships and Offshore Structures)

Wave Energy

$$dE = \frac{\rho}{2} \cdot \left(\frac{g\eta_0 k}{\omega \cosh kh} \right)^2 \left(\cosh^2 k(z+h) \cos^2(kx - \omega t) + \sinh^2 k(z+h) \sin^2(kx - \omega t) \right) dzdx$$

$$\cosh^2 k(z+h) \cos^2(kx-\omega t) + \sinh^2 k(z+h) \sin^2(kx-\omega t)$$

$$= \cosh^2 k(z+h) \cos^2(kx-\omega t) + \sinh^2 k(z+h) \left\{ 1 - \cos^2(kx-\omega t) \right\}$$

$$= \cosh^2 k(z+h) \cos^2(kx-\omega t) - \sinh^2 k(z+h) \cos^2(kx-\omega t) + \sinh^2 k(z+h)$$

$$= \left\{ \cosh^2 k(z+h) - \sinh^2 k(z+h) \right\} \cos^2(kx-\omega t) + \sinh^2 k(z+h)$$

$$= \cos^2(kx-\omega t) + \sinh^2 k(z+h)$$

$$= \frac{1 + \cos 2(kx-\omega t)}{2} + \frac{\cosh 2k(z+h) - 1}{2}$$

$$= \frac{\cos 2(kx-\omega t) + \cosh 2k(z+h)}{2}$$

$$dE = \frac{\rho}{2} \cdot \left(\frac{g\eta_0 k}{\omega \cosh kh} \right)^2 \left(\frac{\cos 2(kx-\omega t) + \cosh 2k(z+h)}{2} \right) dzdx$$

Wave Energy

- 미소 부피에 대한 Kinetic Energy

$$dE=\frac{\rho}{2}\cdot\left(\frac{g\eta_0 k}{\omega\cosh kh}\right)^2\left(\frac{\cos 2(kx-\omega t)+\cosh 2k(z+h)}{2}\right)dzdx$$

- 한 주기 파의 평균 Kinetic Energy

$$KE=\frac{1}{L}\int dE=\frac{1}{L}\int_0^L\int_{-h}^0\frac{\rho}{2}\left(\frac{g\eta_0 k}{\omega\cosh kh}\right)^2\left(\frac{\cos 2(kx-\omega t)+\cosh 2k(z+h)}{2}\right)dzdx$$

$$=\frac{\rho}{4L}\left(\frac{g\eta_0 k}{\omega\cosh kh}\right)^2\int_0^L\int_{-h}^0\left[\cos 2(kx-\omega t)+\cosh 2k(z+h)\right]dzdx$$

$$=\frac{\rho}{4L}\cdot\frac{g^2\eta_0^2k^2}{\omega^2\cosh^2 kh}\int_0^L\left[z\cos 2(kx-\omega t)+\frac{\sinh 2k(z+h)}{2k}\right]_{-h}^0 dx$$

$$=\frac{\rho}{4L}\cdot\frac{g^2\eta_0^2k^2}{\omega^2\cosh^2 kh}\int_0^L\left[h\cos 2(kx-\omega t)+\frac{\sinh 2kh}{2k}\right]dx$$

$$=\frac{\rho}{4L}\cdot\frac{g^2\eta_0^2k^2}{\omega^2\cosh^2 kh}\left[\frac{h\cos 2(kx-\omega t)}{2k}+\frac{x\sinh 2kh}{2k}\right]_0^L$$

$$=\frac{\rho}{4L}\cdot\frac{g^2\eta_0^2k^2}{\omega^2\cosh^2 kh}\left[\frac{L\sinh 2kh}{2k}\right]$$

Wave Energy

(Continue)

$$\sinh 2kh = \frac{e^{2kh} - e^{-2kh}}{2} = \frac{\left(e^{kh} + e^{-kh}\right)\left(e^{kh} - e^{-kh}\right)}{2}$$

$$= 2\frac{\left(e^{kh} - e^{-kh}\right)}{2}\frac{\left(e^{kh} - e^{-kh}\right)}{2} = 2(\cosh kh)(\sinh kh)$$

$$KE = \frac{\rho}{4\cancel{L}} \cdot \frac{g^2 \eta_0^2 k^{\cancel{2}}}{\omega^2 \cosh^2 kh}\left[\frac{\cancel{L}\sinh 2kh}{2\cancel{k}}\right]$$

$$= \frac{\rho}{8} \cdot \frac{g^2 \eta_0^2 k}{\omega^2} \cdot \frac{\sinh 2kh}{\cosh^2 kh}$$

$$= \frac{\rho}{\cancel{8}} \cdot \frac{g^2 \eta_0^2 k}{\omega^2} \cdot \frac{\cancel{2}\cancel{(\cosh kh)}(\sinh kh)}{\cosh^{\cancel{2}} kh}$$

$$= \frac{\rho}{4} \cdot \frac{g^2 \eta_0^2 k}{\omega^2} \cdot \frac{\sinh kh}{\cosh kh}$$

