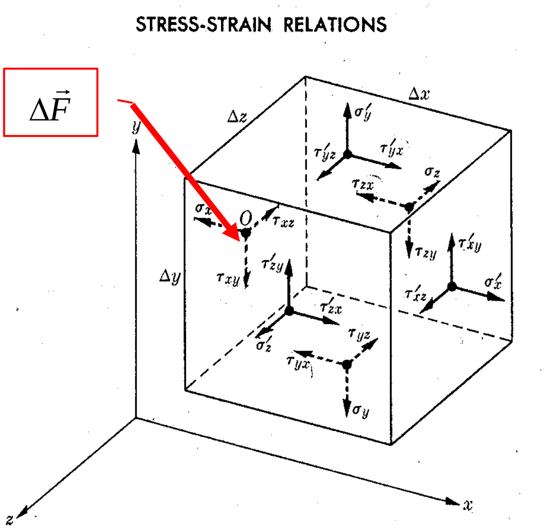
Chapter 5 Stress – Strain Relation

5.1 General Stress – Strain system

Parallelepiped, cube



5.1.1 Surface Stress

Surface stresses:
- normal stress – σ_x
- shear stress – τ_{xy} , τ_{xz}

$$\sigma_{xx} = \sigma_x = \lim_{\Delta A_x \to 0} \frac{\Delta F_x}{\Delta A_x} \qquad (\Delta A_x = \Delta y \Delta z)$$

$$\tau_{xy} = \lim_{\Delta A_x \to 0} \frac{\Delta F_y}{\Delta A_x}$$

$$\tau_{xz} = \lim_{\Delta A_x \to 0} \frac{\Delta F_z}{\Delta A_x}$$

$$\tau_{yx} = \lim_{\Delta A_y \to 0} \frac{\Delta F_x}{\Delta A_y} \qquad (\Delta A_y = \Delta x \Delta z)$$

$$\sigma_{yy} = \sigma_y = \lim_{\Delta A_y \to 0} \frac{\Delta F_y}{\Delta A_y}$$

$$\tau_{yz} = \lim_{\Delta A_y \to 0} \frac{\Delta F_z}{\Delta A_y}$$

$$\tau_{zx} = \lim_{\Delta A_z \to 0} \frac{\Delta F_x}{\Delta A_z} \qquad (\Delta A_z = \Delta x \Delta y)$$

$$\tau_{zy} = \lim_{\Delta A_z \to 0} \frac{\Delta F_y}{\Delta A_z}$$

$$\sigma_{zz} = \sigma_z = \lim_{\Delta A_z \to 0} \frac{\Delta F_z}{\Delta A_z}$$

where ΔF_x , ΔF_y, ΔF_z = component of force vector $\Delta\vec{F}$

ΔF_x – acting in the direction of the x-axis

ΔA_x = area of the x- face of the element = $\Delta y \Delta z$

$\Delta A_y = \Delta x \Delta z$

$\Delta A_z = \Delta x \Delta y$

•subscripts

σ_x : subscript indicates the direction of stress

τ_{xy} : 1st - direction of the normal to the face on which τ acts

2nd - direction in which τ acts

•general stress system: stress tensor

~ 9 scalar components

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

[Re] Tensor:

~ an ordered array of entities which is invariant under coordinate transformation; includes scalars & vectors

~ 3^n

0th order – 1 component, scalar (mass, length, pressure)

1st order – 3 components, vector (velocity, force, acceleration)

2nd order – 9 components (stress, rate of strain, turbulent diffusion)

At three other surfaces,

$$\sigma_x' = \sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x$$

$$\sigma_y' = \sigma_y + \frac{\partial \sigma_y}{\partial y} \Delta y$$

$$\sigma_z' = \sigma_z + \frac{\partial \sigma_z}{\partial z} \Delta z$$

$$\tau_{xy}' = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \Delta x$$

$$\tau_{yx}' = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y$$

$$\tau_{zx}' = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z \qquad (5.1)$$

◈ Shear stress is symmetric.

→ Shear stress pairs with subscripts differing in order are equal.

→ $\tau_{xy} = \tau_{yx}$

[Proof]

In static equilibrium, sum of all moments and sum of all forces equal zero for the element.

