Naval Architecture & Ocean Engineering

[2008][14-1]

Engineering Mathematics 2

December, 2008

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Department of Naval Architecture and Ocean Engineering,
Seoul National University of College of Engineering

Complex (1)
: Complex and Residue Theorem

Complex

Complex numbers

Polar form of Complex Numbers

Complex Functions

Residue Theorem : Overview

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The purpose :

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The purpose :

the evaluation of integrals

$$\oint_C f(z)dz \quad (z : \text{complex})$$

$f(z)$: analytic inside a simple closed path C and on C in a simply connected domain D

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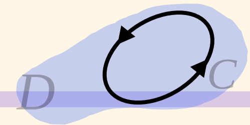

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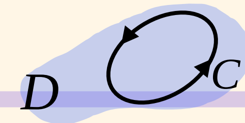

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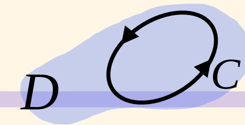

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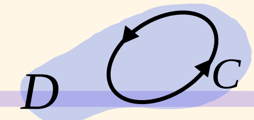

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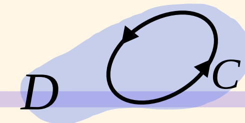

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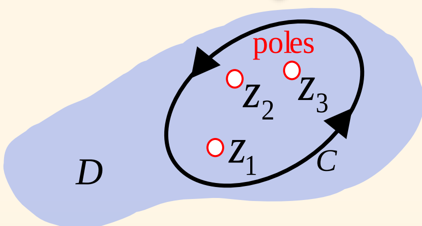

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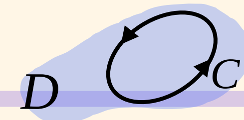

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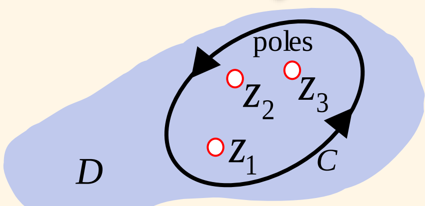

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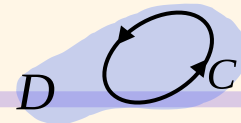

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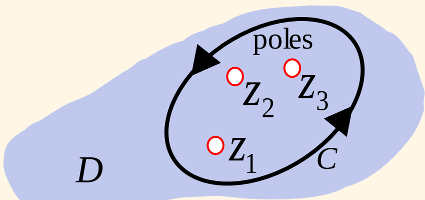

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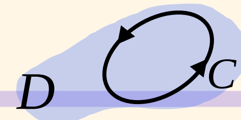

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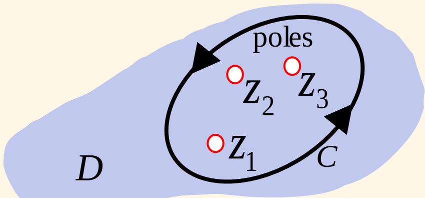

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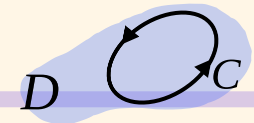

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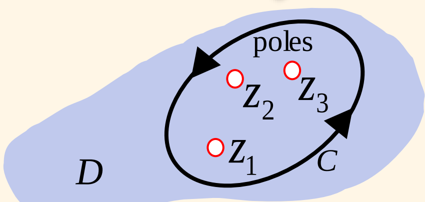

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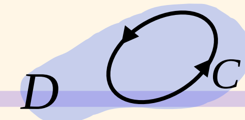

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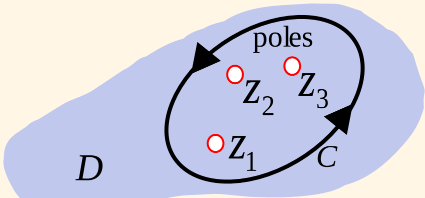

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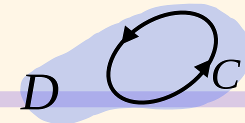

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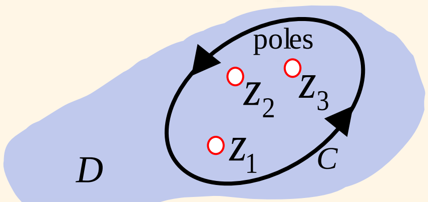

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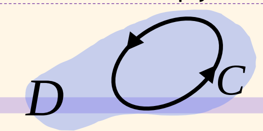

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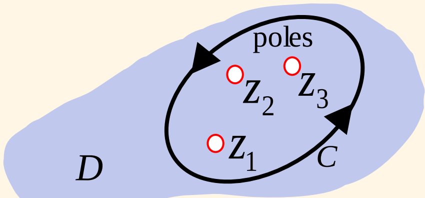

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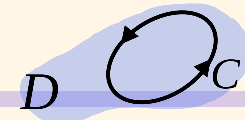

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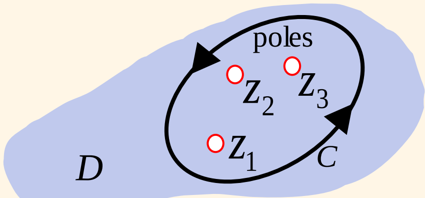

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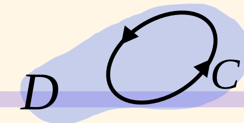

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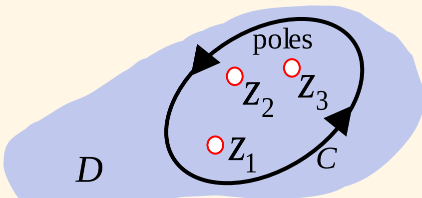

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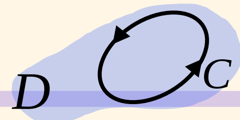

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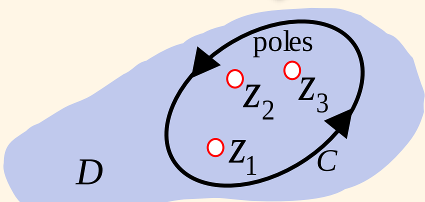

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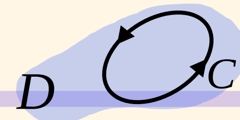

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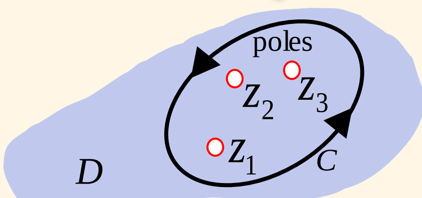

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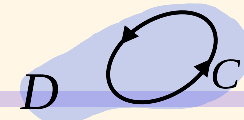

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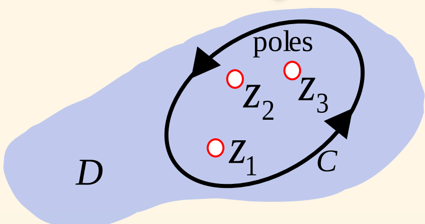

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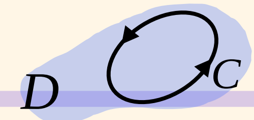

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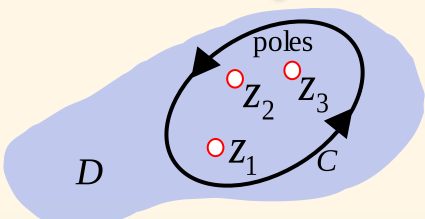

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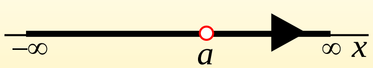

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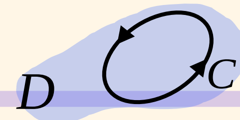

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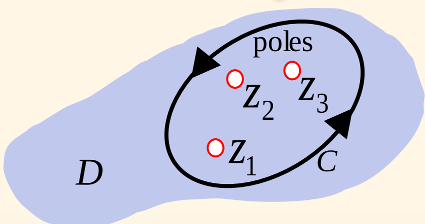

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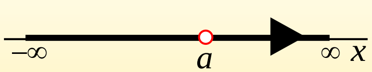

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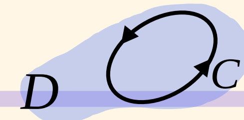

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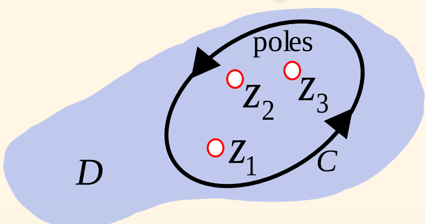

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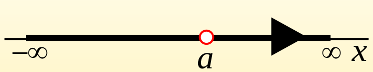

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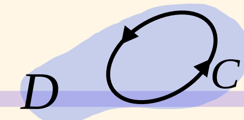

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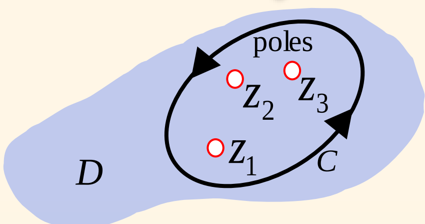

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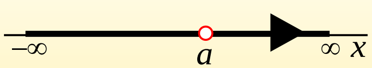

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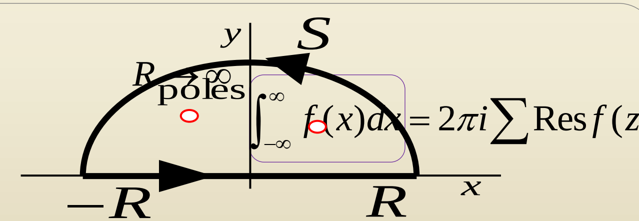

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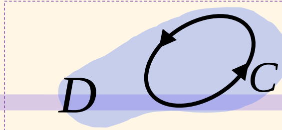

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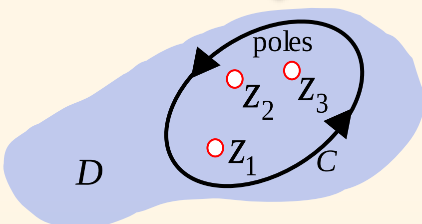

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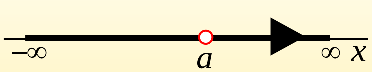

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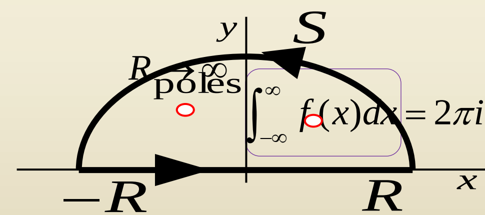

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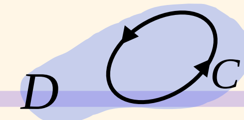

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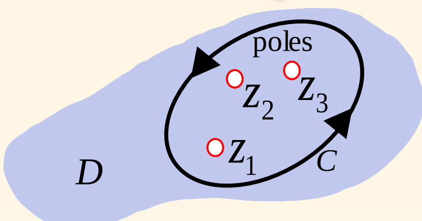

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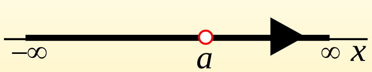

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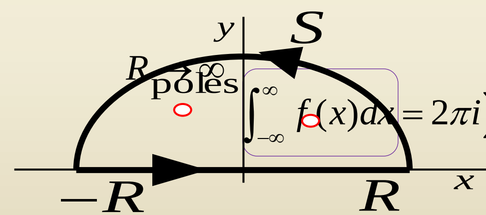

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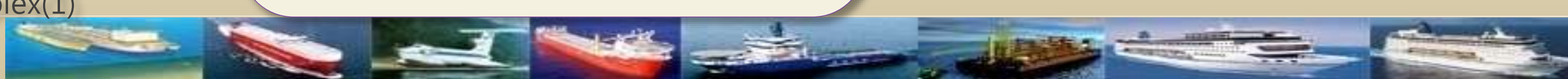

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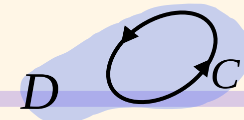

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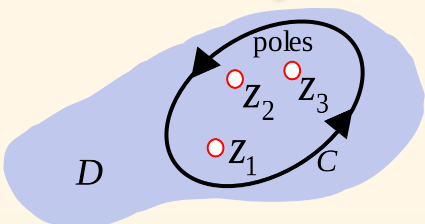

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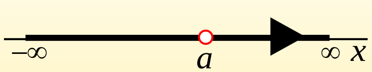

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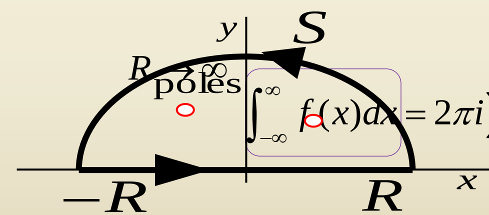

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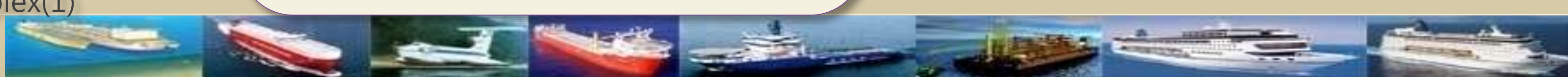

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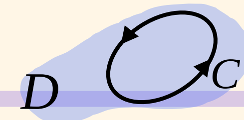

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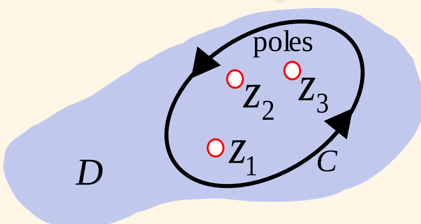

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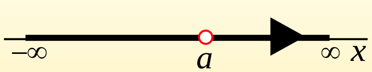

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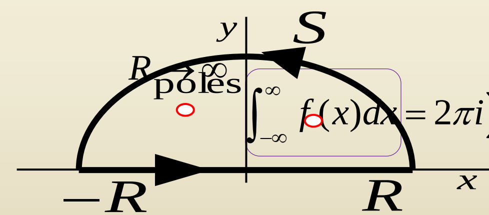

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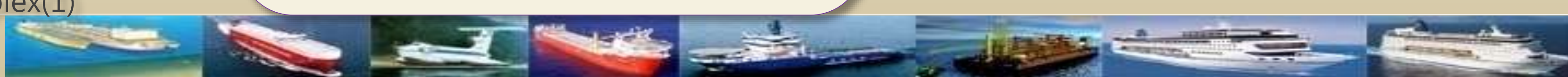

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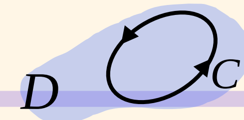

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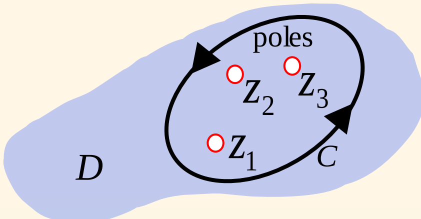

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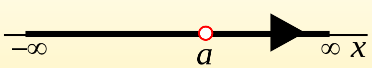

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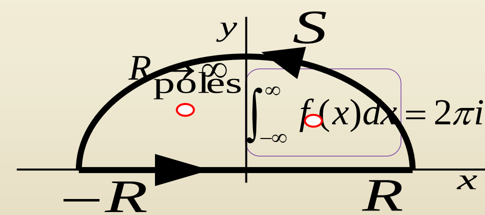
in the upper half-plane **on the real axis**

over all poles



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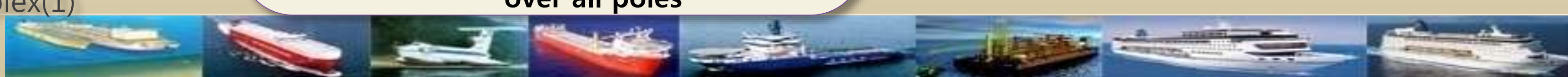
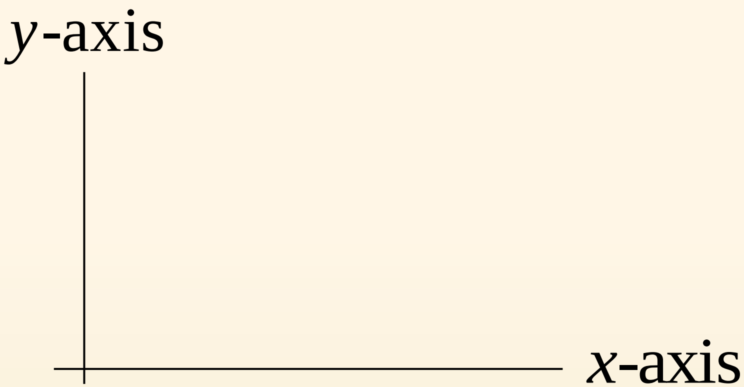


Complex

2008_Complex(1)

Complex Numbers

Complex in Geometry



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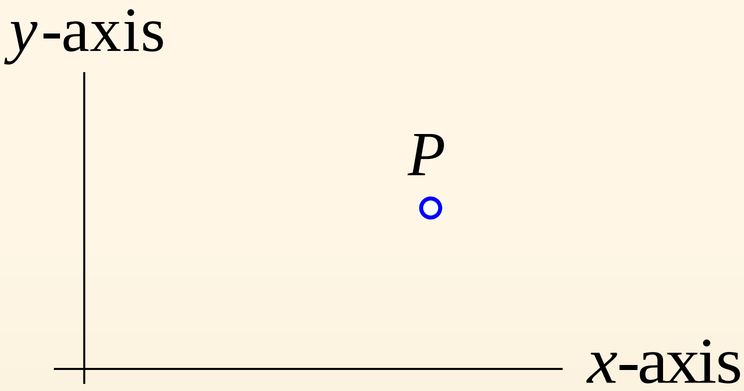

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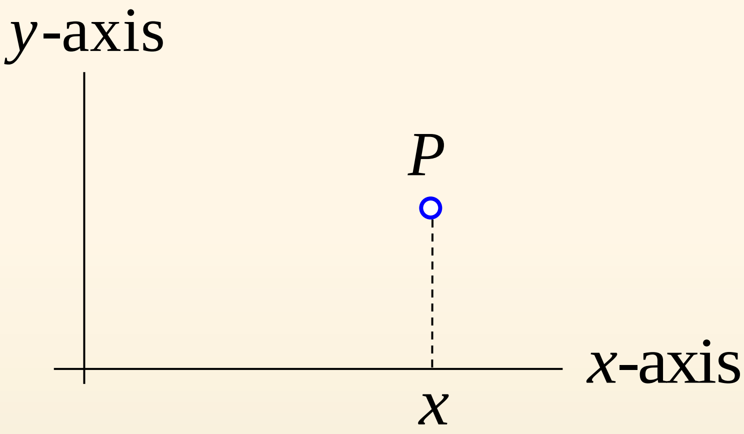

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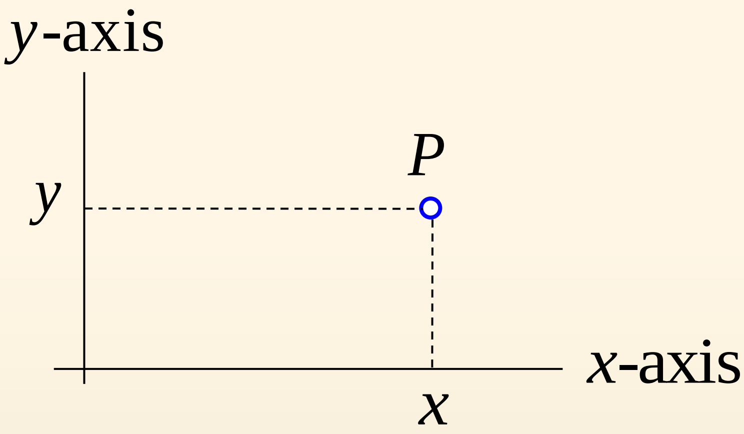

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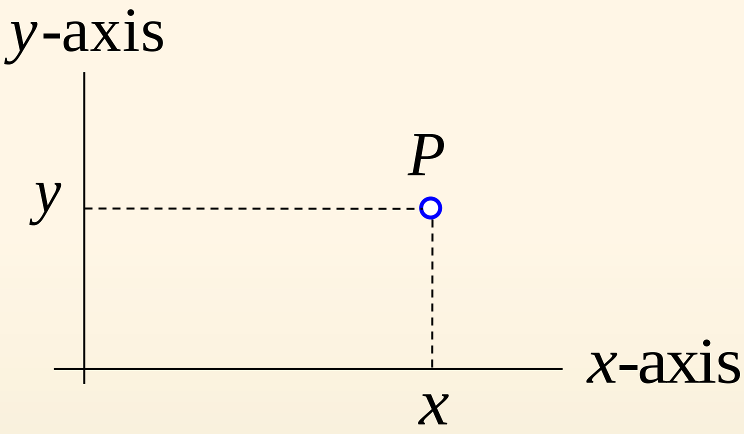
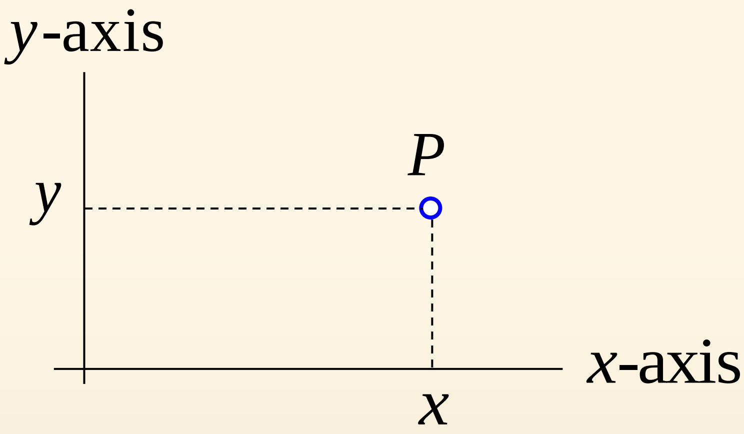


$$P = (x, y)$$

Complex Numbers

Complex in Geometry



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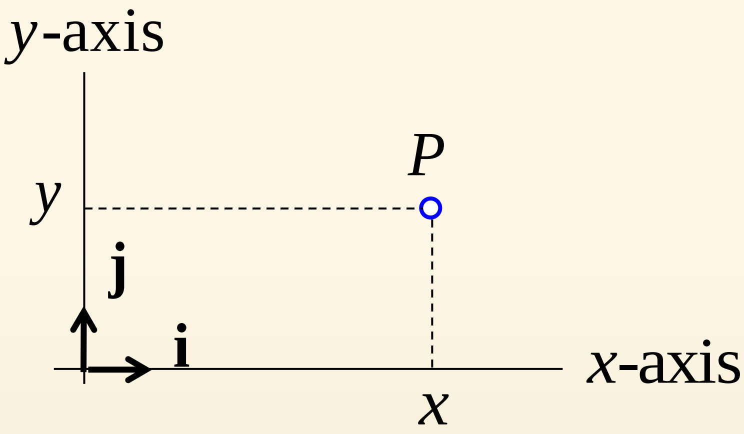

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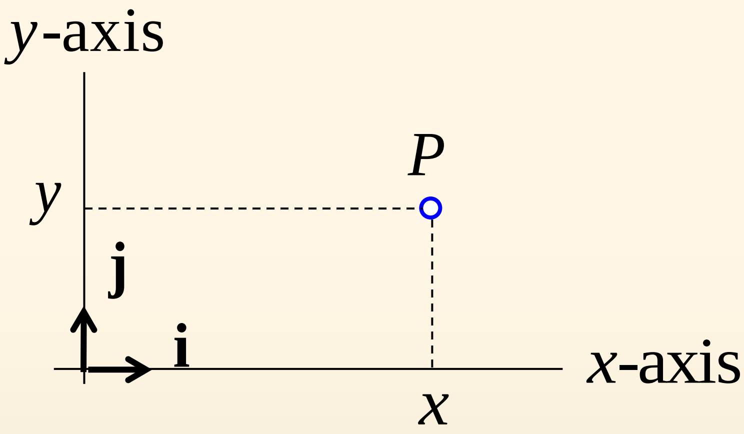

$$P = x\mathbf{i} + y\mathbf{j}$$

Complex Numbers

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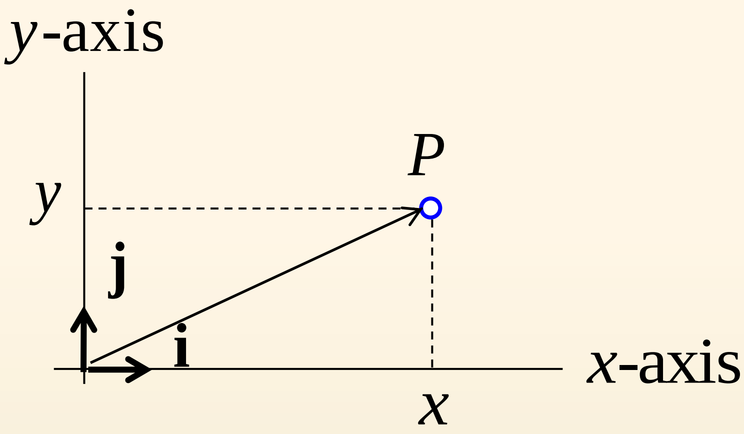

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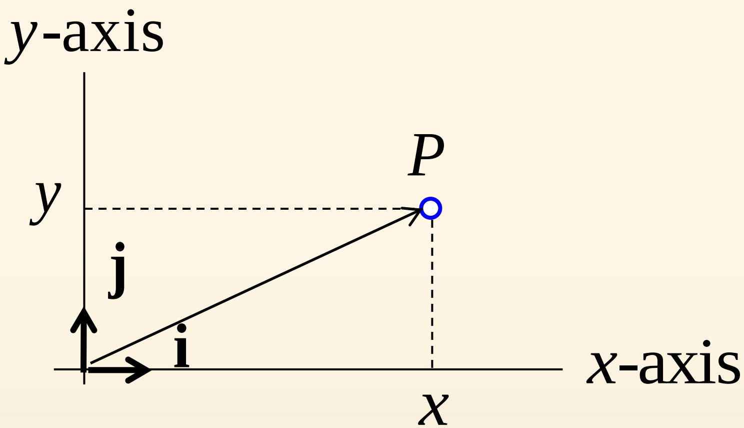

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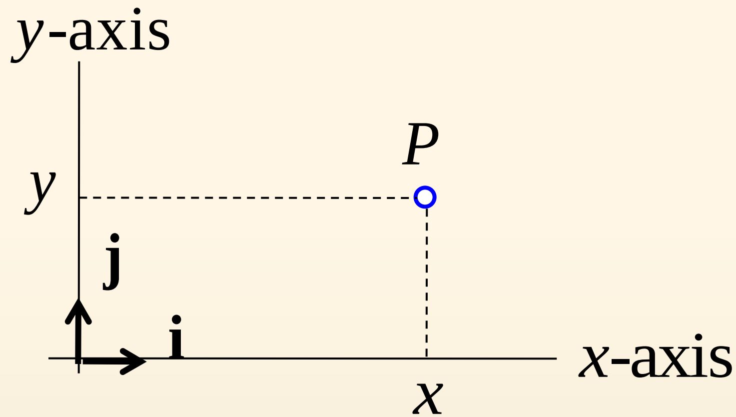

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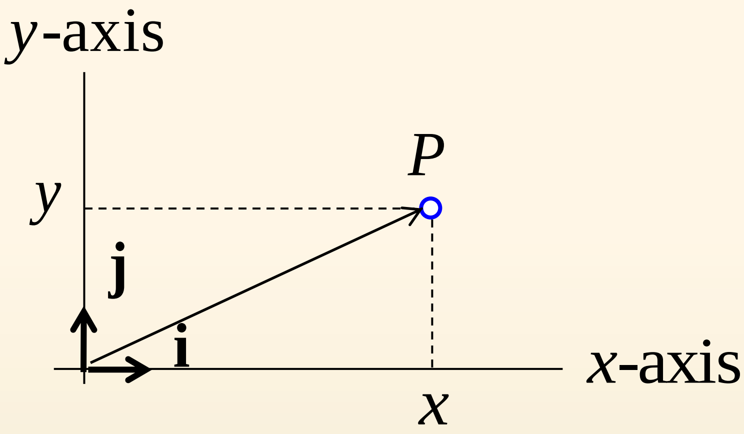

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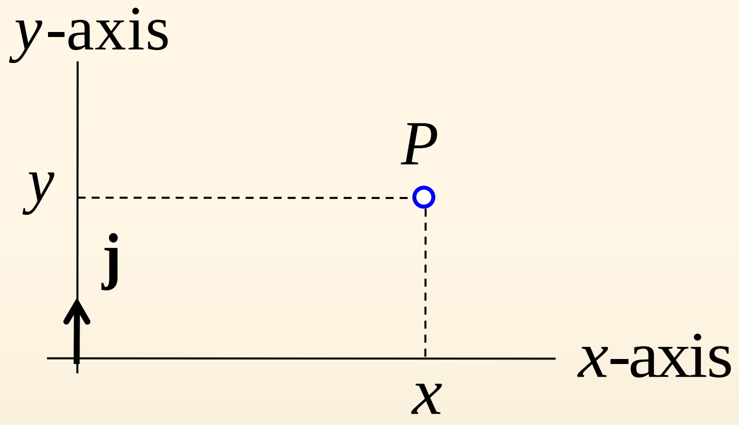

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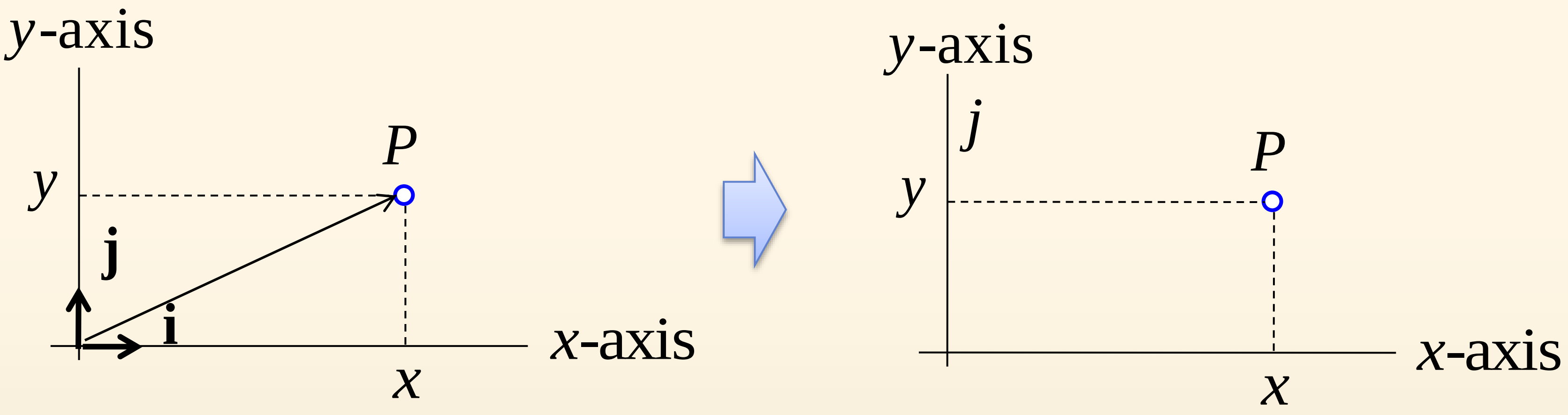

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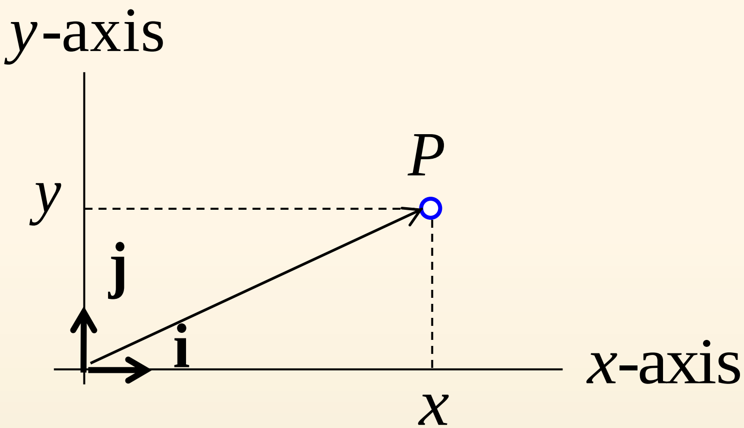

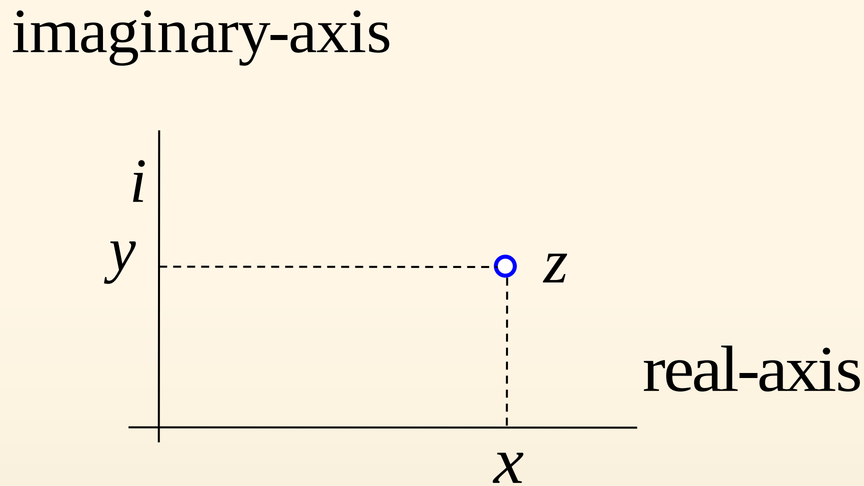

$$P = x\mathbf{i} + y\mathbf{j}$$

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$$z = x + iy$$

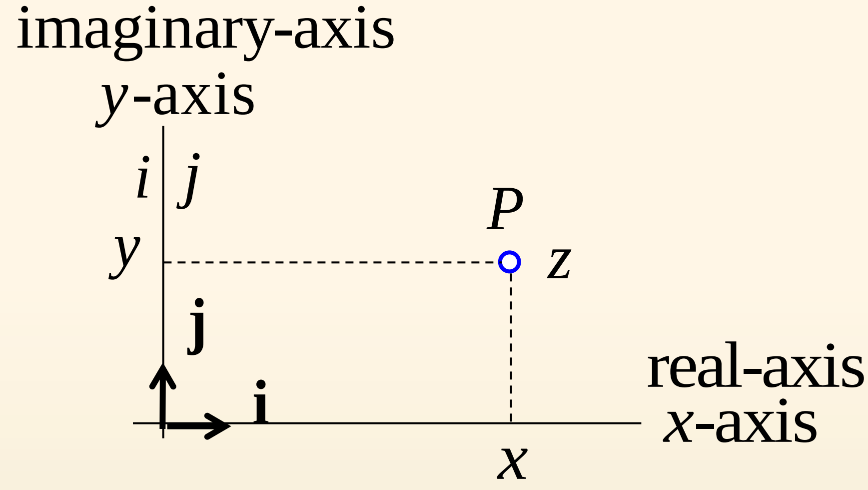
Complex Numbers

- **Complex Number z**

$$z = (x, y)$$

real part imaginary part

$$x = \operatorname{Re} z \qquad y = \operatorname{Im} z$$



- **Imaginary unit i**

$$(1) \quad i = (0,1)$$

$$P = x\mathbf{i} + y\mathbf{j}$$

$$\Downarrow$$

$$P = x + y\mathbf{j}$$

$$\Downarrow$$

$$P = x + jy$$

$$\Downarrow$$

$$z = x + iy$$

Complex Numbers

Addition, Multiplication. Notation *z* = *x* + *iy*

Two complex numbers : $z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2)$

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Addition of two complex numbers :

$$(2) \quad z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \; y_1 + y_2)$$

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In particular, these two definitions imply that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, \; 0) \quad \textbf{and} \quad (x_1, 0)(x_2, 0) = (x_1 x_2, \; 0).$$

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Hence the complex numbers "extend" the real numbers. We can thus write

$(x, 0) = x.$ **Similarly,** $(0, y) = iy.$

Complex Numbers

$(1)\quad i=(0,1)$

$(3)\quad z_1 z_2 = (x_1 x_2 - y_1 y_2,\ \ x_1 y_2 + x_2 y_1)$

Addition, Multiplication. Notation $z = x + iy$

In practice, complex numbers $z = (x, y)$ are written

$$(4)\quad z = x + iy$$

If $x = 0$, then $z = iy$ and is called pure imaginary. Also, (1) and (3) give

$$(5)\quad i^2 = -1$$

because by the definition of multiplication,

$$i^2 = (0,\ 1)(0,\ 1) = (-1,\ 0)$$

Two complex numbers

$$z_1 = (x_1, y_1) = x_1 + iy_1$$

$$z_2 = (x_2, y_2) = x_2 + iy_2$$

- Addition

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

- Multiplication

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2 \\ &= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

Complex Numbers

$$(2) \quad z_1 + z_2 = (x_1 + x_2,\ y_1 + y_2)$$

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Example) Real Part, Imaginary Part, Sum and Product of Complex Numbers

$$z_1 = 8 + 3i, \quad z_2 = 9 - 2i$$

Find the Real Parts, Imaginary Parts, Sum and Product of Complex Numbers

Complex Numbers

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Complex Numbers

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$$z_1 + z_2 = (8 + 3i) + (9 - 2i) = 17 + i$$

$$z_1 z_2 = (8 + 3i)(9 - 2i) = 72 + 6 + i(-16 + 27)$$

Complex Numbers

Subtraction, Division

Two complex numbers : $z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2)$

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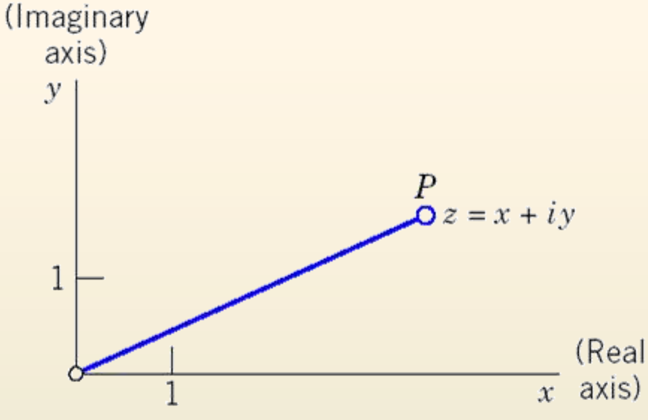
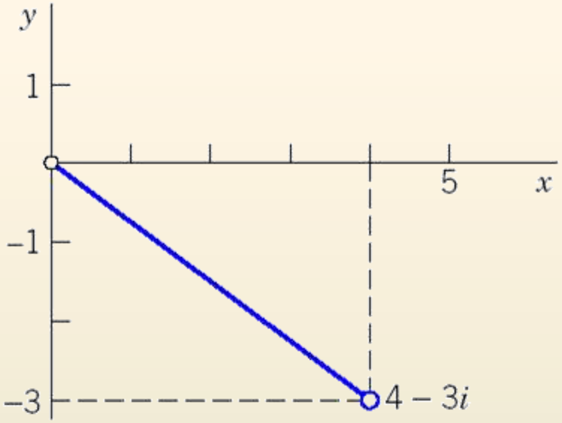
$$= \frac{72 - 6 + i(16+18)}{9^2 + 2^2} = \frac{66}{85} + \frac{43}{85}i$$

Complex plane

Complex plane : two perpendicular coordinate axes, the horizontal x-axis, called the real axis, and the vertical y-axis, called the imaginary axis.