$$= \frac{\rho}{4} \cdot g\eta_0^2 \cdot \frac{gk \tanh kh}{\omega^2}$$

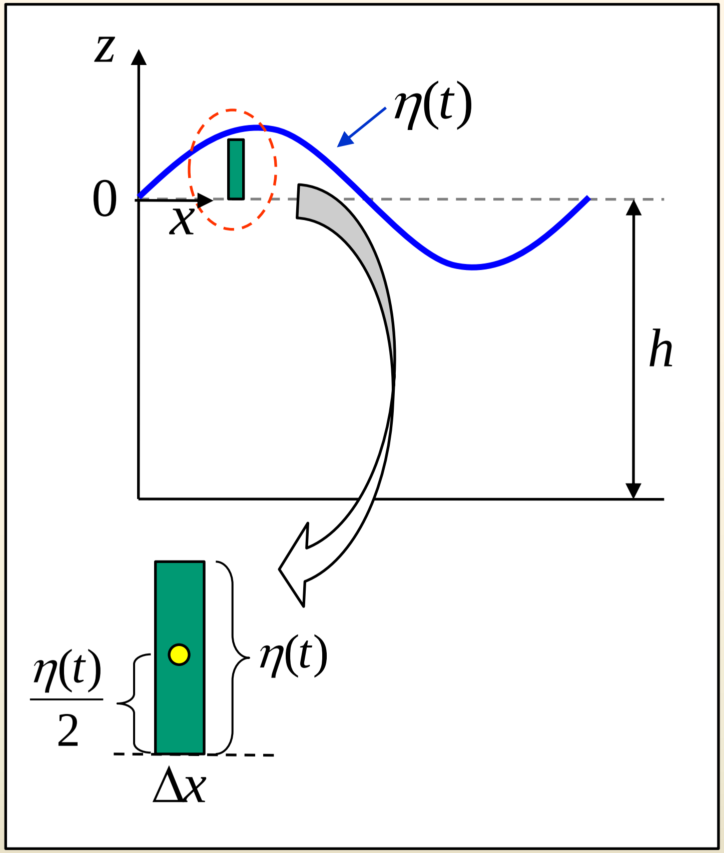
$$= \boxed{\frac{\rho g \eta_0^2}{4}}$$

Dispersion Relation

$$\omega^2 = gk \tanh kh$$

=> 한 주기 파의 평균 Kinetic Energy

Wave Energy



- 한 주기 파의 평균 Potential Energy

$$PE = \frac{1}{4}\rho g \eta_0^2$$

- 한 주기 파의 평균 Kinetic Energy

$$KE = \frac{1}{4}\rho g \eta_0^2$$

- 한 주기 파의 평균 Energy

$$E = PE + KE = \frac{1}{2}\rho g \eta_0^2$$

=> Energy는 (파고)^{2}에 비례함

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

Physical Interpretation : Spectrum

The nature of the representation (7) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies, called a spectral representation.

This name is suggested by optics, where light is such a superposition of colors (frequencies).

In (7), the "spectral density" $\hat{f}(\omega)$ measures the intensity of $f(x)$ in the frequency interval between w and $w+\Delta w$(Δw small, fixed). We claim that in connection with vibrations, the integral

$$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega$$

can be interpreted as the total energy of the physical system. Hence an integral of $\left|\hat{f}(\omega)\right|^2$ from a to b gives the contribution of the frequency w between a and b to the total energy.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

Physical Interpretation : Spectrum

$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega$: **total energy** of the physical system

To make this plausible, we begin with a mechanical system giving a single frequency, namely, the harmonic oscillator (mass on a spring)

$$my''+ky=0.$$ (Here we denote time t by x)

Multiplication by y' gives $my''y'+kyy'=0.$

Integrating with respect to x,

L.H.S: $\int m\frac{d}{dx}(v)\cdot v dx+\int ky\frac{dy}{dx}dx$

$y''=\frac{dv}{dx},\quad v=y'=\frac{dy}{dx},$

$$=\frac{1}{2}mv^2+\frac{1}{2}ky^2+c_1=c_2$$: R.H.S

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

$$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^{2}d\omega$$

Physical Interpretation : Spectrum

$$my''+ky=0,\quad \frac{1}{2}mv^{2}+\frac{1}{2}ky^{2}+c_{1}=c_{2}$$

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kinetic energy + potential energy = total energy of the system

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$$y(x)=a_1\cos\omega_0 x+b_1\sin\omega_0 x=c_1e^{i\omega_0 x}+c_{-1}e^{-i\omega_0 x},\quad \left(\omega_0^2=\frac{k}{m}\right)$$

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$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

$$my'' + ky = 0, \quad \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = E_0 = \text{const}$$

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$$y(x)=A+B,$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

$$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega$$

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$$y(x)=A+B,\quad y'(x)=v=A'+B'=i\omega_0c_1e^{i\omega_0 x}-i\omega_0c_{-1}e^{-i\omega_0 x}$$

$$=i\omega_0(A-B)$$

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$$=i\omega_0(A-B)$$

Substitution of v and y on the left side of the equation for E_0 gives

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

Physical Interpretation : Spectrum

$$my'' + ky = 0, \quad \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = E_0 = \text{const}$$

$$y(x) = a_1 \cos \omega_0 x + b_1 \sin \omega_0 x = c_1 e^{i\omega_0 x} + c_{-1} e^{-i\omega_0 x}, \quad \left(\omega_0^2 = \frac{k}{m} \right)$$

$$c_1 = \frac{a_1 - ib_1}{2}, \quad c_{-1} = \bar{c}_1 = \frac{a_1 + ib_1}{2}$$

Writng simply $A = c_1 e^{i\omega_0 x}, B = c_{-1} e^{-i\omega_0 x}$

$$y(x) = A + B, \quad y'(x) = v = A' + B' = i\omega_0 c_1 e^{i\omega_0 x} - i\omega_0 c_{-1} e^{-i\omega_0 x}$$

$$= i\omega_0 (A - B)$$

Substitution of v and y on the left side of the equation for E_0 gives

$$E_0 = \frac{1}{2} m (iw_0)^2 (A - B)^2 + \frac{1}{2} k (A + B)^2$$

Physical Interpretation : Spectrum

$$A = c_1 e^{i\omega_0 x}, B = c_{-1} e^{-i\omega_0 x}, \quad \left(\omega_0^2 = \frac{k}{m} \right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

$$E_0 = \frac{1}{2} m (i\omega_0)^2 (A - B)^2 + \frac{1}{2} k (A + B)^2$$

Physical Interpretation : Spectrum

$$A=c_1e^{i\omega_0 x},\ B=c_{-1}e^{-i\omega_0 x},\quad \left(\omega_0^2=\frac{k}{m}\right)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$

$$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega$$

$$E_0=\frac{1}{2}m(i\omega_0)^2(A-B)^2+\frac{1}{2}k(A+B)^2$$

$$=\frac{1}{2}m\omega_0^2\left[-(A-B)^2\right]+\frac{1}{2}k(A+B)^2$$

Physical Interpretation : Spectrum

$$A = c_1 e^{i\omega_0 x}, B = c_{-1} e^{-i\omega_0 x}, \quad \left(\omega_0^2 = \frac{k}{m}\right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