First, apply Newton's 2nd law

$$\sum F = m\frac{du}{dt}$$

Then, consider Torque (angular momentum), T

$$\sum T = \frac{d}{dt}(rmu) = \frac{d}{dt}(r^2 m\omega) = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt}$$

where I = moment of inertia = $r^2 m$

r = radius of gyration

$\frac{d\omega}{dt}$ = angular acceleration

Thus,

$$\sum T = mr^2\frac{d\omega}{dt} \qquad \text{(A)}$$

Now, take a moment about a centroid axis in the z-direction

$$LHS = \sum T = \left(\Delta y \Delta z \tau_{xy}\right)\frac{\Delta x}{2} - \left(\tau_{yx}\Delta x \Delta z\right)\frac{\Delta y}{2} = \frac{\Delta x \Delta y \Delta z}{2}\left(\tau_{xy} - \tau_{yx}\right)$$

$$RHS = \rho dvol r^2\frac{d\omega}{dt} = \Delta x \Delta y \Delta z \rho r^2\frac{d\omega}{dt}$$

$$\therefore \left(\tau_{xy} - \tau_{yx}\right)\Delta x \Delta y \Delta z = 2\Delta x \Delta y \Delta z \rho r^2\frac{d\omega}{dt}$$

After canceling terms, this gives

$$\tau_{xy} - \tau_{yx} = 2\rho r^2\frac{d\omega}{dt}$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} r^2 \to 0$$

$$\tau_{xy} - \tau_{yx} = 0$$

$$\therefore \quad \tau_{xy} = \tau_{yx}$$

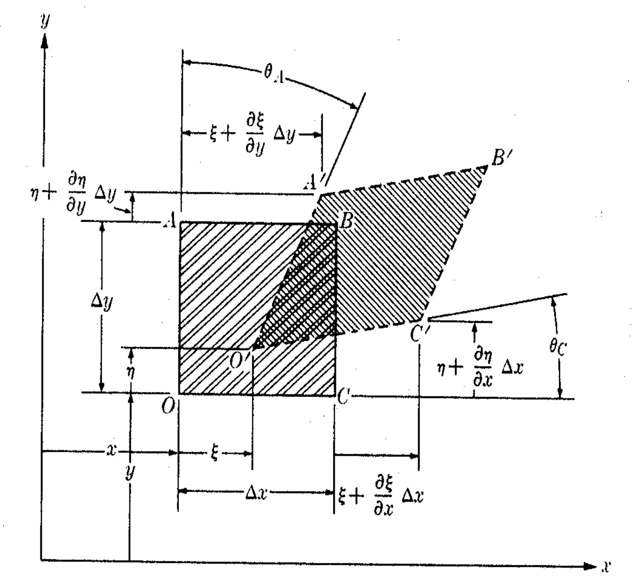
[Homework Assignment-Special work]

Due: 1 week from today

1. Make your own “Stress Cube” using paper box.

5.1.2 Strain components

○ Strain { normal strain: ε ← linear deformation
shear strain: γ ← angular deformation



[Re] displacement vs. deformation

i) Displacement (translation): ξ, η, ζ

$$O(x, y, z) \rightarrow O'(x+\xi,\ y+\eta,\ z+\zeta)$$

ii) Deformation: due to system of external forces

$$OABC \rightarrow O'A'B'C'$$

[Cf] Motion

- translation
- rotation

[Cf] Deformation

- Linear deformation
- Angular deformation

(1) Deformation

1) Normal strain, ε

$$\varepsilon = \frac{\text{change in length}}{\text{original length}}$$

$O'C'$ OC

$$\varepsilon_x = \lim_{\Delta x \to 0} \frac{O'C' - OC}{OC} = \lim_{\Delta x \to 0} \frac{\left\{\left(x + \Delta x + \xi + \frac{\partial \xi}{\partial x}\Delta x\right) - (x + \xi)\right\} - \Delta x}{\Delta x} = \frac{\partial \xi}{\partial x}$$

