Chapter 4 Continuity, Energy, and Momentum Equations

4.1 Conservation of Matter in Homogeneous Fluids

• Conservation of matter in homogeneous (single species) fluid → continuity equation

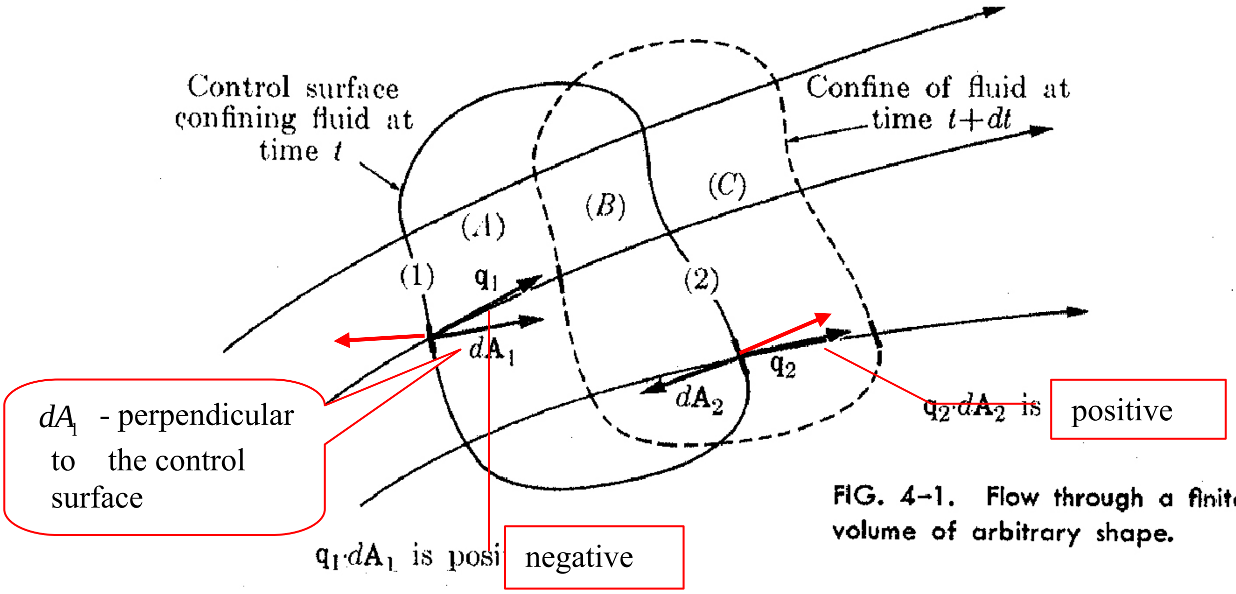


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

4.1.1 Finite control volume method-arbitrary control volume

- Although control volume remains fixed, mass of fluid originally enclosed (regions A+B) occupies the volume within the dashed line (regions B+C).

Since mass m is conserved:

$$(m_A)_t + (m_B)_t = (m_B)_{t+dt} + (m_C)_{t+dt} \tag{4.1}$$

$$\frac{(m_B)_{t+dt} - (m_B)_t}{dt} = \frac{(m_A)_t - (m_C)_{t+dt}}{dt} \tag{4.2}$$

•LHS of Eq. (4.2) = time rate of change of mass in the original control volume in the limit

$$\frac{(m_B)_{t+dt} - (m_B)_t}{dt} \approx \frac{\partial (m_B)}{\partial t} = \frac{\partial}{\partial t}\int_{CV} (\rho\, dV) \tag{4.3}$$

where dV = volume element

•RHS of Eq. (4.2)

= net flux of matter through the control surface

= flux in – flux out

$$= \int \rho q_n \, dA_1 - \int \rho q_n \, dA_2$$

where q_n = component of velocity vector normal to the surface of CV $= |\vec{q}| \cos\phi$

$$\therefore \quad \frac{\partial}{\partial t}\int_{CV} (\rho \, dV) = \int_{CS} \rho q_n dA_1 - \int_{CS} \rho q_n dA_2 \qquad (4.4)$$

※ Flux (= mass/time) is due to velocity of the flow.

• Vector form

$$\frac{\partial}{\partial t}\int_{CV} (\rho \, dV) = -\oint_{CS} \rho \vec{q} \cdot d\vec{A} \qquad (4.5)$$

where $d\vec{A}$ = vector differential area pointing in the outward direction over an enclosed control surface

$$\therefore \quad \vec{q} \cdot d\vec{A} = |\vec{q}| \, |d\vec{A}| \cos\phi$$

$$= \begin{cases} \text{positive for an outflow from cv,} & \phi \le 90^\circ \\ \text{negative for inflow into cv,} & 90^\circ \le \phi \le 180^\circ \end{cases}$$

If fluid continues to occupy the entire control volume at subsequent times

→ time independent

LHS : $\frac{\partial}{\partial t}\int_{CV}(\rho\, dV) \Rightarrow \int_{CV}\frac{\partial \rho}{\partial t}dV$ (4.5a)

Eq. (4.4) becomes

$$\int_{CV}\frac{\partial \rho}{\partial t}dV + \oint_{CS}\rho\vec{q}\cdot d\vec{A} = 0 \tag{4.6}$$

→ General form of continuity equation - integral form

[Re] Differential form

Use Gauss divergence theorem

$$\int_{V}\frac{\partial F}{\partial x_i}dV = \int_{A}F dA_i$$

Transform surface integral of Eq. (4.6) into volume integral

$$\oint_{CS}\rho\vec{q}\cdot d\vec{A} = \int_{CV}\nabla\cdot(\rho\vec{q})\,dV$$

Then, Eq. (4.6) becomes

$$\int_{CV}\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{q})\right]dV = 0 \tag{4.6a}$$

Eq. (4.6a) holds for any volume only if the integrand vanishes at every point.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{q}) = 0 \tag{4.6b}$$

→ Differential form

◈ Simplified form of continuity equation

(1) For a steady flow of a compressible fluid

$$\int_{CV}\frac{\partial \rho}{\partial t}dv = 0$$

Therefore, Eq. (4.6) becomes

$$\oint_{CS} \rho \vec{q} \cdot d\vec{A} = 0 \tag{4.7}$$

(2) For incompressible fluid (for both steady and unsteady conditions)

$$\rho = \text{const.} \rightarrow \frac{\partial \rho}{\partial t} = 0, \frac{d\rho}{dt} = 0$$

Therefore, Eq. (4.6) becomes

$$\oint_{CS} \vec{q} \cdot d\vec{A} = 0 \tag{4.8}$$

[Cf] Non-homogeneous fluid mixture

→ conservation of mass equations for the individual species

→ advection - diffusion equation

= conservation of mass equation $\left(\frac{\partial c}{\partial t} + \frac{\partial q}{\partial x} = 0 \right)$

+ mass flux equation due to advection and diffusion $\left(q = uc - D\frac{\partial c}{\partial x} \right)$

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(uc - D\frac{\partial c}{\partial x} \right) = 0$$

$$\rightarrow \frac{\partial c}{\partial t} + \frac{\partial uc}{\partial x} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x} \right)$$

4.1.2 Stream - tube control volume analysis for steady flow

CONSERVATION OF MATTER IN HOMOGENEOUS FLUIDS

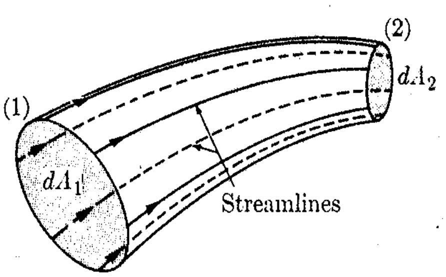


FIG. 4–2. Steady flow stream tube control volume.