The complex plane



The number 4-3*i* in the complex plane

Complex Conjugate Numbers

The complex conjugate $\bar{z}$ **of a complex number** $z = x + iy$

$$\bar{z} = x - iy$$

$$\text{(8)}\quad \operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z})$$

$$\text{(9)}\quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2, \quad \overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

example) $z_1 = 4 + 3i, \quad z_2 = 2 + 5i$

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(9) $\overline{(z_1 z_2)} = \overline{(4+3i)(2+5i)} = \overline{8 - 12 + i(20+6)}$

$$= \overline{-7 + 26i} = -7 - 26i,$$

Complex Conjugate Numbers

The complex conjugate $\bar{z}$ **of a complex number** $z = x + iy$

$$\bar{z} = x - iy$$

$$(8)\quad \operatorname{Re} z = x = \frac{1}{2}(z + \bar{z}), \quad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z})$$

$$(9)\quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2, \quad \overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

example) $z_1 = 4 + 3i, \quad z_2 = 2 + 5i$

(8) $\operatorname{Im} z_1 = \dfrac{1}{2i}\left[(4+3i) - (4-3i)\right] = 3i$

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$$= \overline{-7 + 26i} = -7 - 26i,$$

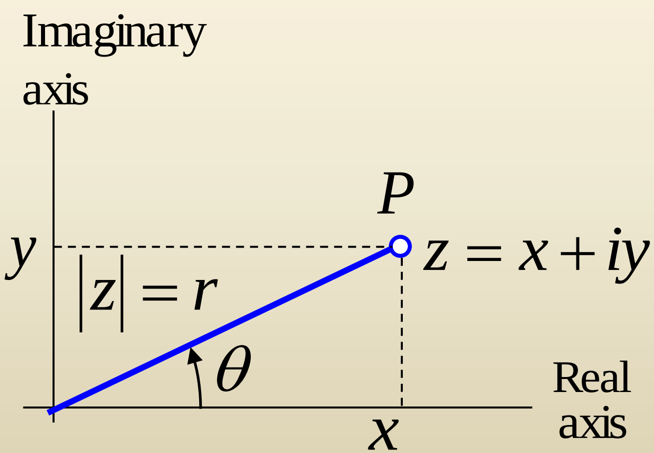
$$\bar{z}_1 \bar{z}_2 = (4-3i)(2-5i) = 8 - 15 + i(-20-6) = -7 - 26i$$

Polar Form of Complex Numbers

Polar Form of Complex Numbers

(1) $x = r\cos\theta, \qquad y = r\sin\theta$

We see that then $z = x+iy$ takes the so-called polar form



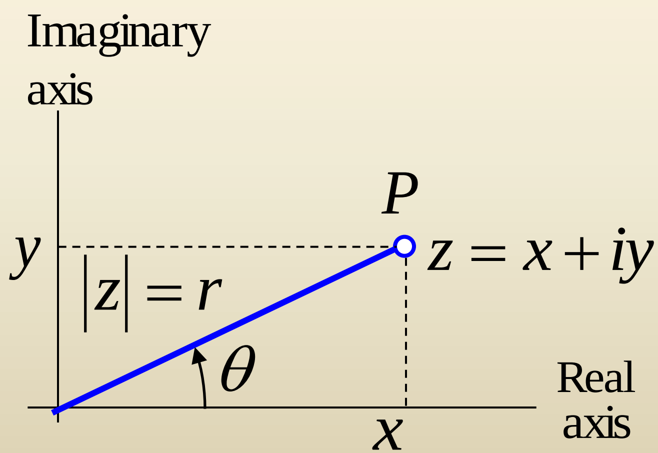
Complex plane, polar form of a complex number

Polar Form of Complex Numbers

$$(1)\quad x = r\cos\theta, \qquad y = r\sin\theta$$

We see that then $z = x+iy$ takes the so-called polar form

$$(2)\quad z = r(\cos\theta + i\sin\theta)$$



Complex plane, polar form of a complex number

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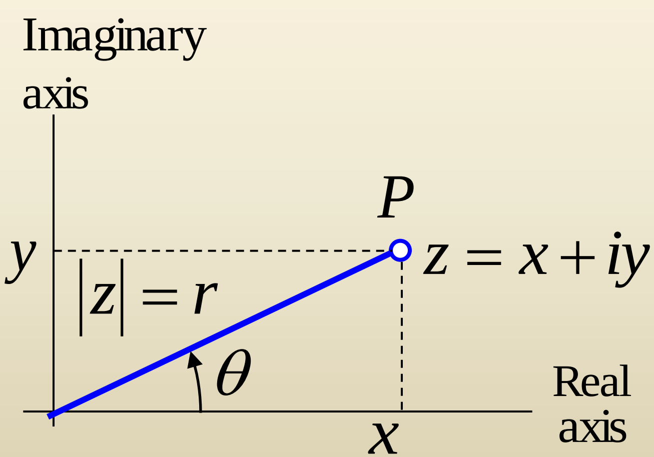
absolute value or modulus of z



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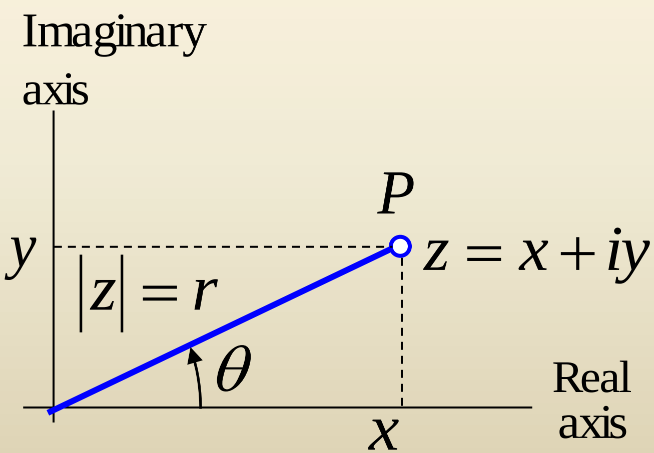
$$(3)\quad |z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$



Complex plane, polar form of a complex number

Polar Form of Complex Numbers

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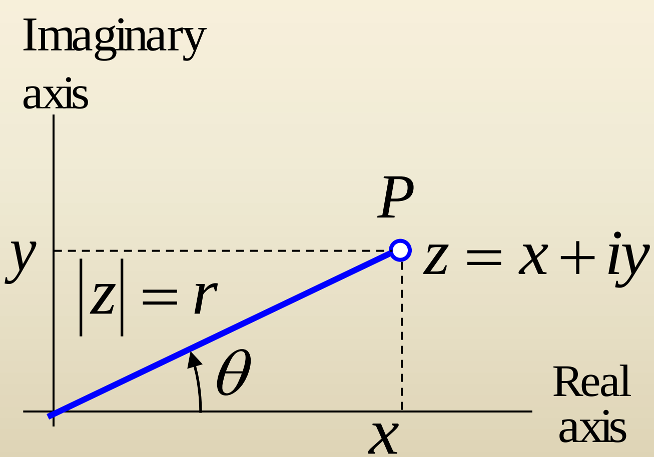
We see that then $z = x+iy$ takes the so-called polar form

$$(2)\quad z = r(\cos\theta + i\sin\theta)$$

absolute value or modulus of z

$$(3)\quad |z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}} \qquad \left(\begin{aligned} z\bar{z} &= (x+iy)(x-iy) \\ &= x^2-(iy)^2 = x^2+y^2 \end{aligned} \right)$$



Complex plane, polar form of a complex number

Polar Form of Complex Numbers

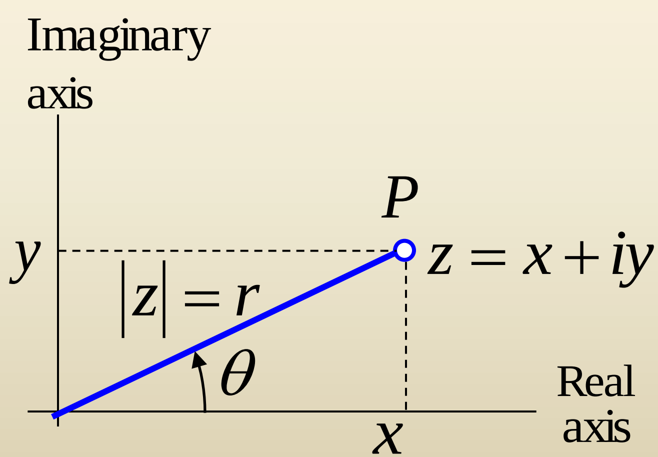
$$(2)\quad z = r(\cos\theta + i\sin\theta)$$

$|z|$: distance of the point z from the origin

θ : argument of z.

the directed angle from the positive x-axis to OP in Fig.

angles are measured in radians and positive in the counterclockwise sense.



Complex plane, polar form of a complex number

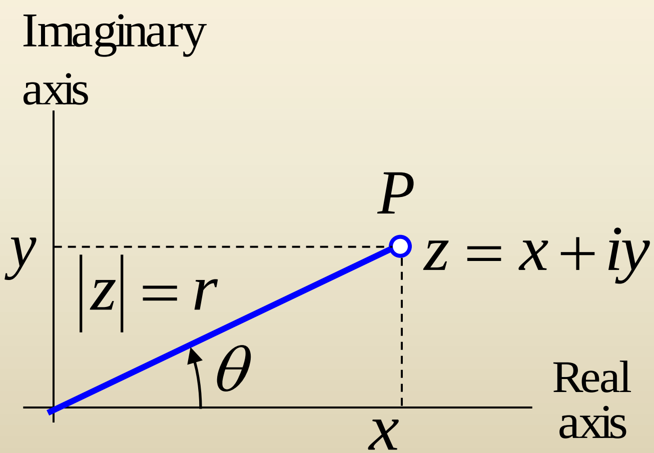
Polar Form of Complex Numbers

$$(2) \quad z = r(\cos\theta + i\sin\theta)$$

- **Principal value Arg z or arg z by the double inequality**

$$(5) \quad -\pi < \mathrm{Arg}\ z \le \pi$$

$$\arg z = \mathrm{Arg}\ z \pm 2n\pi \ \ (n = \pm 1, \pm 2, \cdots)$$



Complex plane, polar form of a complex number

Polar Form of Complex Numbers

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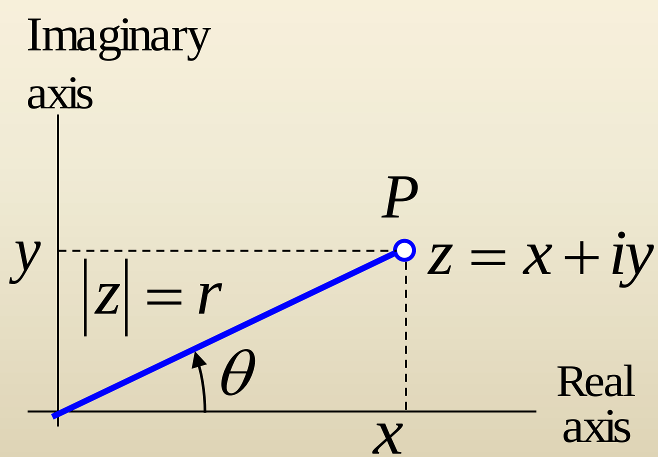

Complex plane, polar form of a complex number

Ex.) $z = \sqrt{3} + i$

Polar Form of Complex Numbers

$$(2)\quad z = r(\cos\theta + i\sin\theta)$$

- **Principal value $\mathbf{Arg}\,z$ or $\mathbf{arg}\,z$ by the double inequality**

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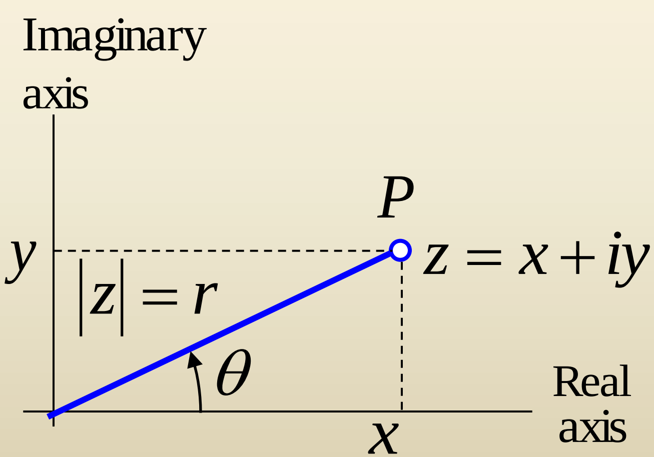

Complex plane, polar form of a complex number

Ex.) $z = \sqrt{3} + i$

$$\operatorname{Arg} z = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$\arg z = \frac{\pi}{6} \pm 2n\pi \quad (n = \pm 1, \pm 2, \cdots)$$

Polar Form of Complex Numbers

$$(2) \quad z = r(\cos\theta + i\sin\theta)$$

$$(5) \quad -\pi < \text{Arg } z \le \pi$$

$$\arg z = \text{Arg } z \pm 2n\pi \quad (n = \pm 1, \pm 2, \cdots)$$

$$z = 1 + i$$

Find Polar Form and Principal Value Arg z

Polar Form of Complex Numbers

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Find Polar Form and Principal Value Arg z

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Polar Form of Complex Numbers

$$(2)\quad z = r(\cos\theta + i\sin\theta)$$

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$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\therefore |z| = \sqrt{2}, \quad \operatorname{Arg} z = \frac{\pi}{4}$$

Polar Form of Complex Numbers

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Polar Form of Complex Numbers

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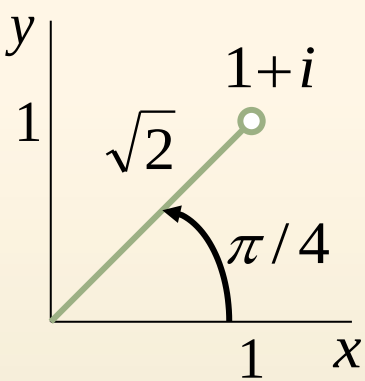

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\therefore |z| = \sqrt{2}, \quad \operatorname{Arg} z = \frac{\pi}{4}$$

$$\arg z = \frac{\pi}{4} \pm 2n\pi \quad (n = 0, 1, \cdots)$$

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

- **Multiplication.**

$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

- **Multiplication.**

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)] \end{aligned}$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$$

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2$$

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

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$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$$

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$$(7) \quad z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

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$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$$

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$$(8)\quad |z_1 z_2| = r_1 r_2 = |z_1||z_2|$$

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

- **Multiplication.**

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$$

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$$(9)\quad \arg|z_1 z_2| = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

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Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_3 = r_3(\cos\theta_3 + i\sin\theta_3),$$
$$z_4 = \cos\theta_4 + i\sin\theta_4$$

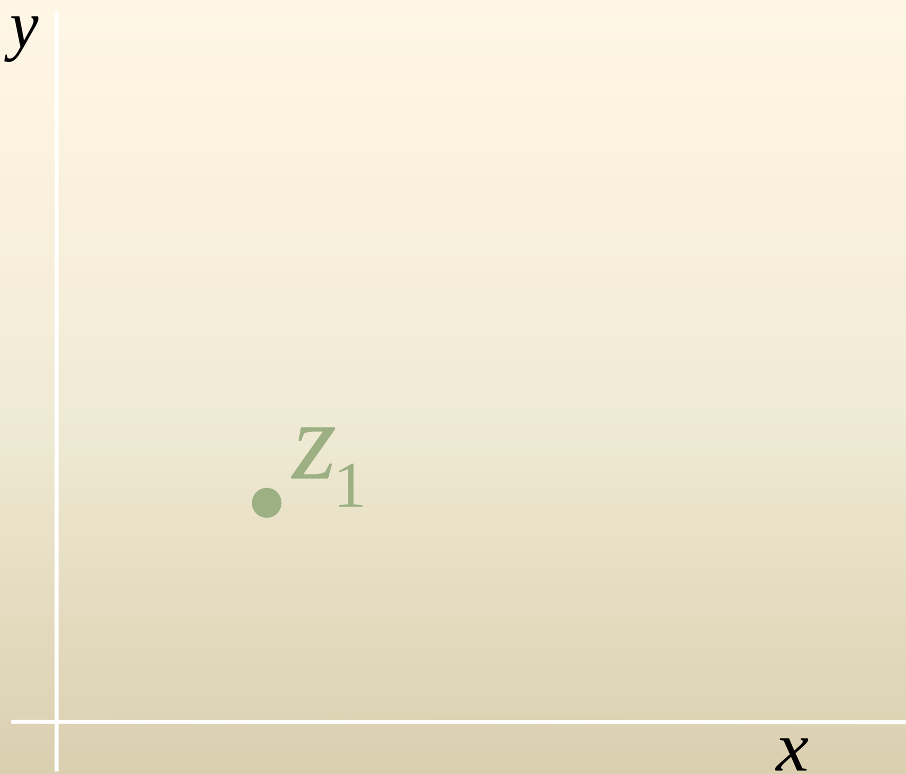
$$(7)\quad z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\,\theta_1),\qquad z_3 = r_3(\cos\theta_3 + i\sin\,\theta_3),$$

$$z_2 = r_2(\cos\theta_2 + i\sin\,\theta_2)\qquad z_4 = \cos\theta_4 + i\sin\,\theta_4$$



$(7)\ \ z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$
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$$z_3 = r_3(\cos\theta_3 + i\sin\theta_3),$$
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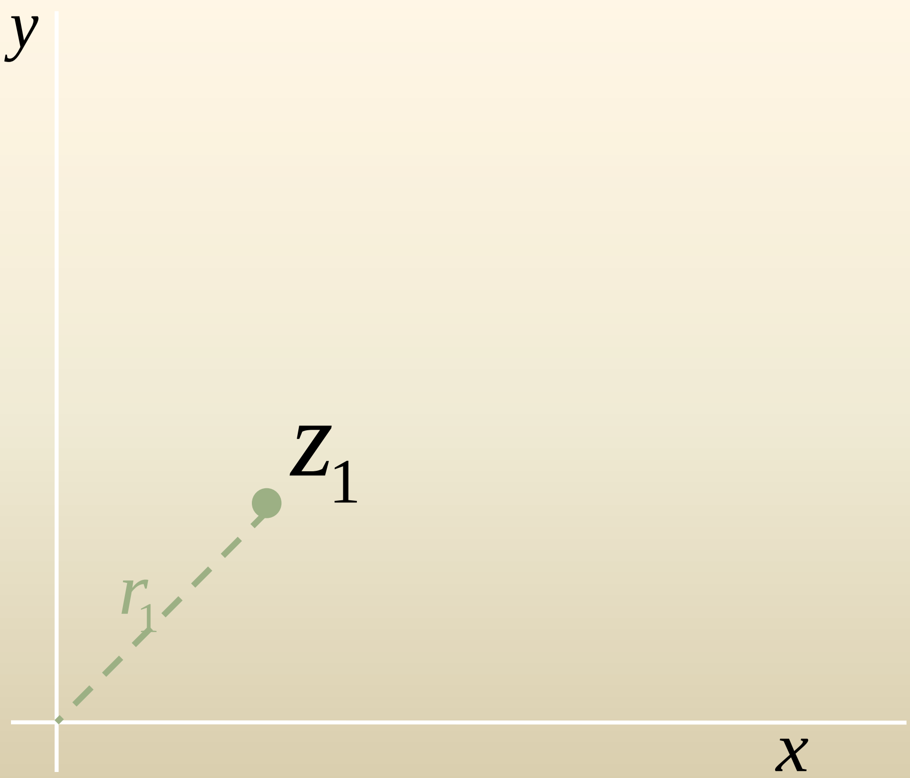

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$
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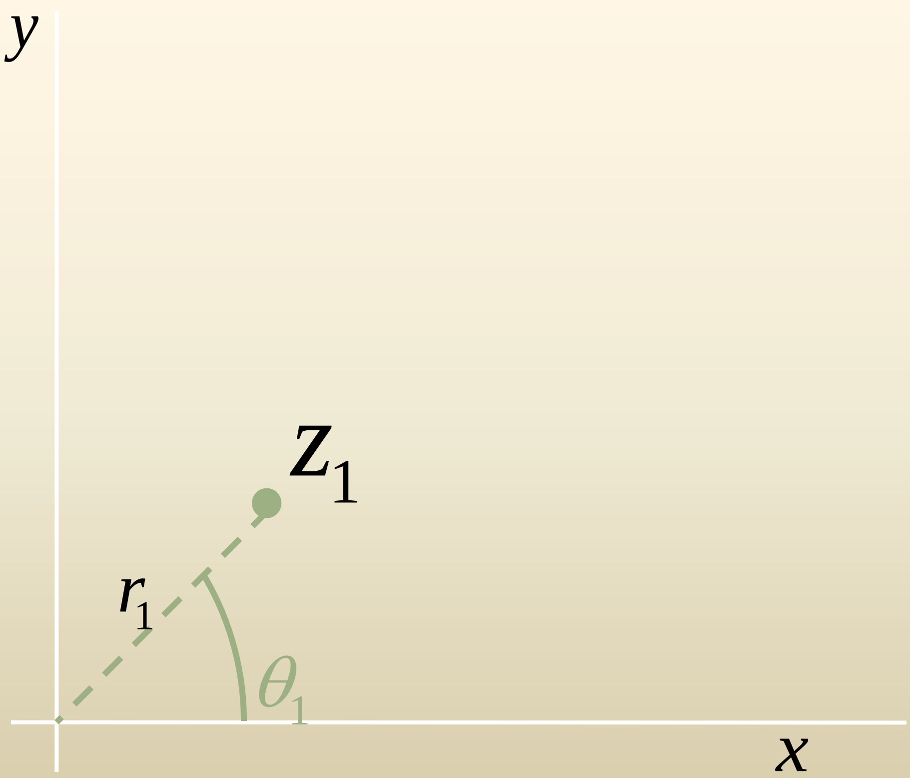

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

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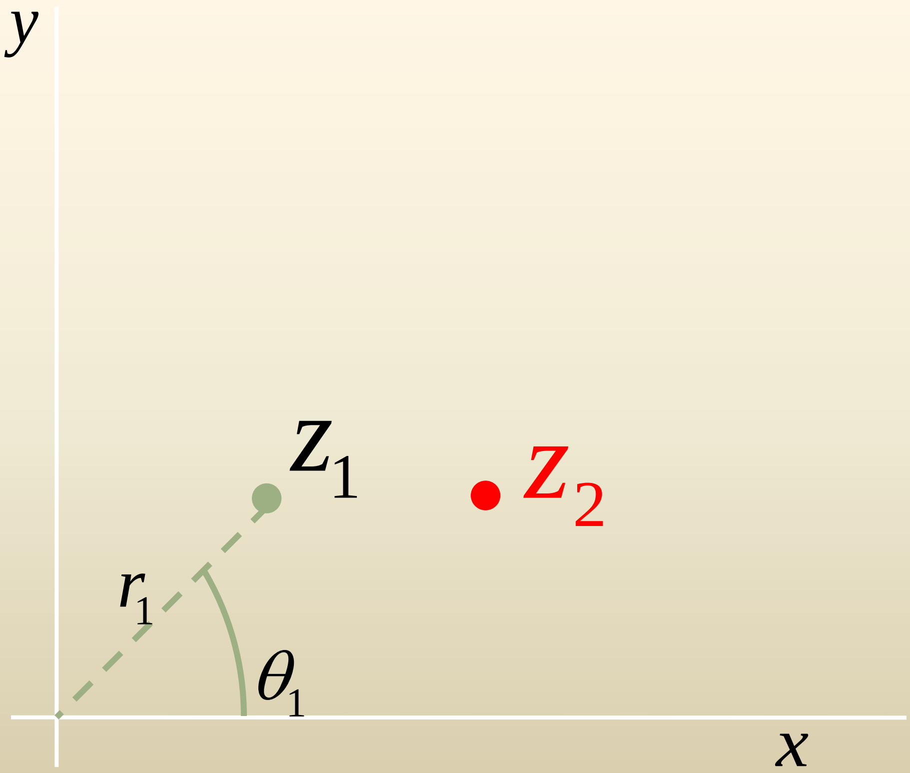

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Multiplication and Division in Polar Form

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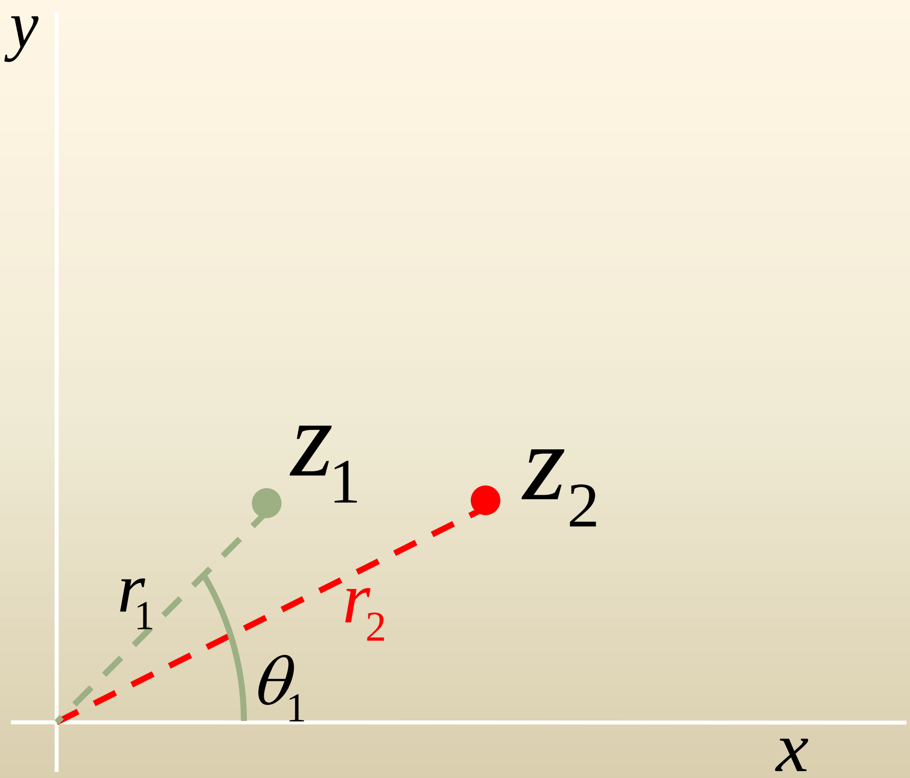

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

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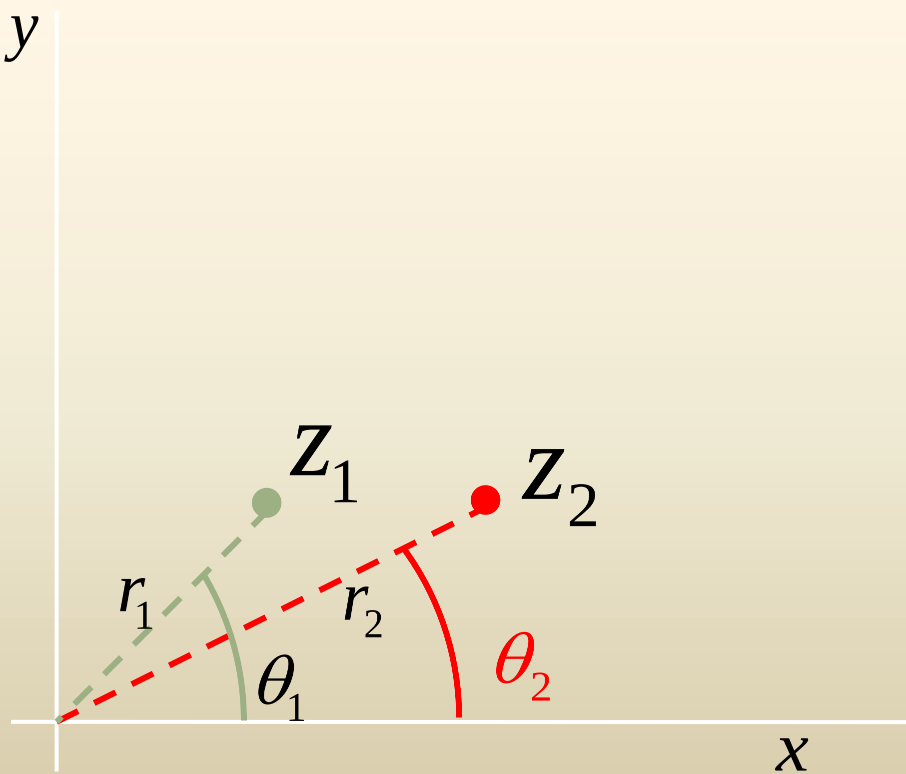

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$
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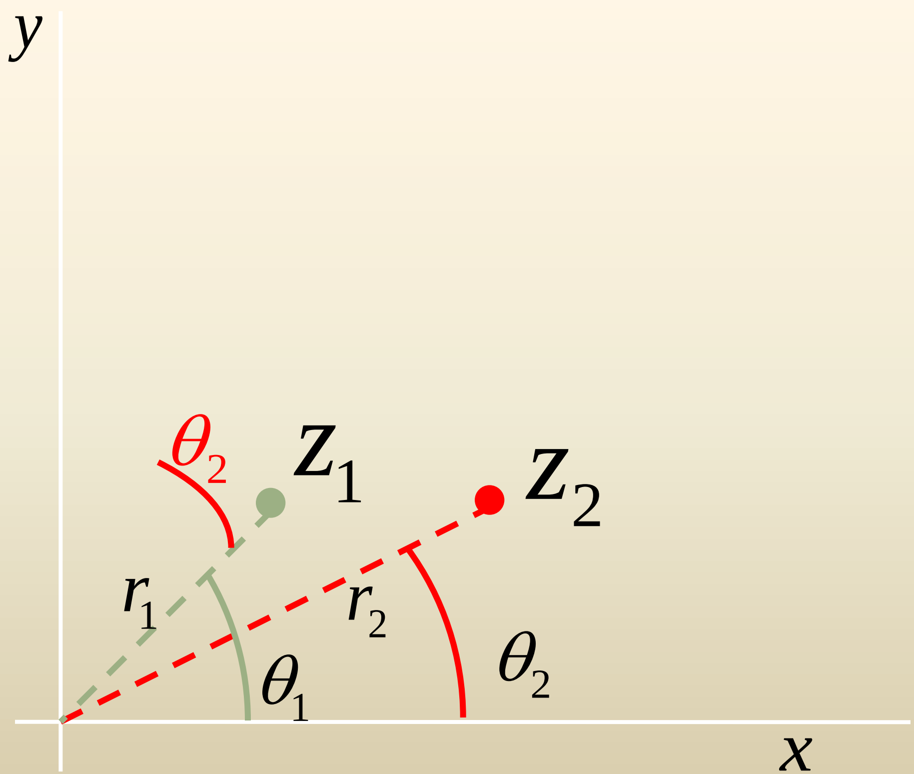

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$

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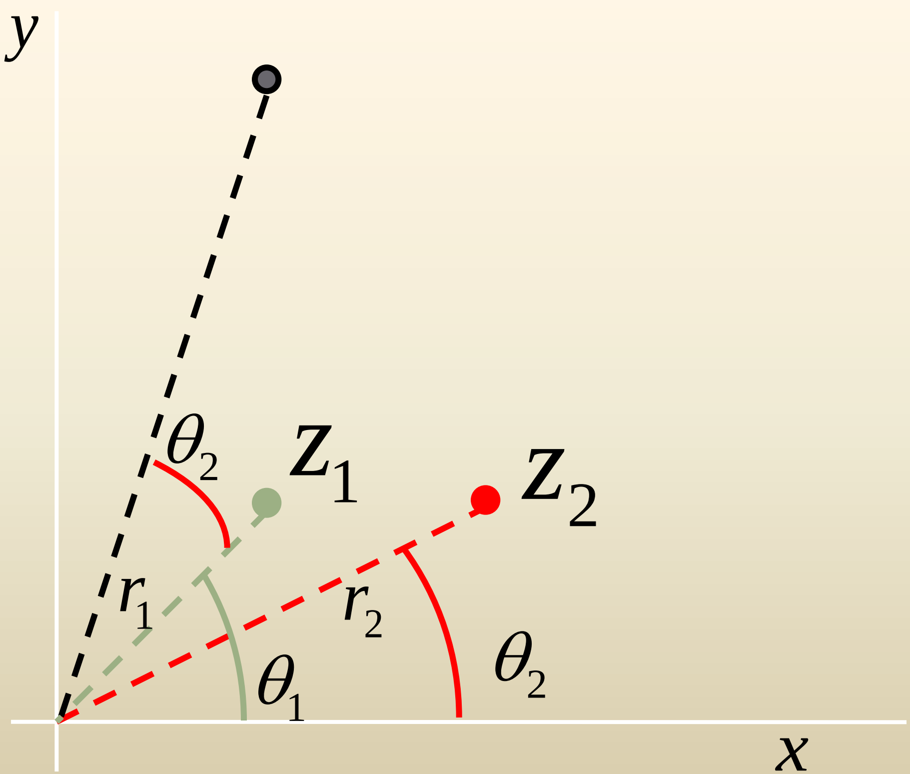

$$(7)\quad z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$
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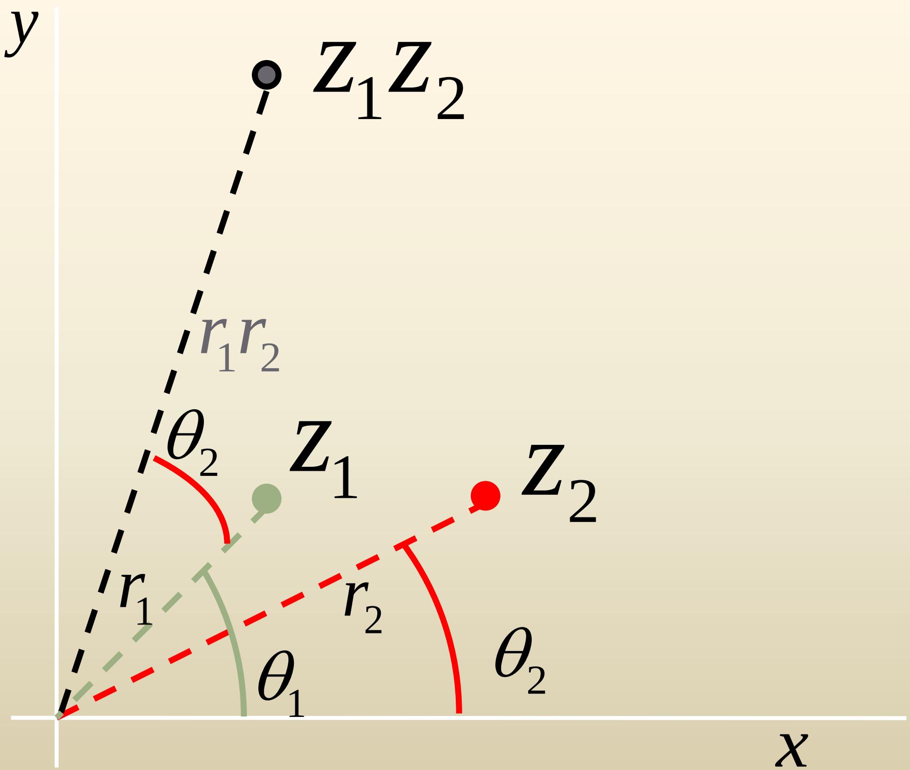
$$z_3 = r_3(\cos\theta_3 + i\sin\theta_3),$$
$$z_4 = \cos\theta_4 + i\sin\theta_4$$



Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1),$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

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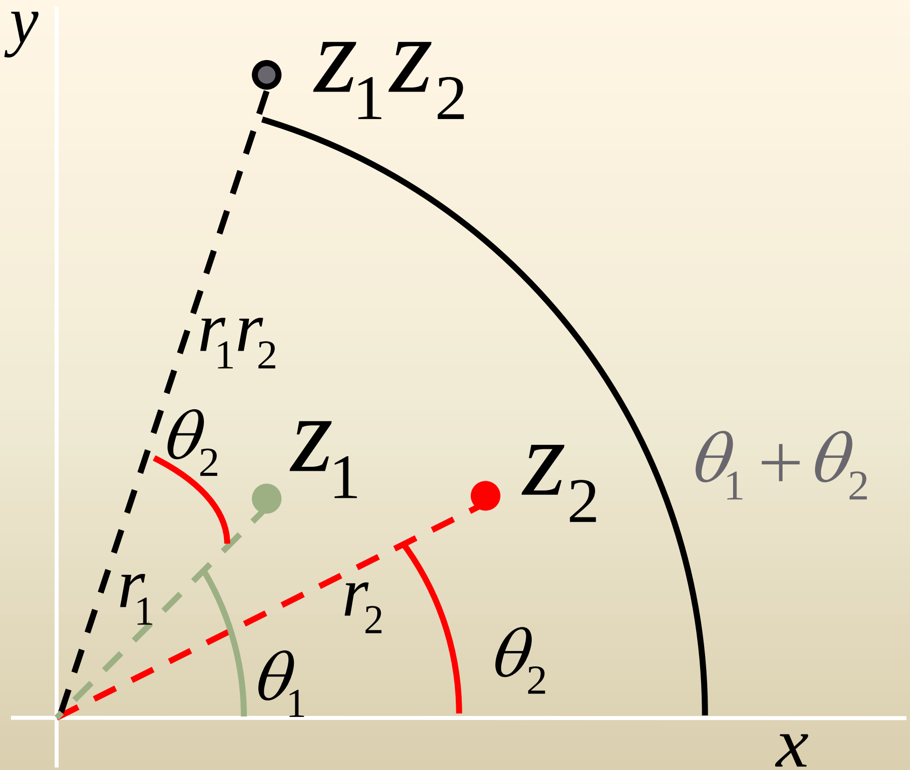
$$z_4 = \cos\theta_4 + i\sin\theta_4$$