$$E_0 = \frac{1}{2} m (i\omega_0)^2 (A - B)^2 + \frac{1}{2} k (A + B)^2$$

$$= \frac{1}{2} m\omega_0^2 \left[-(A - B)^2 \right] + \frac{1}{2} k (A + B)^2$$

$$= \frac{1}{2} k \left[-(A - B)^2 \right] + \frac{1}{2} k (A + B)^2$$

Physical Interpretation : Spectrum

$$A = c_1 e^{i\omega_0 x},\ B = c_{-1} e^{-i\omega_0 x}, \quad \left(\omega_0^2 = \frac{k}{m}\right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

$$E_0 = \frac{1}{2} m (i\omega_0)^2 (A - B)^2 + \frac{1}{2} k (A + B)^2$$

$$= \frac{1}{2} m\omega_0^2 \left[-(A - B)^2 \right] + \frac{1}{2} k (A + B)^2$$

$$= \frac{1}{2} k \left[-(A - B)^2 \right] + \frac{1}{2} k (A + B)^2$$

$$= \frac{1}{2} k \left[(A + B)^2 - (A - B)^2 \right]$$

Physical Interpretation : Spectrum

$$A = c_1 e^{i\omega_0 x}, B = c_{-1} e^{-i\omega_0 x}, \quad \left(\omega_0^2 = \frac{k}{m} \right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

$$E_0 = \frac{1}{2} m (i\omega_0)^2 (A - B)^2 + \frac{1}{2} k (A + B)^2$$

$$= \frac{1}{2} m\omega_0^2 \left[-(A - B)^2 \right] + \frac{1}{2} k (A + B)^2$$

$$= \frac{1}{2} k \left[-(A - B)^2 \right] + \frac{1}{2} k (A + B)^2$$

$$= \frac{1}{2} k \left[(A + B)^2 - (A - B)^2 \right]$$

$$= 2kAB$$

Physical Interpretation : Spectrum

$$A=c_1e^{i\omega_0 x},\; B=c_{-1}e^{-i\omega_0 x}, \quad \left(\omega_0^2=\frac{k}{m}\right)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$

$$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega$$

$$E_0=\frac{1}{2}m(i\omega_0)^2(A-B)^2+\frac{1}{2}k(A+B)^2$$

$$=\frac{1}{2}m\omega_0^2\left[-(A-B)^2\right]+\frac{1}{2}k(A+B)^2$$

$$=\frac{1}{2}k\left[-(A-B)^2\right]+\frac{1}{2}k(A+B)^2$$

$$=\frac{1}{2}k\left[(A+B)^2-(A-B)^2\right]$$

$$=2kAB \;=2kc_1e^{i\omega_0 x}\cdot c_{-1}e^{-i\omega_0 x}$$

Physical Interpretation : Spectrum

$$A = c_1 e^{i\omega_0 x}, B = c_{-1} e^{-i\omega_0 x}, \quad \left(\omega_0^2 = \frac{k}{m} \right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

$$E_0 = \frac{1}{2} m (i\omega_0)^2 (A-B)^2 + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} m\omega_0^2 \left[-(A-B)^2 \right] + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} k \left[-(A-B)^2 \right] + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} k \left[(A+B)^2 - (A-B)^2 \right]$$

$$= 2kAB \quad = 2k c_1 e^{i\omega_0 x} \cdot c_{-1} e^{-i\omega_0 x}$$

$$= 2k c_1 \cdot c_{-1}$$

Physical Interpretation : Spectrum

$$A = c_1 e^{i\omega_0 x},\ B = c_{-1} e^{-i\omega_0 x},\quad \left(\omega_0^2 = \frac{k}{m}\right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

$$E_0 = \frac{1}{2} m (i\omega_0)^2 (A-B)^2 + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} m\omega_0^2 \left[-(A-B)^2 \right] + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} k \left[-(A-B)^2 \right] + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} k \left[(A+B)^2 - (A-B)^2 \right]$$

$$= 2kAB \ = 2k c_1 e^{i\omega_0 x} \cdot c_{-1} e^{-i\omega_0 x}$$

$$= 2k c_1 \cdot c_{-1}$$

$$= 2k \left| c_1 \right|^2$$

Physical Interpretation : Spectrum

$$A = c_1 e^{i\omega_0 x}, B = c_{-1} e^{-i\omega_0 x}, \quad \left(\omega_0^2 = \frac{k}{m} \right)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega$$

$$\int_{-\infty}^{\infty} \left| \hat{f}(\omega) \right|^2 d\omega$$

$$E_0 = \frac{1}{2} m (i\omega_0)^2 (A-B)^2 + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} m\omega_0^2 \left[-(A-B)^2 \right] + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} k \left[-(A-B)^2 \right] + \frac{1}{2} k (A+B)^2$$

$$= \frac{1}{2} k \left[(A+B)^2 - (A-B)^2 \right]$$

$$= 2kAB \quad = 2kc_1 e^{i\omega_0 x} \cdot c_{-1} e^{-i\omega_0 x}$$

$$= 2kc_1 \cdot c_{-1}$$

$$= 2k |c_1|^2 \qquad \because c_{-1} = \overline{c_1}$$

Physical Interpretation : Spectrum

$$A=c_1 e^{i\omega_0 x},\ B=c_{-1}e^{-i\omega_0 x},\quad \left(\omega_0^2=\frac{k}{m}\right)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega$$

$$\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2 d\omega$$

$$E_0=\frac{1}{2}m(i\omega_0)^2(A-B)^2+\frac{1}{2}k(A+B)^2$$

$$=\frac{1}{2}m\omega_0^2\left[-(A-B)^2\right]+\frac{1}{2}k(A+B)^2$$

$$=\frac{1}{2}k\left[-(A-B)^2\right]+\frac{1}{2}k(A+B)^2$$

$$=\frac{1}{2}k\left[(A+B)^2-(A-B)^2\right]$$

$$=2kAB\ =2kc_1e^{i\omega_0 x}\cdot c_{-1}e^{-i\omega_0 x}$$

$$=2kc_1\cdot c_{-1}$$

$$=2k\left|c_1\right|^2\qquad \because c_{-1}=\overline{c}_1$$

the energy is proportional to the square of the amplitude $|c_1|$.