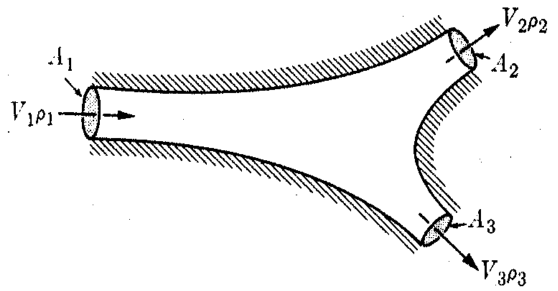


FIG. 4–3. Control volume coincident with boundaries of conduit.

There is no flow across the longitudinal boundary

∴ Eq. (4.7) becomes

$$\oint \rho \vec{q} \cdot d\vec{A} = -\rho_1 q_1 dA_1 + \rho_2 q_2 dA_2 = 0 \qquad (4.9)$$

$$\rho q dA = \text{const.}$$

If density = const.

$$q_1 dA_1 = q_2 dA_2 = dQ \qquad (4.10)$$

where dQ = volume rate of flow

(1) For flow in conduit with variable density

$$V = \frac{\int q dA}{A} \quad \rightarrow \text{ average velocity}$$

$$\rho' = \frac{\int \rho dQ}{Q} \quad \rightarrow \text{ average density}$$

$$\rho_1' V_1 A_1 = \rho_2' V_2 A_2 \tag{4.11}$$

(2) For a branching conduit

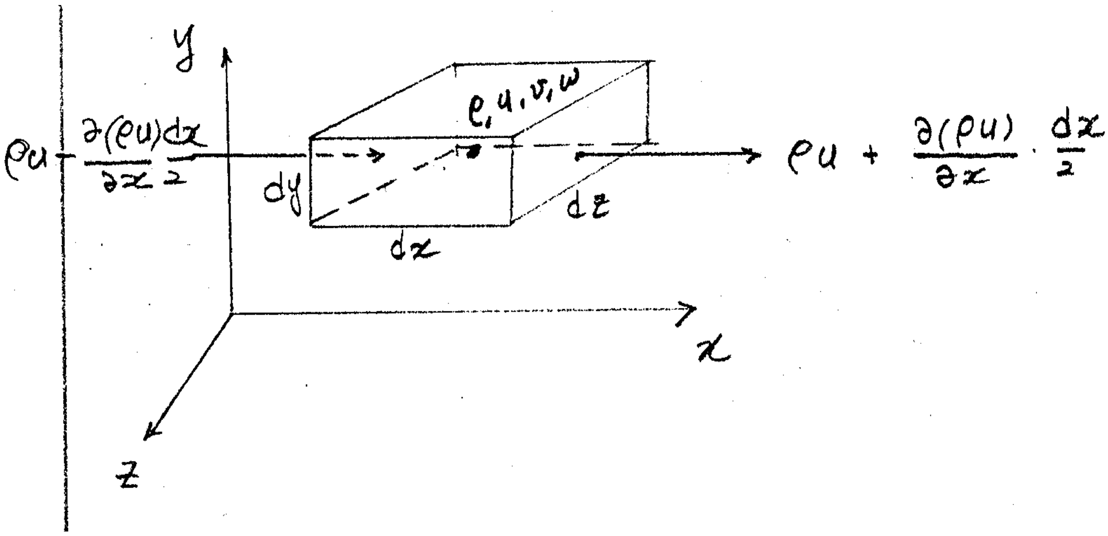
$$\oint \rho \vec{q} \cdot d\vec{A} = 0$$

$$-\int_{A_1} \rho_1 q_1 dA_1 + \int_{A_2} \rho_2 q_2 dA_2 + \int_{A_3} \rho_3 q_3 dA_3 = 0$$

$$\rho_1' V_1 A_1 = \rho_2' V_2 A_2 + \rho_3' V_3 A_3$$

[Appendix 4.1] Equation of Continuity

~ infinitesimal control volume method



At centroid of the control volume: ρ, u, v, w

Rate of mass flux across the surface perpendicular to x is

$$\text{flux in} = \left\{ \rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} \right\} dydz$$

$$\text{flux out} = \left\{ \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} \right\} dydz$$

$$\text{net flux} = \text{flux in} - \text{flux out} = -\frac{\partial(\rho u)}{\partial x} dxdydz$$

net mass flux across the surface perpendicular to y $= -\dfrac{\partial(\rho v)}{\partial y} dydxdz$

net mass flux across the surface perpendicular to z $= -\dfrac{\partial(\rho w)}{\partial z} dzdxdy$

Time rate of change of mass inside the cv

$$= \frac{\partial(\rho dxdydz)}{\partial t}$$

Time rate of change of mass inside = The sum of three net rates

$$\frac{\partial(\rho dxdydz)}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dxdydz$$

By taking limit $dV = dxdydz$

$$-\frac{\partial \rho}{\partial t} = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \nabla \cdot \rho\vec{V} = div\left(\rho\vec{V}\right)$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \nabla \cdot \rho\vec{q} = div\left(\rho\vec{q}\right)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{q} = 0 \qquad \text{(A1)}$$

→ general point (differential) form of Continuity Equation

By the way,

$$\nabla \cdot \rho\vec{q} = \vec{q} \cdot \nabla\rho + \rho\nabla \cdot \vec{q}$$

Thus, (A1) becomes

$$\frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla\rho + \rho\nabla \cdot \vec{q} = 0 \qquad \text{(A2)}$$

1) For incompressible fluid

$$\frac{d\rho}{dt} = 0 \ \ (\rho = \text{const.})$$

$$\rightarrow \quad \frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho = \frac{d\rho}{dt} = 0$$

Therefore Eq. (A2) becomes

$$\rho \nabla \cdot \vec{q} = 0 \quad \rightarrow \quad \boxed{\nabla \cdot \vec{q} = 0} \qquad \text{(A3)}$$

In scalar form,

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \qquad \text{(A4)}$$

→ Continuity Eq. for 3D incompressible fluid

For 2D incompressible fluid,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

2) For steady flow,

$$\frac{\partial \rho}{\partial t} = 0$$

Thus, (A1) becomes

$$\nabla \cdot \rho \vec{q} = \vec{q} \cdot \nabla \rho + \rho \nabla \cdot \vec{q} = 0$$

4.2 The General Energy Equation

4.2.1 The 1st law of thermodynamics

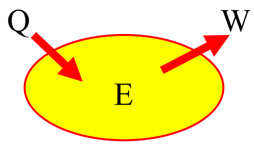
The 1st law of thermodynamics : Continuity + energy → energy equation

– property of a system: location, velocity, pressure, temperature, mass, volume

– state of a system: condition as identified through properties of the system

The difference between the heat added to a system of masses and the work done by the system depends only on the initial and final states of the system(→ change in energy).