Polar Form of Complex Numbers

Multiplication and Division in Polar Form

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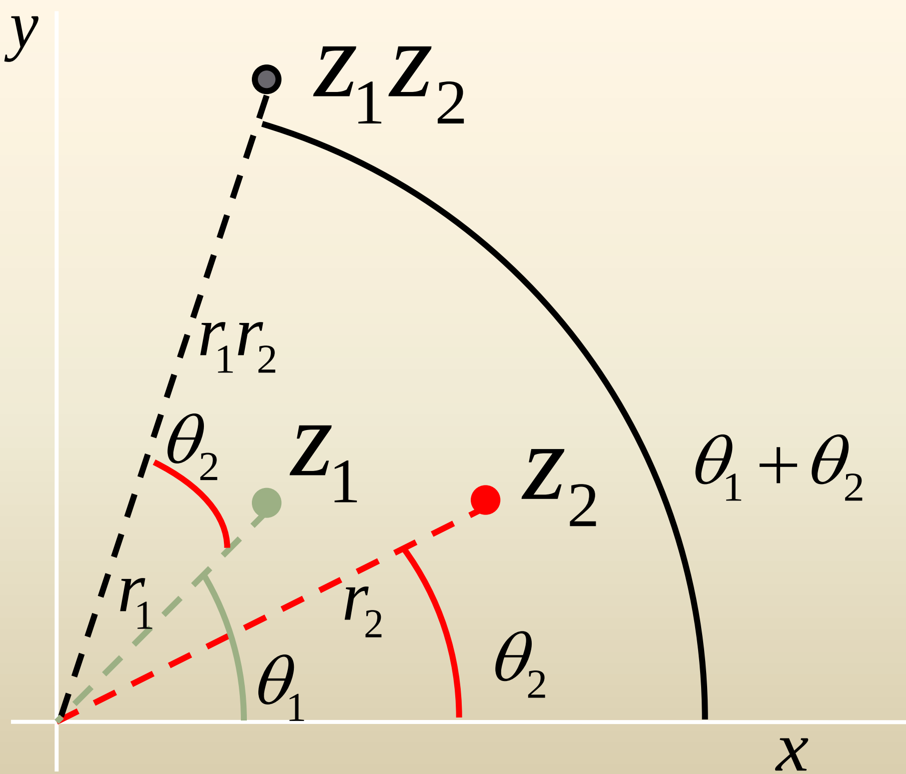
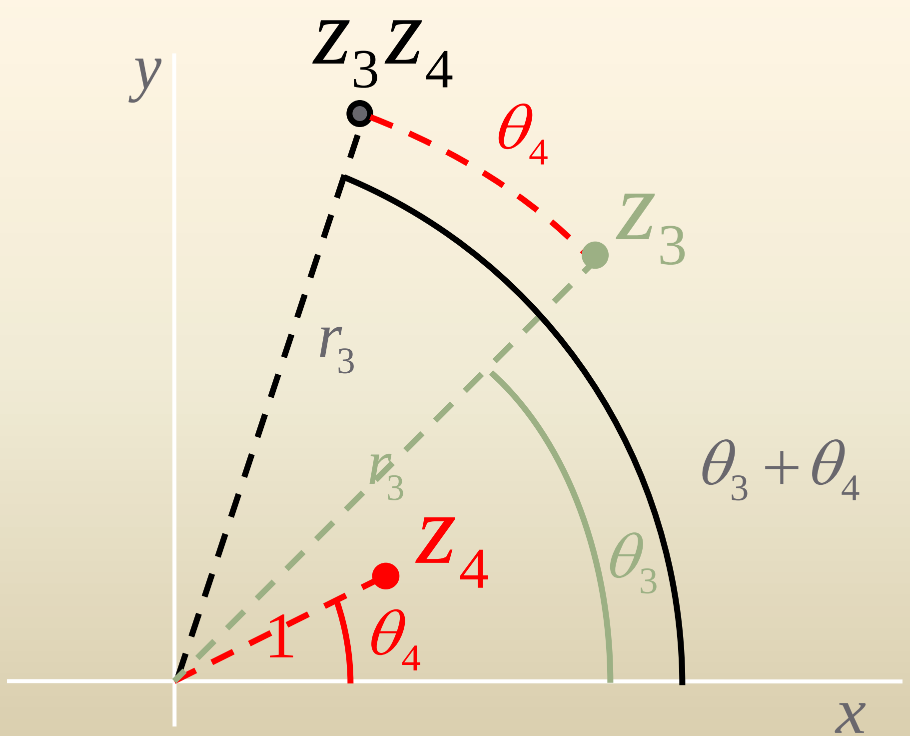
$$z_3 = r_3(\cos\theta_3 + i\sin\theta_3),$$
$$z_4 = \cos\theta_4 + i\sin\theta_4$$





$$(9)\quad \arg(z_1 z_2) = \arg z_1 + \arg z_2$$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

- **Division.**

$$(9) \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2$$

Polar Form of Complex Numbers

Multiplication and Division in Polar Form

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

- **Division.**

$$z_1 = \frac{z_1}{z_2} z_2$$

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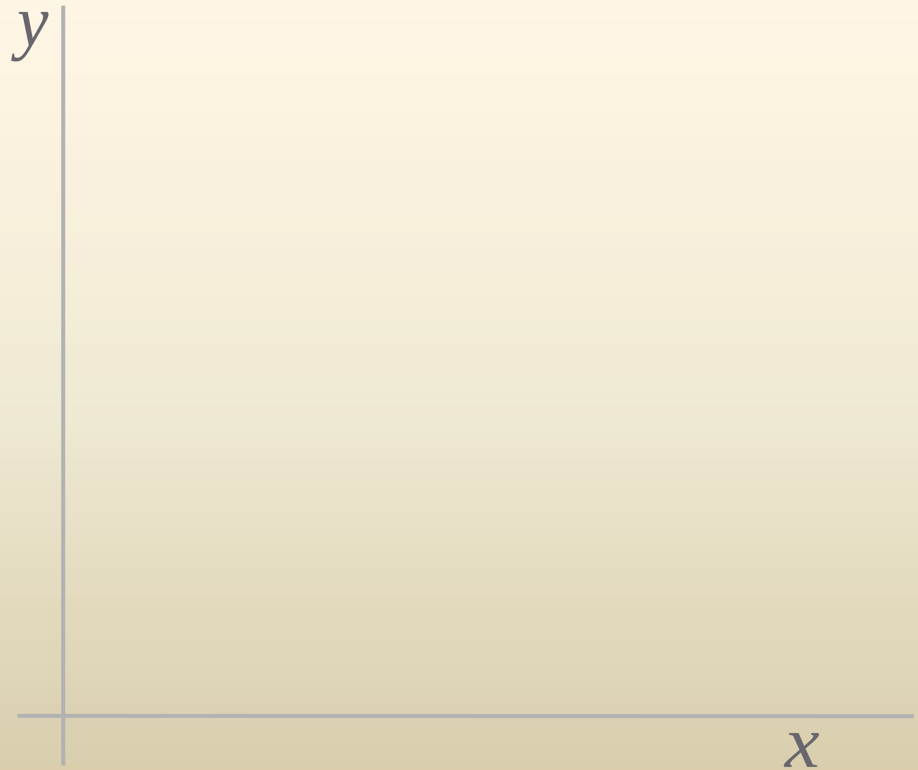

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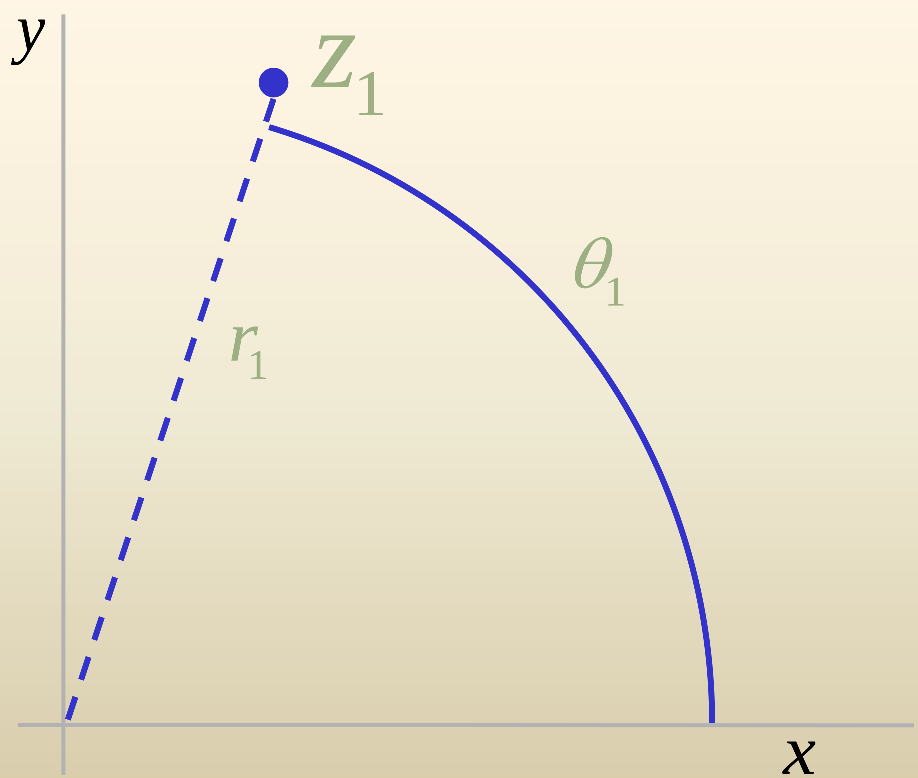

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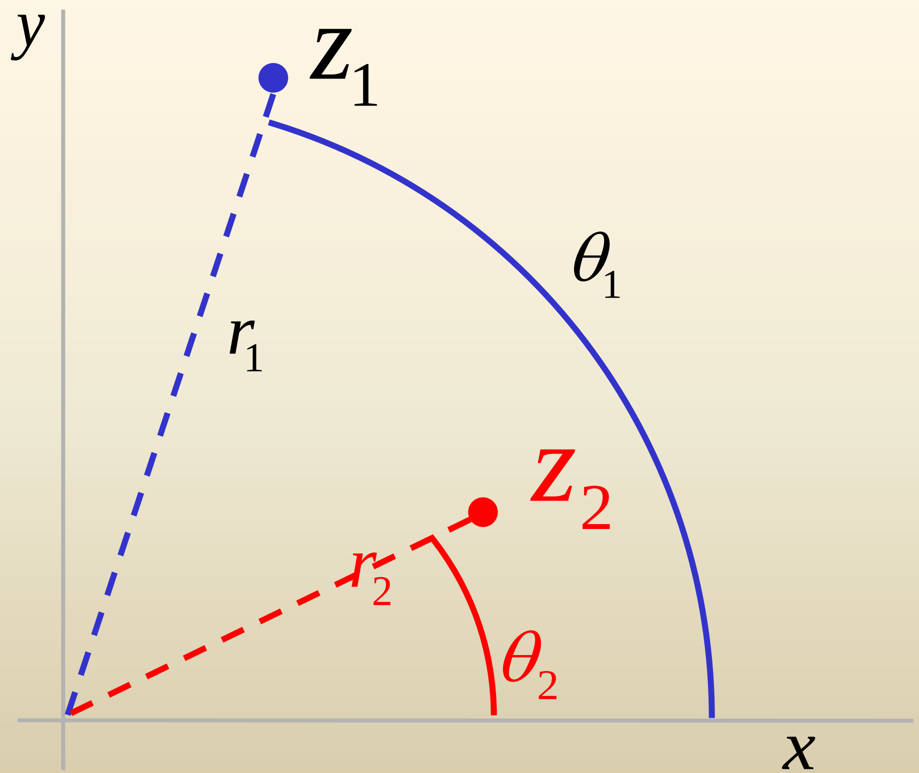

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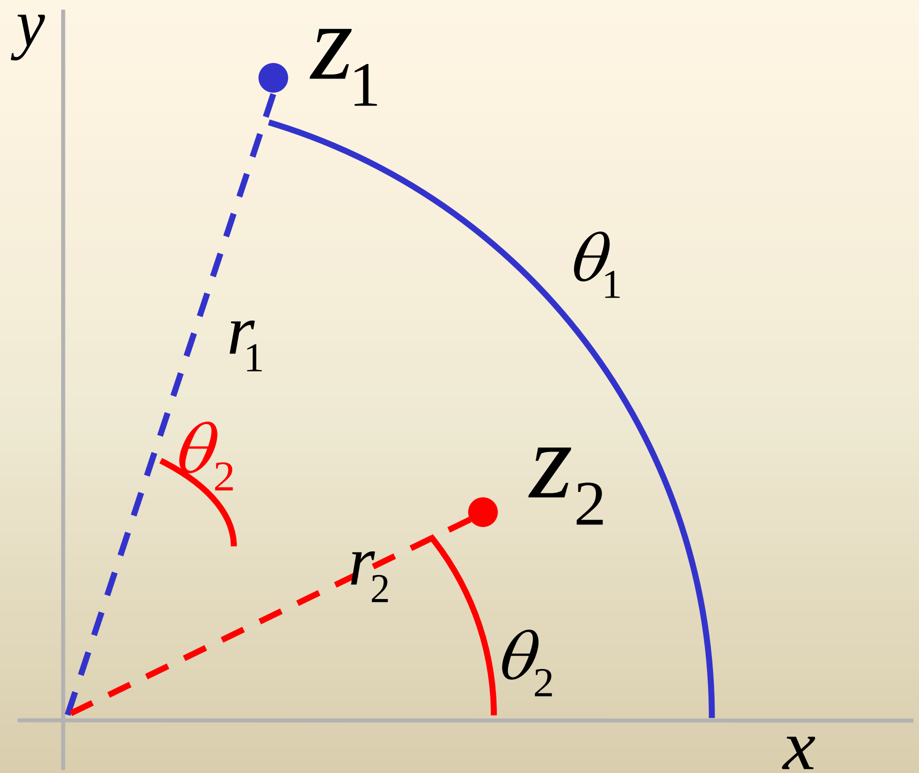

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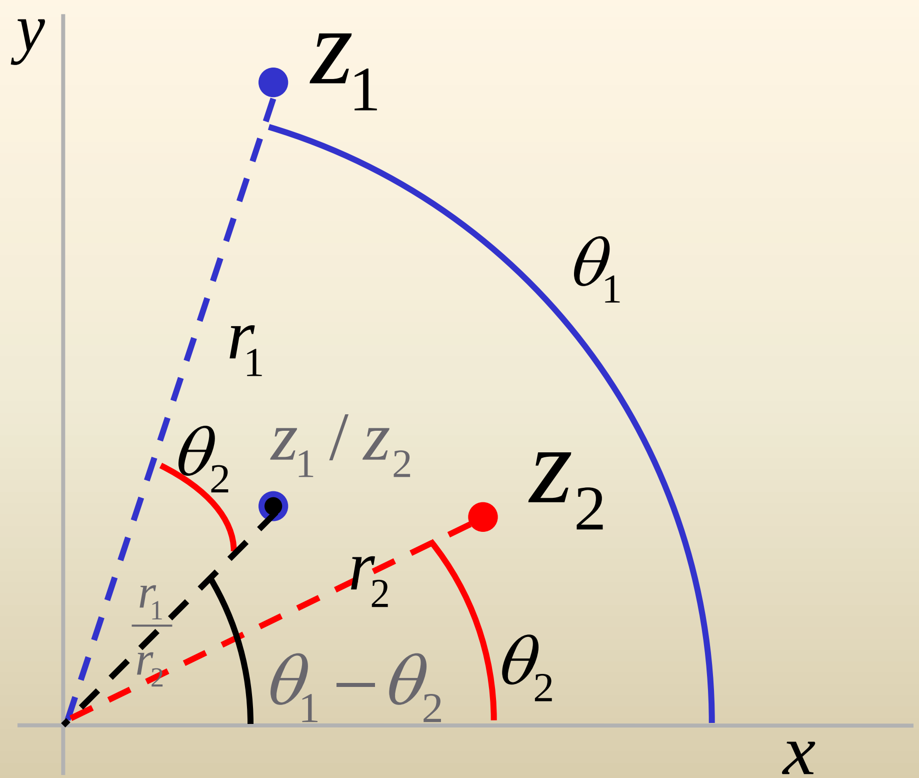


Complex Functions

-Exponential

-Trigonometric and Hyperbolic

-Logarithm

-General Powers

Complex Function

Real function f defined on a set S of real numbers (usually an interval) is a rule that assigns to every x in S a real number $f(x)$, called the value of f at x.

$$f = f(x)$$

Complex function f defined on a set S (a set of complex numbers) is a rule that assigns to every z in S a complex number w, called the value of f at z.

$$w = f(z)$$

z : complex variables

S : domain of definition of f or domain of f.

example) $w = f(z) = z^2 + 3z$

Complex Function

$$w = f(z)$$

range of f : the set of all values of a function f

w is complex, and we write $w = u + iv$ where u and v are the real and imaginary parts, respectively.

Now w depend on $z = x + iy$. Hence u becomes a real function of x and y, so does v.

$$\boxed{w = f(z) = u(x, y) + iv(x, y)}$$

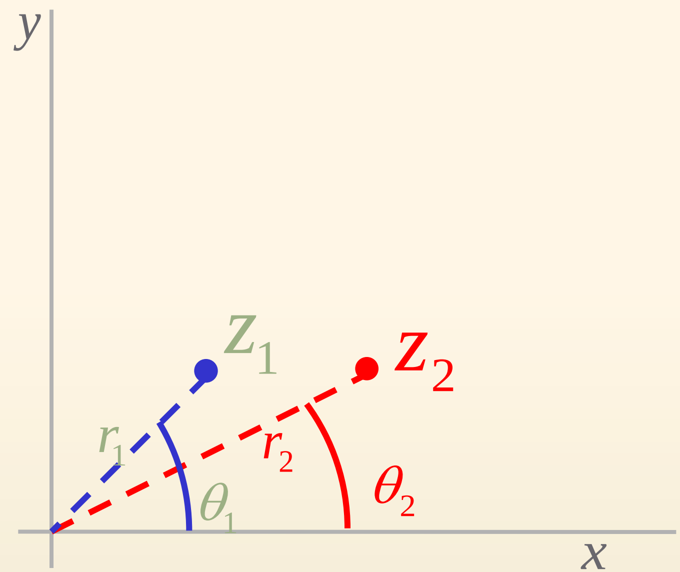
This shows that a complex function $f(z)$ is equivalent to a pair of real functions $u(x,y)$ and $v(x,y)$, each depending on the two real variables x and y.

Complex Function

$$\text{Let } w = f(z) = z_1 z + z_2$$

$$\text{where } z_1, z_2 : \text{constant}$$

Geometrically, what happens if we input a certain complex z into the function f?

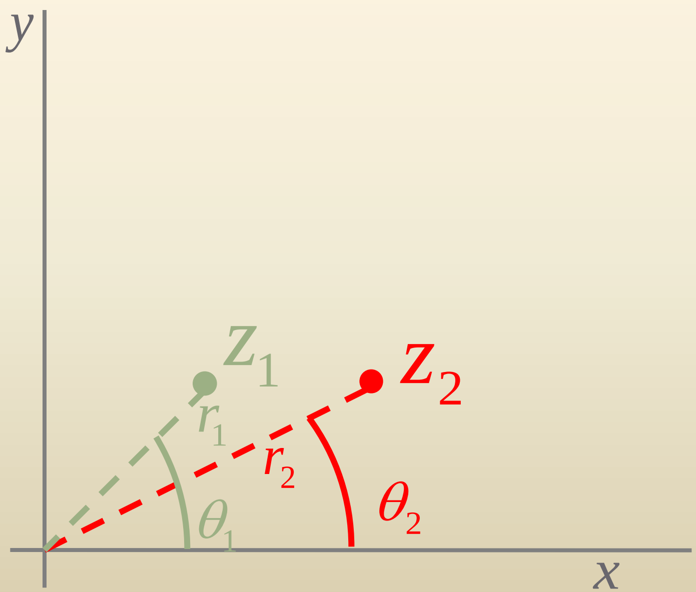


Complex Function

$$w = f(z) = z_1 z + z_2$$

where z_1, z_2 : constant

$$z = r(\cos\theta + i\sin\theta)$$



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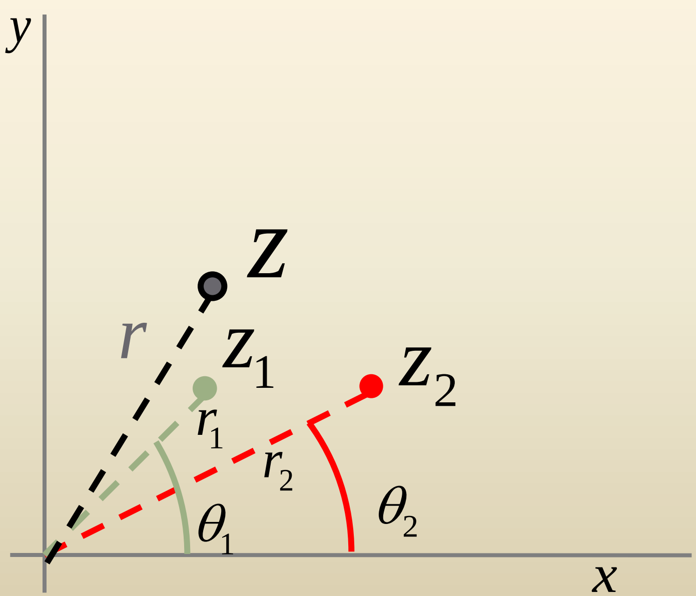

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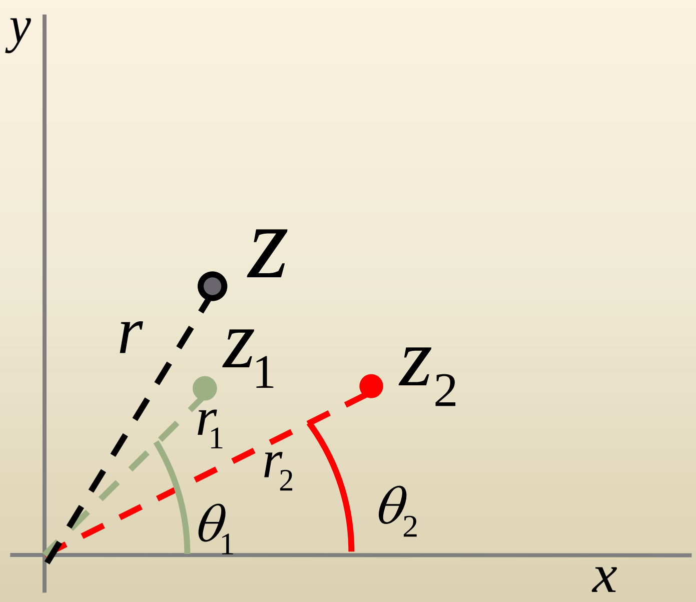
$$z = r(\cos\theta + i\sin\theta)$$

$$z_1 z = r_1 r[\cos(\theta_1 + \theta) + i\sin(\theta_1 + \theta)]$$



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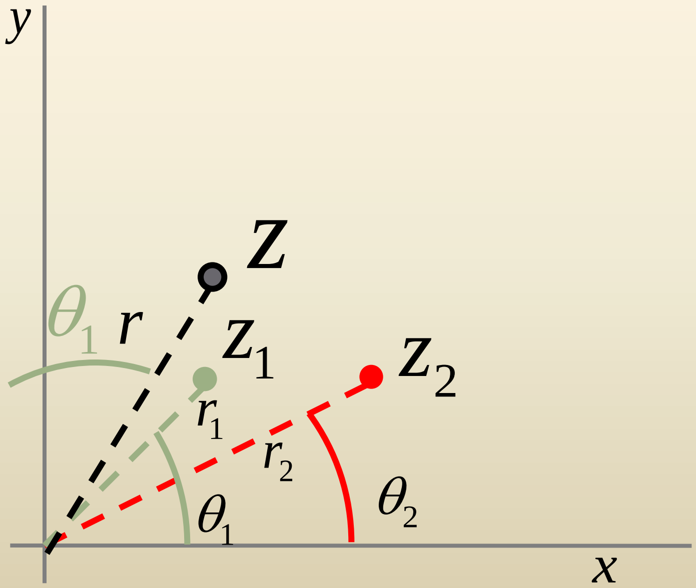

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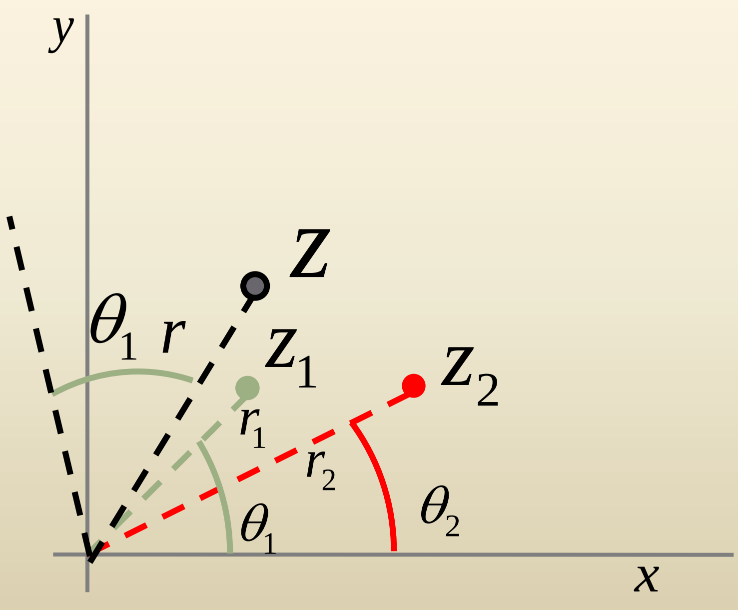

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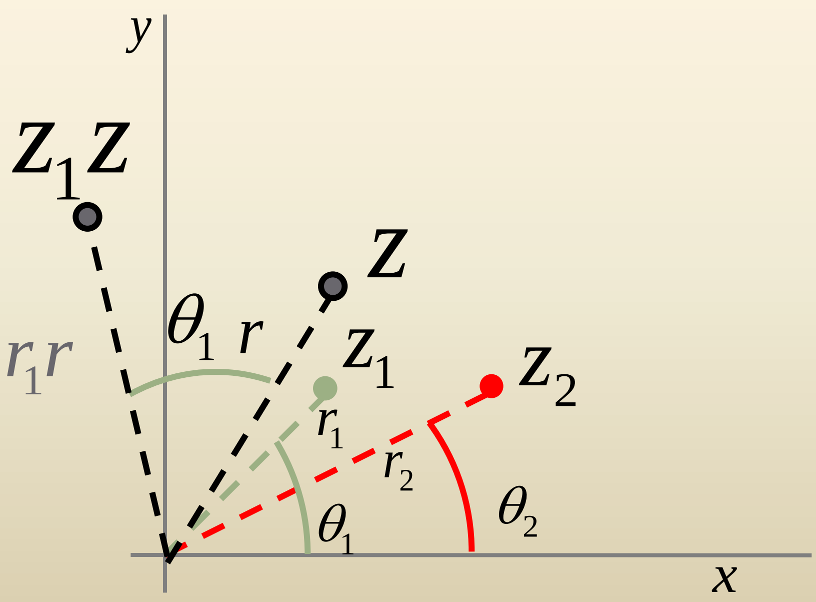

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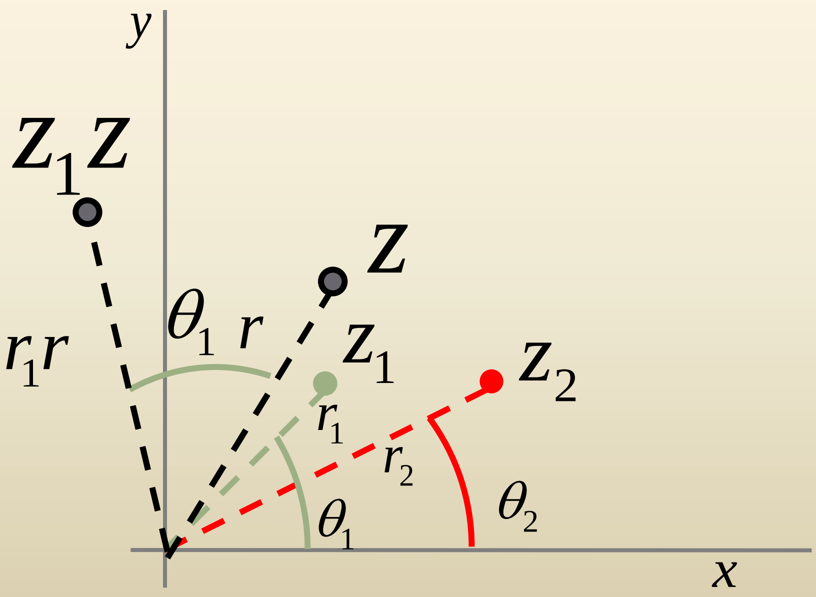

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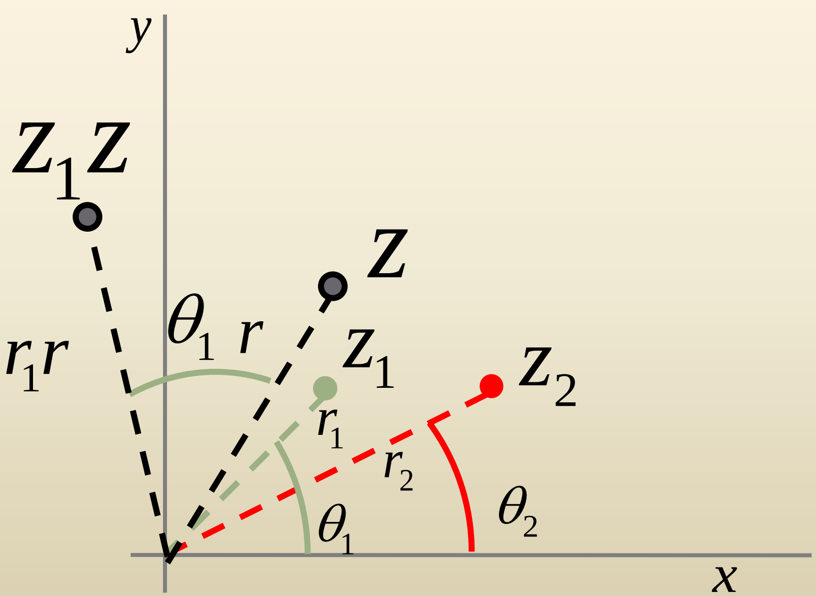
$$z_1 z + z_2 = r_1 r[\cos(\theta_1 + \theta) + i\sin(\theta_1 + \theta)] + r_2\cos\theta_2 + r_2 i\sin\theta_2$$



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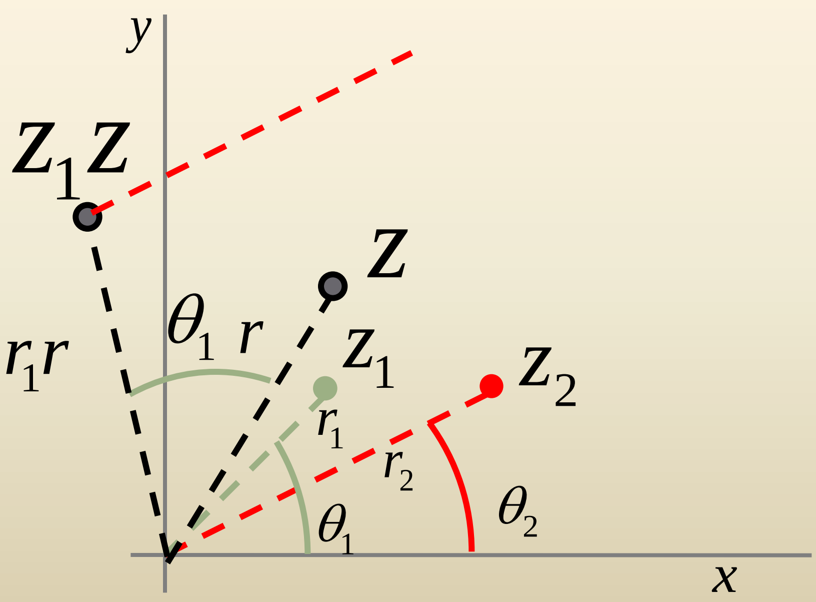

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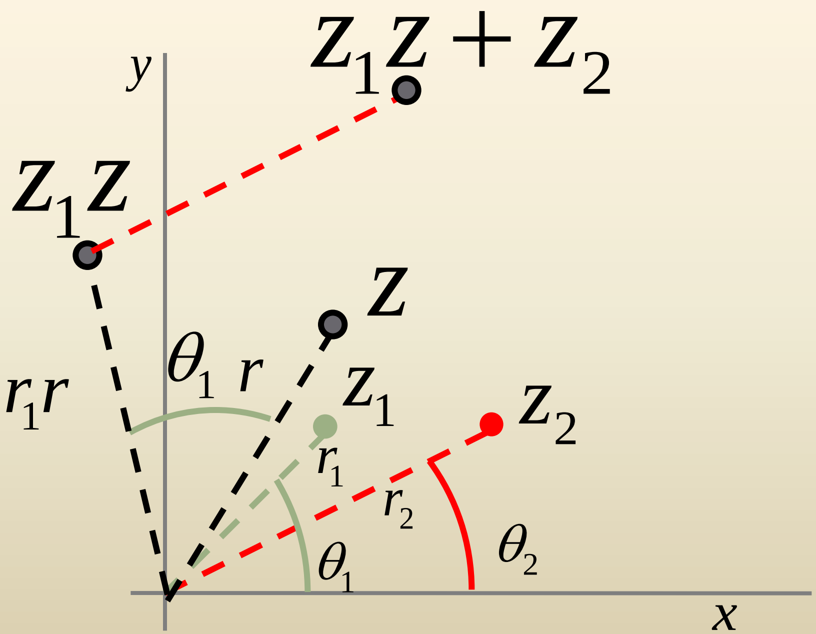

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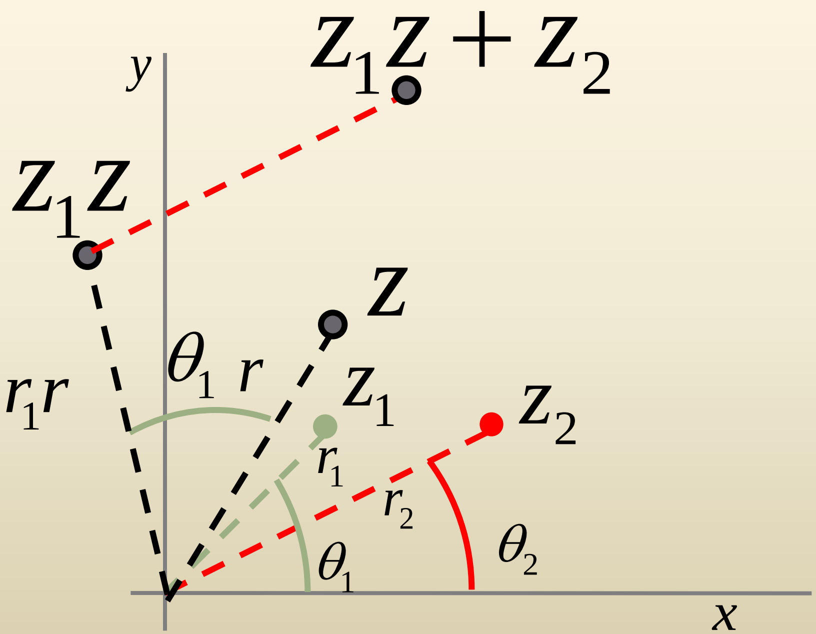
: translation



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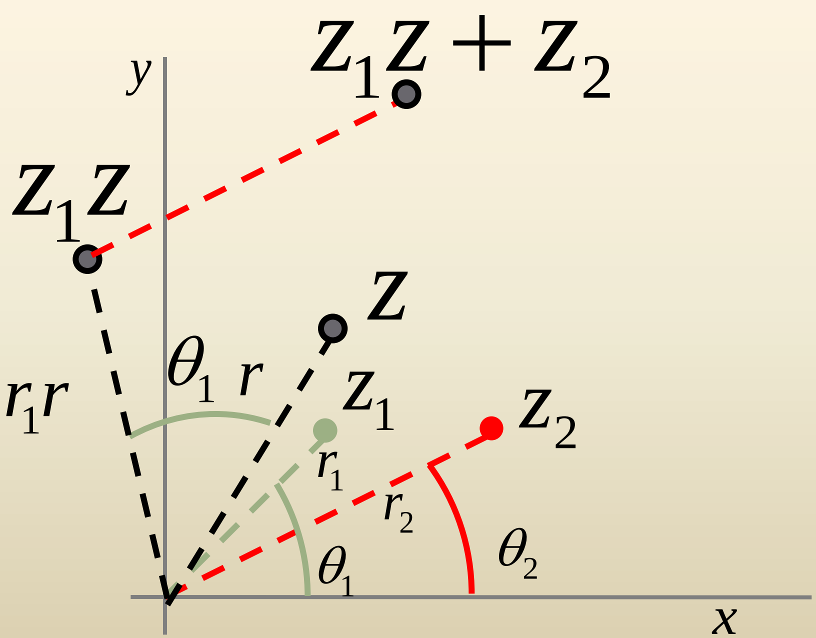
: **translation**



The complex function $f(z)$

- **rotate : θ_1**
- **multiply by : r_1**
- **translate : $r_2\cos\theta_2, r_2\sin\theta_2$**

Exponential Function

The most important analytic functions, the complex exponential function

$$e^z \text{ also written } \exp z. \qquad (z = x + iy)$$

Definition of the complex exponential function

$$(1) \quad e^z = e^x(\cos y + i \sin y) \qquad e^z = e^{x+iy} = e^x e^{iy}$$

This definition is motivated by the fact the e^z extends the real exponential function e^x of calculus in a natural fashion. Namely (A,B,C) :

(A) $e^z = e^x$ for real $z = x$ $\qquad (\because y = 0,\ \cos y = 1,\ \sin y = 0)$

(B) e^z is analytic for all z.

(C) The derivative of e^z is e^z, that is (2) $(e^z)' = e^z$ $\qquad (e^z)' = \dfrac{\partial}{\partial x} e^z$

$$(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x$$

$$= e^x \cos y + i e^x \sin y = e^z$$

Trigonometric and Hyperbolic Functions

We now want to extend the familiar real trigonometric functions to complex trigonometric functions.

Euler formula

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

By addition and subtraction, we obtain the real cosine and sine

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

This suggests the following <u>definitions</u> for complex values $z = x + iy$

$$(1) \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

Differ from a real cosine and sine : non-periodic

Trigonometric and Hyperbolic Functions

(1) $\cos z = \dfrac{1}{2}(e^{iz} + e^{-iz}), \ \sin z = \dfrac{1}{2i}(e^{iz} - e^{-iz})$

Trigonometric and Hyperbolic Functions

$$(1)\quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}),\ \ \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

As in calculus we define

$$(2)\quad \tan z = \frac{\sin z}{\cos z},\quad \cot z = \frac{\cos z}{\sin z}$$ **and** $$(3)\quad \sec z = \frac{1}{\cos z},\quad \csc z = \frac{1}{\sin z}.$$

$$(4)\quad (\cos z)' = -\sin z,\ \ (\sin z) = \cos z,\ \ (\tan z)' = \sec^2 z$$

$$(5)\quad e^{iz} = \cos z + i \sin z \quad \text{for all} \quad z$$

Trigonometric and Hyperbolic Functions

$$(1)\quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}),\quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

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The real and imaginary parts of $\cos z$ and $\sin z$ are needed in computing values, and they also help in displaying properties of our functions

$$(6)\quad \begin{aligned} &(a)\ \cos z = \cos x \cosh y - i \sin x \sinh y \\ &(b)\ \sin z = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

$$(7)\quad \begin{aligned} &(a)\ |\cos z|^2 = \cos^2 x + \sinh^2 y \\ &(b)\ |\sin z|^2 = \sin^2 x + \sinh^2 y \end{aligned}$$

Trigonometric and Hyperbolic Functions

Show that

$$(6)\quad \begin{aligned} &(a)\ \cos z = \cos x \cosh y - i \sin x \sinh y \\ &(b)\ \sin z = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

From $(1)\ \cos z = \frac{1}{2}(e^{iz} + e^{-iz}),\ \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

Trigonometric and Hyperbolic Functions

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From (1) $\cos z = \dfrac{1}{2}(e^{iz} + e^{-iz}),\ \ \sin z = \dfrac{1}{2i}(e^{iz} - e^{-iz})$

$$\cos z = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)})$$

Trigonometric and Hyperbolic Functions

Show that

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$$\cos z = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) = \frac{1}{2}(e^{-y+ix} + e^{y-ix})$$

Trigonometric and Hyperbolic Functions

Show that

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$$= \frac{1}{2}e^{-y}(\cos x + i \sin x) + \frac{1}{2}e^{y}(\cos x - i \sin x)$$

Trigonometric and Hyperbolic Functions

Show that

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$$= \cos x \cosh y - i \sin x \sinh y$$

Trigonometric and Hyperbolic Functions

Show that

(6) $(a)\ \cos z = \cos x \cosh y - i \sin x \sinh y$

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Trigonometric and Hyperbolic Functions

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$$\begin{aligned} \left|\cos z\right|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y \quad (\because 1 = \cosh^2 y - \sinh^2 y) \\ &= \cos^2 x + (\cos^2 x + \sin^2 x) \sinh^2 y \\ &= \cos^2 x + \sinh^2 y \end{aligned}$$

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\end{aligned}$$

$$\cosh y = \frac{e^{y} + e^{-y}}{2}, \quad \sinh y = \frac{e^{y} - e^{-y}}{2}$$

Trigonometric and Hyperbolic Functions

The complex hyperbolic cosine and sine are defined by the formulas

$$\boxed{(11) \quad \cosh z = \frac{1}{2}(e^{z} + e^{-z}), \quad \sinh\ z = \frac{1}{2}(e^{z} - e^{-z})}$$

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The other hyperbolic functions are defined by

$$(11)\quad \begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \\ \operatorname{sech} z &= \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}. \end{aligned}$$

Trigonometric and Hyperbolic Functions

Complex Trigonometric and Hyperbolic Functions Are Related.

$$(1)\quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

$$(11)\quad \cosh z = \frac{1}{2}(e^{z} + e^{-z}), \quad \sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

If in (11), we replace z by iz and then use (1), we obtain

$$\cosh iz = \frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$$

$$\sinh iz = \frac{1}{2}(e^{iz} - e^{-iz}) = i \sin z.$$

$$(14)\quad \cosh iz = \cos z, \quad \sinh iz = i \sin z.$$

Similarly, in (1), we replace z by iz and then use (11), we obtain conversely

$$(15)\quad \cos iz = \cosh z, \quad \sin iz = i \sinh z.$$

$$\cos iz = \frac{1}{2}(e^{-z} + e^{z}) = \cosh z$$

$$\sin iz = \frac{1}{2i}(e^{-z} - e^{z}) = i \sinh z$$

Logarithm

The natural logarithm of $z = x + iy$, $\ln z$

The natural logarithm is defined as the inverse of the exponential function.