$$\varepsilon_y = \lim_{\Delta y \to 0} \frac{\mathrm{O'A'} - OA}{OA} = \lim_{\Delta y \to 0} \frac{\left\{\left(y + \Delta y + \eta + \frac{\partial \eta}{\partial y}\Delta y\right) - (y + \eta)\right\} - \Delta y}{\Delta y} = \frac{\partial \eta}{\partial y}$$

$$\varepsilon_x = \frac{\partial \zeta}{\partial z}$$

~ ε is positive when element elongates under deformation

2) Shear strain, γ

~ <u>change in angle</u> between two originally perpendicular elements

For xy–plane

$$\gamma_{xy} = \lim_{\Delta x, \Delta y \to 0} (\theta_c + \theta_A) \cong \lim_{\Delta x, \Delta y \to 0} (\tan\theta_c + \tan\theta_A)$$

C'D A'E

$$= \lim_{\Delta x, \Delta y \to 0} \left\{ \frac{\frac{\partial \eta}{\partial x}\Delta x}{\Delta x + \cancel{\frac{\partial \xi}{\partial x}\Delta x}} + \frac{\frac{\partial \xi}{\partial y}\Delta y}{\Delta y + \cancel{\frac{\partial \eta}{\partial y}\Delta y}} \right\} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

O'D O'E

$$\left(\therefore \Delta x \cdot \frac{\partial \xi}{\partial x} < \Delta x\right)$$

$$\gamma_{xy} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

$$\gamma_{yz} = \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial x} \qquad (5.4)$$

$$\gamma_{zx} = \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}$$

(2) displacement vector $\vec{\delta}$

$$\vec{\delta} = \xi\vec{i} + \eta\vec{j} + \zeta\vec{k}$$

(3) Volume dilation

$$e = \frac{\text{change of volume of deformed element}}{\text{original volume}}$$

$$e = \frac{d(\Delta V)}{\Delta V} = \frac{\left(\Delta x + \frac{\partial \xi}{\partial x}\Delta x\right)\left(\Delta y + \frac{\partial \eta}{\partial y}\Delta y\right)\left(\Delta z + \frac{\partial \zeta}{\partial z}\Delta z\right) - \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}$$

$$\cong \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z \qquad (5.6)$$

$$e = \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = \nabla \cdot \vec{\delta} \qquad \text{--- divergence} \qquad (5.7)$$

5.2 Relations between Stress and Strain for Elastic Solids

5.2.1 Normal Stresses

Hooke's law: stress is linear with strain

$$\sigma_x = E \cdot \varepsilon_x^{\circ}$$

$$\varepsilon_x^{\circ} = \frac{1}{E}\sigma_x$$

in which E = Young's modulus of elasticity

ε_x° = elongation in the $x-dir$. due to normal stress, σ_x

$$y-dir. \ : \ \varepsilon_y^\circ = \frac{\sigma_y}{E}$$

$$z-dir. \ : \ \varepsilon_z^\circ = \frac{\sigma_z}{E}$$

Now, we have to consider other elongations because of lateral contraction of matter under tension.

$\varepsilon_x{}'$ = elongation in the $x-dir$. due to σ_y

$\varepsilon_x{}''$ = elongation in th $x-dir$. due to σ_z

Now, define

$$\varepsilon_x{}' = -n\varepsilon_y^\circ = -n\frac{\sigma_y}{E} \qquad (5.9)$$

$$\varepsilon_x{}'' = -n\varepsilon_z^\circ = -n\frac{\sigma_z}{E} \qquad (5.10)$$

where n = **Poisson's ratio**

Thus, total strain ε_x is

$$\varepsilon_x = \varepsilon_x^\circ + \varepsilon_x{}' + \varepsilon_x{}'' = \frac{\sigma_z}{E} - \frac{n}{E}\left(\sigma_y + \sigma_z\right) = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y + \sigma_z\right)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - n\left(\sigma_z + \sigma_x\right)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - n\left(\sigma_x + \sigma_y\right)\right] \qquad (5.12)$$