Physical Interpretation : Spectrum

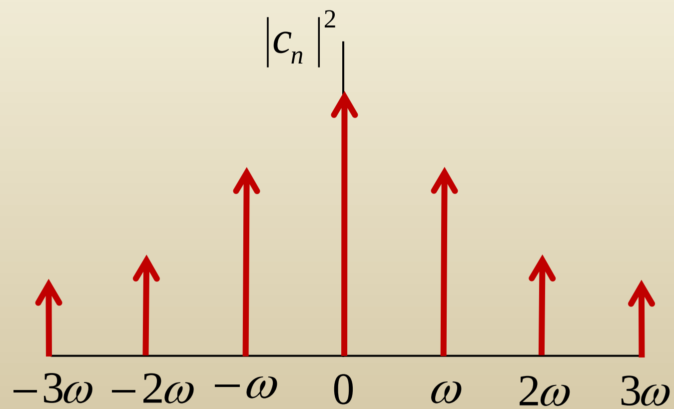
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p}$$

$$c_n = \frac{1}{2p}\int_{-p}^{p} f(x) e^{-in\pi x/p} dx$$

$$E_0 = 2k|c_1|^2$$

Hence the energy is proportional to the square of the amplitude $|c_1|$.

As the next step, if a more complicated system leads to a periodic solution $y = f(x)$ that can be represented by a Fourier series, then instead of the single energy term $|c_1|^2$ we get a series of squares $|c_n|^2$ of Fourier coefficients c_n given by (6). In this case we have a "discrete spectrum" (or "point spectrum") consisting of countably many isolated frequencies (infinitely many, in general), the corresponding $|c_n|^2$ being the contributions to the total energy.



Fourier Transform

Fourier Cosine Transform

For an even function $f(x)$, the Fourier integral is the Fourier cosine integral

$$f(x)=\int_0^{\infty} A(\omega)\cos\omega x\,d\omega\cdots(1a)$$

$$\text{where }\ A(\omega)=\frac{2}{\pi}\int_0^{\infty} f(v)\cos\omega v\,dv\cdots(1b)$$

$$\text{Now we set }\ A(\omega)=\sqrt{2/\pi}\,\hat{f}_c(\omega),$$ **where c suggests "cosine"**

Then from (1b), writing $v = x$

$$\hat{f}_c(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\omega x\,dx\cdots(2)$$

: Fourier cosine transform of $f(x)$

and from (1a),

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}_c(\omega)\cos\omega x\,d\omega\cdots(3)$$

: inverse Fourier cosine transform of $\hat{f}_c(\omega)$

Fourier Sine Transform

For an odd function *f (x)*, the Fourier integral is the Fourier sine integral

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\, d\omega \cdots (4a)$$

$$\text{where } B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\, dv \cdots (4b)$$

Now we set $B(\omega) = \sqrt{2/\pi}\,\hat{f}_s(\omega)$, **where *s* suggests "sine"**

Then from (4b), writing *v* = *x*

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\, dx \cdots (5)$$

: Fourier sine transform of *f* (*x*)

and from (4a),

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin\omega x\, d\omega \cdots (6)$$

: inverse Fourier sine transform of $\hat{f}_s(\omega)$

Fourier Cosine and Sine Transform

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx \cdots (2)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(\omega) \cos \omega x \, d\omega \cdots (3)$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, dx \cdots (5)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(\omega) \sin \omega x \, d\omega \cdots (6)$$

Other notations

$$\mathcal{F}_c(f) = \hat{f}_c(\omega), \quad \mathcal{F}_s(f) = \hat{f}_s(\omega)$$

$\mathcal{F}_c^{-1}(f)$, $\mathcal{F}_s^{-1}(f)$: inverses of $\mathcal{F}_c$ and $\mathcal{F}_s$, respectively.

Fourier Cosine and Sine Transform

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \cdots (5)$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx \cdots (2)$$

Ex.)Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = \begin{cases} k & if\ 0 < x < a \\ 0 & if\ x > a \end{cases}$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} k \int_0^a \cos \omega x \, dx + \sqrt{\frac{2}{\pi}} \int_a^\infty 0 \cdot \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} k \left(\frac{\sin a\omega}{\omega} \right)$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} k \int_0^a \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{1 - \cos a\omega}{\omega} \right)$$

Fourier Cosine and Sine Transform

$$\hat{f}_s(\omega)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\,dx\cdots(5)$$

$$\hat{f}_c(\omega)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx\cdots(2)$$

Ex.)Find the Fourier cosine and Fourier sine transforms of the function

$$f(x)=e^{-x}$$

$$\mathcal{F}_c(e^{-x})=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos\omega x\,dx=\sqrt{\frac{2}{\pi}}\frac{1}{1+\omega^2}$$

$$\mathcal{F}_s(e^{-x})=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\omega x\,dx=\sqrt{\frac{2}{\pi}}\frac{\omega^2}{1+\omega^2}$$

$$\int_0^\infty e^{-x}\cos\omega x\;dx=\left[e^{-x}\frac{\sin\omega x}{\omega}\right]_0^\infty-\int_0^\infty(-e^{-x})\frac{\sin\omega x}{\omega}dx$$

$$=\left[e^{-x}\frac{(-\cos\omega x)}{\omega^2}\right]_0^\infty-\int_0^\infty(-e^{-x})\frac{(-\cos\omega x)}{\omega^2}dx$$

$$=\frac{1}{\omega^2}-\frac{1}{\omega^2}\int_0^\infty e^{-x}\cos\omega x dx$$

$$\therefore\int_0^\infty e^{-x}\cos\omega x dx=\frac{1}{1+\omega^2}$$

$$\int_0^\infty e^{-x}\sin\omega x\;dx=\left[e^{-x}\frac{(-\cos\omega x)}{\omega}\right]_0^\infty-\int_0^\infty(-e^{-x})\frac{(-\cos\omega x)}{\omega}dx$$

$$=1-\left\{\left[e^{-x}\frac{\sin\omega x}{\omega^2}\right]_0^\infty-\int_0^\infty(-e^{-x})\frac{(\sin\omega x)}{\omega^2}dx\right\}$$

$$=1-\left\{0+\frac{1}{\omega^2}\int_0^\infty e^{-x}\sin\omega x dx\right\}$$

$$\therefore\int_0^\infty e^{-x}\sin\omega x dx=\frac{\omega^2}{1+\omega^2}$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, dx \cdots (5)$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx \cdots (2)$$

If $f(x)$ is absolutely integrable on the positive x-axis and piecewise continuous on every finite interval, then the Fourier cosine and sine transforms of f exist.