$w = \ln z$ is defined for $z \neq 0$ by the relation $\underline{e^w = z}$

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If we set $w = u + iv$ and $z = re^{i\theta}$, $e^w = e^{u+iv} = e^{\ln z} = z = re^{i\theta}$.

$$\left.\begin{array}{l} \therefore e^u = r,\ v = \theta \\ e^u = r \text{ gives } u = \ln r, \text{ where } r = |z| \end{array}\right\} \ln z = w = u + iv = \ln r + i\theta$$

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The complex natural logarithm $\ln z\ (z \neq 0)$ is infinitely many-valued.

$$\theta = \arg z = \mathrm{Arg}\, z \pm 2n\pi$$

$$-\pi < \mathrm{Arg}\, z \leq \pi \quad : \text{principal value}$$

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They all have the same real part, and their imaginary parts differ by integer multiples of 2π.

$$(3)\ \ \ln z = \operatorname{Ln} z \pm 2n\pi i = \ln|z| + i\operatorname{Arg} z \pm 2n\pi i$$

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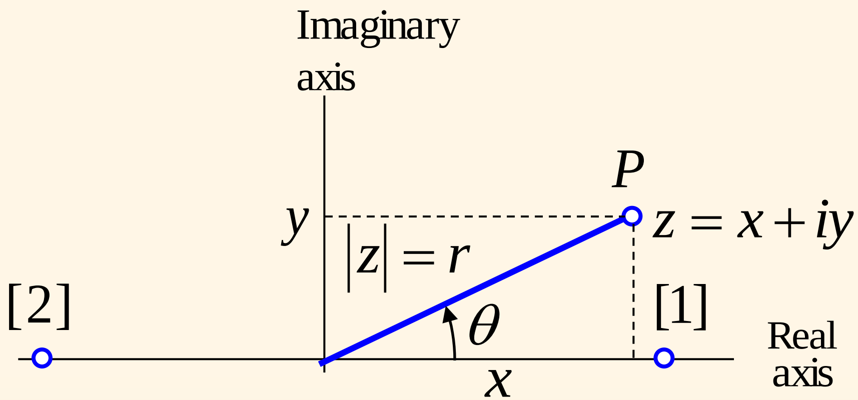

Complex plane, polar form of a complex number

[1] If z is positive real,

[2] If z is negative real,

Ex.)

$$\ln 1 = 0,\ \pm 2\pi i,\ \pm 4\pi i,\ \cdots$$
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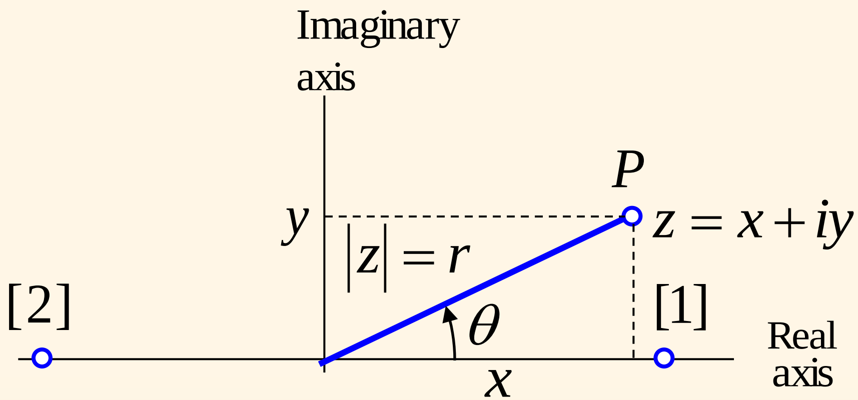

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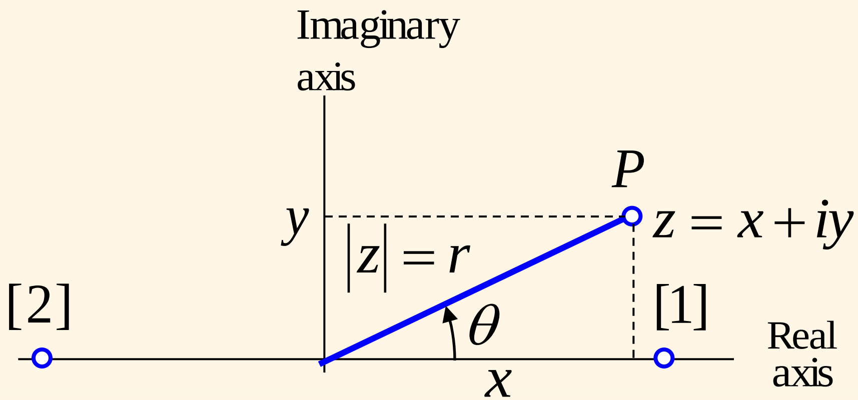

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General Powers

<u>Definition</u> : General powers of a complex number $z = x + iy$

$$(7)\quad z^c = e^{c\ln z} \qquad (c : \text{complex},\quad z \neq 0)$$

Multi-valued $\because \ln z$: **Multi-valued**

Ex.) $i^i = e^{i\ln i} = \exp(i\ln i)$

$$= \exp\left[i\left(\frac{\pi}{2} i \pm 2n\pi i \right) \right]$$

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$$= \exp\left[i \underline{\left(\frac{\pi}{2} i \pm 2n\pi i \right)} \right] \qquad = \exp\left[-\frac{\pi}{2} \mp 2n\pi \right]$$

$$\left(\begin{aligned} & e^w = e^{x+iy} = e^x(\cos y + i \sin y) = i, \\ & \therefore x = 0, y = \frac{\pi}{2}, \quad w = \ln i = \frac{\pi}{2} i \pm 2n\pi i \end{aligned} \right)$$

All these values are real and the principal value (*n* = 0) is $e^{-\pi/2}$.

General Powers

$$(2)\quad \operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z$$

Definition : General powers of a complex number $z = x + iy$

$$(7)\quad z^c = e^{c \ln z} \qquad (c : \text{complex},\quad z \neq 0)$$

Multi-valued $\because \ln z$: **Multi-valued**

Ex.) $(1+i)^{2-i} = \exp\left[(2-i)\ln(1+i)\right]$

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$$= 2e^{\frac{\pi}{4} \pm 2n\pi}\left[\sin\left(\tfrac{1}{2}\ln 2\right) + i \cos\left(\tfrac{1}{2}\ln 2\right)\right]$$

Analytic Functions

Analytic Functions

Definition) Analyticity

A function $f(z)$ is said to be analytic in a domain D
if $f(z)$ is defined and differentiable at all points of D.

The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.
Also, by an analytic function we mean a function that is analytic in some domain.

Theorem 5) Analytic Functions. Their Derivatives.*

A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence.
The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius convergence as the original series. Hence, by the first statement, each of them represents an analytic function.

*proof) Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p681

Cauchy-Riemann Equations. Laplace's Equation

Cauchy-Riemann Equations

$$w = f(z) = u(x,y) + iv(x,y)$$

Analytic

1)

2)

$$u_x(x,y) = v_y(x,y)$$
$$u_y(x,y) = -v_x(x,y)$$

Theorem 1) Cauchy-Riemann Equations

Ref. Cauchy-Riemann Equations

Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann equation (1).

Hence if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (1) at all points of D.

Theorem 2) Cauchy-Riemann Equations

If two real-valued continuous functions $u(x,y)$ and $v(x,y)$ of two real variables x and y have continuous first partial derivatives that satisfy the Cauchy-Riemann equation in some domain D, then the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in D.

Cauchy-Riemann Equations

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Example) $z = x + iy$

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$$u_x = e^x \cos y = v_y$$

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Example) $z = x + iy$

$$f(z) = e^z = e^x(\cos y + i\sin y) = u(x,y) + iv(x,y)$$ **: analytic for all z.**

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$$f(z) = z^2$$

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$$f(z) = \bar{z}$$

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$$f(z) = \bar{z}$$
$$f(z) = x - iy$$
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$$u_x = 1 \neq -1 = v_y$$

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$$f(z) = (x+iy)^2$$
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$$\therefore u = x^2 - y^2, \quad v = 2xy$$
$$u_x = 2x = v_y$$
$$u_y = -2y = -v_x$$

$f(z) = \bar{z}$ **: not analytic for all z.**

$$f(z) = x - iy$$
$$\therefore u = x, \quad v = -y$$
$$u_x = 1 \neq -1 = v_y$$
$$u_y = 0 = -v_x$$

Laplace's Equation. Harmonic Functions

Laplace's Equation. Harmonic Functions

Theorem 3) Laplace's Equation

If $f(z)=u(x,y)+iv(x,y)$ **is analytic in a domain *D*,** $(u_x = v_y,\quad u_y = -v_x)$
then both *u* and *v* satisfy Laplace's equation
in *D* and have continuous second partial derivatives in *D*.

$$(8)\quad \nabla^2 u = u_{xx} + u_{yy} = 0$$

$$(9)\quad \nabla^2 v = v_{xx} + v_{yy} = 0,$$

$u = x^2 - y^2 - y$

Laplace's Equation. Harmonic Functions

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Example) **Verify that** $u = x^2 - y^2 - y$ **is harmonic in the whole complex plane and find a harmonic conjugate function v of u.**

$$u_x = 2x, \; u_y = -2y - 1$$

$$u_{xx} = 2, \; u_{yy} = -2$$

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

So u is harmonic in the whole complex plane.

Laplace's Equation. Harmonic Functions

Theorem 3) Laplace's Equation

If $f(z)=u(x,y)+iv(x,y)$ **is analytic in a domain *D*,** $(u_x=v_y,\quad u_y=-v_x)$
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Example) **Verify that** $u=x^2-y^2-y$ **is harmonic in the whole complex plane and find a harmonic conjugate function v of u.**

By the Cauchy-Riemann Equations

$$v_y=u_x=2x$$

$$v_x=-u_y=2y+1$$

Integrating first equation with respect to y

$$v=\int u_x dy+h(x)\ =2xy+h(x)$$

Differentiating v with respect to x,

$$v_x=2y+\frac{dh}{dx}.$$

Comparing this and v_x = -u_y,

$$\therefore \frac{dh}{dx}=1 \quad \rightarrow h(x)=x+c$$

$$\therefore v=2xy+x+c$$ **(c is any real constant.)**

The corresponding analytic function

$$f(z)=u+iv$$
$$=x^2-y^2-y+i(2xy+x+c)$$
$$=z^2+iz+ic$$

Line Integral in the Complex Plane

Line Integral in the Complex Plane

• **Line integrals**

(Complex definite integrals)

$$\int_C f(z)dz.$$

curve C in the complex plane

Ex.)by a parametric representation.

$$(1) \quad z(t) = x(t) + iy(t), \quad (a \le t \le b)$$

If C is a closed path

$$(3) \quad \int_C f(z)dz, \quad \text{or by} \oint_C f(z)dz$$

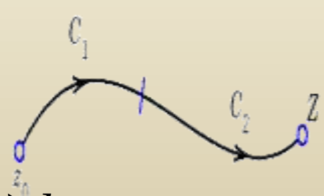
$$\int_C f(z)dz.$$

1. Linearity

$$(4) \quad \int_C [k_1 f_1(z) + k_2 f_2(z)]dz = k_1 \int_C f_1(z)dz + k_2 \int_C f_2(z)dz.$$

2. Sense reversal

$$(5) \quad \int_{z_0}^{Z} f(z)dz = -\int_{Z}^{z_0} f(z)dz$$



3. Partitioning of path

$$(6) \quad \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

Line Integral in the Complex Plane

- **Line integrals**

(Complex definite integrals)

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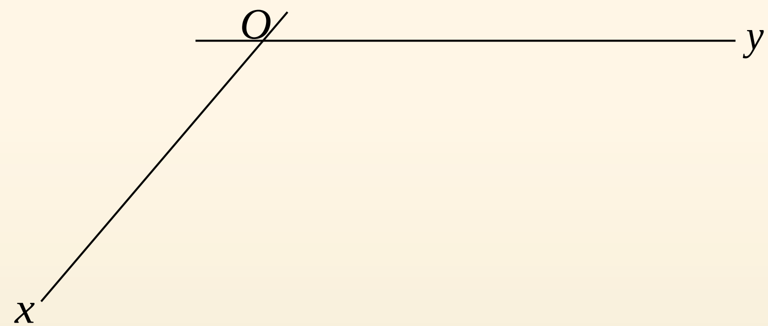

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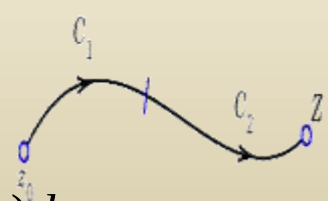

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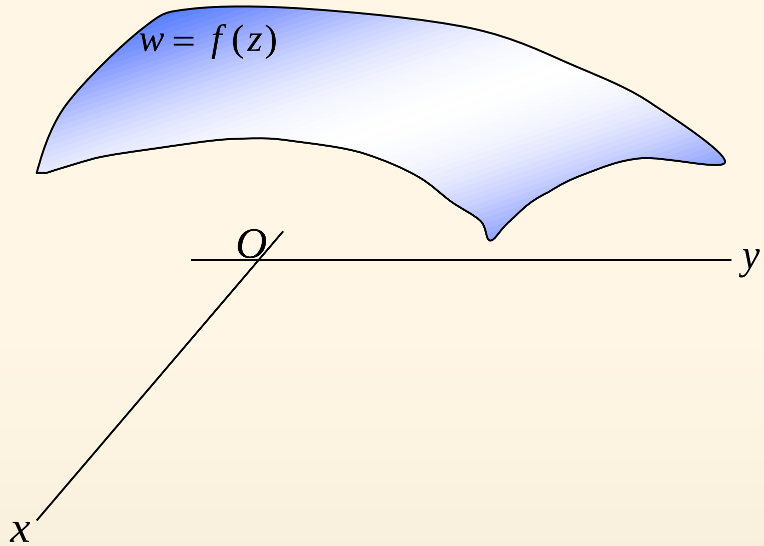
Line Integral in the Complex Plane

• **Line integrals**

(Complex definite integrals)

$$\int_C f(z)dz.$$



curve C in the complex plane

Ex.)by a parametric representation.

$$(1) \quad z(t) = x(t) + iy(t), \quad (a \le t \le b)$$

If C is a closed path

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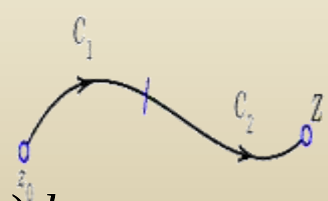

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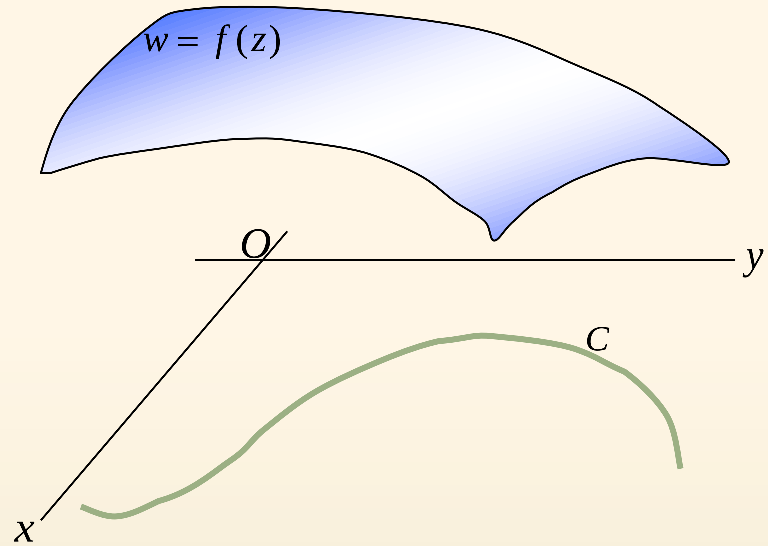

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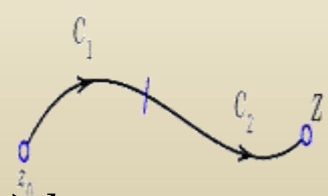

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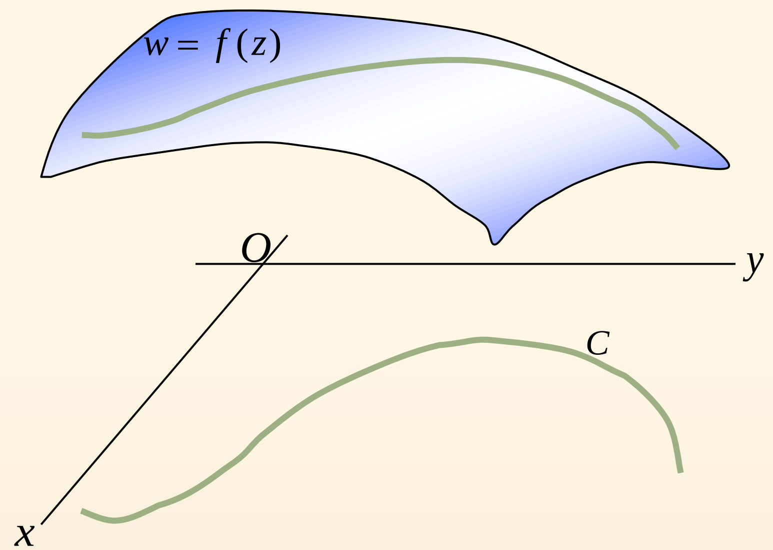

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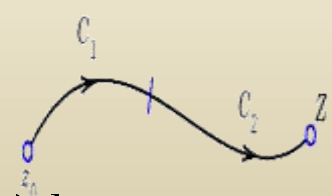

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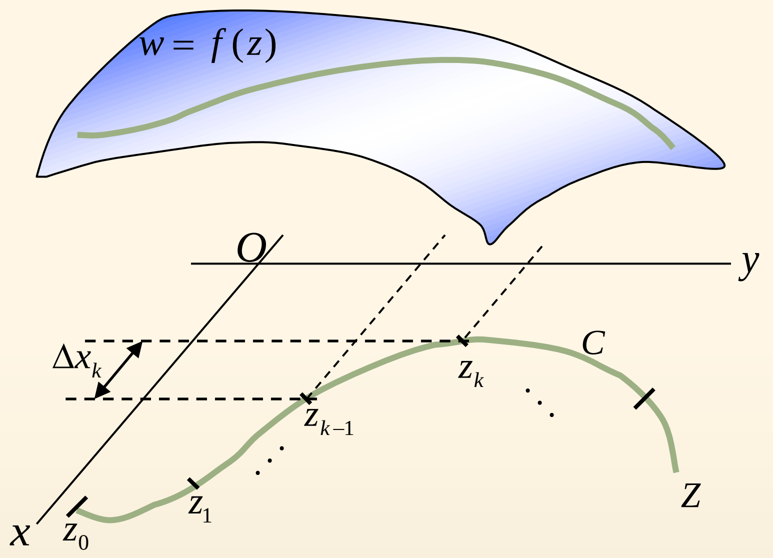

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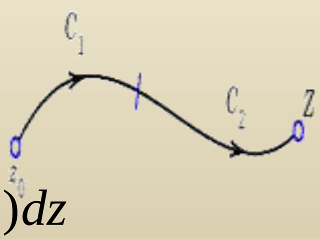

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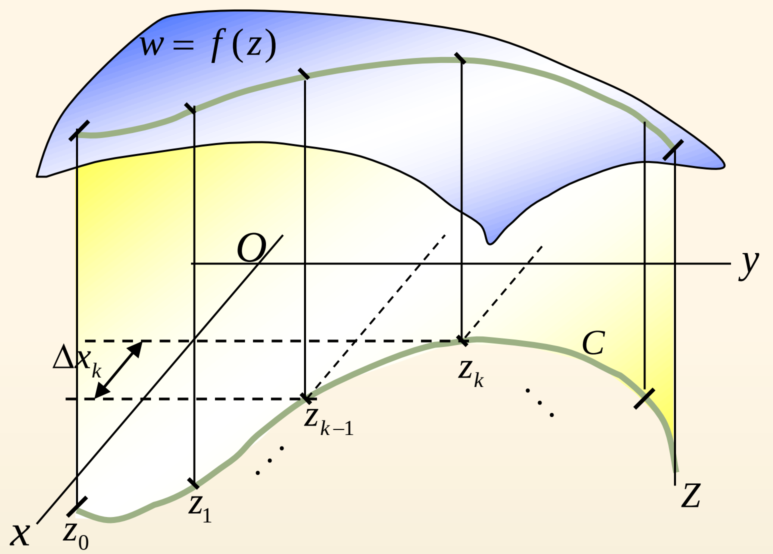

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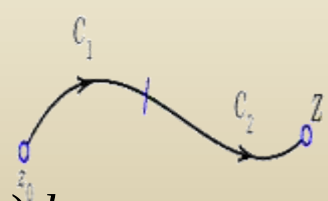

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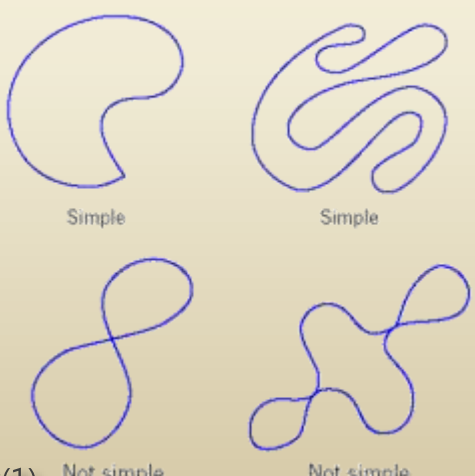
If $f(z)$ is entire, we can take for D the complex plane which is certainly simply connected.

- Simple closed

A simple closed path is a closed path that does not intersect or touch itself



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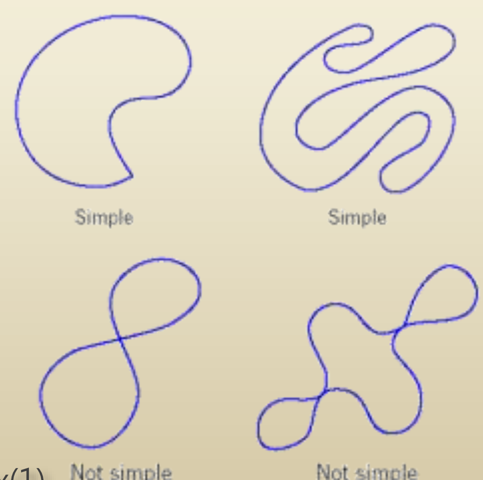

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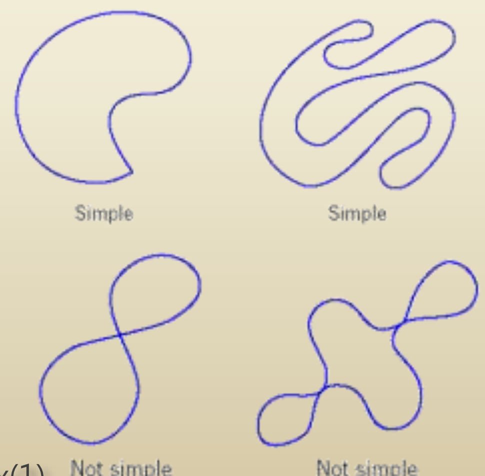

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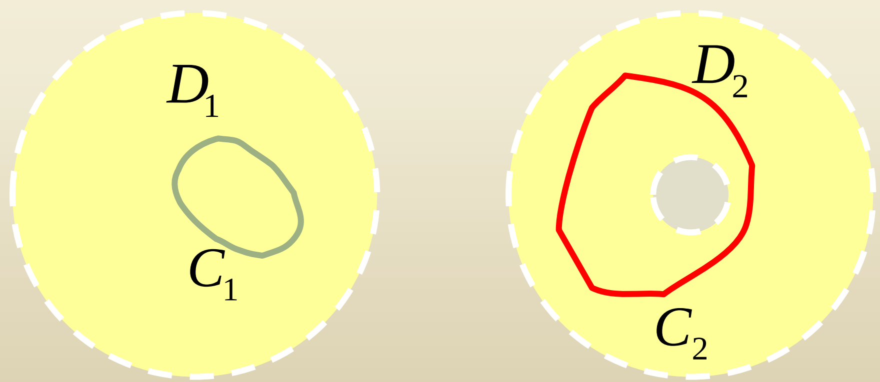

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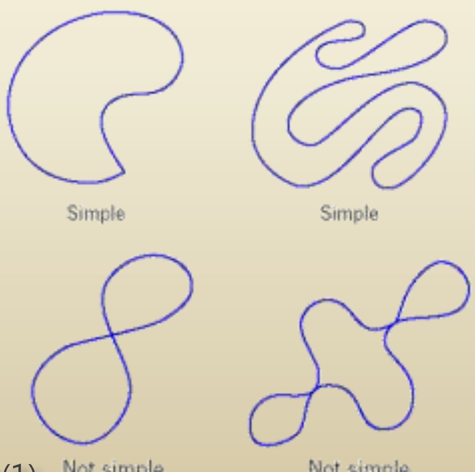

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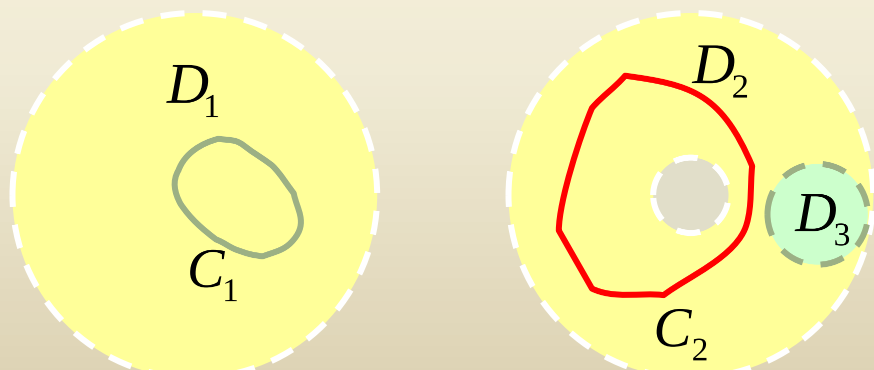

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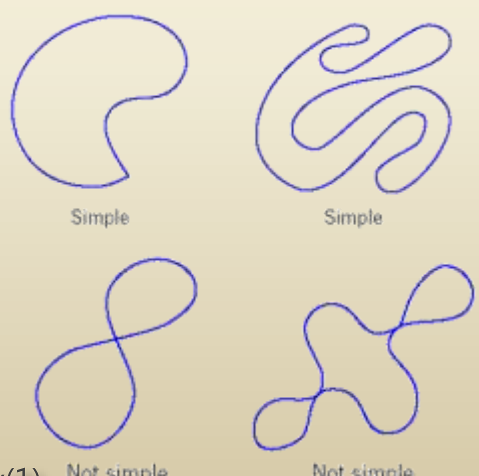

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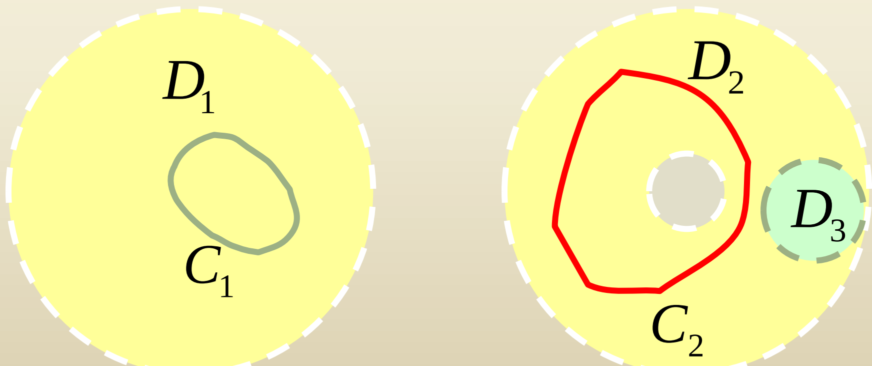

- Simply connected : D_1, D_3
- Not simply connected : D_2

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$$\int_0^{1+i} z^2 dz = ?$$

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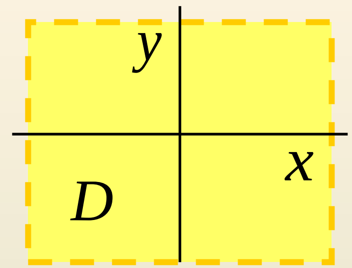

D: simple connected domain

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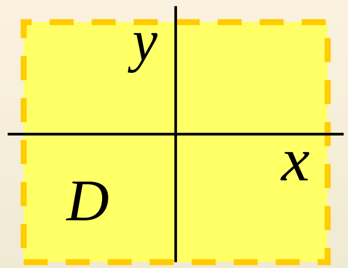

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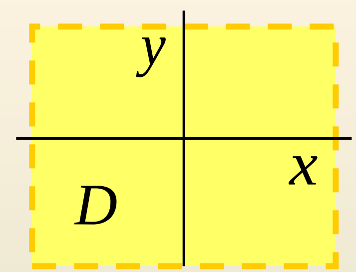

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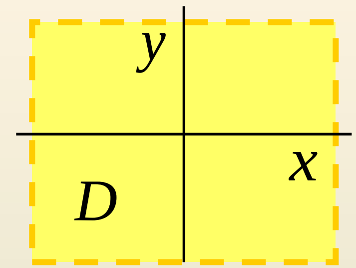

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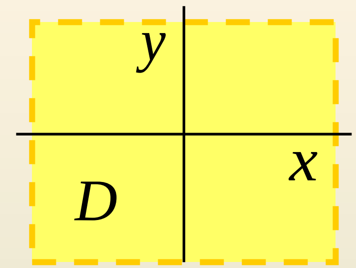

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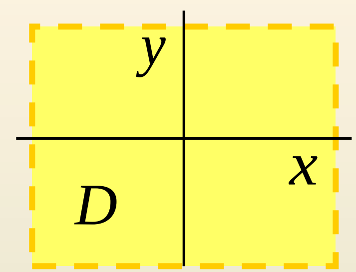

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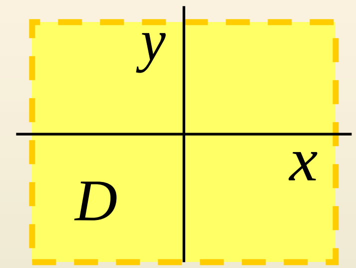
$$= 2\sin \pi i$$

$$= 2(\sin 0 \cosh \pi + i \cos 0 \sinh \pi)$$



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(Simple connectedness is quite essential)

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$$(9) \quad \int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0), \qquad F'(z) = f(z)$$

A function $f(z)$ that is analytic for all z is called an entire function.

If $f(z)$ is entire, we can take for D the complex plane which is certainly simply connected.

Example)

$$\int_0^{1+i} z^2\,dz = ?$$

$$\int_0^{1+i} z^2\,dz = \frac{1}{3} z^3 \Big|_0^{1+i}$$

$$= \frac{1}{3}(1+i)^3$$

$$= -\frac{2}{3} + \frac{2}{3} i$$

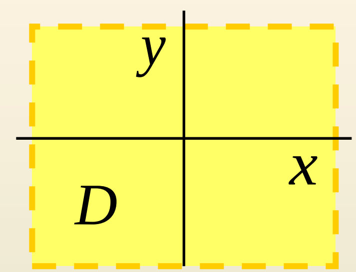
$$\int_{-\pi i}^{\pi i} \cos z\,dz = ?$$

$$\int_{-\pi i}^{\pi i} \cos z\,dz = \sin z \Big|_{-\pi i}^{\pi i}$$

$$= 2\sin \pi i$$

$$= 2(\sin 0 \cosh \pi + i \cos 0 \sinh \pi)$$

$$= 2i \sinh \pi$$



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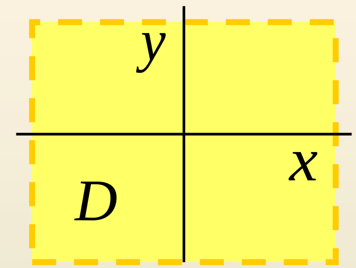
Example)

$$\int_0^{1+i} z^2 dz = ?$$

$$\int_0^{1+i} z^2 dz = \frac{1}{3} z^3 \Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$$

$$\int_{-\pi i}^{\pi i} \cos z dz = ?$$

$$\begin{aligned}\int_{-\pi i}^{\pi i} \cos z dz &= \sin z \Big|_{-\pi i}^{\pi i} \\ &= 2\sin \pi i \\ &= 2(\sin 0 \cosh \pi + i \cos 0 \sinh \pi) \\ &= 2i \sinh \pi \\ &= 23.097i\end{aligned}$$



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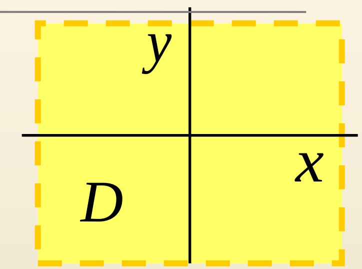
Ex.) $\displaystyle\int_{8+\pi i}^{8-3\pi i} e^{z/2}dz = ?$



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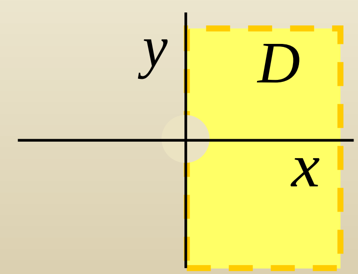

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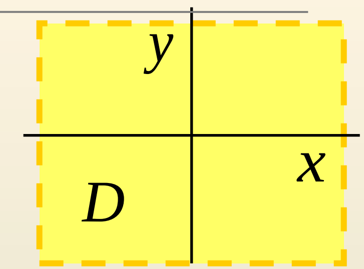
$$\int_{8+\pi i}^{8-3\pi i} e^{z/2} dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i}$$



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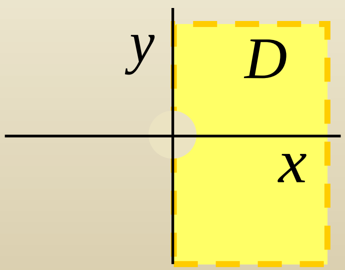

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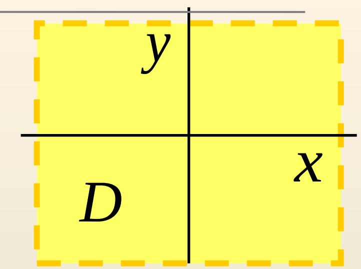

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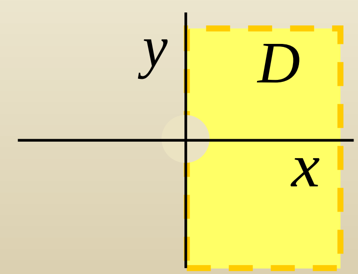

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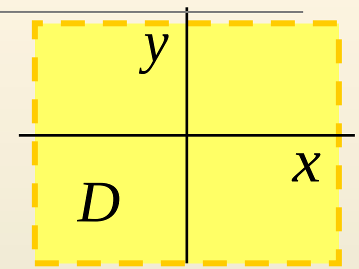
$$= 2e^4(\cos\tfrac{3\pi}{2} - i\sin\tfrac{3\pi}{2})$$



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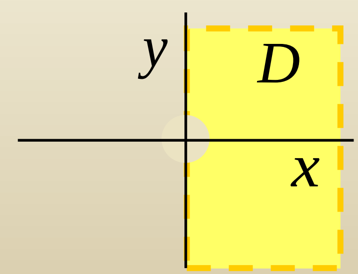

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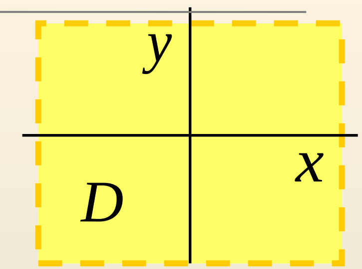

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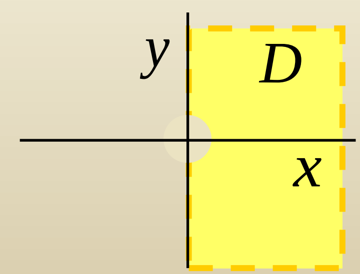

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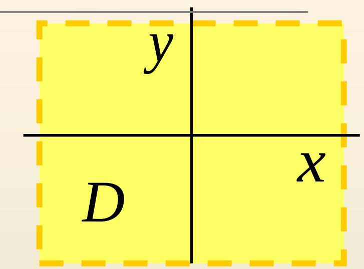

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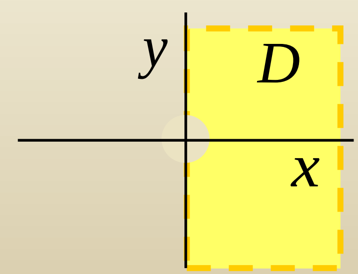
$$\int_{-i}^{i} \frac{dz}{z} = \text{Ln}\, z\Big|_{-i}^{i}$$



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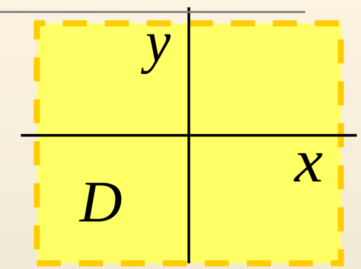

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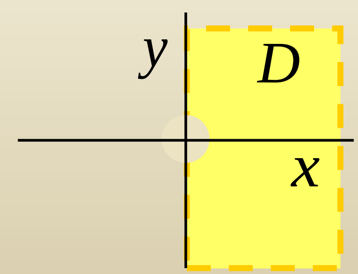

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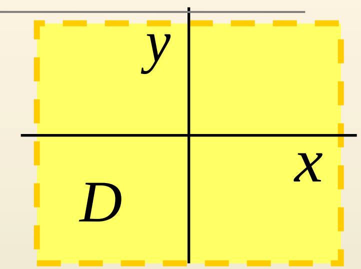

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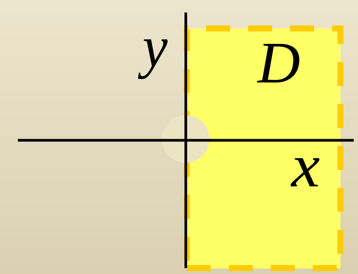

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Theorem 2) Integration by the Use of the Path

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$(10) \quad \int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt, \quad \left(\dot{z} = \frac{dz}{dt}\right).$$

proof)

$$(8)^{**} \quad \int_C f(z)dz = \int_C udx - \int_C vdy + i\left[\int_C udy + \int_C vdx\right] \Leftrightarrow$$

c.f.*

$$F(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P(x, y)dx + Q(x, y)dy$$

$$\int_a^b f[z(t)]\dot{z}(t)dt$$

$$= \int_a^b (u + iv)(\dot{x} + i\dot{y})dt$$

$$= \int_C [u\dot{x} - v\dot{y} + i(u\dot{y} + v\dot{x})]dt = \int_C [udx - vdy + i(udy + vdx)]$$

$$= \int_C udx - \int_C vdy + i\left[\int_C udy + \int_C vdx\right]$$

$$z(t) = x(t) + iy(t), \quad \dot{z}(t) = \dot{x}(t) + i\dot{y}(t)$$

$$\frac{dx}{dt} = \dot{x}, \quad \frac{dy}{dt} = \dot{y}, \quad dx = \dot{x}dt, \; dy = \dot{y}dt$$

$$f(z) = u[x(t), y(t)] + iv[x(t), y(t)]$$

*Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p420-421
**(proof) Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p639-640

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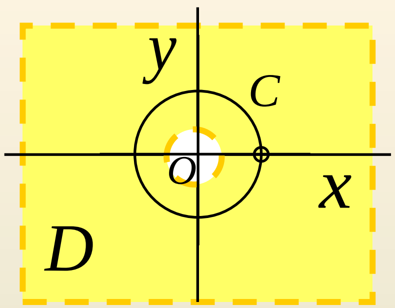
Ex.) Show that by integrating $1/z$ counterclockwise around the unit circle we obtain

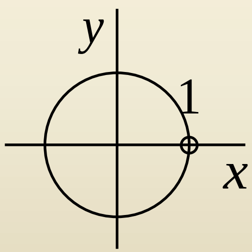
$$\boxed{(11) \quad \oint_C \frac{dz}{z} = 2\pi i}$$

(C the unit circle, counterclockwise)



the function is not analytic at 0.