5.2.2 Shear Stress

~ Hooke's law $\tau_{xy} = G\gamma_{xy}$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} = \frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial z}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} = \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}$$

where G = shear modulus of elasticity

$$G = \frac{E}{2(1+n)} \tag{5.14}$$

■ Volume dialation

$$\begin{aligned}\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z &= \frac{1}{E}\left[\sigma_x - n\left(\sigma_y + \sigma_z\right)\right] \\ &+ \frac{1}{E}\left[\sigma_y - n\left(\sigma_z + \sigma_x\right)\right] \\ &+ \frac{1}{E}\left[\sigma_z - n\left(\sigma_x + \sigma_y\right)\right] \\ &= \frac{1}{E}\left[(1-2n)\left(\sigma_x + \sigma_y + \sigma_z\right)\right] \end{aligned} \tag{5.15}$$

■ $\bar{\sigma}$ = arithmetic mean of 3 normal stresses

$$\bar{\sigma} = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_y\right) \tag{5.16}$$

From Eq. (5.12), (5.14) and (5.15)

$$\sigma_x = 2G\left[\varepsilon_x + \frac{ne}{1-2n}\right] \tag{5.17}$$

Therefore

$$\sigma_x - \overline{\sigma} = 2G\left(\varepsilon_x - \frac{e}{3}\right)$$

$$\sigma_y - \overline{\sigma} = 2G\left(\varepsilon_y - \frac{e}{3}\right)$$

$$\sigma_z - \overline{\sigma} = 2G\left(\varepsilon_z - \frac{e}{3}\right) \tag{5.18}$$

$$\tau_{xy} = \tau_{yx} = G\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right)$$

$$\tau_{zy} = \tau_{yz} = G\left(\frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z}\right)$$

$$\tau_{xz} = \tau_{zx} = G\left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}\right) \tag{5.19}$$

[Proof] Eq. (5.17) & (5.18)

$$(5.15) \rightarrow e = \frac{1}{E}(1-2n)\left(\sigma_x + \sigma_y + \sigma_z\right) \tag{A}$$

$$(5.12) \rightarrow \varepsilon_x = \frac{1}{E}\left[\sigma_x - n\left(\sigma_y + \sigma_z\right)\right] \tag{B}$$

$$(5.14) \rightarrow G = \frac{E}{2(1+n)} \rightarrow E = 2G(1+n) \tag{C}$$

i) Combine (A) and (B)

$$+\left|\begin{array}{lll}\dfrac{n}{(1+2n)}\times e & = \dfrac{n}{(1-2n)}\dfrac{(1-2n)}{E}\left(\sigma_x+\sigma_y+\sigma_z\right)=\dfrac{n}{E}\left(\sigma_x+\sigma_y+\sigma_z\right) \\ \varepsilon_x & =\dfrac{1}{E}\left[\sigma_x-n\left(\sigma_y+\sigma_z\right)\right]\end{array}\right.$$

$$\frac{n}{(1-2n)}e+\varepsilon_x=\frac{1+n}{E}\sigma_x$$

$$\therefore \sigma_x=\frac{E}{1+n}\left[\varepsilon_x+\frac{n}{(1-2n)}e\right] \qquad \text{(D)}$$

Substitute (C) into (D)

$$\therefore \sigma_x=2G\left[\varepsilon_x+\frac{n}{(1-2n)}e\right] \qquad \rightarrow \qquad \text{Eq. (5.17)}$$

ii) Subtract (5.16) from (5.17)

$$\sigma_x-\bar{\sigma}=2G\left[\varepsilon_x+\frac{n}{(1-2n)}e\right]-\frac{1}{3}\left(\sigma_x+\sigma_y+\sigma_z\right) \qquad \text{(E)}$$

Substitute (A) into (E); $\sigma_x+\sigma_y+\sigma_z=\dfrac{E}{(1-2n)}e$

$$\therefore\ RHS\ of\ (E)=2G\left[\varepsilon_x+\frac{n}{(1-2n)}e\right]-\frac{1}{3}\frac{E}{(1-2n)}e$$

$$=2G\varepsilon_x+\left[\frac{2Gn}{(1-2n)}-\frac{1}{3}\frac{2G(1+n)}{(1-2n)}\right]e\ =\ 2G\left\{\varepsilon_x\left[\frac{n}{(1-2n)}-\frac{\frac{1+n}{3}}{(1+2n)}\right]e\right\}$$

$$= 2G\left\{ \varepsilon_x + \frac{-\frac{1}{3}(1-2n)}{(1-2n)} e \right\}$$

$$= 2G\left(\varepsilon_x - \frac{1}{3} e \right) \qquad \rightarrow \text{Eq. (5.18)}$$