If f and g have Fourier cosine and sine transforms, so does $af+bg$ for any constants a and b, and by (2)

$$\mathcal{F}_c(af+bg) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} [af(x)+bg(x)] \cos \omega x \, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx + b\sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x) \cos \omega x \, dx$$

$$= a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$$

Fourier Cosine and Sine Transform

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx \cdots (5)$$

Linearity, Transforms of Derivatives

$$\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$$

Similarly for F_s, by (5)

$$\mathcal{F}_s(af + bg) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left[af(x) + bg(x)\right] \sin \omega x \, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx + b\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \sin \omega x \, dx$$

$$= a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$$

This shows that the Fourier cosine and sine transforms are linear operations,

$$F_c(af + bg) = aF_c(f) + bF_c(g), \cdots (7a)$$

$$\mathcal{F}_s(af + bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g) \cdots (7b)$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

Theorem1) Cosine and Sine Transforms of Derivatives

Let $f(x)$ be continuous and absolutely integrable on the x-axis (Fourier cosine and sine transforms exist), let $f'(x)$ be piecewise continuous on every finite interval, and let

$f(x) \to 0$ as $x \to \infty$. Then

$$\mathcal{F}_c\{f'(x)\} = w\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0), \cdots (8a)$$

$$\mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\} \cdots (8b)$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathcal{F}_c\{f'(x)\} = \omega\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0), \cdots (8a)$$

$$\mathcal{F}_s\{f'(x)\} = -\omega\mathcal{F}_c\{f(x)\} \cdots (8b)$$

proof) $\mathcal{F}_c\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos\omega x\,dx$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathcal{F}_c\{f'(x)\} = \omega \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0), \cdots (8a)$$

$$\mathcal{F}_s\{f'(x)\} = -\omega \mathcal{F}_c\{f(x)\} \cdots (8b)$$

proof) $\mathcal{F}_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \omega x \, dx$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \cos \omega x \Big|_0^\infty + \omega \int_0^\infty f(x) \sin \omega x \, dx \right]$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathcal{F}_c\{f'(x)\} = \omega\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0), \cdots (8a)$$

$$\mathcal{F}_s\{f'(x)\} = -\omega\mathcal{F}_c\{f(x)\} \cdots (8b)$$

proof) $\mathcal{F}_c\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos\omega x\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[f(x)\cos\omega x\Big|_0^\infty + \omega\int_0^\infty f(x)\sin\omega x\,dx \right]$$

$$= -\sqrt{\frac{2}{\pi}}f(0) + \omega F_s\{f(x)\};$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathscr{F}_c\{f'(x)\} = \omega\mathscr{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0), \cdots (8a)$$

$$\mathscr{F}_s\{f'(x)\} = -\omega\mathscr{F}_c\{f(x)\} \cdots (8b)$$

proof) $\mathscr{F}_c\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos\omega x\,dx$

$$= \sqrt{\frac{2}{\pi}}\left[f(x)\cos\omega x\Big|_0^\infty + \omega\int_0^\infty f(x)\sin\omega x\,dx \right]$$

$$= -\sqrt{\frac{2}{\pi}}f(0) + \omega F_s\{f(x)\};$$

$$\mathscr{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin\omega x\,dx$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathcal{F}_c\{f'(x)\} = \omega \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0), \cdots (8a)$$

$$\mathcal{F}_s\{f'(x)\} = -\omega \mathcal{F}_c\{f(x)\} \cdots (8b)$$

proof) $$\mathcal{F}_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \cos \omega x \Big|_0^\infty + \omega \int_0^\infty f(x) \sin \omega x \, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + \omega F_s\{f(x)\};$$

$$\mathcal{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \sin \omega x \Big|_0^\infty - \omega \int_0^\infty f(x) \cos \omega x \, dx \right]$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathcal{F}_c\{f'(x)\} = \omega \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0), \cdots (8a)$$

$$\mathcal{F}_s\{f'(x)\} = -\omega \mathcal{F}_c\{f(x)\} \cdots (8b)$$

proof) $\mathcal{F}_c\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos \omega x \, dx$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \cos \omega x \Big|_0^\infty + \omega \int_0^\infty f(x) \sin \omega x \, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + \omega F_s\{f(x)\};$$

$$\mathcal{F}_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x) \sin \omega x \Big|_0^\infty - \omega \int_0^\infty f(x) \cos \omega x \, dx \right]$$

$$= 0 - \omega F_c\{f(x)\}$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

Formula (8a) with f' instead of *f* gives (when f', f'' satisfy the respective assumptions for f', f'' in Theorem 1)

$$\mathcal{F}_c\{f''(x)\} = \omega \mathcal{F}_s\{f'(x)\} - \sqrt{\frac{2}{\pi}} f'(0);$$

hence, by (8b)

$$\mathcal{F}_c\{f''(x)\} = \omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0) \cdots (9a)$$

Similarly,

$$\mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}} \omega f(0). \cdots (9b)$$

$$\mathcal{F}_c(af+bg)=a\mathcal{F}_c(f)+b\mathcal{F}_c(g),\cdots(7a)$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathcal{F}_c\{f''(x)\}=\omega^2\mathcal{F}_c\{f(x)\}-\sqrt{\frac{2}{\pi}}f'(0)\cdots(9a)$$

$$\mathcal{F}_s\{f''(x)\}=-\omega^2\mathcal{F}_s\{f(x)\}+\sqrt{\frac{2}{\pi}}\omega f(0).\cdots(9b$$