→ Conservation of energy



$$\delta Q - \delta W = dE \tag{4.14}$$

where δQ = heat added to the system from surroundings

δW = work done by the system on its surroundings

dE = increase in energy of the system

~ consider time rate of change

$$\frac{\delta Q}{dt} - \frac{\delta W}{dt} = \frac{dE}{dt} \tag{4.15}$$

(1) Work

$W_{pressure}$ = work of normal stresses acting on the system boundary

W_{shear} = work of tangential stresses done at the system boundary on adjacent external fluid in motion

W_{shaft} = shaft work done on a rotating element in the system

(2) Energy

Consider e = energy per unit mass = E/mass

e_u = internal energy associated with fluid temperature = u

e_p = potential energy per unit mass = gh

where h = local elevation of the fluid

e_q = kinetic energy per unit mass = $\frac{q^2}{2}$

$u + \frac{p}{\rho}$ = enthalpy

$$e = e_u + e_p + e_q = u + gh + \frac{q^2}{2} \tag{4.16}$$

- Internal energy

= activity of the molecules comprising the substance

= force existing between the molecules

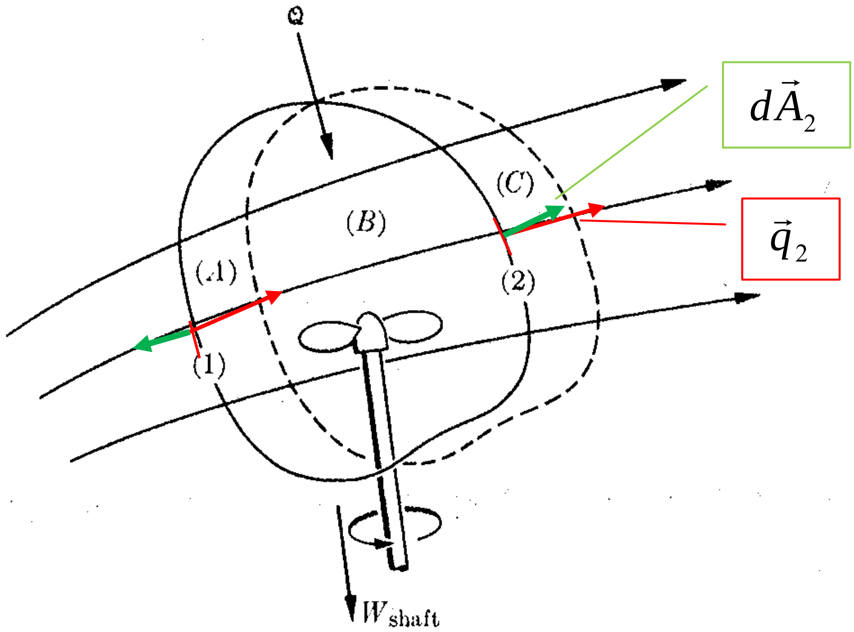
~ depend on temperature and change in phase

4.2.2 General energy equation

$$\frac{\delta Q}{dt} - \frac{\delta W}{dt} = \frac{dE}{dt} \tag{4.15}$$

$$\frac{\delta W}{dt} = \frac{\delta W_{pressure}}{dt} + \frac{\delta W_{shaft}}{dt} + \frac{\delta W_{shear}}{dt} \qquad (4.15a)$$

THE GENERAL ENERGY EQUATION



$\frac{\delta W_{pressure}}{dt}$ = net rate at which work of pressure is done by the fluid on the surroundings

= work flux$_{out}$ – work flux$_{in}$

$$= \oint_{CS} p\left(\vec{q}\cdot d\vec{A}\right)$$

p = pressure acting on the surroundings = $F/A = F/L^2$

$$\vec{q}\cdot d\vec{A} = \begin{cases} \text{Positive for outflow into CV} \\ \text{Negative for inflow} \end{cases}$$

$$\vec{q}\cdot d\vec{A} = Q = L^3/t$$

$$p\left(\vec{q}\cdot d\vec{A}\right) = \frac{F}{L^2}\frac{L^3}{t} = FL/t = E/t$$

Thus, (4.15a) becomes

$$\frac{\delta W}{dt} = \oint_{CS} p\left(\vec{q}\cdot d\vec{A}\right)+\frac{\delta W_{shaft}}{dt}+\frac{\delta W_{shear}}{dt} \tag{4.15b}$$

$\dfrac{dE}{dt}$ = total rate change of stored energy

= net rate of energy flux through C.V.

$(energy\ @\ t = t + \Delta t\ -\ energy\ @\ t = t)$

+ time rate of change inside C.V.

$$= \oint_{CS} e\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV}\left(e\rho\, dV\right) \tag{4.15c}$$

$e = E\,/\,mass;\ \rho\left(\vec{q}\cdot d\vec{A}\right) = mass\,/\,time$

$e\rho\left(\vec{q}\cdot d\vec{A}\right) = E\,/\,t$

Substituting (4.15b) and (4.15c) into Eq. (4.15) yields

$$\frac{\delta Q}{dt} - \frac{\delta W_{shaft}}{dt} - \frac{\delta W_{shear}}{dt} - \left[\oint_{CS} p\left(\vec{q}\cdot d\vec{A}\right)\right]$$

$$= \oint_{CS} e\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV}\left(e\rho\, dvol.\right)$$

$$\frac{\delta Q}{dt} - \frac{\delta W_{shaft}}{dt} - \frac{\delta W_{shear}}{dt}$$

$$= \oint_{CS}\left(\frac{p}{\rho} + e\right)\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV}\left(e\rho\, dV\right) \tag{4.17}$$

Assume potential energy $e_p = gh$ (due to gravitational field of the earth)

Then $e = u + gh + \dfrac{q^2}{2}$

Then, Eq. (4.17) becomes

$$\frac{\delta Q}{dt} - \frac{\delta W_{shaft}}{dt} - \frac{\delta W_{shear}}{dt} = \oint_{CS}\left(\frac{p}{\rho} + u + gh + \frac{q^2}{2}\right)\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} e\rho\, dV \qquad (4.17)$$

◈ Application: generalized apparatus

At boundaries normal to flow lines → no shear

→ $W_{shear} = 0$

EQUATIONS FOR FINITE CONTROL VOLUMES

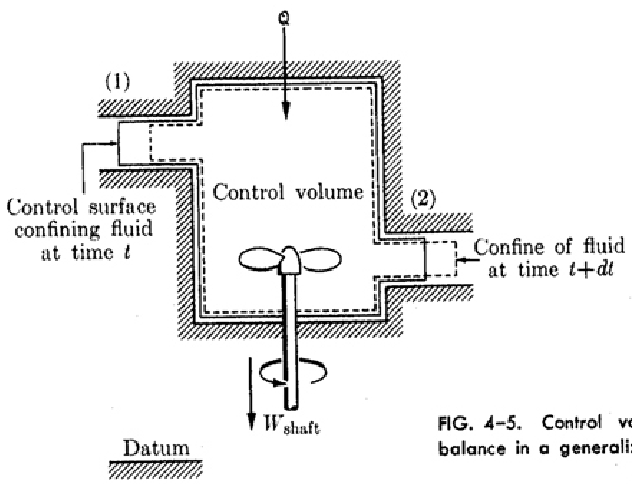


FIG. 4-5. Control volume for energy balance in a generalized apparatus.

$$\frac{\delta Q}{dt} - \frac{\delta W_{shaft}}{dt} = \oint_{CS}\left(\frac{p}{\rho} + u + gh + \frac{q^2}{2}\right)\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} e\rho\, dV \qquad (4.19)$$

For steady motion,

$$\frac{\delta Q}{dt} - \frac{\delta W_{shaft}}{dt} = \oint_{CS} \left(\frac{p}{\rho} + u + gh + \frac{q^2}{2} \right) \rho \left(\vec{q} \cdot d\vec{A} \right) \tag{4.20}$$

◈ Effect of friction

~ This effect is accounted for implicitly.