(A) Represent the unit circle C



$$z(t) = \cos t + i \sin t = e^{it}$$

$$(0 \le t \le 2\pi)$$

counterclockwise integration corresponds to an increase of t from 0 to 2π

So, we will use (10) to find the integral.

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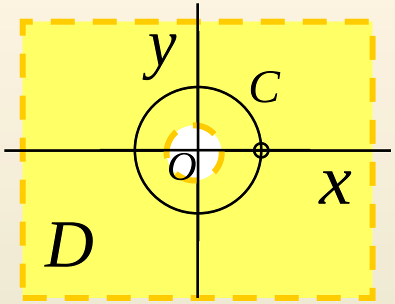

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(A) Represent the unit circle C $\quad z(t) = \cos t + i\sin t = e^{it}$

(B) Calculate the derivative $\quad (0 \le t \le 2\pi)$

(C) Substitute $z(t)$ for every z in $f(z)$

(D) Integrate

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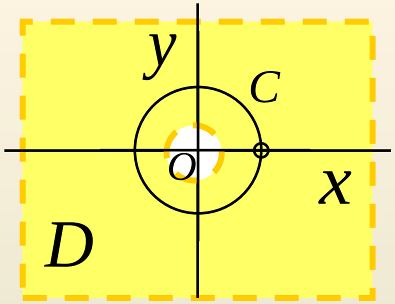

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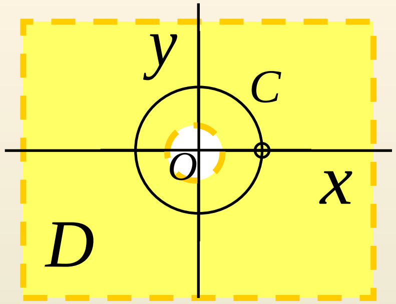

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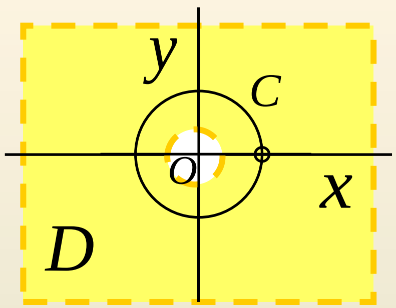
Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

$$(10) \quad \int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt, \quad \left(\dot{z} = \frac{dz}{dt}\right).$$

Ex.) Show that by integrating $1/z$ counterclockwise around the unit circle we obtain

$$\boxed{(11) \quad \oint_C \frac{dz}{z} = 2\pi i}$$

(C the unit circle, counterclockwise)



the function is not analytic at 0.

(A) Represent the unit circle C $\quad z(t) = \cos t + i \sin t = e^{it}$

(B) Calculate the derivative $\quad (0 \le t \le 2\pi)$

$$\dot{z}(t) = ie^{it}$$

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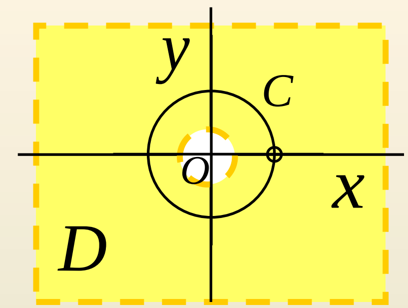

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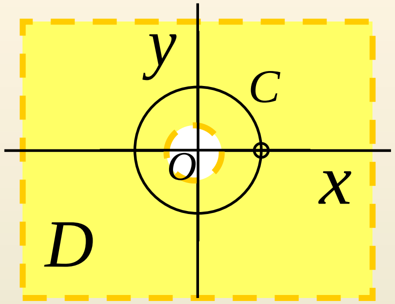

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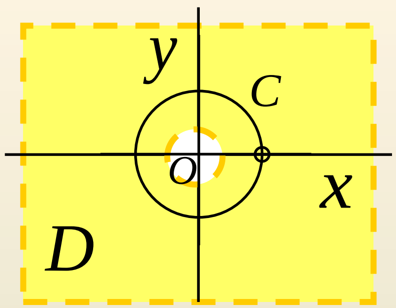

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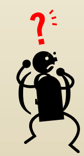

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$$

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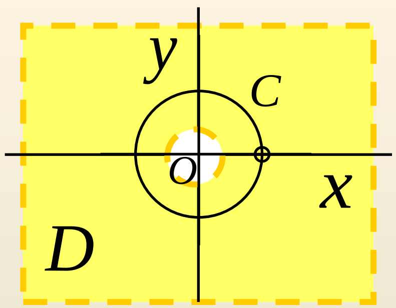

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unit circle must contain $z = 0$,
not simply connected

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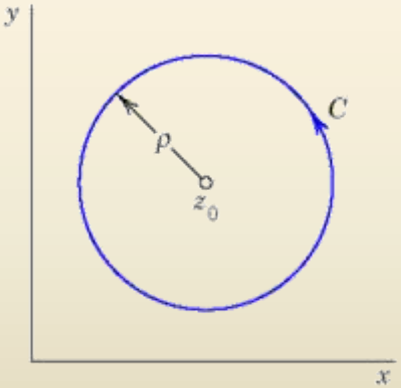

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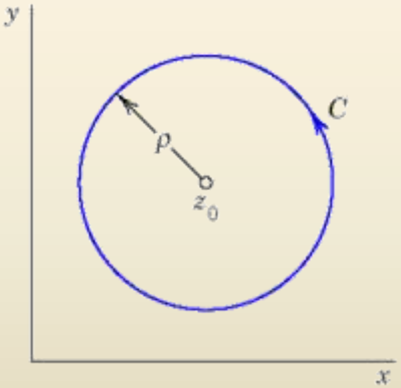

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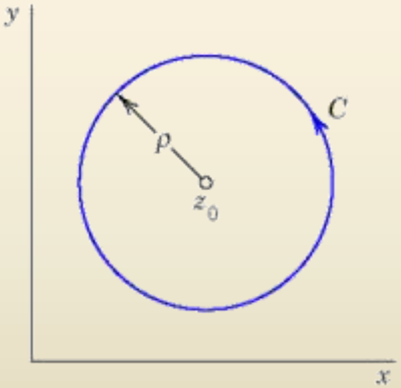

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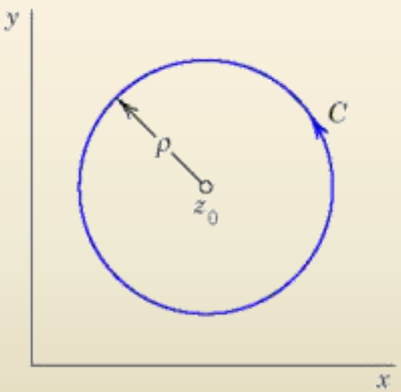

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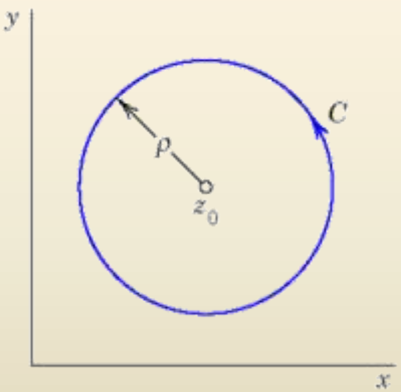

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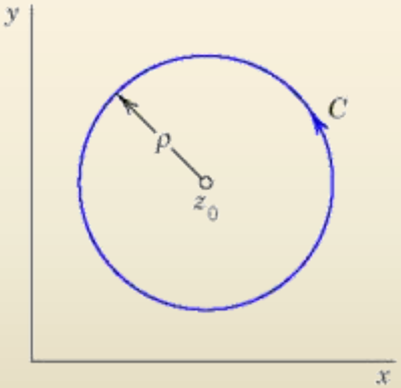

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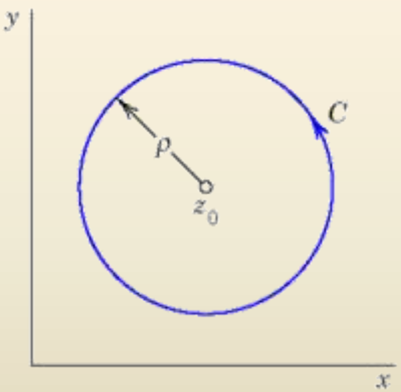

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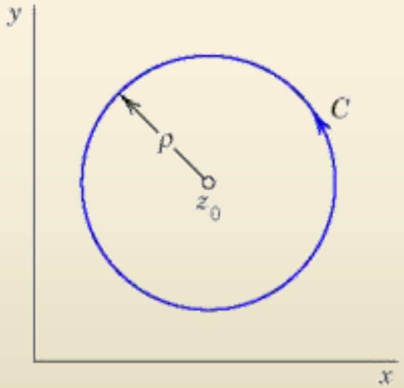

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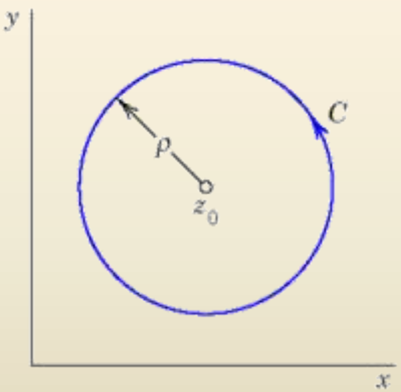

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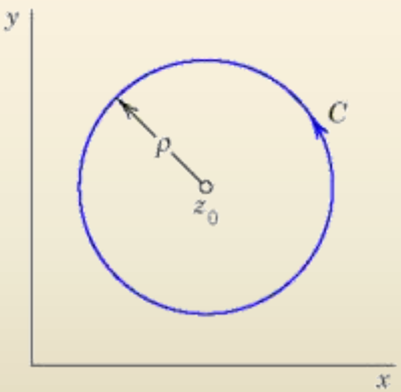

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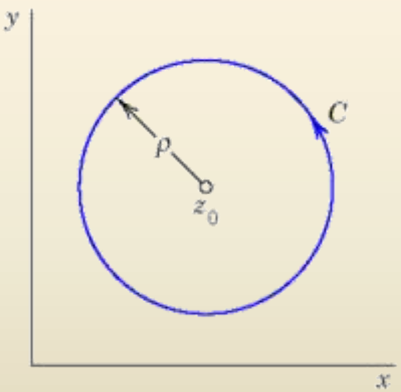

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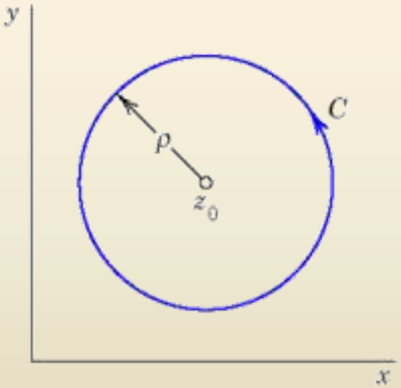

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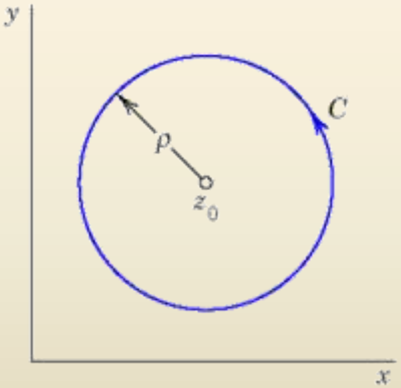

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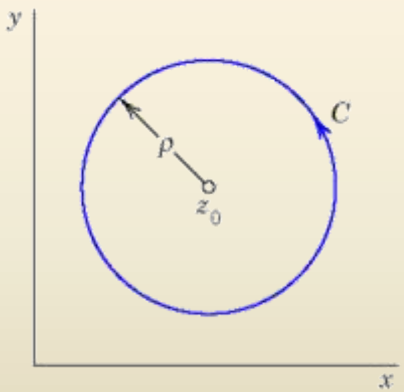

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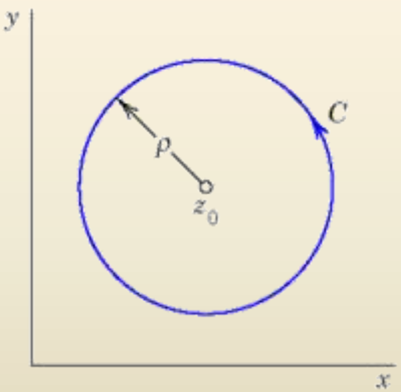

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(A) Represent the unit circle C

$$z = z_0 + \rho(\cos t + i \sin t) \;\; (0 \le t \le 2\pi)$$

$$z = z_0 + \rho e^{it}$$

(B) Calculate the derivative

$$dz = i\rho e^{it} dt$$

(C) Substitute $z(t)$ for every z in $f(z)$

$$f(z) = \rho^m e^{imt}$$

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Line Integral in the Complex Plane

Theorem 2) Integration by the Use of the Path

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then

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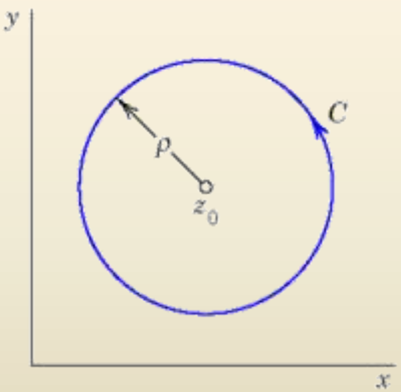

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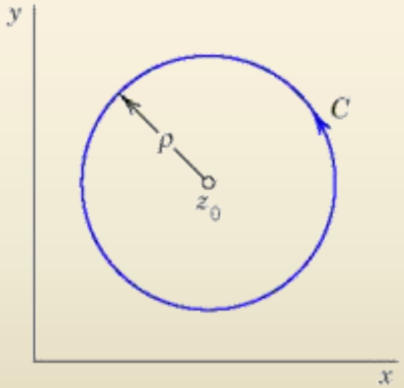

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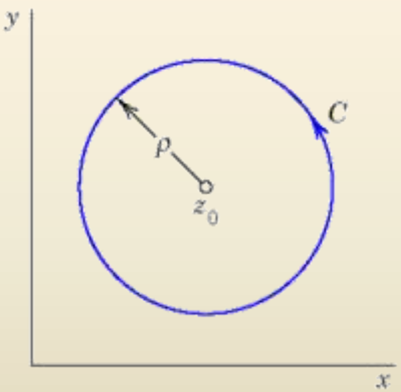

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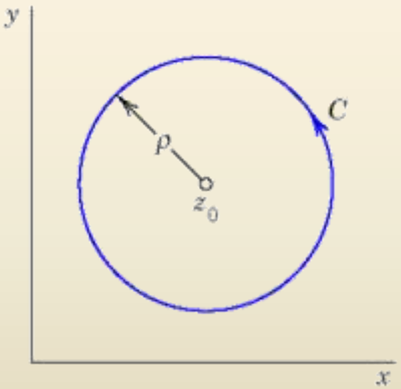

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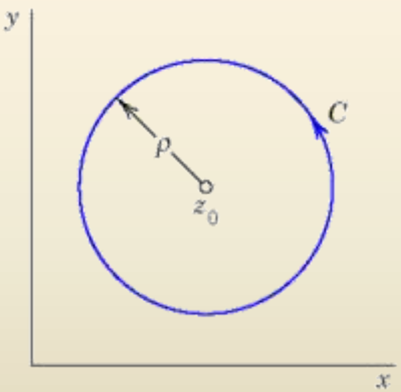

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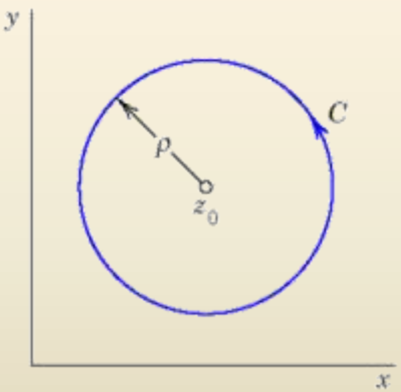

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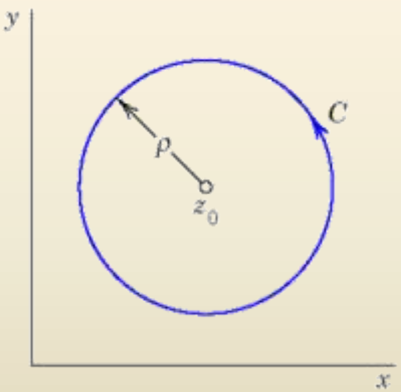

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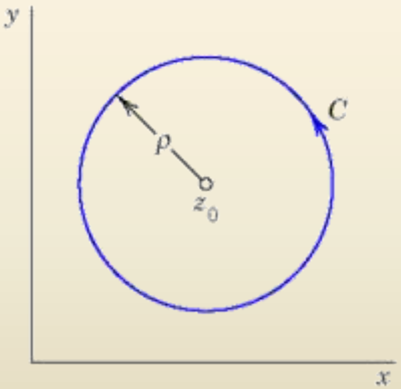

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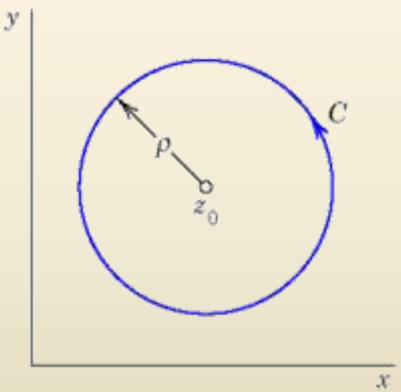

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$$\frac{i\rho^{m+1}}{m+1}\left[\sin(m+1)t - i\cos(m+1)t\right]_0^{2\pi} = 0$$

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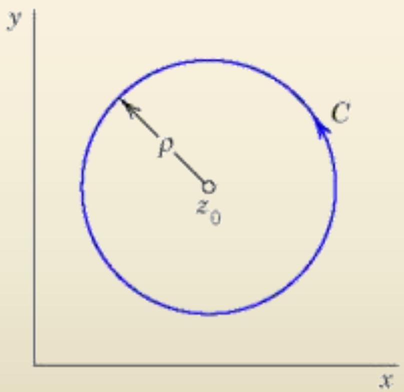

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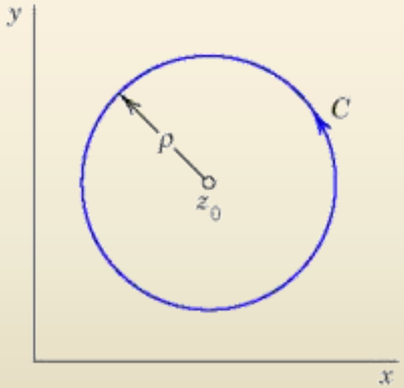

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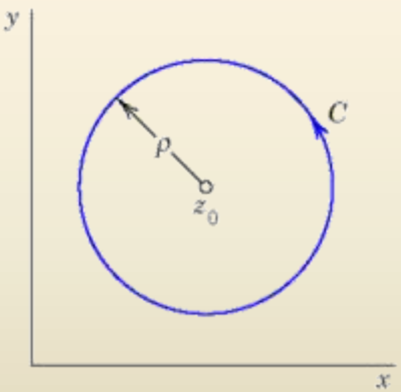

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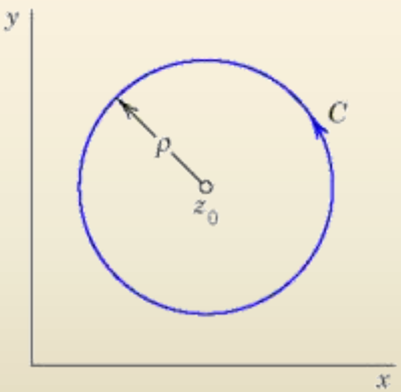

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Simple connectedness and analytic function is quite essential (충분조건) in Theorem 1.

Cauchy’s Integral Theorem

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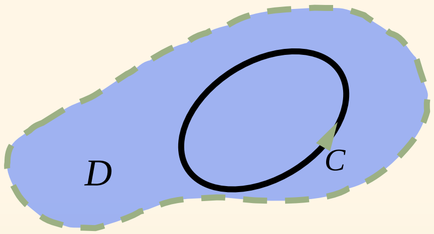
Let $f(z)$ be analytic in a simply connected domain D, then for every simple closed path C in D,

$$(1) \quad \oint_C f(z)\,dz = 0$$



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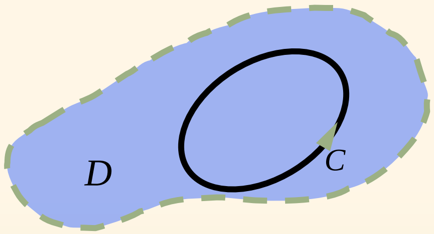

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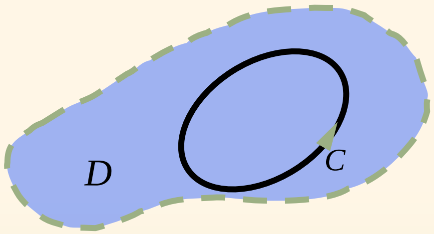

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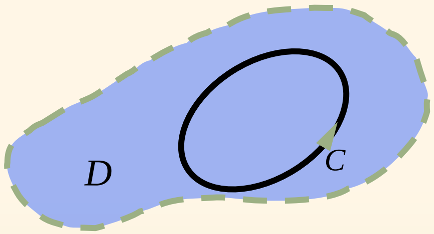

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$$\int_{z_0}^{z_1} f(z)\,dz = 0 \quad (\because \text{closed path}, \quad z_1 = z_0)$$

Cauchy - Riemann equation

$(1) \quad u_x = v_y, \quad u_y = -v_x$

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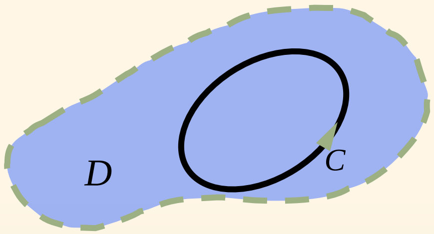

Example) No Singularities (Entire Functions)

$$\oint_C e^z dz = 0,$$

$$\oint_C \cos z dz = 0,$$

$$\oint_C z^n dz = 0 \quad (n = 0, 1, \cdots)$$

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

$$u = e^x \cos y, \; v = e^x \sin y$$

$$u_x = e^x \cos y = v_y$$

$$u_y = -e^x \sin y = -v_x$$

Integrals are zero for any closed path, since these functions are entire (analytic for all z).

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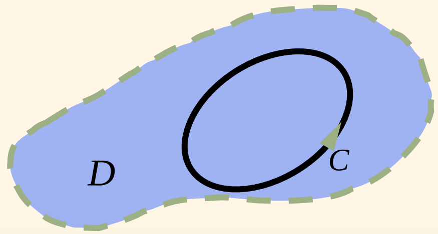

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$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

$$u = \cos x \cosh y, \; v = -\sin x \sinh y$$

$$u_x = -\sin x \cosh y = v_y$$

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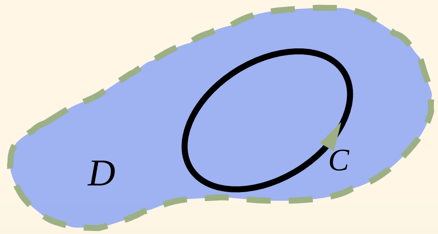


Example) Singularities outside the contour

$$\oint_C \sec z\,dz = 0$$

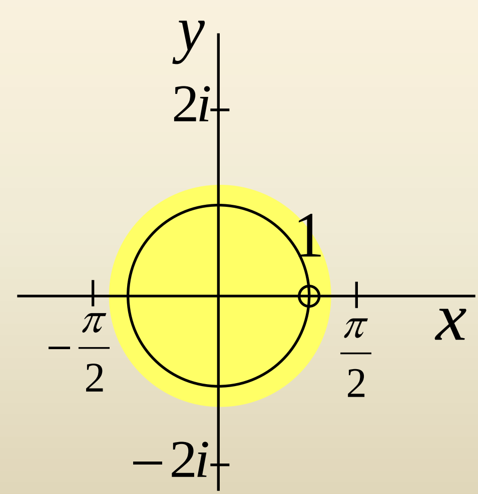
$$\oint_C \frac{dz}{z^2+4} = 0$$

$$\sec z = \frac{1}{\cos z} \text{ is not analytic}$$

$$\text{at } z = \pm\pi/2, \ \pm 3\pi/2, \cdots$$

$$(\because \cos z = 0),$$

but all these points lie outside C; non lies on C or inside C.



$(C : \text{unit circle})$

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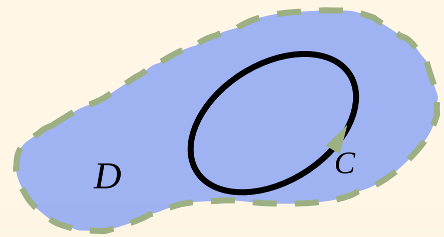

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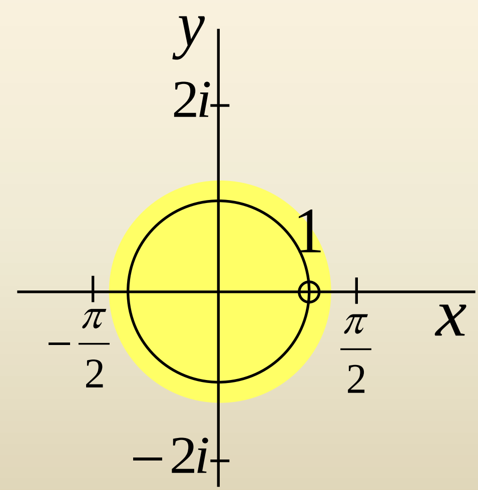
$\dfrac{1}{z^2+4}$ is not analytic at $z = \pm 2i$ outside C.



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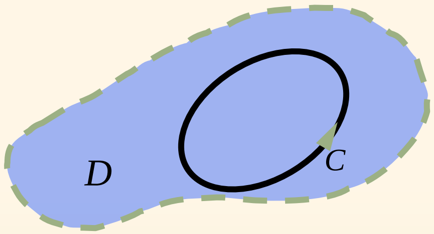
Theorem 1) Cauchy's Integral Theorem

Let $f(z)$ be analytic in a simply connected domain D, then for every simple closed path C in D,

$$(1) \quad \oint_C f(z)dz = 0$$



Example) Not analytic function $\left(C : z(t) = e^{it} \text{ is the unit circle}\right)$ $\qquad \int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt$

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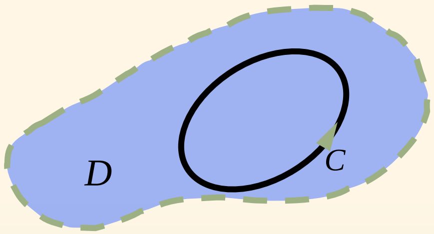

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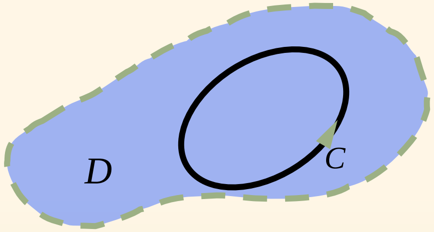

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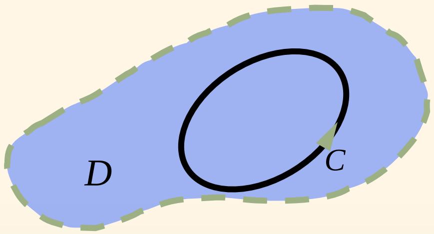

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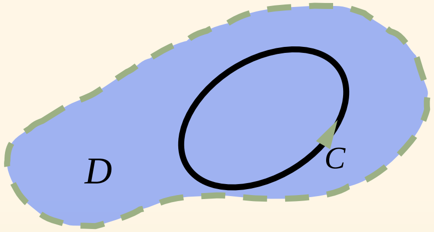

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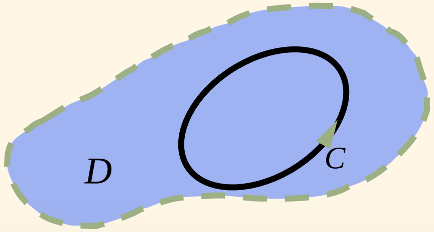

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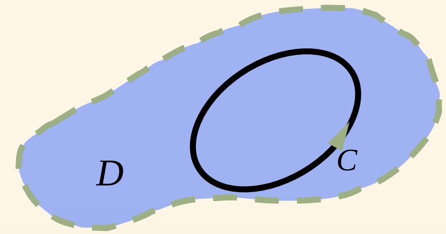

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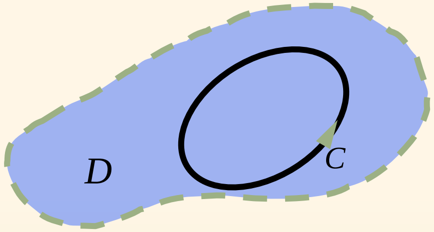

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This does not contradict Cauchy's theorem because is not analytic

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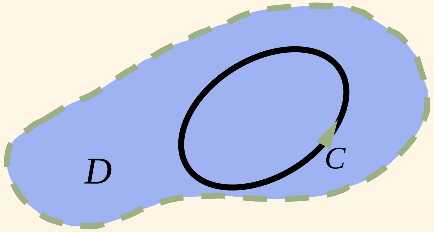

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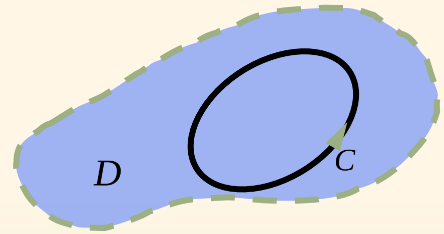

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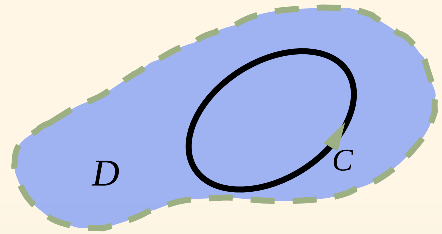

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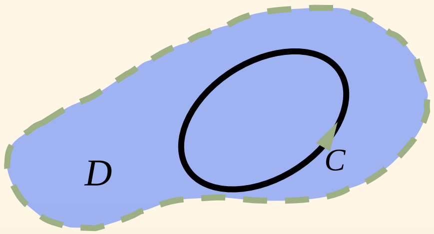

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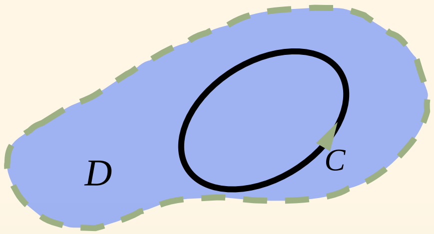

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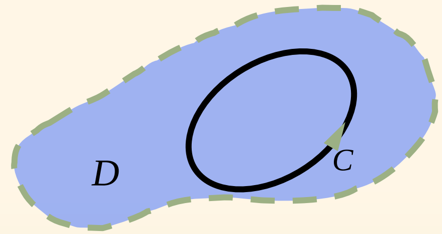

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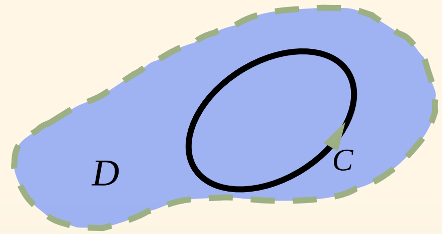

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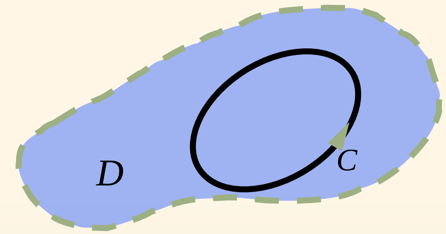

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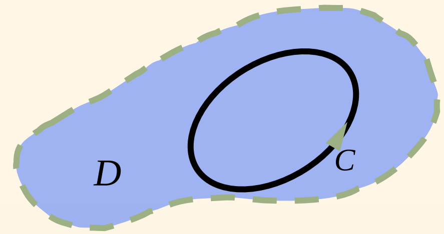

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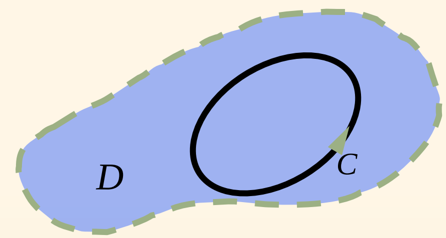

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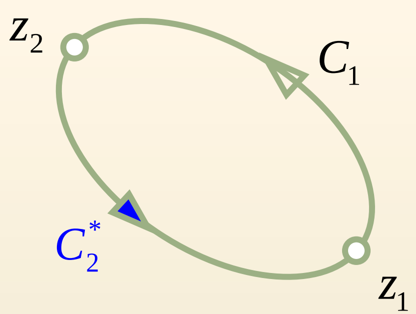
This result does not follow from Cauchy's integral theorem, because $f(z)$ is not analytic at z=0. Hence the condition that f be analytic in D is sufficient(충분조건) rather than necessary for (1) to be true.

$$(9)\quad \int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0), \qquad F'(z) = f(z)$$

Independence of path

Theorem 2) Independence of Path

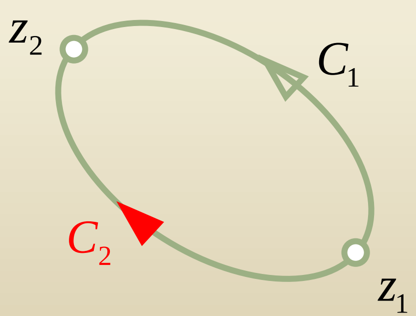
If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.



$$(1)\quad \oint_C f(z)dz = 0$$

$$(1)\oint_C f(z)dz = \int_{C_1} fdz + \int_{C_2^*} fdz = 0$$

$$\therefore \int_{C_1} fdz = -\int_{C_2^*} fdz$$



$$(2)\quad \int_{C_1} fdz = \int_{C_2} fdz$$

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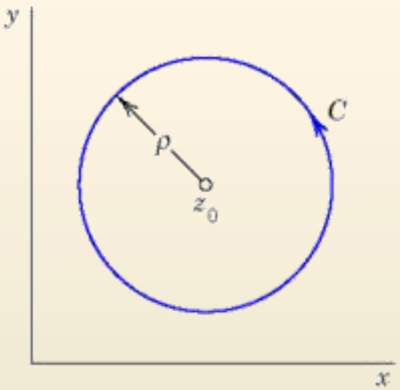
If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Example) Basic Result: Integral of Integer Powers



$$f(z) = (z - z_0)^m$$

m : integer,
z_0 : constant

if $m < 0$,

$$f(z) = \frac{1}{(z - z_0)^p}$$

is not analytic at z_0.

if $m \geq 0$,

$$f(z) = (z - z_0)^m$$

is analytic for all z.

$$\oint_C (z - z_0)^m dz = \begin{cases} 2\pi i & (m = -1), \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

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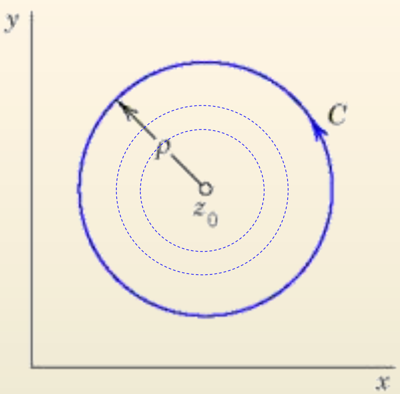

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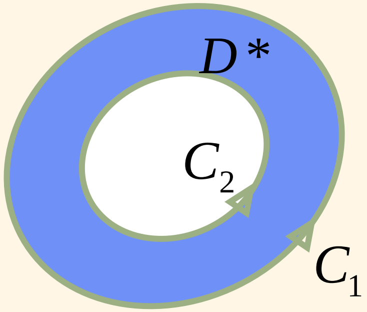
The integral is independent of the radius ρ.