5.3 Relations between Stress and Rate of Strain for Newtonian Fluids

Experimental evidence suggests that, in fluid, stress is linear with time rate of strain.

$$\rightarrow \quad stress \propto \frac{\partial}{\partial t}(strain)$$

→ Newtonian fluid (**Newton's law of viscosity**)

[Cf] For solid,

$$stress \propto strain$$

5.3.1 Normal stress

For solid, Eq. (5.18) can be used as

Hookeian elastic solid: $\sigma_x - \overline{\sigma} = 2\left(\frac{F}{L^2}\right)\left(\varepsilon_x - \frac{e}{3}\right)$

By analogy,

Newtonian fluid: $\sigma_x - \overline{\sigma} = 2\left(\frac{Ft}{L^2}\right)\frac{\partial}{\partial t}\left(\varepsilon_x - \frac{e}{3}\right)$ (5.20)

Now set $\mu \equiv \frac{Ft}{L^2}$ = dynamic viscosity

$$\sigma_x - \overline{\sigma} = 2\mu \frac{\partial \varepsilon_x}{\partial t} - \frac{2}{3}\mu \frac{\partial e}{\partial t} \tag{5.21}$$

By the way,

$$\varepsilon_x = \frac{\partial \xi}{\partial x}\ ;\ e = \nabla \cdot \vec{\delta}$$

$$u = \frac{\partial \xi}{\partial t},\quad v = \frac{\partial \eta}{\partial x},\quad w = \frac{\partial \xi}{\partial t}\quad (\xi, \eta, \zeta = \text{displacement})$$

Therefore,

$$\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial t}\right) = \frac{\partial u}{\partial x} \tag{5.22}$$

$$\frac{\partial e}{\partial t} = \nabla \cdot \frac{\partial \vec{\delta}}{\partial t} = \nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{5.23}$$

$$\vec{\delta} = \xi \vec{i} + \eta \vec{j} + \zeta \vec{k}$$

$$\vec{q} = \frac{\partial \vec{\delta}}{\partial t} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$\nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Eq. (5.21) becomes

$$\sigma_x = \overline{\sigma} + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

For compressible fluid,

$$\sigma_x = \overline{\sigma} + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

$$\sigma_y = \overline{\sigma} + 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right)$$

$$\sigma_z = \bar{\sigma} + 2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\nabla \cdot \vec{q}\right) \tag{5.24}$$

For incompressible fluid,

$$\frac{de}{dt} = \nabla \cdot \vec{q} = 0 \quad \leftarrow \text{ time rate of volume expansion=0}$$

$$\rightarrow \nabla \cdot \vec{q} = 0 \quad \rightarrow \text{ Continuity Eq.}$$

Therefore, Eq. (5.24) becomes

$$\sigma_x = \bar{\sigma} + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_y = \bar{\sigma} + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_z = \bar{\sigma} + 2\mu \frac{\partial w}{\partial z}$$

5.3.2. Shear stress

By following the same analogy

$$\mu$$

$$\tau_{xy} = G\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right) = \left(\frac{Ft}{L^2}\right)\frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\right)$$

$$= \mu \frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial t}\right) = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial \eta}{\partial t} = v \qquad \frac{\partial \xi}{\partial t} = u$$

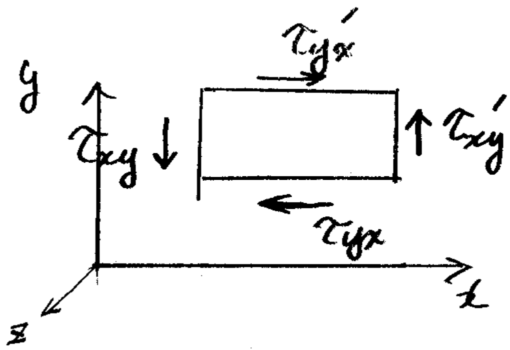
$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

$$\tau_{zy}=\tau_{yz}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) \qquad (5.25)$$