~ This results in a degradation of mechanical energy into heat which may be transferred away (Q, heat transfer), or may cause a temperature change

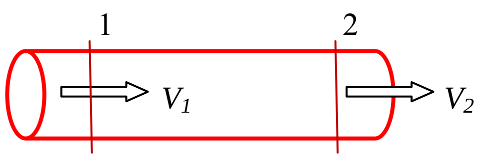
→ modification of internal energy.

→ Thus, Eq. (4.20) can be applied to both viscous fluids and non-viscous fluids (ideal frictionless processes).

4.2.3 1 D Steady flow equations

For flow through conduits, properties are uniform normal to the flow direction.

→ one - dimensional flow



Integrated form of Eq. (4.20) = ② − ①

$$\frac{\delta Q}{dt} - \frac{\delta W_{shaft}}{dt} = \left[u + \frac{p}{\rho} + gh + \frac{V^2}{2} \right]_{②} \rho Q - \left[u + \frac{p}{\rho} + gh + \frac{V^2}{2} \right]_{①} \rho Q$$

where $\frac{V^2}{2}$ = average kinetic energy per unit mass

$$Section\,1: \int_1 \rho \left(\vec{q} \cdot d\vec{A} \right) = -\rho Q = \text{mass flow rate into CV}$$

$$Section\ 2: \int_2 \rho \left(\vec{q} \cdot d\vec{A}\right) = \rho Q = \text{mass flow from CV}$$

Divide by ρQ (mass/time)

$$\frac{heat\ transfer}{mass} - \frac{W_{shaft}}{mass} = \left[u + \frac{p}{\rho} + gh + \frac{V^2}{2}\right]_{②} - \left[u + \frac{p}{\rho} + gh + \frac{V^2}{2}\right]_{①}$$

Divide by g

$$\frac{heat\ transfer}{weight} - \frac{W_{shaft}}{weight} = \left[\frac{u}{g} + \frac{p}{\gamma} + h + \frac{V^2}{2g}\right]_{②} - \left[\frac{u}{g} + \frac{p}{\gamma} + h + \frac{V^2}{2g}\right]_{①}$$

(4.21)

◈ Energy Equation for 1-D steady flow: Eq. (4.21)

~ use average values for p, γ, h, u and V at each flow section

~ use K_e (energy correction coeff.) to account for non-uniform velocity distribution over flow cross section

$$K_e \frac{\rho}{2} V^2 Q = \int \frac{\rho}{2} q^2 dQ \quad \text{---- kinetic energy/time} = \frac{1}{2}\frac{mV^2}{t}$$

$$K_e = \frac{\int \frac{\rho}{2} q^2 dQ}{\frac{\rho}{2} V^2 Q} \geq 1$$

$$\frac{heat\ transfer}{weight} - \frac{W_{shaft}}{weight} = \left[\frac{p}{\gamma} + h + K_e \frac{V^2}{2g}\right]_{②} - \left[\frac{p}{\gamma} + h + K_e \frac{V^2}{2g}\right]_{①} + \frac{u_2 - u_1}{g}$$

(4.23)

$K_e = \begin{cases} 2, \text{ for laminar flow (parabolic velocity distribution)} \\ 1.06, \text{ for turbulent flow (smooth pipe)} \end{cases}$

For a fluid of uniform density γ

$$\frac{p_1}{\gamma} + h_1 + K_{e_1}\frac{V_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + K_{e_2}\frac{V_2^2}{2g} + \frac{W_{shaft}}{weight} - \frac{heat\ transfer}{weight} + \frac{u_2 - u_1}{g}$$

(4.24)

→ unit: m (energy per unit weight)

For viscous fluid;

$$-\frac{heat\ transfer}{weight} + \frac{u_2 - u_1}{g} = H_{L_{1-2}}$$

→ loss of mechanical energy

~ irreversible in liquid

Then, Eq. (4.24) becomes

$$\frac{p_1}{\gamma} + h_1 + K_{e_1}\frac{V_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + K_{e_2}\frac{V_2^2}{2g} + \Delta H_M + \Delta H_{L_{1-2}} \quad (4.24a)$$

where ΔH_M = shaft work transmitted from the system to the outside

$$H_1 = H_2 + \Delta H_M + \Delta H_{L_{1-2}} \quad (4.24b)$$

where H_1, H_2 = weight flow rate average values of total head

◈ Bernoulli Equation

Assume

①ideal fluid → friction losses are negligible

② no shaft work → $\Delta H_M = 0$

③ no heat transfer and internal energy is constant → $\Delta H_{L_{1-2}} = 0$

$$\frac{p_1}{\gamma} + h_1 + K_{e_1}\frac{V_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + K_{e_2}\frac{V_2^2}{2g} \qquad (4.25)$$

$$H_1 = H_2$$

Pressure head

Potential head

Velocity head

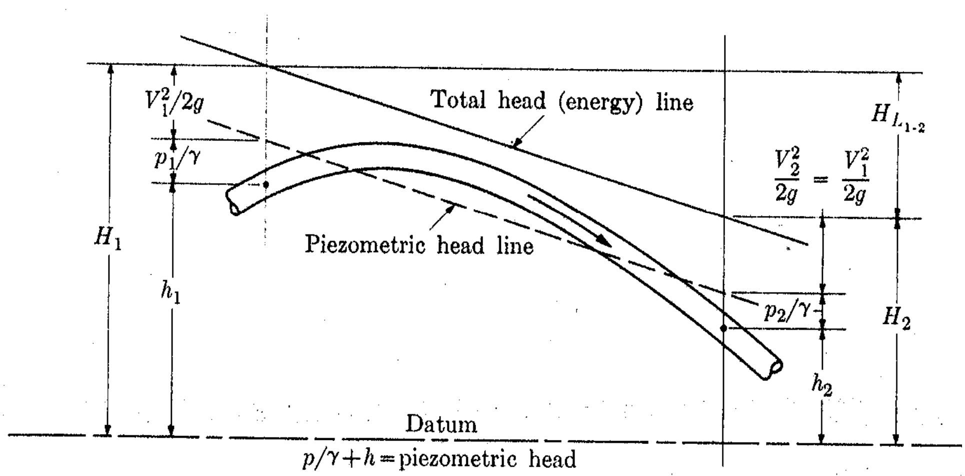
If $K_{e_1} = K_{e_2} = 1$, then Eq. (4.25) reduces to

$$H = \frac{p_1}{\gamma} + h_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + \frac{V_2^2}{2g} \qquad (4.26)$$

~ total head along a conduct is constant

◈ Grade lines

THE GENERAL ENERGY EQUATION



1) Energy (total head) line (E.L) ~ H above datum

2) Hydraulic (piezometric head) grade line (H.G.L.)