Cauchy's Integral Theorem

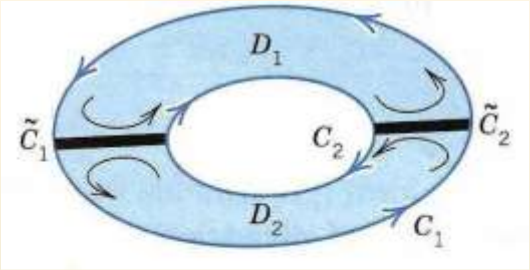
(1) $\oint_C f(z)dz = 0$

Multiply Connected Domains



If a function $f(z)$ is analytic in any domain D^* that contains D and its boundary curves, we claim that

$$(6)\quad \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$



positive direction : counterclockwise

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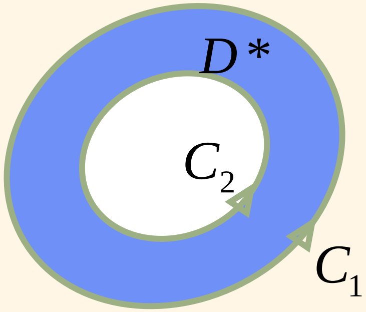
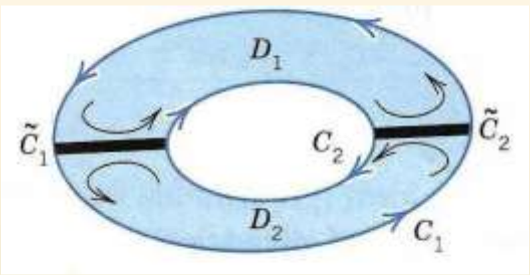




$$D_1 \;:\; \int_{C_{1up}} f(z)dz + \int_{\tilde{C}_1} f(z)dz - \int_{C_{2up}} f(z)dz + \int_{\tilde{C}_2} f(z)dz = 0$$

positive direction : counterclockwise

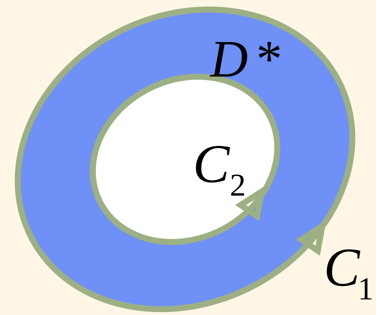
Cauchy's Integral Theorem

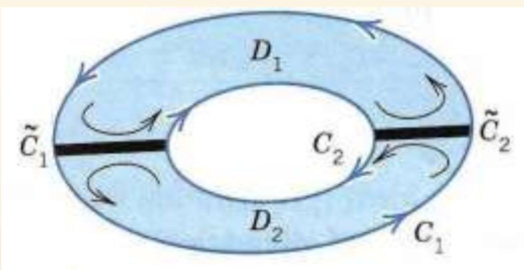
$$(1)\quad \oint_C f(z)dz = 0$$

Multiply Connected Domains

If a function $f(z)$ is <u>analytic</u> in any domain D^* that contains D and its boundary curves, we claim that

$$(6)\quad \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$





$$D_1 \;:\; \int_{C_{1up}} f(z)dz + \int_{\tilde{C}_1} f(z)dz - \int_{C_{2up}} f(z)dz + \int_{\tilde{C}_2} f(z)dz = 0$$

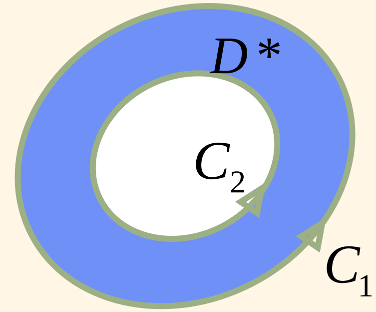
$$D_2 \;:\; \int_{C_{1down}} f(z)dz - \int_{\tilde{C}_2} f(z)dz - \int_{C_{2down}} f(z)dz - \int_{\tilde{C}_1} f(z)dz = 0$$

positive direction : counterclockwise

Cauchy's Integral Theorem

$$(1)\quad \oint_C f(z)dz = 0$$

Multiply Connected Domains



If a function $f(z)$ is <u>analytic</u> in any domain D^* that contains D and its boundary curves, we claim that

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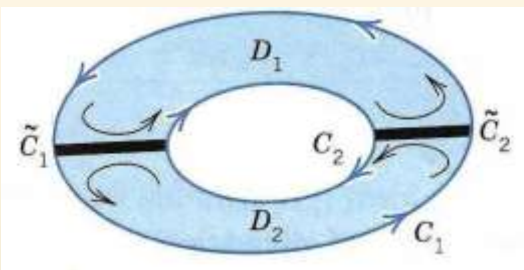


$$D_1 \;:\; \int_{C_{1up}} f(z)dz + \cancel{\int_{\tilde{C}_1} f(z)dz} - \int_{C_{2up}} f(z)dz + \cancel{\int_{\tilde{C}_2} f(z)dz} = 0$$

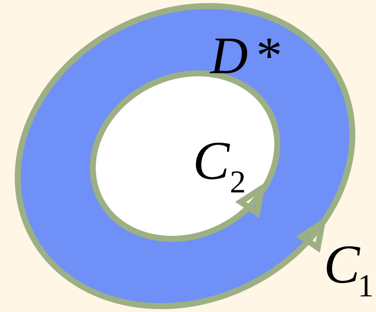
$$D_2 \;:\; \int_{C_{1down}} f(z)dz - \int_{\tilde{C}_2} f(z)dz - \int_{C_{2down}} f(z)dz - \cancel{\int_{\tilde{C}_1} f(z)dz} = 0$$

positive direction : counterclockwise

Cauchy's Integral Theorem

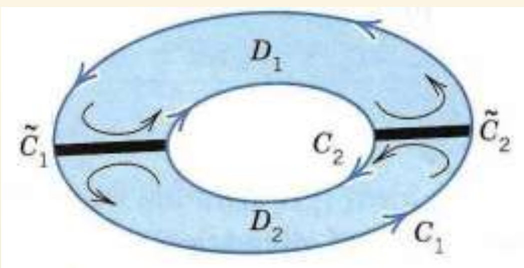
$$(1)\quad \oint_C f(z)dz = 0$$

Multiply Connected Domains



If a function *f* (*z*) is <u>analytic</u> in any domain *D that contains *D* and its boundary curves, we claim that**

$$(6)\quad \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$



$$D_1 \;:\; \int_{C_{1_{up}}} f(z)dz + \cancel{\int_{\tilde{C}_1} f(z)dz} - \int_{C_{2_{up}}} f(z)dz + \cancel{\int_{\tilde{C}_2} f(z)dz} = 0$$

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positive direction : counterclockwise

Cauchy's Integral Theorem

$$(1)\quad \oint_C f(z)dz = 0$$

Multiply Connected Domains

If a function $f(z)$ is <u>analytic</u> in any domain D^* that contains D and its boundary curves, we claim that

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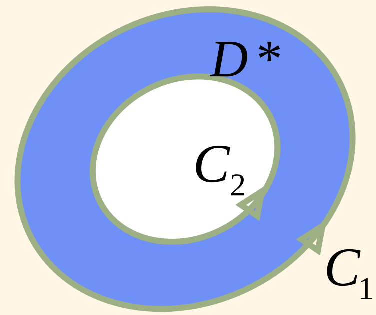

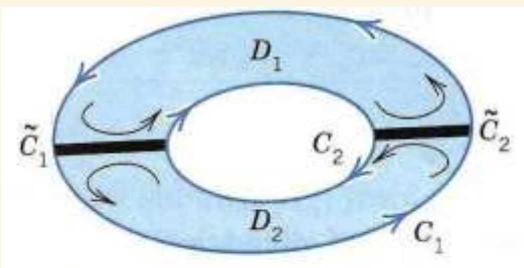

positive direction : counterclockwise

$$D_1 \;:\; \int_{C_{1_{up}}} f(z)dz + \cancel{\int_{\tilde{C}_1} f(z)dz} - \int_{C_{2_{up}}} f(z)dz + \cancel{\int_{\tilde{C}_2} f(z)dz} = 0$$

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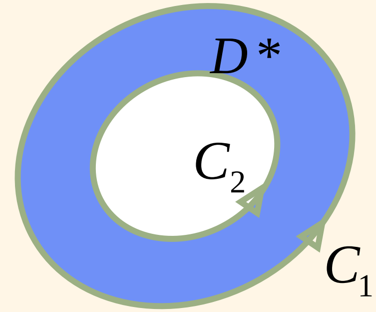
$$D_1 + D_2 \;:\; \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz = 0$$

Cauchy's Integral Theorem

$$(1)\quad \oint_C f(z)dz = 0$$

Multiply Connected Domains



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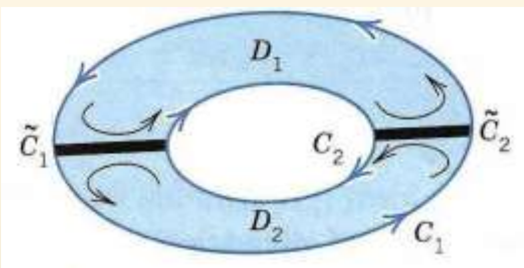

positive direction : counterclockwise

$$D_1\ :\ \int_{C_{1up}} f(z)dz + \cancel{\int_{\tilde{C}_1} f(z)dz} - \int_{C_{2up}} f(z)dz + \cancel{\int_{\tilde{C}_2} f(z)dz} = 0$$

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$$D_1 + D_2\ :\ \oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz = 0$$

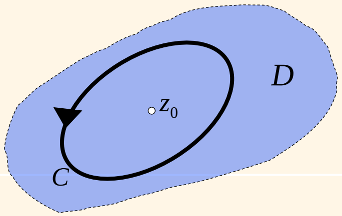
$$\therefore (6)\quad \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

Cauchy's Integral Formula

Cauchy's Integral Formula

Theorem 1) Cauchy's Integral Formula*

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0



$$(1)\quad \oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$(1^*)\quad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

*Theorem & proof) Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p654-655

Cauchy's Integral Theorem $\oint_C f(z)dz = 0$

Cauchy's Integral Formula

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Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0

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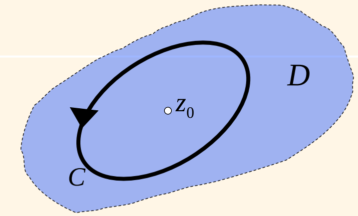
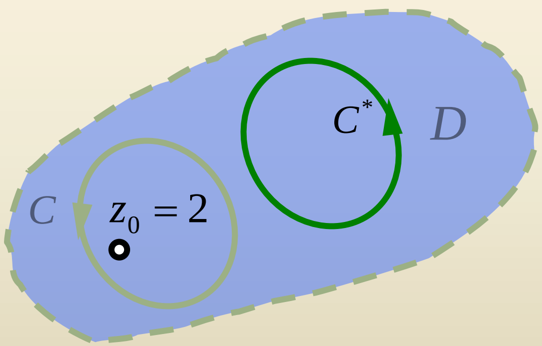


Cauchy's integral formula

For any contour C enclosing $z_0 = 2$

$$\oint_C \frac{e^z}{z-2} dz = ?$$



$$\oint_C \frac{e^z}{z-2} dz = 2\pi i \, f(z_0) = 2\pi i e^{z_0} = 2\pi i e^2$$

$$= 46.4268i$$

For any contour C^* for which $z_0 = 2$ lies outside

$$\oint_{C^*} \frac{e^z}{z-2} dz = 0$$

, since e^z is entire.

Cauchy's Integral Formula

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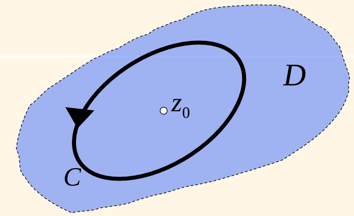
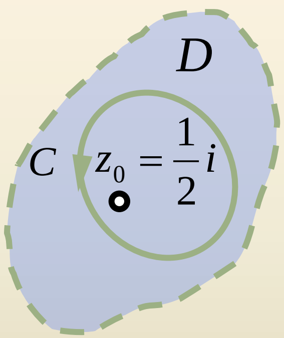


Cauchy's integral formula



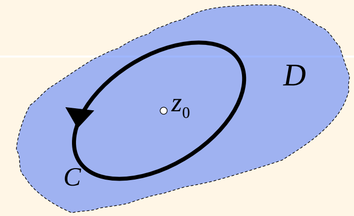
For any contour C enclosing $z_0 = ½\, i$

$$\oint_C \frac{z^3-6}{2z-i}\,dz = ?$$

Cauchy's Integral Formula

Theorem 1) Cauchy's Integral Formula*

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0



Cauchy's integral formula

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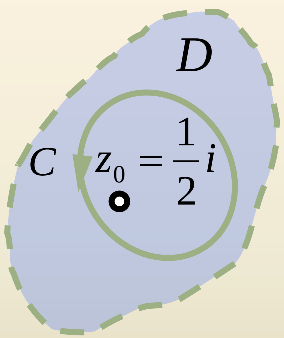

For any contour C enclosing $z_0 = \frac{1}{2} i$

$$\oint_C \frac{z^3 - 6}{2z - i} dz = ?$$

$$\oint_C \frac{z^3 - 6}{2z - i} dz = \oint_C \frac{\frac{1}{2} z^3 - 3}{z - \frac{1}{2} i} dz$$

Cauchy's Integral Formula

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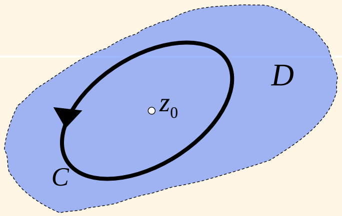
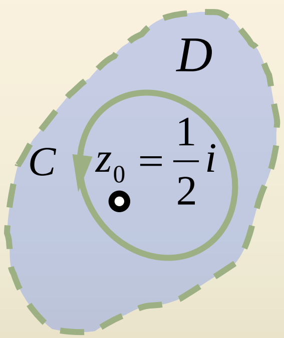


Cauchy's integral formula



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Cauchy's Integral Formula

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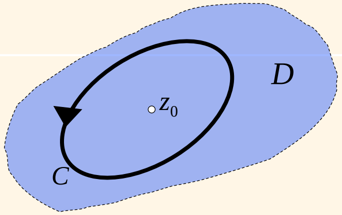
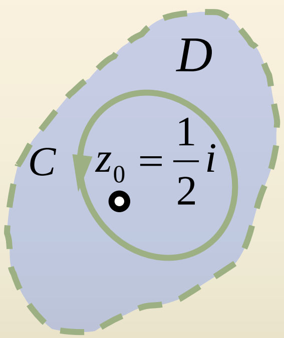


Cauchy's integral formula



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$$= 2\pi i(\tfrac{1}{2}(\tfrac{1}{2}i)^3-3)$$

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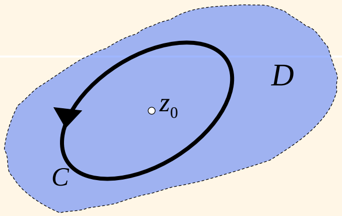
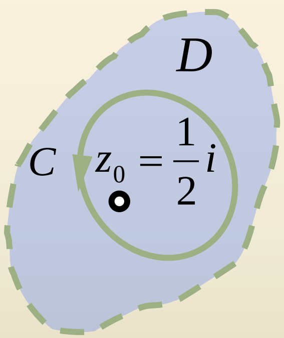


Cauchy's integral formula



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$$= 2\pi i(\tfrac{1}{2}(\tfrac{1}{2}i)^3-3) = \frac{\pi}{8}-6\pi i$$

Cauchy's Integral Formula

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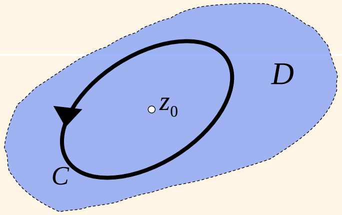
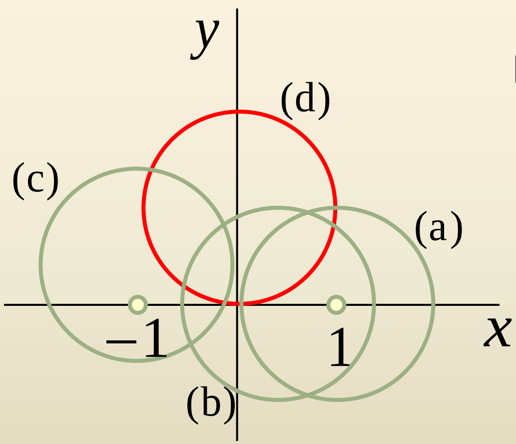


Cauchy's integral formula



Integrate $g(z) = \dfrac{z^2+1}{z^2-1} = \dfrac{z^2+1}{(z+1)(z-1)}$ counterclockwise around each of the four circles.

Singular points : (-1,0), (1,0)

Circles (a), (b), (c) enclose a singular point.

Cauchy's Integral Formula

Theorem 1) Cauchy's Integral Formula*

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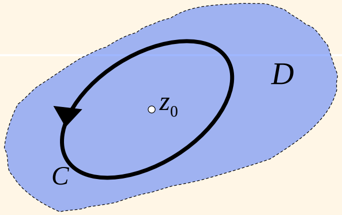
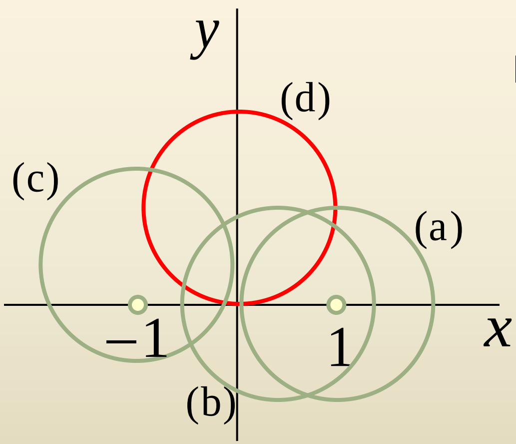


Cauchy's integral formula



Integrate $g(z) = \dfrac{z^2+1}{z^2-1} = \dfrac{z^2+1}{(z+1)(z-1)}$ counterclockwise around each of the four circles.

(d) has no singular point, so the integral is zero by Cauchy's Integral Theorem.

Singular points : (-1,0), (1,0)

Circles (a), (b), (c) enclose a singular point.

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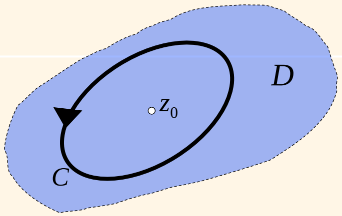
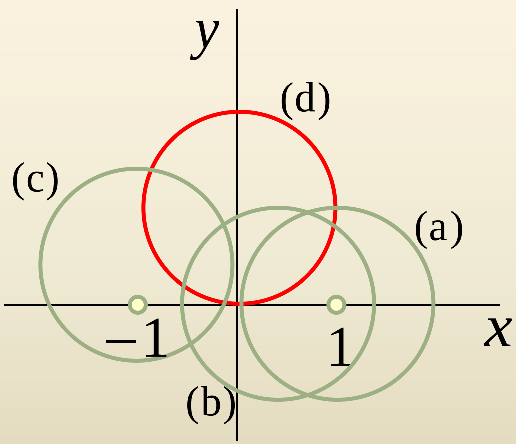


Cauchy's integral formula



Integrate $g(z) = \dfrac{z^2+1}{z^2-1} = \dfrac{z^2+1}{(z+1)(z-1)}$ counterclockwise around each of the four circles.

(d) has no singular point, so the integral is zero by Cauchy's Integral Theorem.

$$\therefore \oint_{C(d)} f(z)dz = 0$$

Singular points : (-1,0), (1,0)

Circles (a), (b), (c) enclose a singular point.

Cauchy's Integral Formula

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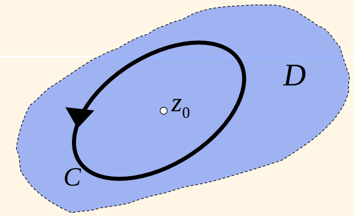
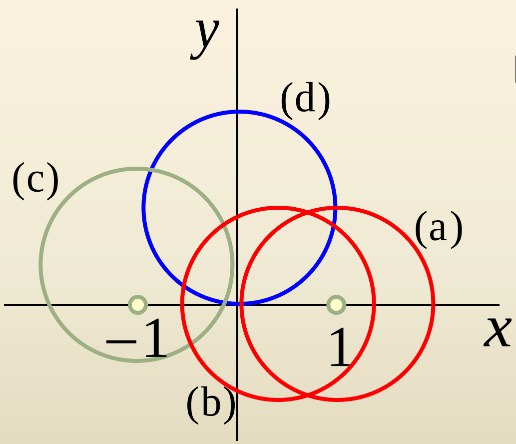


Cauchy's integral formula



Integrate $g(z) = \dfrac{z^2+1}{z^2-1} = \dfrac{z^2+1}{(z+1)(z-1)}$ counterclockwise around each of the four circles.

The circles (a) and (b) enclose the point $z_0 = 1$ where $g(z)$ is not analytic.

Hence in (1) we have to write

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1} \cdot \frac{1}{z-1}$$

Singular points : (-1,0), (1,0)

Circles (a), (b), (c) enclose a singular point.

Cauchy's Integral Formula

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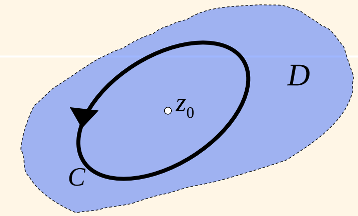
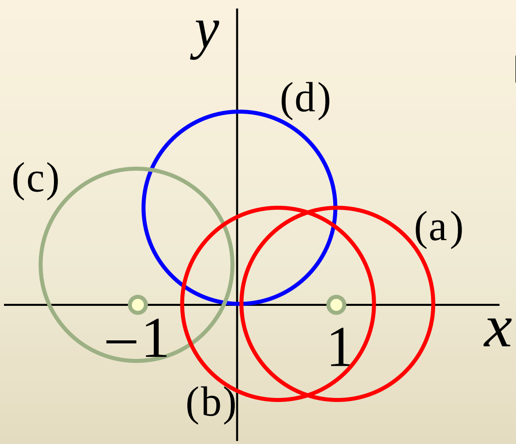


Cauchy's integral formula



Integrate $g(z) = \dfrac{z^2+1}{z^2-1} = \dfrac{z^2+1}{(z+1)(z-1)}$ counterclockwise around each of the four circles.

The circles (a) and (b) enclose the point $z_0 = 1$ where $g(z)$ is not analytic.

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Singular points : (-1,0), (1,0)

Circles (a), (b), (c) enclose a singular point.

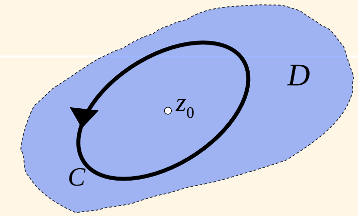
Cauchy's Integral Formula

Theorem 1) Cauchy's Integral Formula*

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0

$$(1) \quad \oint_C \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0)$$

$$(1^*) \quad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$



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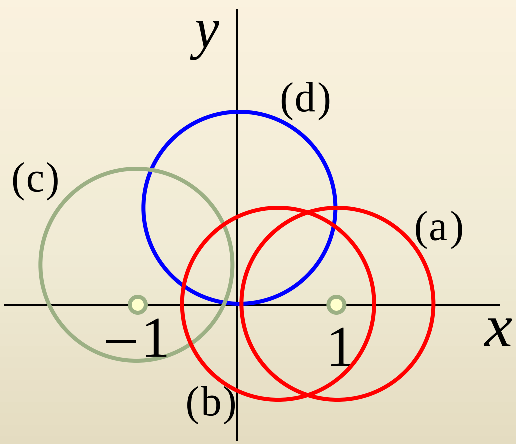

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$$\therefore \oint_C \frac{z^2+1}{z^2-1}dz = 2\pi i\, f(1) = 2\pi i.$$

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Cauchy's Integral Theorem $\oint_C f(z)dz = 0$

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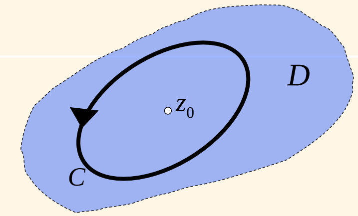

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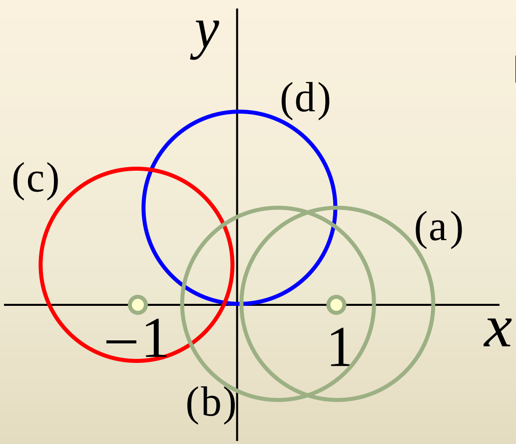

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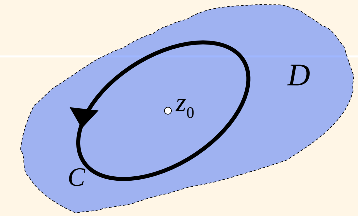

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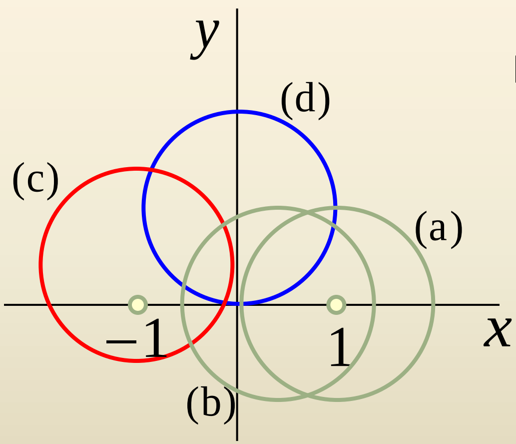

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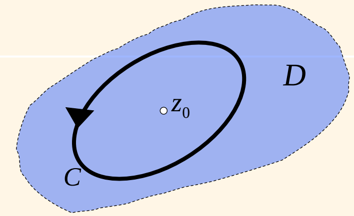

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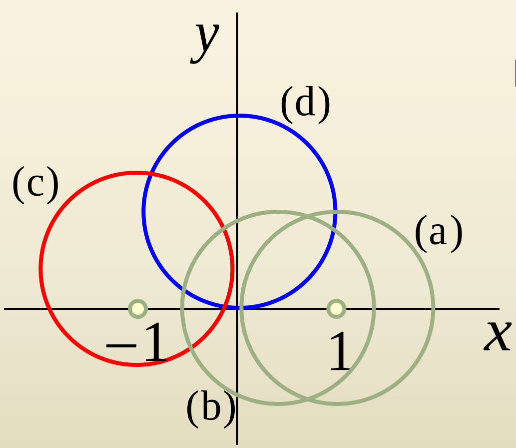

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$$\therefore \oint_C \frac{z^2+1}{z^2-1}dz = 2\pi i\, f(-1) = -2\pi i.$$

Taylor and Maclaurin Series

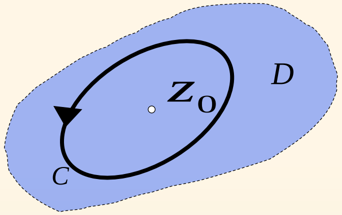
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Taylor and Maclaurin Series

- **Taylor series**

The Taylor series of a function $f(z)$, the complex analog of the real Taylor series is

$$(1)\quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(z_0)$$



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*Theorem & proof) Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p658-660

Derivatives of Analytic Functions

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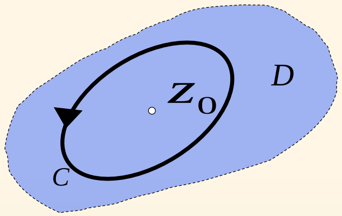

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Writing our the corresponding partial sum of (1), we thus have

$$(4)\quad f(z) = f(z_0) + \frac{z-z_0}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + R_n(z).$$

Taylor and Maclaurin Series

Derivatives of Analytic Functions

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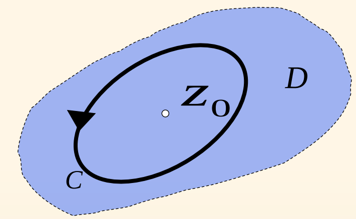

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$$(3)\quad \boxed{R_n(z)} = \frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)}\,dz^*$$

The remainder of the Taylor series (1) after the term $a_n(z-z_0)^n$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

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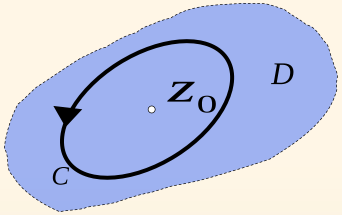

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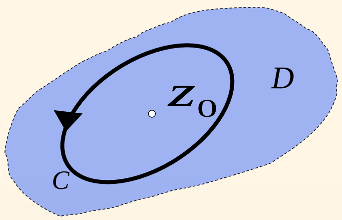

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Recall, **Theorem 5) Analytic Functions. Their Derivatives.***

A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence.

*proof) Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p681

Derivatives of Analytic Functions

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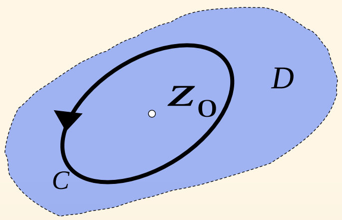

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And we now show that every analytic function can be represented by power series, namely, by Taylor series (with various centers).

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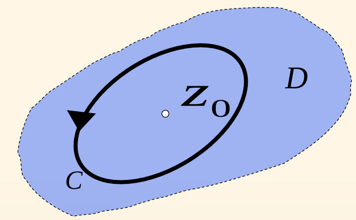

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And we now show that every analytic function can be represented by power series, namely, by Taylor series (with various centers).

- Comparison with Real Functions.

This is not always true always in general for real functions; there are real functions that have derivatives of all orders but cannot be represented by a power series.

(Example : this function cannot be represented by a Maclaurin series in an open disk with center 0 because all its derivatives at 0 are zero.)

$$f(x) = e^{-\frac{1}{x^2}}$$

$$f'(x) = e^{-\frac{1}{x^2}}\left(2x^{-3}\right)$$

$$f''(x) = e^{-\frac{1}{x^2}}\left(2x^{-3}\right)\left(2x^{-3}\right) + e^{-\frac{1}{x^2}}\left(-6x^{-4}\right)$$

Taylor and Maclaurin Series

- **Maclarurin series**

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \quad a_n=\frac{1}{n!}f^{(n)}(0) \ \text{ or } \ a_n=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{{z^*}^{n+1}}dz^*$$

Important Special Taylor (Maclauin) Series

$$f(z)=(1-z)^{-1}$$

$$f'(z)=-1(1-z)^{-2}(-1)=(1-z)^{-2}$$

$$f''(z)=-2(1-z)^{-3}(-1)=2!(1-z)^{-3}$$

$$f^{(n)}(z)=n!(1-z)^{-(n+1)}=\frac{n!}{(1-z)^{n+1}}$$

Let $f(z)=\dfrac{1}{1-z}$ **then we have** $f^{(n)}(z)=\dfrac{n!}{(1-z)^{n+1}}$, $f^{(n)}(0)=n!$.

Hence the Maclaurin expansion of $1/(1-z)$ is the geometric series

$$(11)\quad \frac{1}{1-z}=\sum_{n=0}^{\infty} z^n = 1+z+z^2+\cdots. \quad \left(\because a_n=\frac{1}{n!}f^{(n)}(0)=\frac{1}{n!}n!=1\right)$$

$f(z)$ is singular at $z = 1$; this point lies on the circle of convergence.

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Important Special Taylor (Maclauin) Series

$$f(z)=e^z$$

We know that the exponential function e^z **is analytic for all z, and** $(e^z)'=e^z$

$$\text{(12)}\quad e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots.\quad \left(\because a_n=\frac{1}{n!}f^{(n)}(z_0)=\frac{1}{n!}e^0=\frac{1}{n!}\right)$$

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Important Special Taylor (Maclauin) Series

$$(12) \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

Furthermore, by setting $z = iy$ in (12) and separating the series into the real and imaginary parts ,

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = 1 + iy + - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} - \cdots.$$

$$= \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!}} + i\boxed{\sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}}$$

$$\because e^{iy} = \cos y + i \sin y$$

Euler formula

Maclaurin series of $\cos y$ **Maclaurin series of $\sin y$**

Derivatives of Analytic Functions

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

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$$(12) \quad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

Find the Maclaurin series of <u>$\cos z$</u> and <u>$\sin z$.</u>

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - + \cdots$$

$$e^{-iz} = \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!} = 1 - iz - \frac{z^2}{2!} + i\frac{z^3}{3!} - \frac{z^4}{4!} - i\frac{z^5}{5!} + - \cdots$$

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$$e^{iz}=\sum_{n=0}^{\infty}\frac{(iz)^n}{n!}=1+iz-\frac{z^2}{2!}-i\frac{z^3}{3!}+\frac{z^4}{4!}+i\frac{z^5}{5!}-+\cdots$$

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Find the Maclaurin series of <u>cos z</u> and <u>sin z.</u>

$$\text{(14)} \quad \begin{aligned} \cos z &= \frac{1}{2}(e^{iz}+e^{-iz})=1-\frac{z^2}{2!}+\frac{z^4}{4!}-+\cdots=\sum_{k=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!} \\ \sin z &= \frac{1}{2i}(e^{iz}-e^{-iz})=z-\frac{z^3}{3!}+\frac{z^5}{5!}-+\cdots=\sum_{k=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} \end{aligned}$$

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$$\text{(12)}\quad e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots.$$

Find the Maclaurin series of $\underline{\cosh z}$ and $\underline{\sinh z}$.

$$e^{-z}=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!}=1+(-z)+\frac{(-z)^2}{2!}+\frac{(-z)^3}{3!}+\cdots\ =1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\frac{z^4}{4!}-\frac{z^5}{5!}+-\cdots$$

$$\text{(15)}\quad \begin{aligned}\cosh z&=\frac{1}{2}(e^z+e^{-z})=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots=\sum_{k=0}^{\infty}\frac{z^{2n}}{(2n)!}\\ \sinh z&=\frac{1}{2}(e^z-e^{-z})=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots=\sum_{k=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}\end{aligned}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Taylor and Maclaurin Series

- **Maclarurin series**

A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{n!} f^{(n)}(0) \text{ or } a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^{*\,n+1}}\,dz^*$$

Important Special Taylor (Maclauin) Series

Find the Maclaurin series of $\mathrm{Ln}(1+z)$

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\end{aligned}$$

$$a_n=\frac{1}{n!}f^{(n)}(0)=\frac{1}{n!}(-1)^{n+1}(n-1)!=\frac{(-1)^{n+1}}{n}$$

$$\therefore \mathrm{Ln}\,(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+-\cdots=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}z^n$$

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$$R=\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}}{n}}{\dfrac{(-1)^{n+2}}{n+1}}\right|=\lim_{n\to\infty}\left|\frac{n+1}{n}\right|=1$$

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Derivatives of Analytic Functions

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$$(16) \quad \mathrm{Ln}\,(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + - \cdots$$

Replacing z by $-z$ and multiplying both sides by -1, we get

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Taylor and Maclaurin Series

- Practical Methods

The following examples show ways of obtaining Taylor series more quickly than by the use of the coefficient formulas. Regardless of the method used, the result will be the same This follows from the uniqueness

  - Substitution
  - Integration
  - Development by Using the Geometric Series

Taylor and Maclaurin Series

$$(11)\quad \frac{1}{1-z}=\sum_{n=0}^{\infty} z^n = 1+z+z^2+\cdots$$

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$$(19)\quad \frac{1}{1+z^2}=\frac{1}{1-(-z^2)}$$

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$$(\arctan x)' = \frac{1}{1+z^2}$$

solution) We have $f'(z) = \dfrac{1}{1+z^2}$

Integrating (19) term by term and using $f(0) = 0$ we get

Taylor and Maclaurin Series

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$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$$

Taylor and Maclaurin Series

$$(19)\quad \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots$$

- Practical Methods

The following examples show ways of obtaining Taylor series more quickly than by the use of the coefficient formulas. Regardless of the method used, the result will be the same This follows from the uniqueness

- Substitution
- Integration
- Development by Using the Geometric Series

Find the Maclaurin series of $f(z) = \arctan z$

$$(\arctan x)' = \frac{1}{1+z^2}$$

solution) We have $f'(z) = \dfrac{1}{1+z^2}$

Integrating (19) term by term and using $f(0) = 0$ we get

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots$$

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$$(11)\quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots.$$

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*(proof) Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p684-686

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$$1+q+\cdots+q^n = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q} \quad (q \neq 1)$$

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$$q = \frac{z-z_0}{c-z_0} < 1$$

$$\downarrow$$

$$\frac{1}{1-q} = 1 + q + \cdots + q^n + \frac{q^{n+1}}{1-q}.$$

$$\frac{1}{c-z} = \frac{1}{(c-z_0)\left(1 - \dfrac{z-z_0}{c-z_0}\right)}$$

$$= \frac{1}{(c-z_0)}\left[1 + \frac{z-z_0}{c-z_0} + \cdots + \left(\frac{z-z_0}{c-z_0}\right)^n\right] + \frac{1}{c-z}\left(\frac{z-z_0}{c-z_0}\right)^{n+1}$$

$$\because \lim_{n\to\infty} \frac{1}{c-z}\left(\frac{z-z_0}{c-z_0}\right)^{n+1} = 0$$

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$$\frac{1}{c-z} = \frac{1}{(c-z_0)\left(1 - \dfrac{z-z_0}{c-z_0}\right)}$$

$$= \frac{1}{(c-z_0)}\left[1 + \frac{z-z_0}{c-z_0} + \cdots + \frac{z-z_0}{c-z_0}^{n}\right] + \frac{1}{c-z}\frac{z-z_0}{c-z_0}^{n+1} = \frac{1}{(c-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{c-z_0}\right)^n \quad \because \lim_{n\to\infty}\frac{1}{c-z}\frac{z-z_0}{c-z_0}^{n+1} = 0$$

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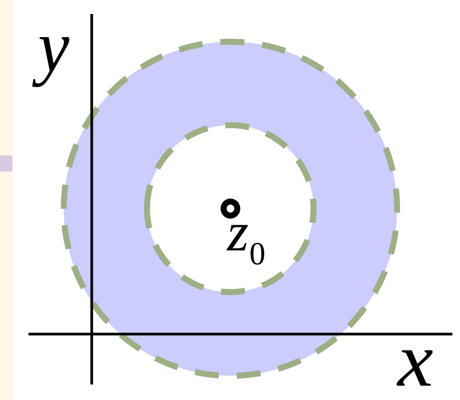
$$\frac{1}{c-z}=\frac{1}{(c-z_0)\left(1-\dfrac{z-z_0}{c-z_0}\right)} \longleftarrow$$

$$=\frac{1}{(c-z_0)}\left[1+\frac{z-z_0}{c-z_0}+\cdots+\frac{z-z_0}{c-z_0}^{n}\right]+\frac{1}{c-z}\frac{z-z_0}{c-z_0}^{n+1}=\frac{1}{(c-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{c-z_0}\right)^n \quad \because \lim_{n\to\infty}\frac{1}{c-z}\frac{z-z_0}{c-z_0}^{n+1}=0$$

The series converges for $\left|\dfrac{z-z_0}{c-z_0}\right|<1$

Laurent Series

Laurent Series. Residue Integration



Laurent series generalize Taylor series.