Ex.)Find the Fourier cosine transform of $\mathcal{F}_c(f),\ f(x)=e^{-ax},\ a>0$

$$\mathcal{F}_c(af+bg)=a\mathcal{F}_c(f)+b\mathcal{F}_c(g),\cdots(7a)$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

$$\mathcal{F}_c\{f''(x)\}=\omega^2\mathcal{F}_c\{f(x)\}-\sqrt{\frac{2}{\pi}}f'(0)\cdots(9a)$$

$$\mathcal{F}_s\{f''(x)\}=-\omega^2\mathcal{F}_s\{f(x)\}+\sqrt{\frac{2}{\pi}}\omega f(0).\cdots(9b)$$

Ex.)Find the Fourier cosine transform of $\mathcal{F}_c(f),\ f(x)=e^{-ax},\ a>0$

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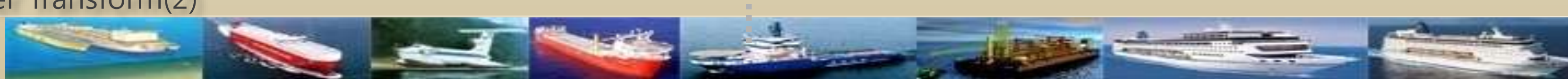

$$\mathcal{F}_c(af + bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g), \cdots (7a)$$

Fourier Cosine and Sine Transform

Linearity, Transforms of Derivatives

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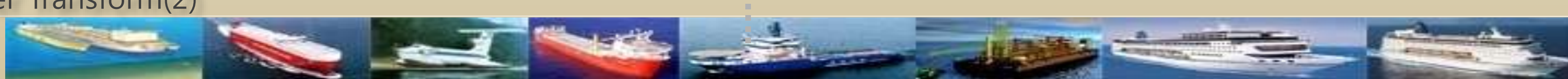

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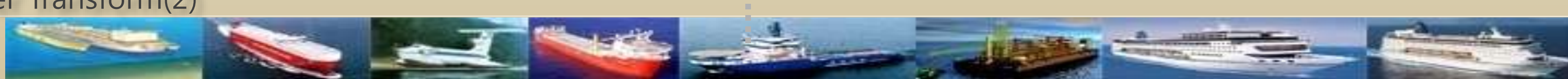

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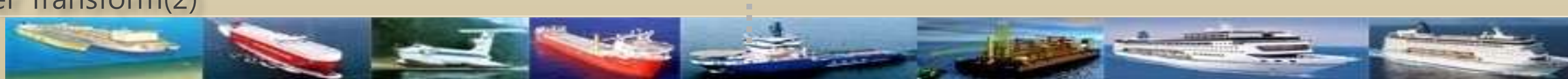

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Hence,

$$a^2 \mathcal{F}_c(f) = \mathcal{F}_c(f'') = -\omega^2 \mathcal{F}_c(f) + a\sqrt{\frac{2}{\pi}}$$

$$(a^2 + \omega^2)\mathcal{F}_c(f) = a\sqrt{\frac{2}{\pi}}$$

$$\mathcal{F}_c(af+bg)=a\mathcal{F}_c(f)+b\mathcal{F}_c(g),\cdots(7a)$$

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$$(a^2+\omega^2)\mathcal{F}_c(f)=a\sqrt{\frac{2}{\pi}}$$

$$\therefore \mathcal{F}_c(e^{-ax})=\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\omega^2}\right)\quad(a>0)$$

Fourier Transform

Fourier transform of $f(x)$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

inverse Fourier transform of $\hat{f}(w)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

Another notation for the Fourier transform is

$$\hat{f} = \mathcal{F}(f)$$

: Fourier transform of $f(x)$ or Fourier transform method.

so that

$$f = \mathcal{F}^{-1}(\hat{f})$$

Fourier Transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

Theorem1) Existence of the Fourier Transform

Let *f* (*x*) be continuous and absolutely integrable on the *x*-axis and piecewise continuous on every finite interval, then the Fourier transform $\hat{f}(\omega)$ of *f* (*x*) given by (6) exists.

Fourier Transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

Ex.)Find the Fourier transform of $f(x)$

$$f(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left. \frac{e^{-i\omega x}}{-i\omega} \right|_{-1}^{1}$$

$$= \frac{1}{-i\omega\sqrt{2\pi}} \left(e^{-i\omega} - e^{i\omega} \right)$$

By Euler formula $(e^{\pm i\omega} = \cos\omega \pm i \sin\omega)$

$$\hat{f}(\omega) = \frac{1}{-i\omega\sqrt{2\pi}} \left[(\cos\omega - i\sin\omega) - (\cos\omega + i\sin\omega) \right]$$

$$= \frac{1}{-i\omega\sqrt{2\pi}} (2i\sin\omega)$$

$$\therefore \hat{f}(\omega) = \sqrt{\frac{\pi}{2}} \frac{\sin\omega}{\omega}$$

Fourier Transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

Find the Fourier transform of

$$f(x) = \begin{cases} e^{-ax}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases},$$

$$(a > 0)$$

Fourier Transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

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$$f(x) = \begin{cases} e^{-ax}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases},$$

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$$\mathscr{F}\left(e^{-ax}\right) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ax} e^{-i\omega x} dx$$

Fourier Transform

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\cdots(6)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

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$$\mathcal{F}\left(e^{-ax}\right)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ax}e^{-i\omega x}dx =\frac{1}{\sqrt{2\pi}}\left.\frac{e^{-(a+i\omega)x}}{-(a+i\omega)}\right|_{x=0}^{\infty}$$

Fourier Transform

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Fourier Transform : Linearity

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

Theorem2) Linearity of the Fourier Transform

The Fourier transform is a linear operation; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants a and , the Fourier transform of $af+bg$ exists, and

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g) \cdots (8)$$

proof)

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$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\cdots(6)$$

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$$\mathcal{F}(af+bg)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[af(x)+bg(x)\right]e^{-i\omega x}dx$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

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proof)

$$\mathcal{F}(af+bg) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{-i\omega x} dx$$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx + b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx$$

Fourier Transform : Linearity

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\cdots(6)$$

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$$\mathcal{F}(af+bg)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x)+bg(x)]e^{-i\omega x}dx$$

$$=a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx+b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx$$

$$=a\mathcal{F}(f)+b\mathcal{F}(g)$$

Fourier Transform : Derivative

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Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$.

Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

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$$= \frac{1}{\sqrt{2\pi}} \left[f(x) e^{-i\omega x} \Big|_{-\infty}^{\infty} - (-i\omega) \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \right]$$

$$= 0 + i\omega \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

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$$=0+i\omega\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \quad =i\omega F\{f(x)\}$$

Fourier Transform : Derivative

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

$$\mathcal{F}\{f'(x)\} = i\omega \mathcal{F}\{f(x)\} \cdots (9)$$

Two successive application of (9) give

$$\mathcal{F}\{f''(x)\} = i\omega \mathcal{F}\{f'(x)\}$$

$$= i\omega i\omega \mathcal{F}\{f(x)\}$$

$$\therefore \mathcal{F}\{f''(x)\} = -\omega^2 \mathcal{F}\{f(x)\} \cdots (10)$$

Fourier Transform : Derivative

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}\{f(x)\} \cdots (9)$$

Find the Fourier transform of xe^{-x^2}

Fourier Transform : Derivative

$$\mathcal{F}\left\{f'(x)\right\} = i\omega\mathcal{F}\left\{f(x)\right\} \cdots (9)$$

Find the Fourier transform of xe^{-x^2}

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$$= -\frac{1}{2}i\omega\mathcal{F}\left(e^{-x^2}\right)$$

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$$= -\frac{1}{2}\mathcal{F}\left\{\left(e^{-x^2}\right)'\right\}$$

$$= -\frac{1}{2}i\omega\mathcal{F}\left(e^{-x^2}\right)$$

$$= -\frac{1}{2}i\omega\frac{1}{\sqrt{2}}e^{-\omega^2/4}$$

Fourier Transform : Derivative

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$$= -\frac{1}{2}\mathcal{F}\left\{\left(e^{-x^2}\right)'\right\}$$

$$= -\frac{1}{2}i\omega\mathcal{F}\left(e^{-x^2}\right)$$

$$= -\frac{1}{2}i\omega\frac{1}{\sqrt{2}}e^{-\omega^2/4}$$

$$= -\frac{i\omega}{2\sqrt{2}}e^{-\omega^2/4}$$

Fourier Transforms : P.D.E

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}(\omega)$$
$$\mathcal{F}\{f''(x)\} = -\omega^2\mathcal{F}(\omega)$$

☑ Example 1 Using the Fourier Transform

Solve the heat equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t},\ -\infty < x < \infty,\ t > 0$

Subject to $u(x,0) = f(x),$ **where** $f(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

---

define $\mathcal{F}\{u(x,t)\} = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}dx = U(\omega,t)$

Transforming the equation

$$\mathcal{F}\left\{k\frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{F}\left\{\frac{\partial u}{\partial t}\right\}$$

$$-k\omega^2 U = \frac{dU}{dt} \quad or \quad \frac{dU}{dt} + k\omega^2 U = 0 \xrightarrow{\text{solution}} U = ce^{-k\omega^2 t}$$

Fourier Transforms : P.D.E

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}(\omega)$$

$$\mathcal{F}\{f''(x)\} = -\omega^2\mathcal{F}(\omega)$$

☑ Example 1 Using the Fourier Transform

Solve the heat equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \ -\infty < x < \infty, \ t > 0$

Subject to $u(x,0) = f(x)$, where $f(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

$$\frac{dU}{dt} + k\omega^2 U(\omega,t) = 0 \quad \Rightarrow \quad U(\omega,t) = ce^{-k\omega^2 t}$$

Initial condition transformation

$$\mathcal{F}\{u(x,0)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} u_0 e^{-i\omega x}dx = u_0\frac{e^{i\omega} - e^{-i\omega}}{\sqrt{2\pi}i\omega}$$

$$U(\omega,0) = \mathcal{F}\{u(x,0)\} = \frac{2}{\sqrt{2\pi}}u_0\frac{\sin\omega}{\omega}$$

$$\therefore U(\omega,t) = \frac{2}{\sqrt{2\pi}}u_0\frac{\sin\omega}{\omega}e^{-k\omega^2 t}$$

Fourier Transforms : P.D.E

$$\mathcal{F}\{f'(x)\} = i\omega \mathcal{F}(\omega)$$
$$\mathcal{F}\{f''(x)\} = -\omega^2 \mathcal{F}(\omega)$$

☑ Example 1 Using the Fourier Transform

Solve the heat equation $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \; -\infty < x < \infty, \; t > 0$

Subject to $u(x,0) = f(x)$, where $f(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

$$U(\omega,t) = \frac{2}{\sqrt{2\pi}} u_0 \frac{\sin\omega}{\omega} e^{-k\omega^2 t}$$

Inverse transformation

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{2}{\sqrt{2\pi}} u_0 \frac{\sin\omega}{\omega} e^{-k\omega^2 t} \right) e^{i\omega x} d\omega$$

$$\because \frac{u_0}{\pi} i \int_{-\infty}^{\infty} \left(\left(\frac{1}{\omega}\right) \sin\omega \, e^{-k\omega^2 t} \sin\omega x \right) d\omega = 0$$

odd, odd, even, odd

odd function of w

$$= \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} e^{-k\omega^2 t} e^{i\omega x} d\omega = \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} e^{-k\omega^2 t} (\cos\omega x + i\sin\omega x) d\omega$$

$$= \frac{u_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin\omega \cos\omega x}{\omega} e^{-k\omega^2 t} d\omega$$

Fourier Transforms

$$\mathcal{F}_c\{f''(x)\} = \omega^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0) \cdots (9a)$$

$$\mathcal{F}_s\{f''(x)\} = -\omega^2 \mathcal{F}_s\{f(x)\} + \sqrt{\frac{2}{\pi}} \omega f(0). \cdots (9b)$$

☑ Example 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < \pi,\ y > 0$$

Subject to

$$u(0, y) = 0,\ u(\pi, u) = e^{-y},\ y > 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad 0 < x < \pi.$$