~ $\left(\dfrac{p}{\gamma}+h\right)$ above datum

For flow through a pipe with a constant diameter

$$V_1 = V_2 \quad \rightarrow \quad \frac{V_1^2}{2g} = \frac{V_2^2}{2g}$$

1) If the fluid is real (viscous fluid) and if no energy is being added, then the energy line may never be horizontal or slope upward in the direction of flow.

2) Vertical drop in energy line represents the head loss or energy dissipation.

4.3 Linear Momentum Equation for Finite Control Volumes

4.3.1 Momentum Principle

Newton's 2nd law of motion

$$\vec{F} = m\vec{a} = m\frac{d\vec{q}}{dt} = \frac{d\left(m\vec{q}\right)}{dt} = \frac{d\vec{M}}{dt} \tag{4.27}$$

$\vec{M}$ = linear momentum vector = $m\vec{q}$

$\vec{F}$ = external force

= boundary (surface) forces: normal to boundary - pressure, $\vec{F}_p$

tangential to boundary - shear, $\vec{F}_s$

body forces - force due to gravitational or magnetic fields, $\vec{F}_b$

$$\vec{F}_p + \vec{F}_s + \vec{F}_b = \frac{d\vec{M}}{dt} \tag{4.28}$$

$\vec{F}_b = \int_{CV} f_b\left(\rho\, dv\right)$, where f_b = body force per unit mass

4.3.2 The general linear momentum equation

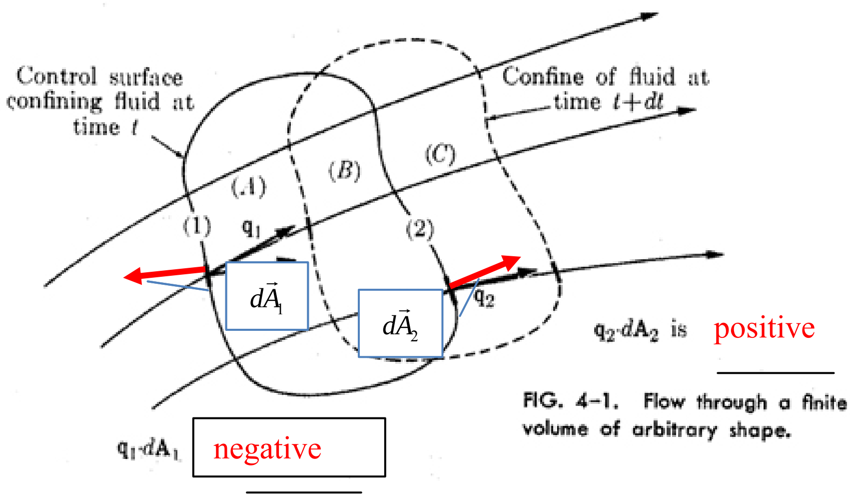


FIG. 4-1. Flow through a finite control volume of arbitrary shape.

$\dfrac{d\vec{M}}{dt}$ = total rate of change of momentum

= net momentum flux across the CV boundaries

+ time rate of increase of momentum within CV

$$= \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dv \tag{4.29}$$

where $\vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right)$ = flux of momentum = velocity × mass per time

$d\vec{A}$ = vector unit area pointing <u>outward</u> over the control surface

Substitute (4.29) into (4.28)

$$\vec{F}_p + \vec{F}_s + \vec{F}_b = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dv \tag{4.30}$$

For steady flow and negligible body forces

$$\vec{F}_p + \vec{F}_s = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) \tag{4.30a}$$

- Eq. (4.30)

1) It is applicable to both ideal fluid systems and viscous fluid systems involving <u>friction and energy dissipation</u>.

2) It is applicable to both compressible fluid and incompressible fluid.

3) Combined effects of friction, energy loss, and heat transfer appear implicitly in the magnitude of the external forces.

- Eq. (4.30a)

1) Knowledge of the <u>internal conditions is not necessary.</u>

2) We can consider only external conditions.

4.3.3 Inertial control volume for a generalized apparatus

• Three components of the forces

$$x-dir.:\ \vec{F}_{p_x}+\vec{F}_{s_x}+\vec{F}_{b_x}=\oint_{CS}u\rho\left(\vec{q}\cdot d\vec{A}\right)+\frac{\partial}{\partial t}\int_{CV}u\rho\,dV$$

$$y-dir.:\ \vec{F}_{p_y}+\vec{F}_{s_y}+\vec{F}_{b_y}=\oint_{CS}v\rho\left(\vec{q}\cdot d\vec{A}\right)+\frac{\partial}{\partial t}\int_{CV}v\rho\,dV$$

$$z-dir.:\ \vec{F}_{p_z}+\vec{F}_{s_z}+\vec{F}_{b_z}=\oint_{CS}w\rho\left(\vec{q}\cdot d\vec{A}\right)+\frac{\partial}{\partial t}\int_{CV}w\rho\,dV \qquad (4.32)$$

• For flow through generalized apparatus

$$x-dir.:\ \vec{F}_{p_x}+\vec{F}_{s_x}+\vec{F}_{b_x}=\int_2 u\rho\,dQ-\int_1 u\rho\,dQ+\frac{\partial}{\partial t}\int_{CV}u\rho\,dV$$

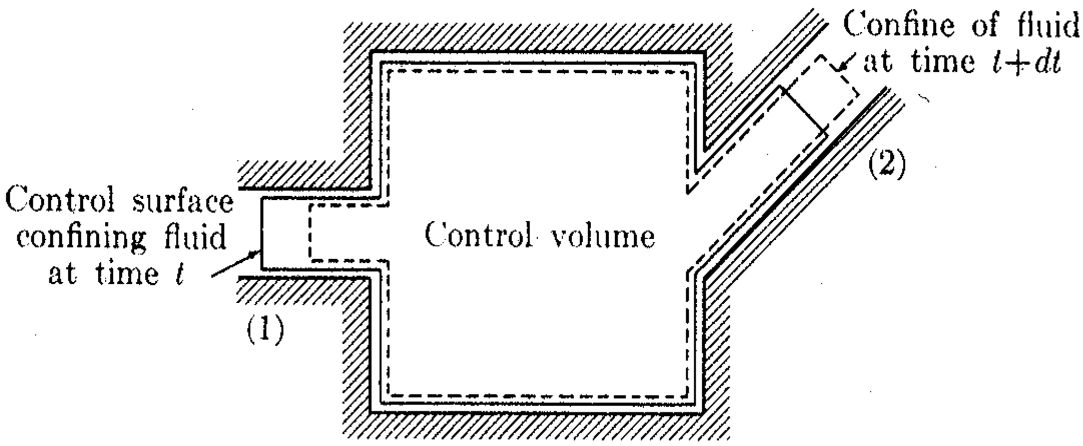


FIG. 4-9. Control volume for generalized apparatus.