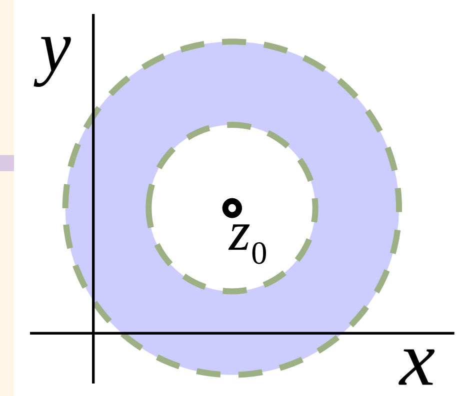
Indeed, whereas a Taylor series has positive integer powers (and a constant term) and converges in a disk, a Laurent series (Sec. 16.1) is a series of positive and negative integer powers of $z - z_0$ and converges in an annulus (a circular ring) with center z_0.

Hence by a Laurent series we can represent a given function $f(z)$ that is analytic in an annulus and may have singularities outside the ring as well as in the "hole" of the annulus.

We know that for a given function the Taylor series with a given center z_0 is unique. We shall see that, in contrast, a function $f(z)$ can have several Laurent series with the same center z_0 and valid in several concentric annuli.

The most important of these series is that which converges for $0 < |z - z_0| < R$, that is, everywhere near the center z_0 except at z_0 itself, where z_0 is a singular point of $f(z)$.

Laurent Series



Laurent series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

$$=a_0+a_1(z-z)+a_2(z-z)^2+\cdots\boxed{+\frac{b_1}{(z-z_0)}+\frac{b_2}{(z-z_0)^2}}+\cdots$$

The series (or finite sum) of the negative powers of this Laurent series is called the principal part of the singularity of $f(z)$ at z_0, and is used to classify this singularity (Sec. 16.2). The coefficient of the power $1/(z-z_0)$ of this series is called the residue of $f(z)$ at z_0.

If in an application we want to develop a function $f(z)$ in powers of $z-z_0$ when $f(z)$ is singular at z_0, we cannot use a Taylor series. Instead we may use Laurent series, consisting of positive integer powers of $z-z_0$ (and a constant) as well as negative integer powers of $z-z_0$; this is the new feature.

Laurent Series

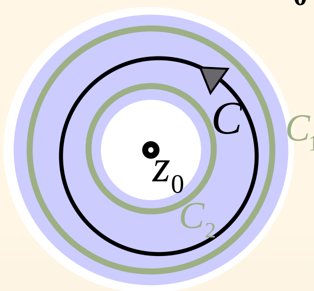
Taylor Series

(1) $f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n, a_n = \frac{1}{n!} f^{(n)}(z_0)$ (2) $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*$

Theorem 1) Laurent's Theorem

Let $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2, with center z_0 and the annulus between them (blue in Fig).

Then $f(z)$ can be represented by the Laurent series consisting of nonnegative and negative powers.



Laurent's theorem

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \quad \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

The coefficients of this Laurent series are given by the integrals [The variable of integration is denoted by z^* since z is used in (1).]

$$(2) \quad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*, \quad b_n = \frac{1}{2\pi i}\oint_C (z^*-z_0)^{n-1} f(z^*) dz^*,$$

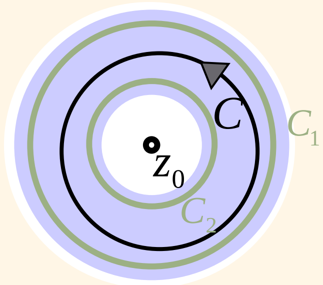
we may write (denoting b_n by a_{-n})

$$(1') \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad (2') \quad a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*, \quad (n = 0, \pm 1, \pm 2, \cdots)$$

Laurent Series

Theorem 1) Laurent's Theorem

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

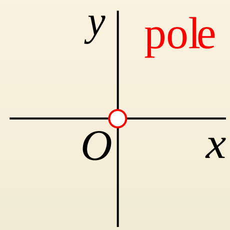


Laurent's theorem

Example)

Find the Laurent series of $z^{-5} \sin z$ with center 0.

solution)



Sec. 15.4. (14)

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

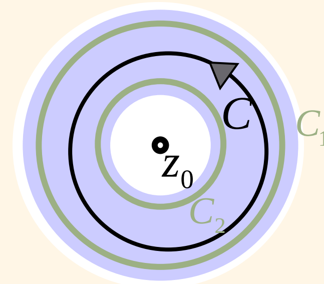
By (14) in sec. 15.4, we obtain

$$z^{-5} \sin z = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} \quad = \frac{1}{z^4} - \frac{1}{3!\,z^2} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots$$

Laurent Series

Theorem 1) Laurent's Theorem

$$(1)\quad f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$



Laurent's theorem

Example)

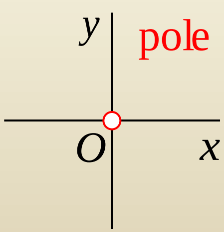
Find the Laurent series of $z^2 e^{1/z}$ with center 0.

solution)

Sec. 15.4. (12)

$$e^{z}=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots.$$

By (12) in sec. 15.4, with z replaced by $1/z$ we obtain a Laurent series whose principal part is an infinite series,

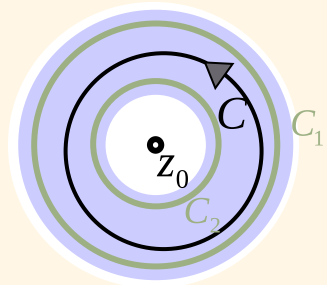


$$z^2 e^{1/z}=z^2\sum_{n=0}^{\infty}\frac{(1/z)^n}{n!}=\sum_{n=0}^{\infty}\frac{z^{2-n}}{n!}=z^2+z+\frac{1}{2}+\frac{1}{3!\,z}+\frac{1}{4!\,z^2}\cdots\quad(|z|>0)$$

Laurent Series

Theorem 1) Laurent's Theorem

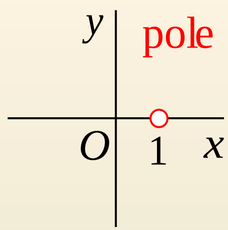
$$(1)\quad f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$



Laurent's theorem

Develop $1/(1-z)$ **(a) in nonnegative powers of z, (b) in negative powers of z.**

solution
(harmonic series)



(a) $\dfrac{1}{1-z}$

(b) $\dfrac{1}{1-z}$

Laurent Series

Theorem 1) Laurent's Theorem

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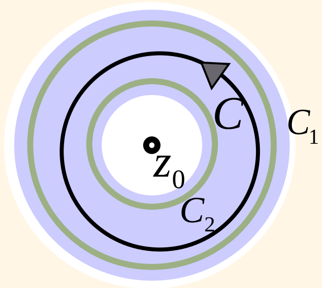

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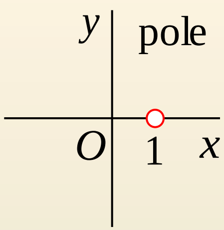
Develop $1/(1-z)$ **(a) in nonnegative powers of z, (b) in negative powers of z.**

(valid if $|z|<1$).

solution
(harmonic series)



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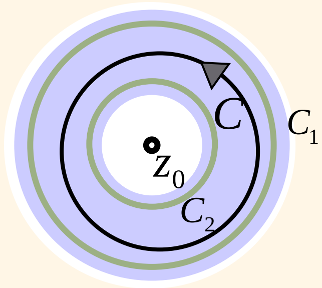
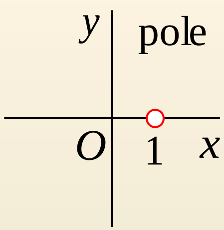


Laurent's theorem

Develop $1/(1- z)$ (a) in nonnegative powers of z, (b) in negative powers of z.

(valid if $|z| < 1$). (valid if $|z| > 1$).

solution
(harmonic series)



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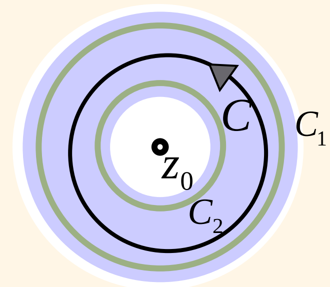

Laurent's theorem

Develop $1/(1-z)$ **(a) in nonnegative powers of z, (b) in negative powers of z.**

(valid if $|z|<1$). (valid if $|z|>1$).

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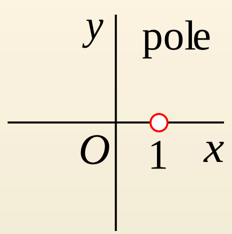

$$\text{(a)}\quad \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n=1+z+z^2+\cdots\quad (\text{valid if } |z|<1).$$

$$\text{(b)}\quad \frac{1}{1-z}$$

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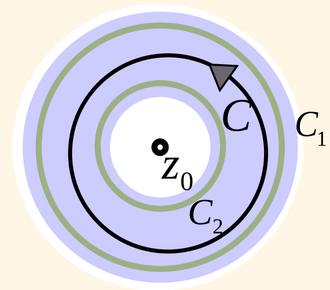

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(valid if $|z| < 1$). (valid if $|z| > 1$).

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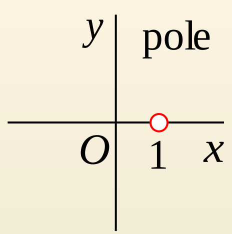

$$\text{(a)} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \quad (\text{valid if } |z| < 1).$$

$$\text{(b)} \quad \frac{1}{1-z} = \frac{-1}{z} \cdot \frac{1}{1 - z^{-1}}$$

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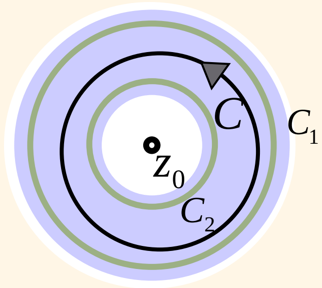

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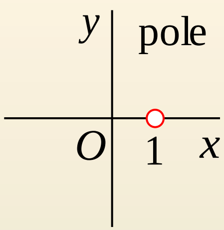
(valid if $|z|<1$). (valid if $|z|>1$).

solution
(harmonic series)



$$\text{(a)}\quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+\cdots \quad (\text{valid if } |z|<1).$$

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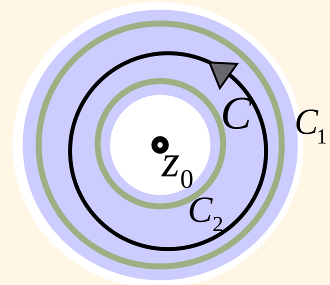
Theorem 1) Laurent's Theorem

$$(1)\quad f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$



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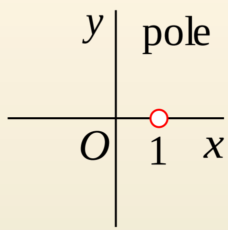

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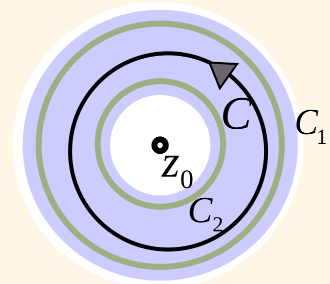

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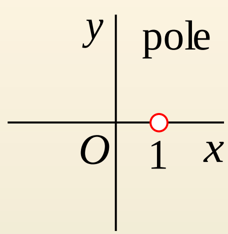

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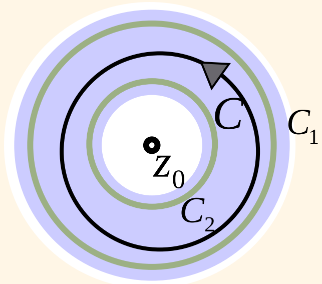

Laurent's theorem

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solution
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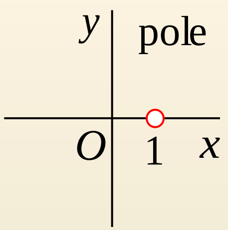

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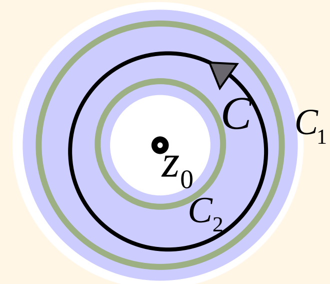

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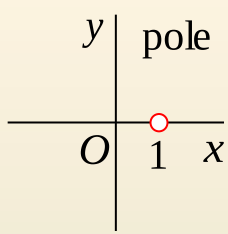

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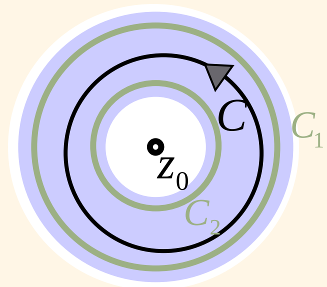
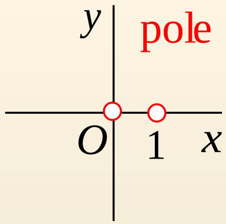


Laurent's theorem

Find all Laurent series of $1/(z^3 - z^4)$ with center 0.

solution)



Example 16.1-3

$$\text{(a)} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

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Laurent Series

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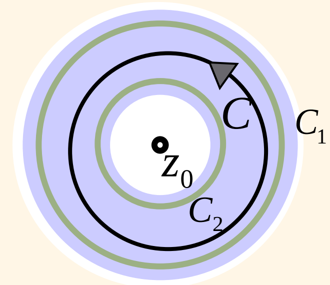

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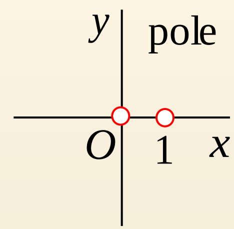

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Multiplying by $1/z^3$, we get from Example 3

Laurent Series

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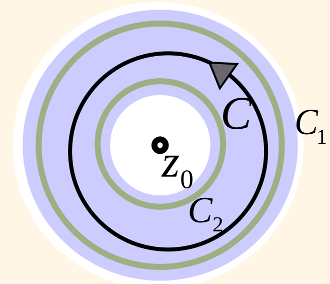

Laurent's theorem

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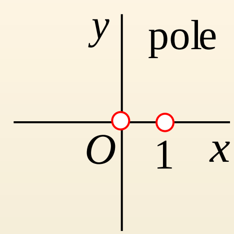
Find all Laurent series of $1/(z^3 - z^4)$ with center 0.



solution)

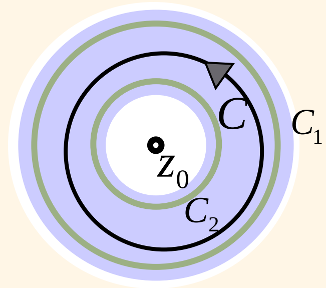
Multiplying by $1/z^3$, we get from Example 3

$$\text{(I)} \qquad \frac{1}{z^3}\frac{1}{(1-z)} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1)$$

Laurent Series

Theorem 1) Laurent's Theorem

$$(1)\quad f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$



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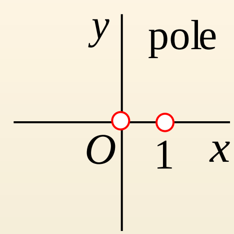

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$$\text{(I)}\quad \frac{1}{z^3}\frac{1}{(1-z)}=\sum_{n=0}^{\infty}z^{n-3}=\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+\cdots \quad (0<|z|<1)$$

$$\text{(II)}\quad \frac{1}{z^3}\frac{1}{(1-z)}=-\sum_{n=0}^{\infty}\frac{1}{z^{n+4}}=-\frac{1}{z^4}-\frac{1}{z^5}-\cdots \quad (|z|>1)$$

Laurent Series

Theorem 1) Laurent's Theorem

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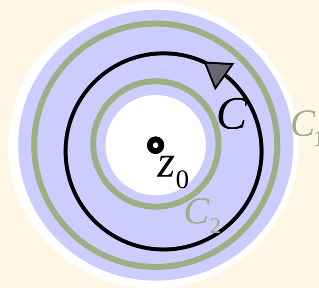

Laurent's theorem

Find all Taylor and Laurent series of $f(z)$ with center 0.

$$f(z)=\frac{-2z+3}{z^2-3z+2}$$

solution)

In terms of partial fraction

$$f(z)=-\frac{1}{z-1}-\frac{1}{z-2}=\frac{1}{1-z}-\frac{1}{z-2}.$$

(a) and (b) in Example 3 take care of the first fraction.

$$(a)\quad \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n\quad(|z|<1)$$

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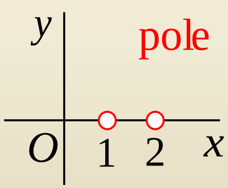

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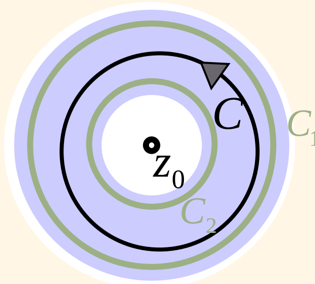

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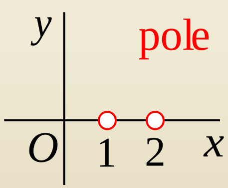

For the second fraction,

$$\text{(a)}\quad \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n\quad(|z|<1)$$

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Laurent Series

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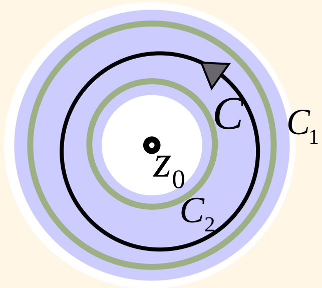

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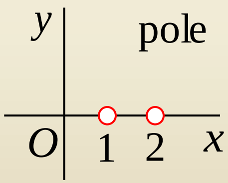

For the second fraction,

$$\text{(c)} \quad -\frac{1}{z-2} = \frac{1}{2\left(1 - \frac{z}{2}\right)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \quad (|z| < 2)$$

$$\text{(a)} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

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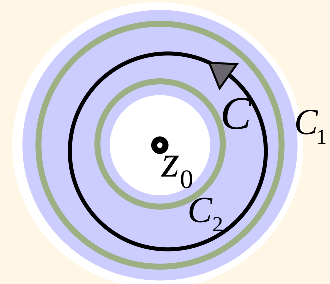

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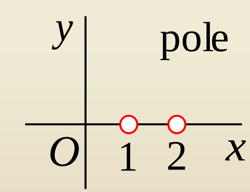

$$(a)\quad \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n\quad(|z|<1)$$

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For the second fraction,

$$(c)\quad -\frac{1}{z-2}=\frac{1}{2\left(1-\frac{z}{2}\right)}=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n\quad(|z|<2)$$

$$(d)\quad -\frac{1}{z-2}=-\frac{1}{z\left(1-\frac{2}{z}\right)}=-\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}\quad(|z|>2)$$

Laurent Series

Theorem 1) Laurent's Theorem

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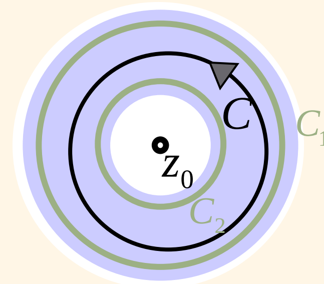

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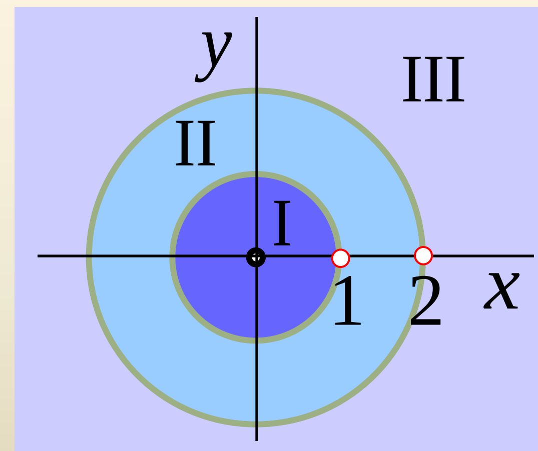
$$\text{(a)} \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \ (|z| < 1), \quad \text{(b)} \quad \frac{1}{1-z} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \quad (|z| > 1)$$

$$\text{(c)} \quad -\frac{1}{z-2} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \ (|z| < 2), \quad \text{(d)} -\frac{1}{z-2} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \quad (|z| > 2)$$



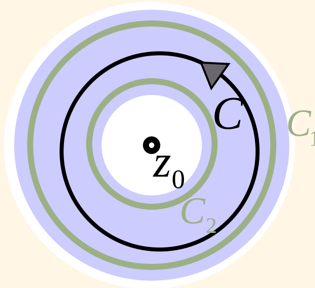
(I) From (a) and (c), valid for $|z| < 1$,

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n = \frac{3}{2} + \frac{5}{4} z + \frac{9}{8} z^2 + \cdots$$

Laurent Series

Theorem 1) Laurent's Theorem

$$(1)\quad f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$



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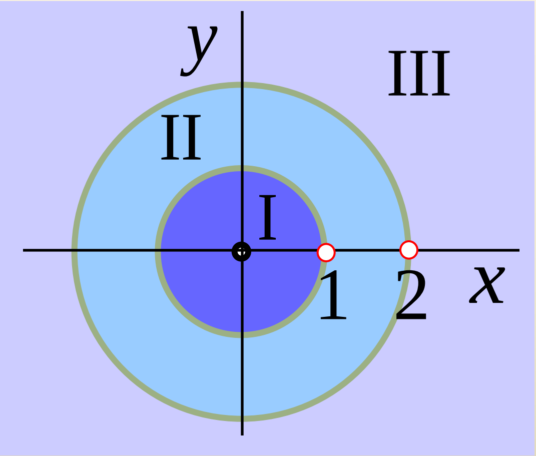

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$$\text{(a)}\ \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n\ (|z|<1),\quad \text{(b)}\ \frac{1}{1-z}=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\ \ (|z|>1)$$

$$\text{(c)}\ -\frac{1}{z-2}=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n\ (|z|<2),\quad \text{(d)}\ -\frac{1}{z-2}=-\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}\ \ (|z|>2)$$

(II) From (c) and (b), valid for $1<|z|<2$,

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}=\frac{1}{2}+\frac{1}{4}z+\cdots-\frac{1}{z}-\frac{1}{z^2}-\cdots$$

Laurent Series

Theorem 1) Laurent's Theorem

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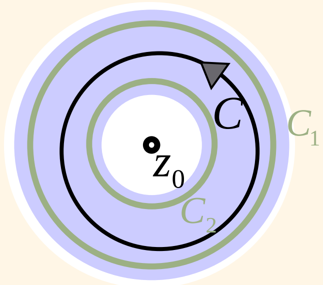

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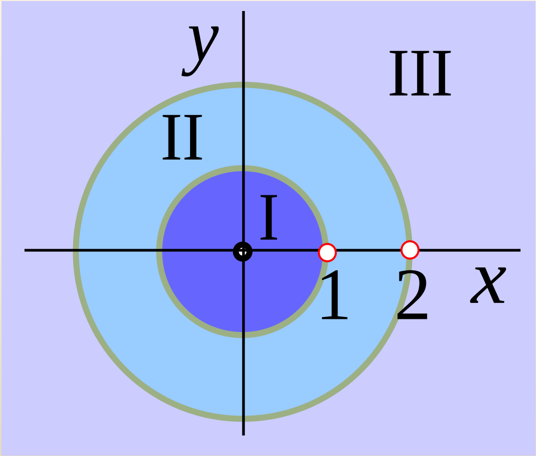

solution)

$$\text{(a)}\ \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n\ (|z|<1),\quad \text{(b)}\ \frac{1}{1-z}=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\quad (|z|>1)$$

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(III) From (d) and (b), valid for $|z|>2$,

$$f(z)=-\sum_{n=0}^{\infty}(2^n+1)\frac{1}{z^{n+1}}=-\frac{2}{z}-\frac{3}{z^2}-\frac{5}{z^3}-\frac{9}{z^4}\cdots$$

Singularities and Zeros. Infinity

Singularities and Zeros

A zero of an analytic function $f(z)$ in a domain D is a $z = z_0$ in D such that $f(z_0) = 0$.

A zero has order n if not only if but also the derivatives

$f',\ f'',\ \cdots,\ f^{(n-1)}$ are all 0 at $z = z_0$ but $f^{(n)} \neq 0.$

A first-order zero is also called a simple zero.

For a second-order zero, $f(z_0) = f'(z_0) = 0$ but $f''(z_0) \neq 0$
and so on.

We call $z = z_0$ an isolated singularity of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$.

Isolated singularities of $f(z)$ at $z = z_0$ can be classified by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

valid in the immediate neighborhood of the singular point $z = z_0$, except at z_0 itself, that is, in a region of the form

$$0 < |z - z_0| < R.$$

Singularities and Zeros

$$(1)\quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

The sum of the first series is analytic at $z = z_0$, as we know from the last section. The second series, containing the negative powers, is called <u>the principal part of (1)</u>, as we remember from the last section. If it has <u>only finitely many terms</u>, it is of the form

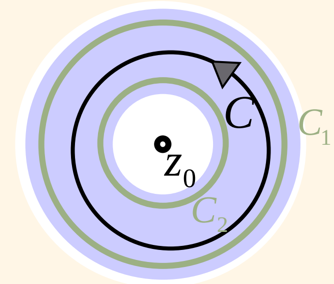
$$(2)\qquad \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0).$$

Then the singularity of $f(z)$ at $z = z_0$ is called a pole(극), and m is called its order. Poles of the first order are also known as simple poles.

Singularities and Zeros

Theorem 1) Poles

If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

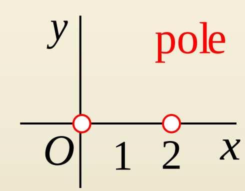


Laurent's theorem

Theorem 4) Poles and Zeros

Let $f(z)$ be analytic at $z = z_0$ and have a zero of nth order at $z = z_0$. then $1/f(z)$ has a pole of nth order at $z = z_0$; and so does $h(z)/f(z)$, provided $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$.

Example)

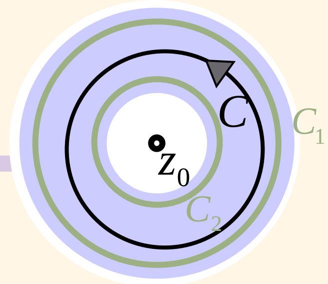


The function $$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$.

Residue Integration Method

Residue Integration Method



Laurent's theorem

The purpose of Cauchy's residue integration: the evaluation of integrals

$$\oint_C f(z)dz$$

If $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C and inside C, then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

that converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z - z_0| < R$

Now comes the key idea. The coefficient b_1 of the first negative power $1/(z - z_0)$ of this Laurent series is given by the integral formula (2) with $n = 1$, namely,

$$b_1 = \frac{1}{2\pi i}\oint_C f(z)dz. \quad \Rightarrow \quad \oint_C f(z)dz = 2\pi i \cdot b_1$$

The coefficient b_1 is called the residue of $f(z)$ at $z = z_0$.

Sec.16.1 (2) $a_n = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z).$$

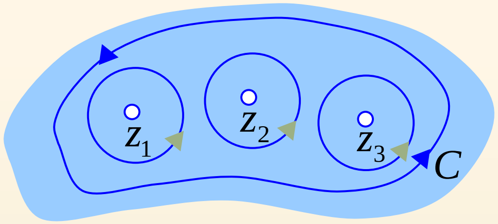
$$b_n = \frac{1}{2\pi i}\oint_C (z^* - z_0)^{n-1} f(z^*) dz^*$$

Several Singularities Inside the Contour

Theorem 1) Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, z_2, \cdots, z_k$:

$$(6) \quad \oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$



Several Singularities Inside the Contour

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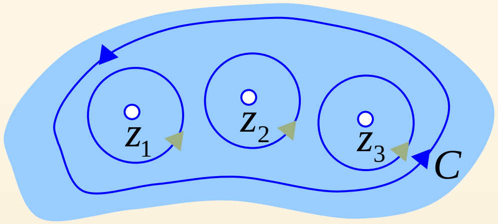


Example) Integrate the function $f(z) = z^{-4} \sin z$ counterclockwise around the unit circle C.

solution) From (14) in Sec. 15.4 we obtain the Laurent series

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} - \frac{z^3}{7!} + - \cdots$$

$$(1) \quad \oint_C f(z)\,dz = 2\pi i b_1.$$

Sec 15.4 (14)

$$\sin z = \sum_{k=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

which converges for $|z| > 0$ (that is, for all $z \neq 0$). This series shows that $f(z)$ has a pole of third order at $z = 0$ and the residue $b_1 = -1/3!$. From (1) we thus obtain the answer

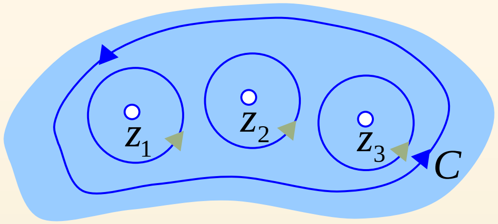
$$\oint_C \frac{\sin z}{z^4}\,dz = 2\pi i b_1 = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}$$

Several Singularities Inside the Contour

Theorem 1) Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, z_2, \cdots, z_k$:

$$(6) \quad \oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$



Example 16.1-4

$$\text{(I)} \quad \frac{1}{z^3}\frac{1}{(1-z)} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \quad (0 < |z| < 1)$$

$$\text{(II)} \quad \frac{1}{z^3}\frac{1}{(1-z)} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots \quad (|z| > 1)$$

Example) Integrate the function $f(z) = 1 / (z^3 - z^4)$ counterclockwise around the circle C: $|z| = 1/2$

solution) $z^3 - z^4 = z^3(1 - z)$ shows that $f(z)$ is singular at $z = 0$ and $z = 1$.

Now $z = 1$ lies outside C. Hence it is of no interest here.
$0 < |z| < 1$. This is series (I) in Example 4, Sec. 16.1,

We see from it that this residue is 1. Clockwise integration thus yields

$$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i$$

CAUTION! Had we used the wrong series (II) in Example 4, Sec. 16.1,
we would have obtained the wrong answer, 0, because this series has no power $1/z$.

Formulas for Residues

To calculate a residue at a pole, we need not produce a whole Laurent series, but, more economically, we can derive formulas for residues once and for all.

<u>Simple Poles</u>. Two formulas for the residue of $f(z)$ at a simple pole at z_0 are

$$(3)\quad \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z-z_0) f(z)$$

and, assuming that $f(z) = p(z)/q(z)$, $p(z_0)\neq 0$, and $q(z)$ has a simple zero at z_0 (so that $f(z)$ has at z_0 a simple pole, by Theorem 4 in Sec. 16.2)*

$$(4)\quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

*(proof) Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p713-714

Formulas for Residues

Simple Poles. Two formulas for the residue of $f(z)$ at a simple pole at z_0 are

$$(3)\quad \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0}(z-z_0)f(z)$$

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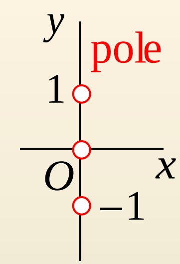
$$(4)\quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Example) $f(z)$ has some simple poles, and (3) gives the residues at the poles. Find the all residues of $f(z)$.

$$f(z) = \frac{9z+i}{z^3+z} = \frac{9z+i}{z(z+i)(z-i)}$$



solution) $\text{Poles}: z = 0,\ i,\ -i$

By (4),

$$z=0,\quad \operatorname*{Res}_{z=0} \frac{9z+i}{z^3+z} = \left[\frac{9z+i}{3z^2+1}\right]_{z=0} = i$$

$$z=i,\quad \operatorname*{Res}_{z=i} \frac{9z+i}{z^3+z} = \left[\frac{9z+i}{3z^2+1}\right]_{z=i} = -5i$$

$$z=-i,\quad \operatorname*{Res}_{z=-i} \frac{9z+i}{z^3+z} = \left[\frac{9z+i}{3z^2+1}\right]_{z=-i} = 4i$$

By (3),

$$\operatorname*{Res}_{z=i} \frac{9z+i}{z^3+z}$$

$$= \lim_{z\to i}(z-i)\frac{9z+i}{z(z+i)(z-i)}$$

$$= \frac{9i+i}{i(i+i)} = -5i.$$

Formulas for Residues

<u>Simple Poles</u>. Two formulas for the residue of $f(z)$ at a simple pole at z_0 are

$$(3)\quad \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0}(z-z_0)f(z)$$

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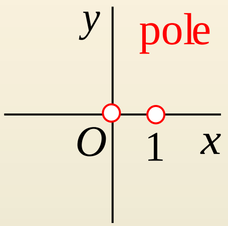
$$(4)\quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Example) Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside (d) 0 and 1 are outside.



$$\oint_C \frac{4-3z}{z^2-z}\,dz = \oint_C \frac{4-3z}{z(z-1)}\,dz$$

solution) The integrand has simple poles at 0 and 1, with residues [by (3)]

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{(z-1)}\right]_{z=0} = -4,$$

$$\operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left[\frac{4-3z}{z}\right]_{z=1} = 1.$$

Formulas for Residues

Simple Poles. Two formulas for the residue of $f(z)$ at a simple pole at z_0 are

$$(3)\quad \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0}(z-z_0)f(z)$$

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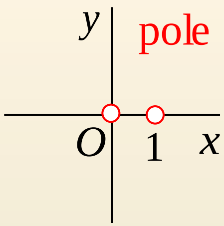
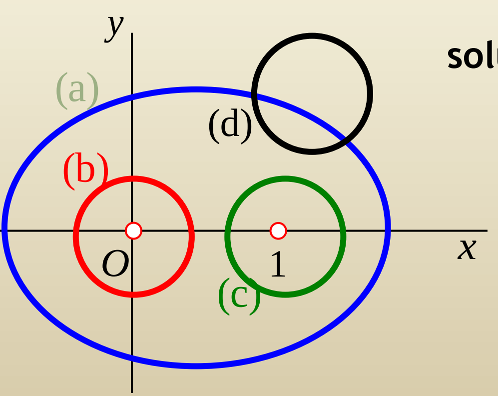


$$\oint_C \frac{4-3z}{z^2-z}\,dz = \oint_C \frac{4-3z}{z(z-1)}\,dz$$



solution) $\operatorname*{Res}_{z=0} \dfrac{4-3z}{z(z-1)} = -4,\qquad \operatorname*{Res}_{z=1} \dfrac{4-3z}{z(z-1)} = 1.$

(a) $2\pi i(-4+1) = -6\pi i$ (b) $2\pi i(-4) = -8\pi i$

(c) $2\pi i(1) = 2\pi i$ (d) 0

Residue Integration of Real Integrals

Residue Integration of Real Integrals

Integrals of Rational Functions of cosθ and sinθ

certain classes of complicated real integrals can be integrated by the residue theorem, as we shall see.

We first consider integrals of the type

$$(1) \quad J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta \qquad Ex.)\ \frac{\sin^2\theta}{5-4\cos\theta}$$

where $F(\cos\theta, \sin\theta)$ **is a real rational function of** $\cos\theta$ **and** $\sin\theta$

Setting $e^{i\theta} = z \quad , dz/d\theta = ie^{i\theta} \quad , d\theta = dz/iz$

$$(2) \quad \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Then,

$$(3) \quad J = \oint_C f(z)\frac{dz}{iz}$$

and, as θ **ranges from 0 to 2π in (1), the variable** $z = e^{i\theta}$ **ranges counterclockwise once around the unit circle** $|z| = 1$**.**

Residue Integration of Real Integrals

Integrals of Rational Functions of cosθ and sinθ

$$(1) \quad J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \quad \Rightarrow \quad (3) \quad J = \oint_C f(z) \frac{dz}{iz}$$

real rational function

Show by the present method that $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi.$

solution) We use $\cos\theta = \frac{1}{2}(z + 1/z)$ **and** $e^{i\theta} = z \quad (d\theta = dz/iz)$

Then the integral becomes

Residue Integration of Real Integrals

Integrals of Rational Functions of cosθ and sinθ

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Then the integral becomes

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = \oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)}$$

Residue Integration of Real Integrals

Integrals of Rational Functions of cosθ and sinθ

$$(1) \quad J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \quad \Rightarrow \quad (3) \quad J = \oint_C f(z) \frac{dz}{iz}$$

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solution) We use $\cos\theta = \frac{1}{2}(z + 1/z)$ **and** $e^{i\theta} = z \quad (d\theta = dz/iz)$

Then the integral becomes

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = \oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} = -\frac{2}{i} \oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}$$

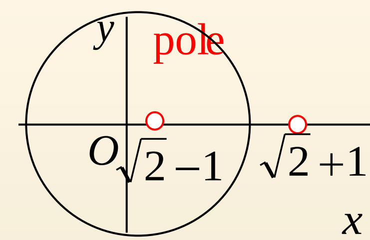
C : counterclockwise once around the unit circle $|z| = 1$

Residue Integration of Real Integrals

Integrals of Rational Functions of cosθ and sinθ

$$(1)\quad J=\int_0^{2\pi} F(\cos\theta,\sin\theta)\,d\theta \quad\Rightarrow\quad (3)\quad J=\oint_C f(z)\frac{dz}{iz}$$

real rational function



Show by the present method that $\displaystyle\int_0^{2\pi}\frac{d\theta}{\sqrt{2}-\cos\theta}=2\pi.$

Solution) $\displaystyle -\frac{2}{i}\oint_C \frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}$ **C : counterclockwise once around the unit circle $|z|$ = 1**

We see that the integrand has a simple pole at $z_1=\sqrt{2}+1$ outside the unit circle C, so that it is of no interest here, and another simple pole at $z_2=\sqrt{2}-1$

(where $z-\sqrt{2}+1=0$) inside C with

$$\operatorname*{Res}_{z=z_2}\frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}=\left[\frac{1}{z-\sqrt{2}-1}\right]_{z=\sqrt{2}-1}=-\frac{1}{2}$$

$$\therefore -\frac{2}{i}\oint_C\frac{dz}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}=-\frac{2}{i}\cdot 2\pi i\cdot\operatorname*{Res}_{z=z_2}\frac{1}{(z-\sqrt{2}-1)(z-\sqrt{2}+1)}=-\frac{2}{i}\cdot 2\pi i\cdot\left(-\frac{1}{2}\right)=2\pi$$

Residue Integration of Real Integrals

Improper Integral

As another large class, let us consider real integrals of the form

$$(4) \qquad \int_{-\infty}^{\infty} f(x)dx.$$

Such an integral, whose interval of integration is not finite is called an improper integral(이상적분), and it has the meaning

$$(5') \qquad \int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_{a}^{0} f(x)dx + \lim_{b\to\infty}\int_{0}^{b} f(x)dx$$

If both limits exist, we may couple the two independent passages to $-\infty$ and ∞, and write

$$(5) \qquad \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

The limit in (5) is called the Cauchy principal value of the integral. It is written

$$\text{pr.v.} \int_{-\infty}^{\infty} f(x)dx$$

Residue Integration of Real Integrals

Improper Integral

$$(5') \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^0 f(x)dx + \lim_{b\to\infty}\int_0^b f(x)dx$$

$$(5) \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

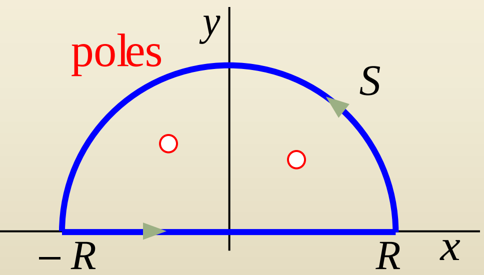
We assume that the function $f(x)$ in (5') is a real rational function whose denominator is different from zero for all x and is of degree at least two units higher than the degree of the numerator. Then the limits in (5') exist, and we may start from (5).