Cosine transform is suitable

Define:

$$\mathcal{F}_c\{u(x,y)\} = \int_0^\infty u(x,y) \cos \omega y \, dy = U(x,\omega)$$

$$u(x,y) = \frac{2}{\pi} \int_0^\infty U(x,\omega) \cos \omega y \, d\omega$$

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \mathcal{F}_c\{0\}$$

Fourier Transforms

$$\mathcal{F}_c\{f''(x)\}=\omega^2\mathcal{F}_c\{f(x)\}-\sqrt{\frac{2}{\pi}}f'(0)\cdots(9a)$$

$$\mathcal{F}_s\{f''(x)\}=-\omega^2\mathcal{F}_s\{f(x)\}+\sqrt{\frac{2}{\pi}}\omega f(0).\cdots(9b)$$

☑ Example 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ 0<x<\pi,\ y>0$$

Subject to

$$u(0,y)=0,\ \ u(\pi,u)=e^{-y},\ \ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

Cosine transform is suitable

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}+\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial y^2}\right\}=\mathcal{F}_c\{0\}$$

$$\frac{d^2U}{dx^2}-\omega^2U(x,\omega)-u_y(x,0)=0\ \Rightarrow\ \frac{d^2U}{dx^2}-\omega^2U=0\ (\because \text{boundary condition})$$

$$\therefore U(x,\omega)=c_1\cosh\omega x+c_2\sinh\omega x$$

solution

$$\hat{f}_c(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\omega x\,dx\cdots(2)$$

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Subject to

$$u(0,y)=0,\ u(\pi,u)=e^{-y},\ \ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

$$U(x,\omega)=c_1\cosh\omega x+c_2\sinh\omega x$$

Boundary condition

$$\mathscr{F}_c\{u(0,y)\}=U(0,\omega)=\mathscr{F}_c\{0\}$$

$$\therefore U(0,\omega)=0$$

$$\mathscr{F}_c\{u(\pi,y)\}=U(\pi,\omega)=\mathscr{F}_c\{e^{-y}\}$$

$$\therefore U(\pi,\omega)=\frac{1}{1+\omega^2}$$

$$\int_0^{\infty} e^{-x}\cos\omega x\,dx=\left[e^{-x}\frac{\sin\omega x}{\omega}\right]_0^{\infty}-\int_0^{\infty}(-e^{-x})\frac{\sin\omega x}{\omega}dx$$

$$=\left[e^{-x}\frac{(-\cos\omega x)}{\omega^2}\right]_0^{\infty}-\int_0^{\infty}(-e^{-x})\frac{(-\cos\omega x)}{\omega^2}dx$$

$$=\frac{1}{\omega^2}-\frac{1}{\omega^2}\int_0^{\infty}e^{-x}\cos\omega x dx$$

$$\therefore\int_0^{\infty}e^{-x}\cos\omega x dx=\frac{1}{1+\omega^2}$$

$$\hat{f}_c(\omega)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\omega x\,dx\cdots(2)$$

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Subject to

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$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

$$U(x,\omega)=c_1\cosh\omega x+c_2\sinh\omega x$$

Boundary condition

$$\mathcal{F}_c\{u(0,y)\}=U(0,\omega)=\mathcal{F}_c\{0\}$$

$$\therefore U(0,\omega)=0$$

$$\mathcal{F}_c\{u(\pi,y)\}=U(\pi,\omega)=\mathcal{F}_c\{e^{-y}\}$$

$$\therefore U(\pi,\omega)=\frac{1}{1+\omega^2}$$

$$\int_0^{\infty} e^{-x}\cos\omega x\,dx=\left[e^{-x}\frac{\sin\omega x}{\omega}\right]_0^{\infty}-\int_0^{\infty}(-e^{-x})\frac{\sin\omega x}{\omega}dx$$

$$=\left[e^{-x}\frac{(-\cos\omega x)}{\omega^2}\right]_0^{\infty}-\int_0^{\infty}(-e^{-x})\frac{(-\cos\omega x)}{\omega^2}dx$$

$$=\frac{1}{\omega^2}-\frac{1}{\omega^2}\int_0^{\infty}e^{-x}\cos\omega x dx$$

$$\therefore\int_0^{\infty}e^{-x}\cos\omega x dx=\frac{1}{1+\omega^2}$$

Fourier Transforms

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\cdots(6)$$

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}d\omega\cdots(7)$$

☑ Example 2 Using the Cosine Transform

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Subject to

$$u(0,y)=0,\ u(\pi,u)=e^{-y},\ \ y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0,\quad 0<x<\pi.$$

$$U(x,\omega)=c_1\cosh\omega x+c_2\sinh\omega x$$

Boundary condition

$$U(0,\omega)=0\ ,U(\pi,\omega)=\frac{1}{1+\omega^2}$$

$$U(0,\omega)=c_1=0$$

$$U(\pi,\omega)=c_2\sinh\omega\pi=\frac{1}{1+\omega^2}\quad \therefore c_2=\frac{1}{(1+\omega^2)\sin\omega\pi}$$

$$\therefore U(x,\omega)=\frac{\sinh\omega x}{(1+\omega^2)\sinh\omega\pi}$$

Fourier Transforms

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \cdots (6)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \cdots (7)$$

☑ Example 2 Using the Cosine Transform

The steady-state temperature in a semi-infinite plate is determined from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < \pi,\ y > 0$$

Subject to

$$u(0, y) = 0,\ u(\pi, u) = e^{-y},\ \ y > 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad 0 < x < \pi.$$

$$U(x,\omega) = \frac{\sinh \omega x}{(1+\omega^2)\sinh \omega\pi}$$

Recall, definition

$$\mathcal{F}_c\{u(x,y)\} = \int_0^\infty u(x,y)\cos \omega y\, dy = U(x,\omega)$$

$$u(x,y) = \frac{2}{\pi}\int_0^\infty U(x,\omega)\cos\omega y\, d\omega$$

$$\therefore u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\sinh \omega x}{(1+\omega^2)\sinh\omega\pi}\cos\omega y\, d\omega$$