For 1 - D steady flow, $\frac{\partial}{\partial t}\int_{CV}q\rho\,dV=0$

~ Velocity and density are constant normal to the flow direction.

$$x-dir.:\ \vec{F}_{p_x}+\vec{F}_{s_x}+\vec{F}_{b_x}=\sum F_x=\left(V_x\rho Q\right)_2-\left(V_x\rho Q\right)_1$$

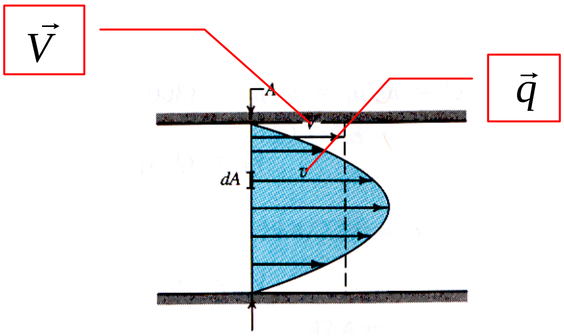
$$= V_{x_2}\rho_2 Q_2 - V_{x_1}\rho_1 Q_1 = Q\rho\left(V_{x_2} - V_{x_1}\right) = Q\rho\left(V_{x_{out}} - V_{x_{in}}\right)$$

$$y-dir.: \sum F_y = \left(V_y \rho Q\right)_2 - \left(V_y \rho Q\right)_1$$

$$z-dir.: \sum F_z = \left(V_z \rho Q\right)_2 - \left(V_z \rho Q\right)_1 \qquad (4.33)$$

where V = average velocity in flow direction

If velocity varies over the cross section, then introduce momentum flux coefficient



$$\int \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) = K_m \vec{V}\left(\rho V A\right)$$

$$\int \vec{q}\rho\, dQ = K_m \vec{V}\rho Q$$

$$K_m = \frac{\int \vec{q}\rho dQ}{\vec{V}\rho Q}$$

where

V = magnitude of average velocity over cross section $= Q/A$

$\vec{V}$ = average velocity vector

K_m = momentum flux coefficient ≥ 1

= 1.33 for laminar flow (pipe flow)
 1.03-1.04 for turbulent flow (smooth pipe)

[Cf] Energy correction coeff.

$$K_e = \frac{\int \frac{\rho}{2} q^2 dQ}{\frac{\rho}{2} \vec{V} Q}$$

$$\sum F_x = (K_m V_x \rho Q)_2 - (K_m V_x \rho Q)_1$$

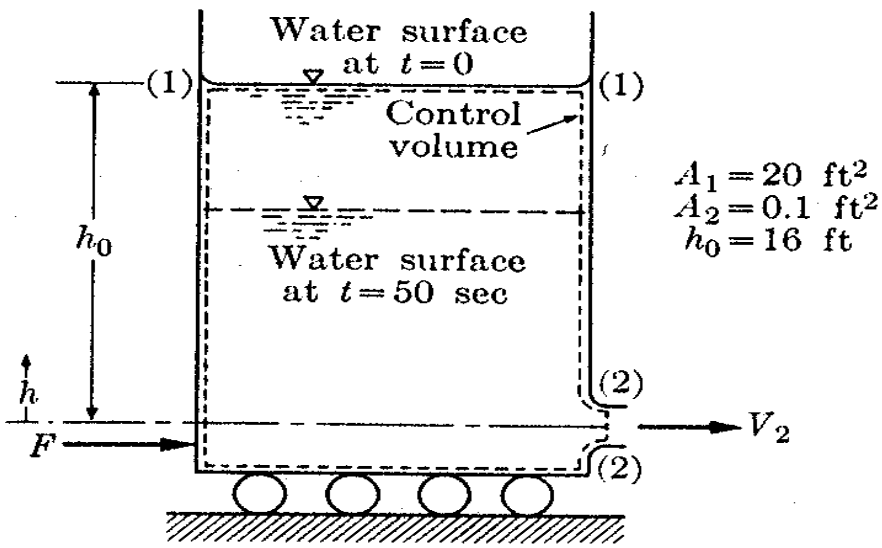
$$\sum F_y = (K_m V_y \rho Q)_2 - (K_m V_y \rho Q)_1$$

$$\sum F_z = (K_m V_z \rho Q)_2 - (K_m V_z \rho Q)_1 \qquad (4.34)$$

[Example 4-4] Continuity, energy, and linear momentum with unsteady flow

$A_1 = 20 \text{ ft}^2, \quad A_2 = 0.1 \text{ ft}^2, \quad h_0 = 16'$

At time $t=0$ valve on the discharge nozzle is opened. Determine depth h, discharge rate Q, and force F necessary to keep the tank stationary after $t = 50$ sec.



i) Continuity equation, Eq. (4.4)

$$\frac{\partial}{\partial t}\int_{CV}\rho\, dV = \int \rho q_n dA_1 - \int \rho q_n dA_2$$

$dV = A_1 dh$, $\rho q_n dA_1 = 0$ (because no inflow across the section (1))

$$\therefore \quad \rho A_1 \frac{\partial}{\partial t}\int_0^h dh = -\rho V_2 A_2$$

$$A_1 \frac{dh}{dt} = -V_2 A_2 \qquad \text{(A)}$$

ii) Energy equation, Eq. (4.19)

~ no shaft work

~ heat transfer and temperature changes due to friction are negligible

$$\cancel{\frac{\delta Q}{dt}} - \cancel{\frac{\delta W_{shaft}}{dt}} - \cancel{\frac{\delta W_{shear}}{dt}}$$

$$= \underbrace{\oint_{CS}\left(\frac{p}{\rho} + u + gh + \frac{q^2}{2}\right)\rho\left(\vec{q}\cdot d\vec{A}\right)}_{\text{I}} + \underbrace{\frac{\partial}{\partial t}\int_{CV} e\rho\, dV}_{\text{II}}$$

e = energy per unit mass = $u + gh + \dfrac{q^2}{2}$

$$\text{I} = \oint_{CS}\left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)\rho\left(\vec{q}\cdot d\vec{A}\right)$$

$$= \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)_2 \rho V_2 A_2 - \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)_1 \rho V_1 A_1$$

$$= \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)_2 \rho V_2 A_2 \qquad \left(V_1 \approx 0\right)$$

$$\text{II} = \frac{\partial}{\partial t}\int_{CV} e\rho\, dV = \frac{\partial}{\partial t}\int_{CV}\left(u + gh + \cancel{\frac{q^2}{2}}\right)\rho dV$$

$A_1\, dh$

∵ nearly constant in the tank except near the nozzle

$$= A_1 \rho \frac{\partial}{\partial t}\int_0^h (u + gh)\, dh$$

$$\therefore\ 0 = \left(u + \frac{p}{\rho} + gh + \frac{q^2}{2}\right)_2 \rho V_2 A_2 + A_1 \rho \frac{\partial}{\partial t}\int_0^h (u + gh)\, dh$$

Assume $\rho = \text{const.}$, $p_2 = p_{atm} = 0$, $h_2 = 0$ (datum)

$$0 = uV_2 A_2 + \frac{V_2^2}{2} V_2 A_2 + uA_1 \frac{dh}{dt} + A_1 gh \frac{dh}{dt} \qquad \text{(B)}$$