We consider the corresponding contour integral

$$(5^*) \quad \oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx$$

around a path C

$$\oint_C f(z)dz = 2\pi i \sum \operatorname{Res} f(z)$$



Path C of the contour integral in (5*)

Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper half-plane, and if we choose R large enough, then C encloses all these poles. By the residue theorem we then obtain

$$(6) \quad \int_{-R}^{R} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)dz$$

Residue Integration of Real Integrals

Improper Integral

$$(5') \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_{a}^{0} f(x)dx + \lim_{b\to\infty}\int_{0}^{b} f(x)dx$$

$$(5) \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

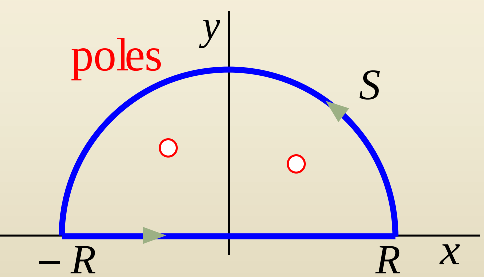
$$(5^*) \quad \oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx$$

around a path C

$$(6) \quad \int_{-R}^{R} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)dz$$

We prove that, if $R\to\infty$, the value of the integral over the semicircle S approaches zero.*

$$(7) \quad \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z)$$



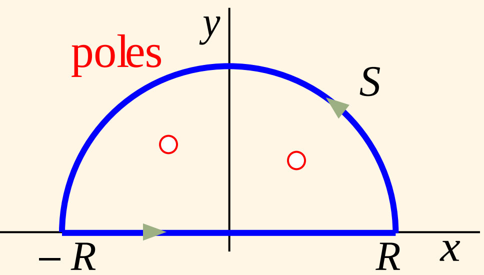
Path C of the contour integral in (5*)

where we sum over all the residues of $f(z)$ at the poles of $f(z)$ in the upper half-plane.

*(proof) Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p719-720

Residue Integration of Real Integrals

Improper Integral

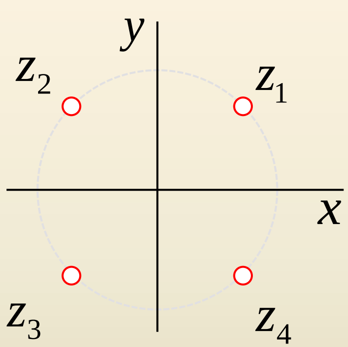


$$(7) \quad \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z)$$

Using (7) show that

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Sec.16.3 (4) $\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z)}{q'(z)}$



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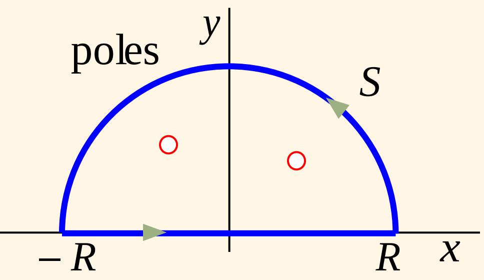
Residue Integration of Real Integrals

Improper Integral



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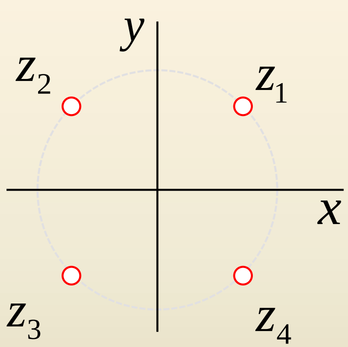

solution) $f(z) = \dfrac{1}{1+z^4}$ **has four simple poles at the points**

$$z_1 = e^{\pi i/4}, \quad z_2 = e^{3\pi i/4}, \quad z_3 = e^{-3\pi i/4}, \quad z_4 = e^{-\pi i/4}$$

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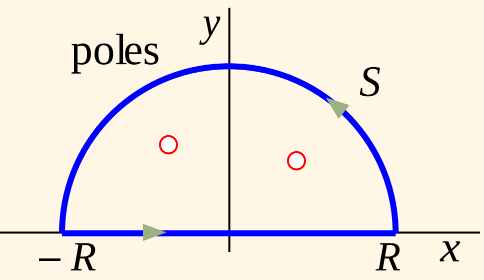

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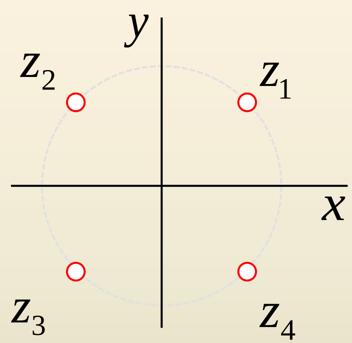

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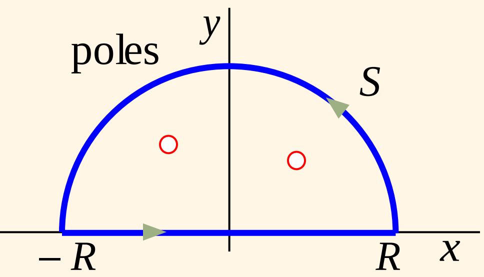

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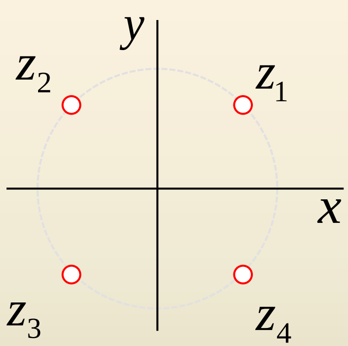

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Residue Integration of Real Integrals

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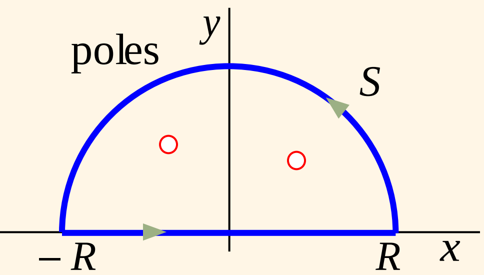

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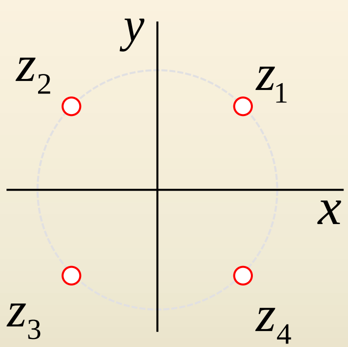

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Residue Integration of Real Integrals

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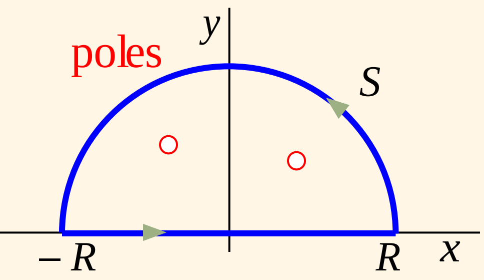

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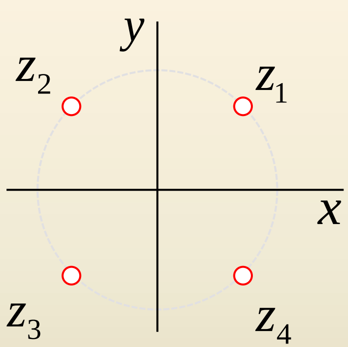

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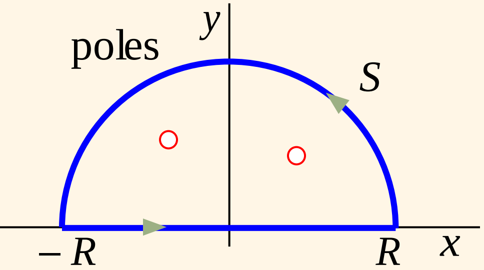

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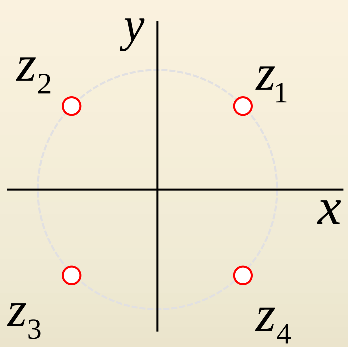

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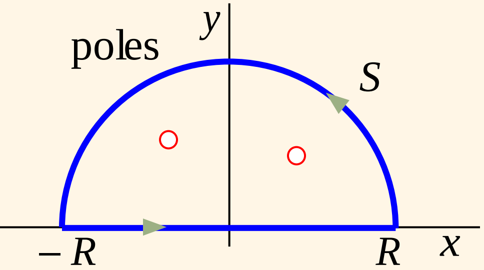

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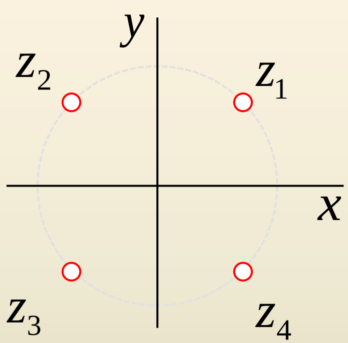

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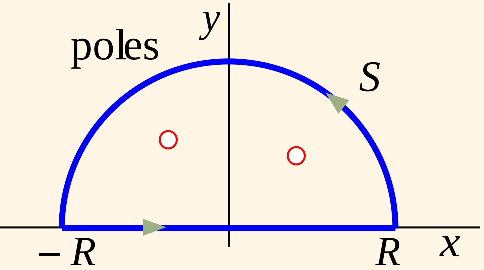

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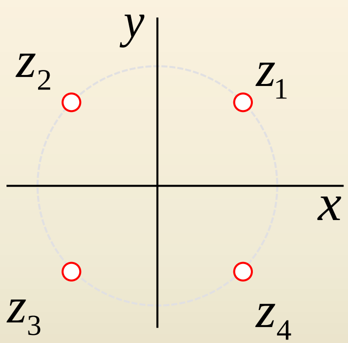

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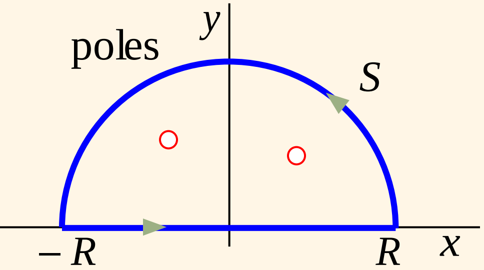

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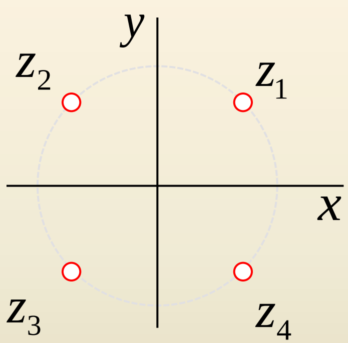

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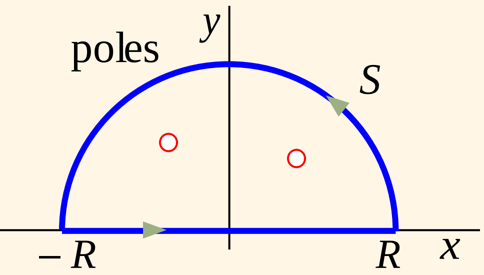

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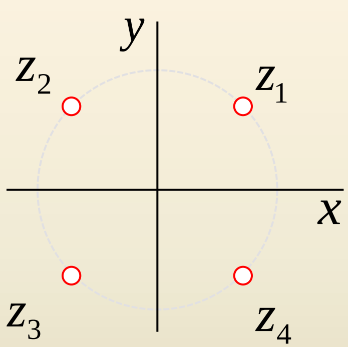

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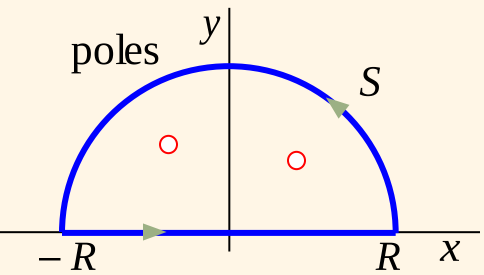

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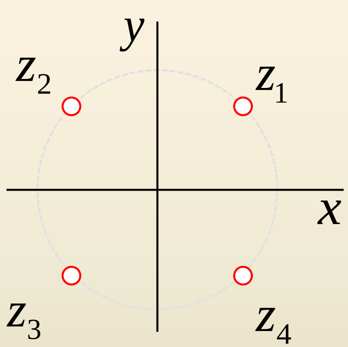

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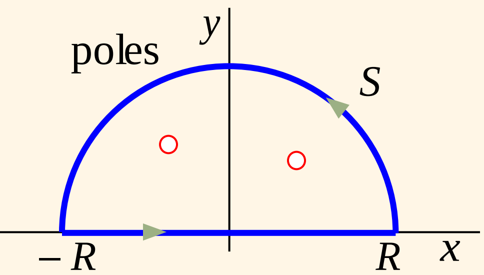

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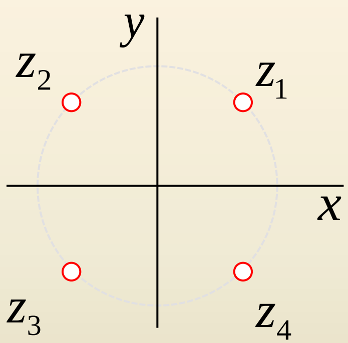

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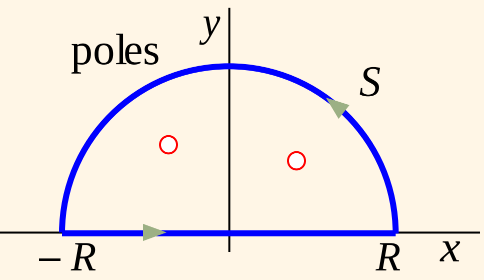

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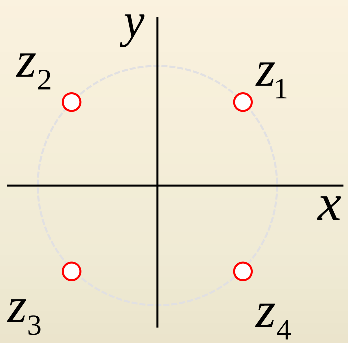

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Since 1 / (1 + x^4) is an even function, we thus obtain, as asserted,

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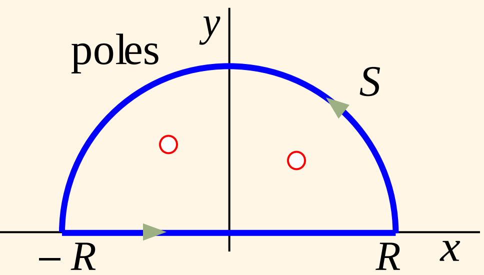
Improper Integral



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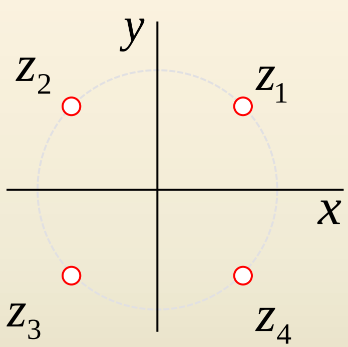

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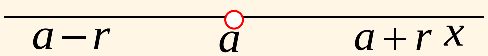
Since 1 / (1 + x^4**) is an even function, we thus obtain, as asserted,**

$$\therefore \int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

Residue Integration of Real Integrals

Another Kind of Improper Integral



$$(11)\quad \int_A^B f(x)dx$$

$$(12)\quad \int_A^B f(x)dx = \lim_{\varepsilon\to 0}\int_A^{a-\varepsilon} f(x)dx + \lim_{\eta\to 0}\int_{a+\eta}^{B} f(x)dx$$

$$(13)\quad \lim_{\varepsilon\to 0}\left[\int_A^{a-\varepsilon} f(x)dx + \int_{a+\varepsilon}^{B} f(x)dx\right]$$

This is called the Cauchy principal value of the integral. It is written

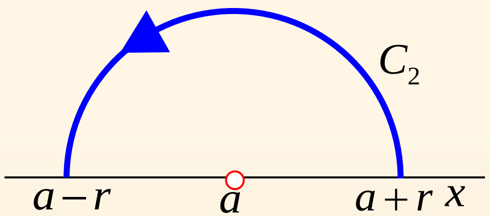
$$\text{pr. v.} \int_A^B f(x)dx.$$

Residue Integration of Real Integrals

Another Kind of Improper Integral

Theorem 1) Simple Poles on the Real Axis*

If $f(z)$ has a simple pole at $z = a$ on the real axis, then



$$\lim_{r \to 0} \int_{C_2} f(z)\,dz = \pi i \operatorname*{Res}_{z=a} f(z)$$

*(proof) Kreyszig E. Advanced Engineering Mathematics, 9th edition, Wiley, 2006, p723

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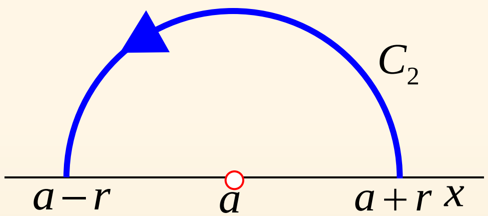

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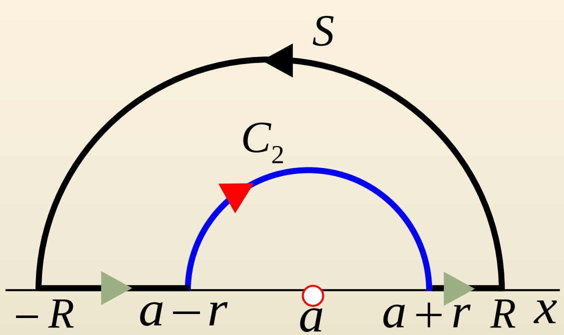

Fig. 374 shows the idea of applying Theorem 1 to obtain the principal value of the integral of a rational function $f(z)$ from $-\infty$ to ∞.

For sufficiently large R the integral over the entire contour in Fig. 374 has the value J given by $2\pi i$ times the sum of the residues of $f(z)$ at the singularities in the upper half-plane.

$$J = 2\pi i \sum \operatorname{Res} f(z)$$

We assume that $f(z) \to 0$, as x goes infinite then the value of the integral over the large semicircle S approaches 0 as $R \to \infty$. For $r \to 0$ the integral over C_2 (clockwise!) approaches the value

$$K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

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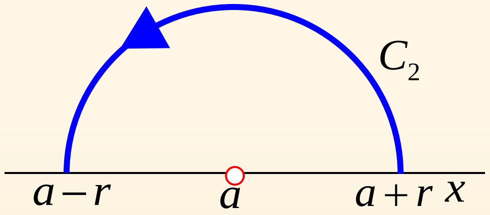
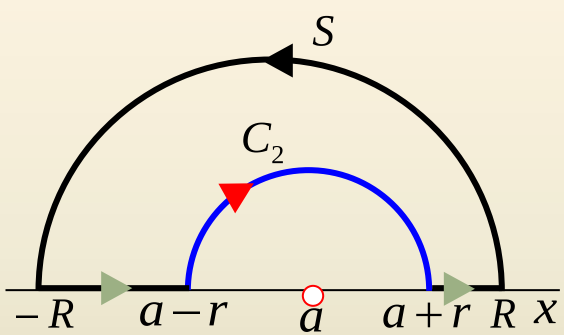


$$\lim_{r \to 0} \int_{C_2} f(z)dz = \pi i \operatorname*{Res}_{z=a} f(z)$$



$$J = 2\pi i \sum \operatorname{Res} f(z) \qquad K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

Together this show that the principal value P of the integral from -∞ to ∞ Plus K equals J.

Hence $P = J - K = 2\pi i \sum \operatorname{Res} f(z) + \pi i \operatorname*{Res}_{z=a} f(z).$

If $f(z)$ has several simple poles on the real axis, then

$$K = -\pi i \sum \operatorname{Res} f(z).$$

Residue Integration of Real Integrals

Another Kind of Improper Integral

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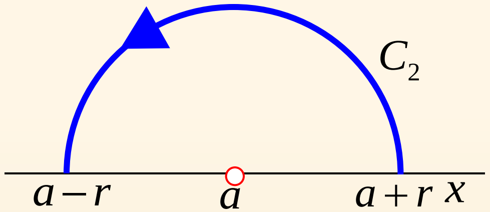

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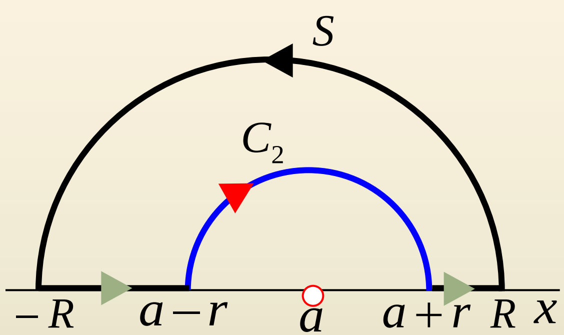

$$P = J - K = 2\pi i \sum \operatorname{Res} f(z) + \pi i \operatorname*{Res}_{z=a} f(z).$$

$$J = 2\pi i \sum \operatorname{Res} f(z) \qquad K = -\pi i \sum \operatorname{Res} f(z).$$

Hence the desired formula is

$$(14) \quad \text{pr. v.}\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

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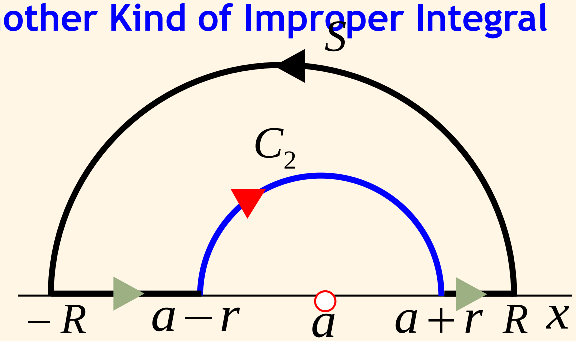

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solution)

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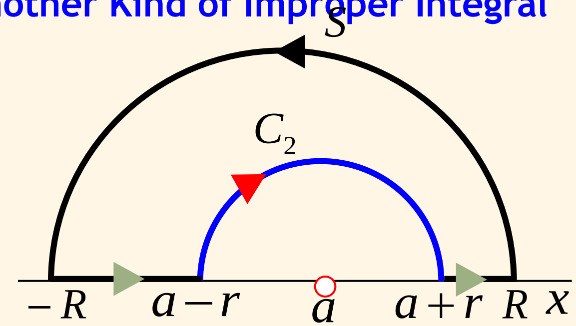

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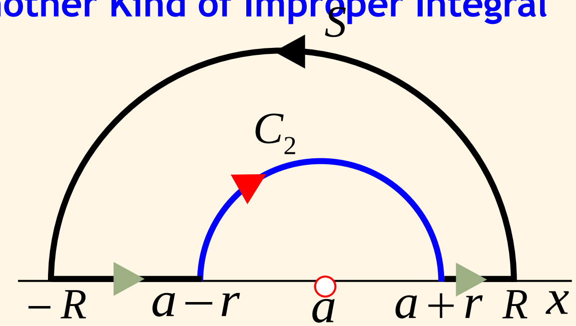

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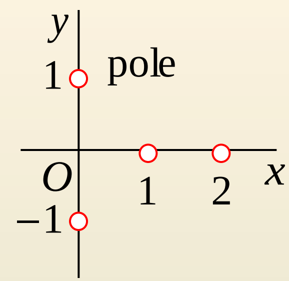

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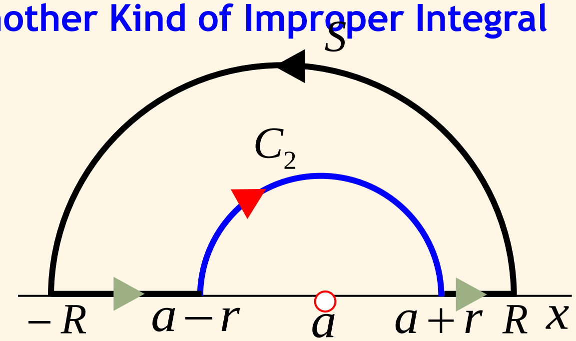

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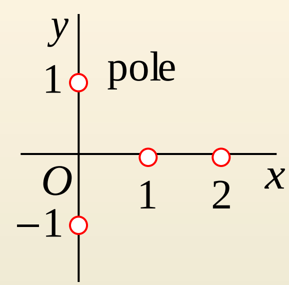

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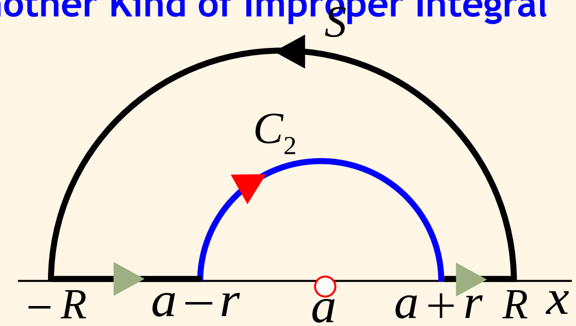

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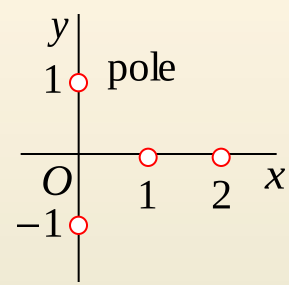

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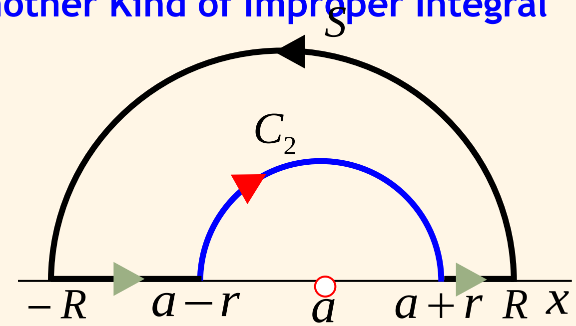

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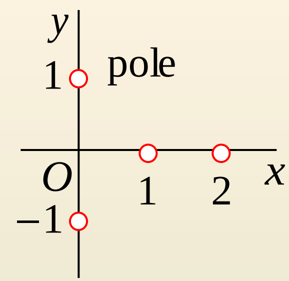

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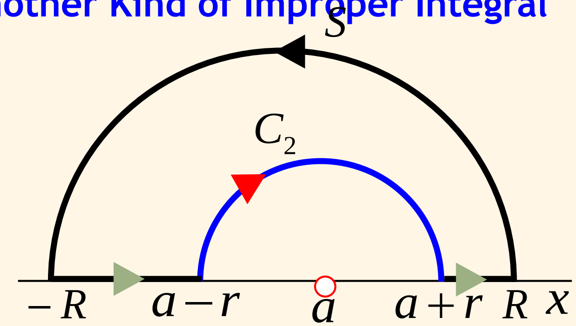

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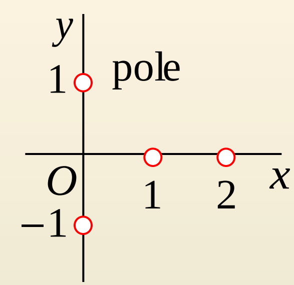

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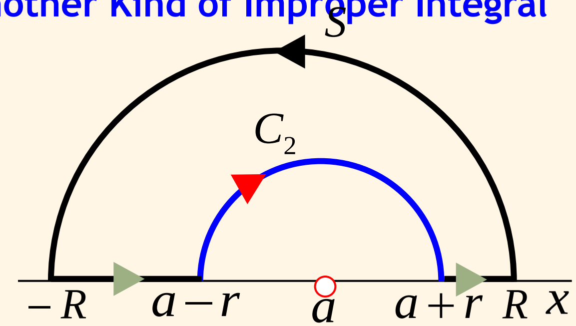

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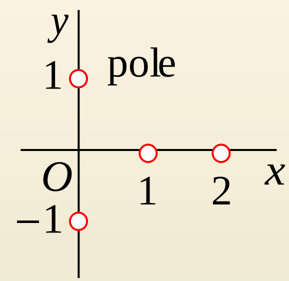

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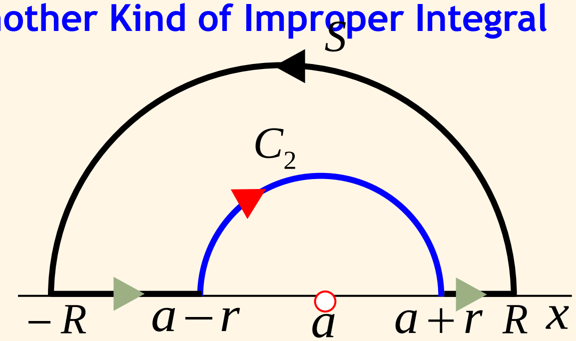

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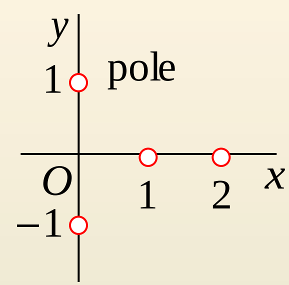

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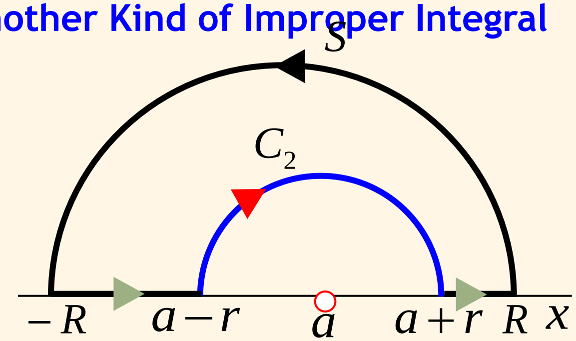

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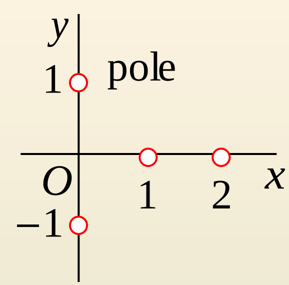

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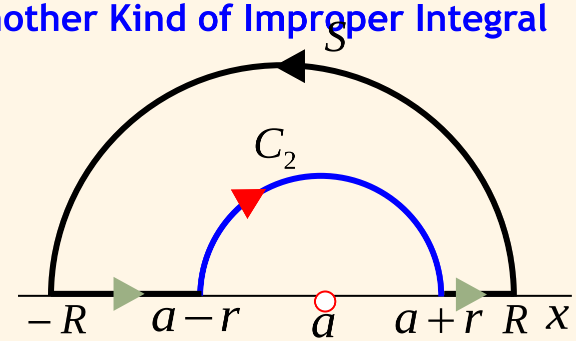

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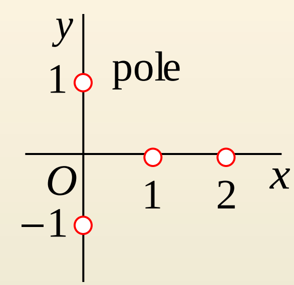

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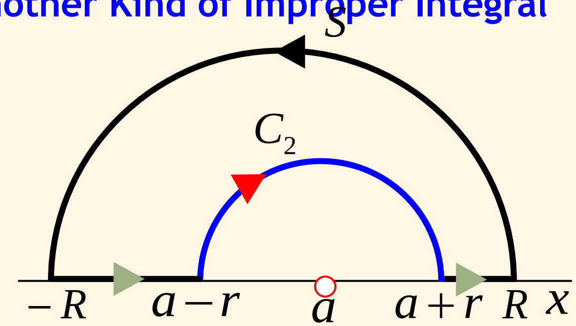

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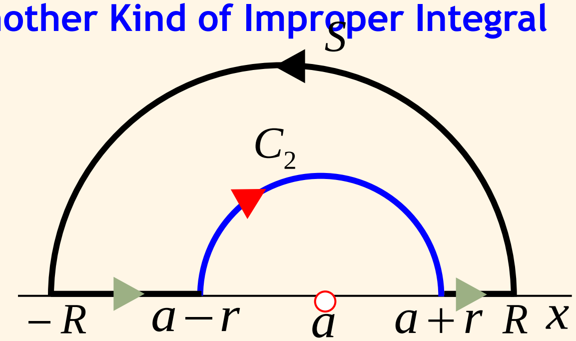

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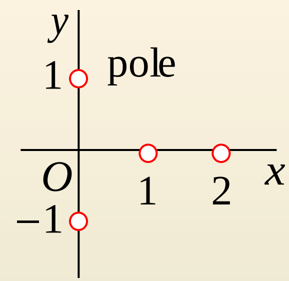

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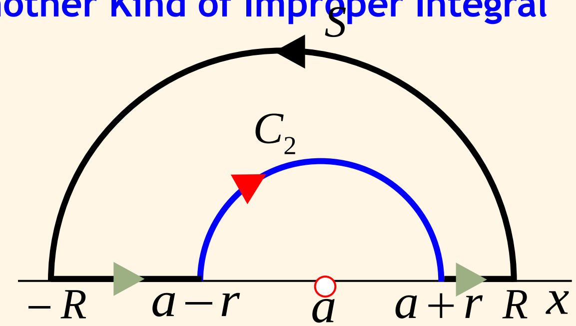

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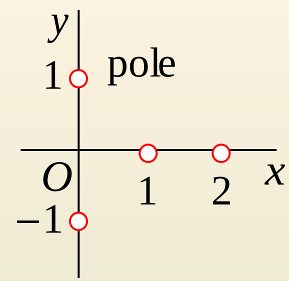

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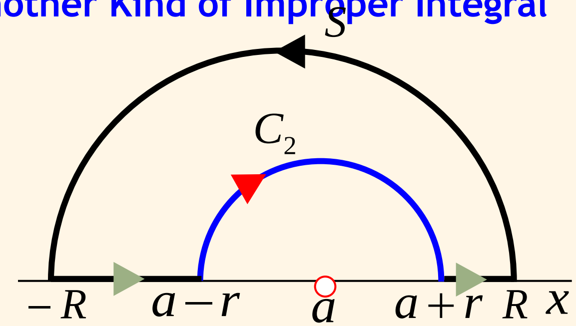

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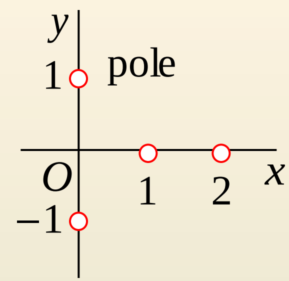

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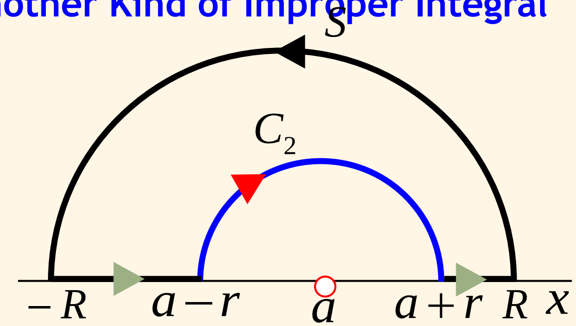
$$\underbrace{\operatorname*{Res}_{z=1} f(z) = -\frac{1}{2}, \quad \operatorname*{Res}_{z=2} f(z) = \frac{1}{5},}_{\textbf{real axis}} \quad \underbrace{\operatorname*{Res}_{z=i} f(z) = \frac{3-i}{20}}_{\textbf{upper half-plane}}$$

z = - i in the lower half-plane, which is of no interest.

Residue Integration of Real Integrals

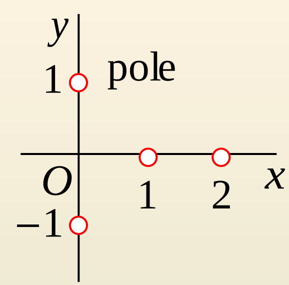
Another Kind of Improper Integral



$$\text{(14)}\quad \text{pr. v.}\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \text{Res}\, f(z) + \pi i \sum \text{Res}\, f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

Find the principal value $\text{pr. v.}\int_{-\infty}^{\infty} \dfrac{dx}{(x^2-3x+2)(x^2+1)}$



solution) Since $f(x) = \dfrac{1}{(x^2-3x+2)(x^2+1)} = \dfrac{1}{(x-1)(x-2)(x+i)(x-i)}$,

$$\underset{z=1}{\text{Res}}\, f(z) = -\frac{1}{2},\quad \underset{z=2}{\text{Res}}\, f(z) = \frac{1}{5},\quad \underset{z=i}{\text{Res}}\, f(z) = \frac{3-i}{20}$$

real axis **upper half-plane**

$z = -i$ in the lower half-plane, which is of no interest.

$$\therefore\ \text{pr. v.}\int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)} = 2\pi i\left(\frac{3-i}{20}\right) + \pi i\left(-\frac{1}{2}+\frac{1}{5}\right) = \frac{\pi}{10}$$

Reference slides

Cauchy-Riemann Equations

Cauchy-Riemann Equations

Theorem 1) Cauchy-Riemann Equations

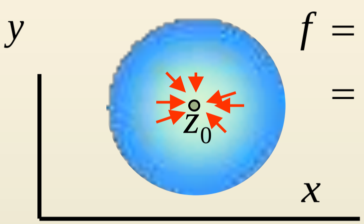
Let $f(z) = u(x,y) + iv(x,y)$ **be defined and continuous in some neighborhood of a point** $z = x + iy$ **and differentiable at** z **itself. Then at that point, the first-order partial derivatives of** u **and** v **exist and satisfy the Cauchy-Riemann equation (1).**

Hence if $f(z)$ **is analytic in a domain** D**, those partial derivatives exist and satisfy (1) at all points of** D**.**

$$(1) \quad u_x = v_y, \quad u_y = -v_x$$

Proof) **By assumption, the derivative** $f'(z)$ **at** z **exist. It is given by**

$$(2) \quad f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$



$$f = f(z) = f(x, y)$$

$$\text{Write} \quad \Delta z = \Delta x + i\Delta y$$

$$\text{then} \quad z + \Delta z = x + \Delta x + i(y + \Delta y)$$

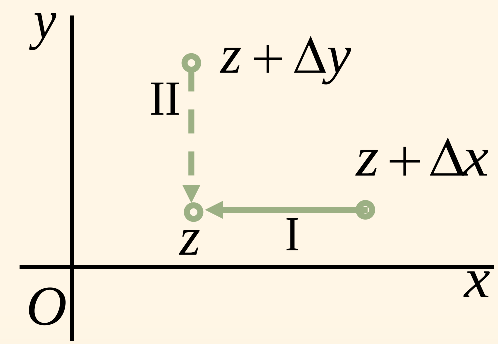
in terms of u **and** v **the derivative in (2) becomes**

$$(3) \quad f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

$$\Delta z = \Delta x + i\Delta y$$

Cauchy-Riemann Equations

(3) $$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$



• Path I

We let $\Delta y \to 0$ first and then $\Delta x \to 0$ After Δy is zero, $\Delta z = \Delta x$

Then (3) becomes

$$f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [u(x,y) + iv(x,y)]}{\Delta x}$$

if we first write the two u-terms and then the two v-terms

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

The two real limits exist. (4) $f'(z) = u_x + iv_x$

• Path II

We let $\Delta x \to 0$ first and then $\Delta y \to 0$ After Δx is zero, $\Delta z = \Delta y$

Then (3) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) + iv(x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{i\Delta y}$$

if we first write the two u-terms and then the two v-terms

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

The two real limits exist. (5) $f'(z) = -iu_y + v_y$

→ $f(z)$ differentiable at z

$$(4) = (5)$$

$$\therefore\ u_x = v_y, \quad u_y = -v_x$$