Substitute (A) into (B)

$$0 = \cancel{uV_2A_2} + \frac{V_2^2}{2} V_2 A_2 + \cancel{u(-V_2A_2)} + gh(-V_2 A_2)$$

$$\therefore\quad \frac{V_2^2}{2} V_2 A_2 = ghV_2 A_2$$

$$V_2 = \sqrt{2gh} \qquad \text{(C)}$$

Substitute (C) into (A)

$$A_2\sqrt{2gh} = -A_1 \frac{dh}{dt}$$

$$\frac{dh}{\sqrt{h}} = -\frac{A_2}{A_1}\sqrt{2g}\, dt$$

Integrate

$$\int_{h_0}^{h} \frac{dh}{\sqrt{h}} = \int_0^t -\frac{A_2}{A_1}\sqrt{2g}\,dt \qquad \left\{ \int_{h_0}^{h} h^{-\frac{1}{2}} dh = \left[2h^{\frac{1}{2}} \right]_{h_0}^{h} \right\}$$

$$h = \left(h_0^{\frac{1}{2}} - \frac{A_2}{A_1}\frac{\sqrt{2g}}{2} t \right)^2$$

$$h = \left(\sqrt{16} - \frac{0.1}{20}\frac{\sqrt{2(32.2)}}{2} t \right)^2$$

$$= (4 - 0.0201t)^2$$

At $t = 50\sec$, $h = (4 - 0.0201 \times 50)^2 = 8.98\,ft$

$$V_2 = \sqrt{2gh} = \sqrt{2(32.2)(8.98)} = 24.05 \text{ fps}$$

$$Q_2 = (VA)_2 = 24.05(0.1) = 2.405 \text{ cfs}$$

iii) Momentum equation, Eq. (4.30)

I II

$$\vec{F}_p + \cancel{\vec{F}_s} + \cancel{\vec{F}_b} = \underbrace{\oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right)}_{\text{I}} + \underbrace{\frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV}_{\text{II}}$$

II = Time rate of change of momentum inside CV is negligible

if tank area (A_1) is large compared to the nozzle area (A_2).

$$\text{I} = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) = \int q_n \rho q_n dA_2 - \cancel{\int q_n \rho q_n dA_1} = V_2 \rho V_2 A_2$$

$$\therefore F_{px} = V_2 \rho V_2 A_2 = V_2 \rho Q_2$$

$$F_{px} = (1.94)(24.05)(2.405) = 112 \text{ lb}$$

4.4 The Moment of Momentum Equation for Finite Control Volumes

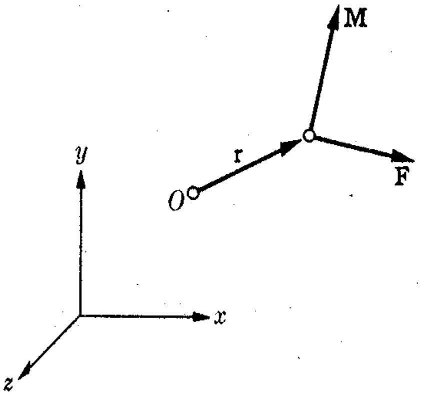


FIG. 4–12. Position, force and momentum vectors.

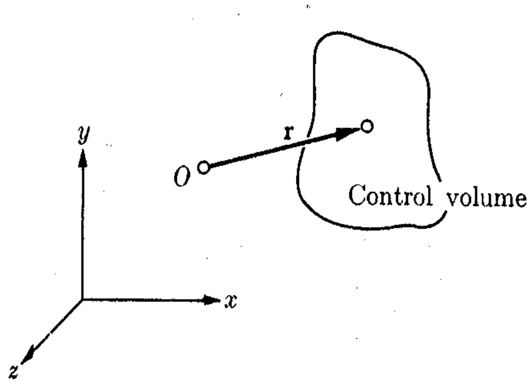


FIG. 4–13. Control volume for moment of momentum analysis.

4.4.1 The Moment of momentum principle for inertial reference systems

Apply Newton's 2nd law to rotating fluid masses

→ The vector sum $\left(\vec{r}\times\vec{F}\right)$ of all the external moments acting on a fluid mass equals the time rate of change of the moment of momentum (angular momentum) vector $\left(\vec{r}\times\vec{M}\right)$ of the fluid mass.

$$T = \vec{r}\times\vec{F} = \frac{d}{dt}\left(\vec{r}\times\vec{M}\right)$$

where $\vec{r}$ = position vector of a mass in an arbitrary curvilinear motion

$\vec{M}$ = linear momentum

[Derivation]

Eq. (4.27) : $\vec{F} = \dfrac{d\vec{M}}{dt}$

Take the vector cross product of $\vec{r}$

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{M}}{dt}$$

By the way,

$$\frac{d}{dt}\left(\vec{r} \times \vec{M}\right) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{M}}_{\text{I}} + \vec{r} \times \frac{d\vec{M}}{dt}$$

$$I = \frac{d\vec{r}}{dt} \times \vec{M} = \vec{q} \times mass \cdot \vec{q} = 0 \quad \left(\because \frac{d\vec{r}}{dt} = \vec{q}\right)$$

$$\left(\therefore \ \vec{q} \times \vec{q} = |\vec{q}|\,|\vec{q}| \sin 0^{\circ} = 0\right)$$

$$\therefore \quad \left(\vec{r} \times \frac{d\vec{M}}{dt}\right) = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right)$$

$$\therefore \therefore \quad \vec{r} \times \vec{F} = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right)$$

where $\vec{r} \times \vec{M}$ = angular momentum (moment of momentum)

[Cf] Torque

$$\vec{T} = \vec{r} \times \vec{F}$$

translational motion → Force – linear acceleration

rotational motion → Torque – angular acceleration

[Re] Vector Product

$$\vec{V} = \vec{a} \times \vec{b}$$

magnitude = $|\vec{V}| = |\vec{a}| \times |\vec{b}| \sin \gamma$ = area of parallelogram

direction = perpendicular to both $\vec{a}$ and $\vec{b}$ (plane of $\vec{a}$ and $\vec{b}$)

→ use right-handed triple

$$\vec{b} \times \vec{a} = -\left(\vec{a} \times \vec{b}\right)$$

$$(k\vec{a}) \times \vec{b} = k\left(\vec{a} \times \vec{b}\right)$$

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{b}\right)$$

• External moments arise from external forces

	Forces	Moment
Boundary (surface) force	$\vec{F}_p,\ \vec{F}_s$	$\vec{T}_p,\ \vec{T}_s$
Body force	$\vec{F}_b$	$\vec{T}_b$

$$\underbrace{\left(\vec{r} \times \vec{F}_p\right)}_{\vec{T}_p} + \underbrace{\left(\vec{r} \times \vec{F}_s\right)}_{\vec{T}_s} + \underbrace{\left(\vec{r} \times \vec{F}_b\right)}_{\vec{T}_b} = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right)$$

$$\therefore \quad \vec{T}_p + \vec{T}_s + \vec{T}_b = \frac{d}{dt}\left(\vec{r} \times \vec{M}\right)$$

where $\vec{T}_p,\ \vec{T}_s,\ \vec{T}_b$ = external torque

4.4.2 The general moment of momentum equation

$$\frac{d\vec{M}}{dt} = \oint_{CS} \vec{q}\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \vec{q}\rho\, dV \qquad (4.30)$$

$$\therefore \quad \frac{d}{dt}\left(\vec{r} \times \vec{M}\right) = \oint_{CS} \left(\vec{r} \times \vec{q}\right)\rho\left(\vec{q}\cdot d\vec{A}\right) + \frac{\partial}{\partial t}\int_{CV} \left(\vec{r} \times \vec{q}\right)\rho\, dV$$

$$\vec{T}_p + \vec{T}_s + \vec{T}_b = \oint_{CS} (\vec{r} \times \vec{q}) \rho (\vec{q} \cdot d\vec{A}) + \frac{\partial}{\partial t} \int_{CV} (\vec{r} \times \vec{q}) \rho \, dV$$

$$x - dir.: \quad \left| (\vec{r} \times \vec{q})_{yz} \right| = r_{yz} \, q_{yz} \sin\left(\frac{\pi}{2} - \alpha_{yz} \right) = (rq \cos\alpha)$$

angle between q_{yz} and r_{yz}

$$\therefore \; \vec{T}_{px} + \vec{T}_{sx} + \vec{T}_{bx} = \oint_{CS} (rq \cos\alpha)_{yz} \, \rho (\vec{q} \cdot d\vec{A}) + \frac{\partial}{\partial t} \int_{CV} (rq \cos\alpha)_{yz} \, \rho \, dV$$

$$y - dir.: \quad \vec{T}_{py} + \vec{T}_{sy} + \vec{T}_{by} = \oint_{CS} (rq \cos\alpha)_{zx} \, \rho (\vec{q} \cdot d\vec{A}) + \frac{\partial}{\partial t} \int_{CV} (rq \cos\alpha)_{zx} \, \rho \, dV$$

$$z - dir.: \quad \vec{T}_{pz} + \vec{T}_{sz} + \vec{T}_{bz} = \oint_{CS} (rq \cos\alpha)_{xy} \, \rho (\vec{q} \cdot d\vec{A}) + \frac{\partial}{\partial t} \int_{CV} (rq \cos\alpha)_{xy} \, \rho \, dV$$

4.4.3 Steady-flow equations for Turbomachinery

■ Turbomachine

~ pumps, turbines, fans, blows, compressors

~ dynamic reaction occurs between a rotating vaned element (runner) and fluid passing through the element

If rotating speed is constant → Fig. 4.14

$$T_r = \int_{A_2} r_2 V_2 \cos\alpha_2 \rho \, dQ - \int_{A_1} r_1 V_1 \cos\alpha_1 \rho \, dQ \quad \text{– Euler's Eq.} \qquad (4.39)$$

Where r_1, r_2 = radii of fluid elements at the entrance and exist of the runner

V_1, V_2 = Velocities relative to the fixed reference frame (absolute)

α_1, α_2 = angles of absolute velocities with tangential direction

A_1, A_2 = entrance and exit area of CV which encloses the runner

T_r = torque exerted on fluid by the runner

• T_r { positive for pump, compressor
negative for turbine

If $r_2 V_2 \cos\alpha_2$ and $r_1 V_1 \cos\alpha_1$ are constant, ρ is constant

$$T_r = \rho Q\left(r_2 V_2 \cos\alpha_2 - r_1 V_1 \cos\alpha_1\right) \tag{4.41}$$

If $r_2 V_2 \cos\alpha_2 = r_1 V_1 \cos\alpha_1$

$$\rho Q r V \cos\alpha = \rho Q r v_\theta = \text{const.}$$

~ zero torque, constant angular momentum

If $\alpha = 0$ → free vortex flow

Homework Assignment # 2

Due: 1 week from today

4-11. Derive the equation for the volume rate of flow per unit width for the sluice gate shown in Fig. 4-20 in terms of the geometric variable b, y_1, and c_c. Assume the pressure in hydrostatic at y_1 and $c_c b$ and the velocity is constant over the depth at each of these sections.

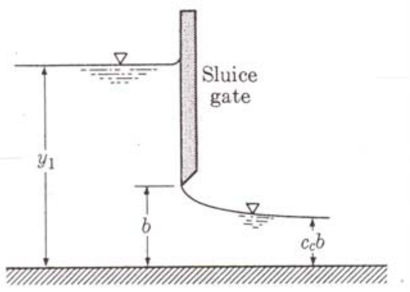


FIGURE 4–20

4-12. Derive the expression for the total force per unit width exerted by the above sluice gate on the fluid in terms of vertical distances shown in Fig. 4-20.

4-14. Consider the flow of an incompressible fluid through the Venturi meter shown in Fig. 4-22. Assuming uniform flow at sections (1) and (2) neglecting all losses, find the pressure difference between these sections as a function of the flow rate Q, the diameters of the sections, and the density of the fluid, ρ. Note that for a given configuration, Q is a function of only the pressure drop and fluid density. The meter is named for Venturi, who investigated its principle in about 1791. However, in 1886 Clemens Herschel first used the meter to measure discharge, and he is usually credited with its invention.

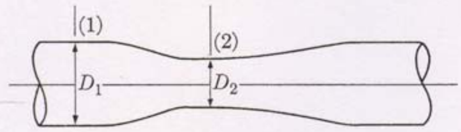


FIGURE 4–22

4-15. Water flows into a tank from a supply line and out of the tank through a horizontal pipe as shown in Fig. 4-23. The rates of inflow and outflow are the same, and the water surface in the tank remains a distance h above the discharge pipe centerline. All velocities in the tank are negligible compared to those in the pipe. The head loss between the tank and the pipe exit is H_L. (a) Find the discharge Q in terms of h, A, and H_L. (b) What is the horizontal force, F_x, required to keep the tank from moving? (c) If the supply line has an area A', what is the vertical force exerted on the water in the tank by the vertical jet?

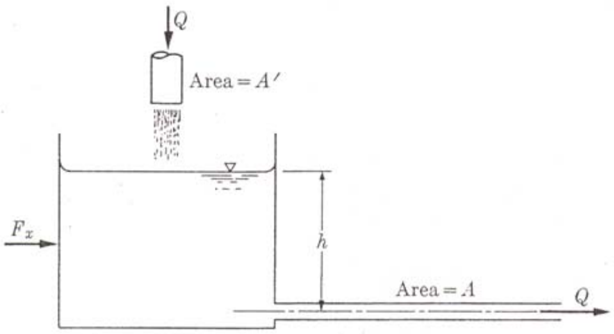


FIGURE 4–23

4-28. Derive the one-dimensional continuity equation for the unsteady, nonuniform flow of an incompressible liquid in a horizontal open channel as shown in Fig. 4-29. The channel has a rectangular cross section of a constant width, b. Both the depth, y_0 and the mean velocity, V are functions of x and t.

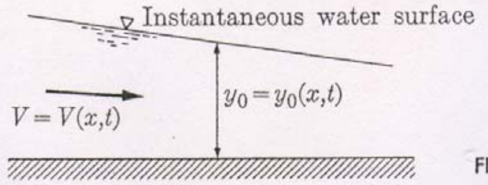


FIGURE 4–29