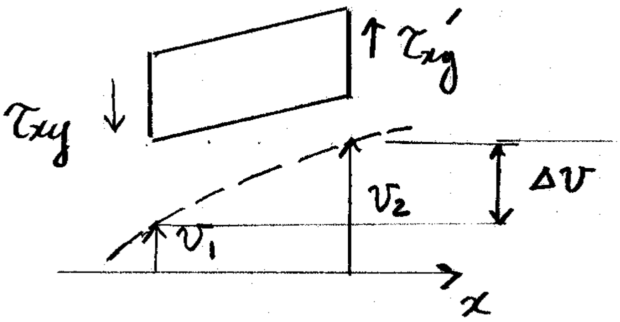
$$\tau_{xz}=\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

[**Appendix 1]**

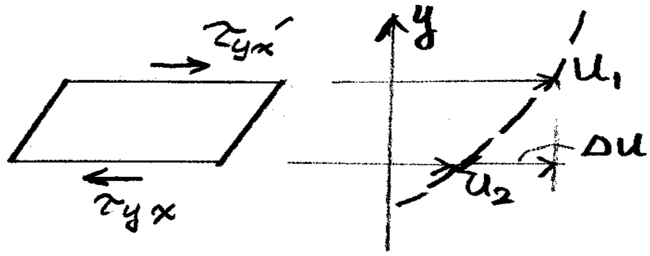
$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$



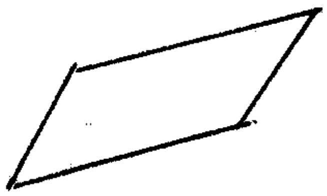
i) τ_{xy} , $\tau_{yx}{}'$



ii) τ_{yx} , τ_{yx}'



iii) composition



5.3.3 Relation between thermodynamic pressure p and mean normal stress $\bar{\sigma}$

1) Assume viscous effects are completely represented by the viscosity μ for

incompressible fluid

$$\bar{\sigma} = -p = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) \tag{5.26}$$

~ minus sign accounts for pressure (compression)

2) For compressible fluid

$$\bar{\sigma} = -p + \mu'\left(\nabla \cdot \vec{q}\right)$$

in which μ' = 2nd coefficient of viscosity associated solely with dilation

= bulk viscosity

Since, dilation effect is small for most cases

$$\mu'\left(\nabla\cdot\vec{q}\right)\rightarrow 0$$

$$\therefore \bar{\sigma}=-p$$

For zero-dilation viscosity effects (μ' = 0), (5.24) becomes

$$\sigma_x=-p+2\mu\frac{\partial u}{\partial x}-\left(\frac{2}{3}\right)\mu\left(\nabla\cdot\vec{q}\right)$$

$$\sigma_y=-p+2\mu\frac{\partial v}{\partial y}-\left(\frac{2}{3}\right)\mu\left(\nabla\cdot\vec{q}\right)$$

$$\sigma_z=-p+2\mu\frac{\partial w}{\partial z}-\left(\frac{2}{3}\right)\mu\left(\nabla\cdot\vec{q}\right) \qquad (5.29)$$

Normal stress

pressure

Viscous effects

■ Shear stresses in a real fluid

$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

$$\tau_{zy}=\tau_{yz}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$$

$$\tau_{xz}=\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \qquad (5.30)$$

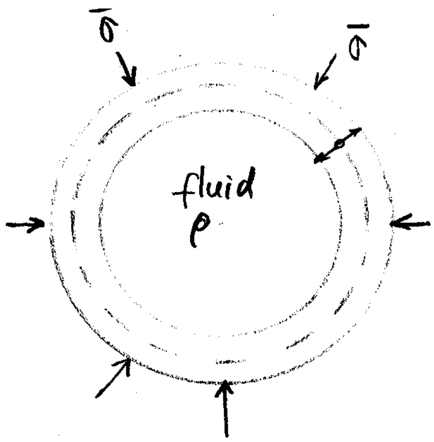
For zero viscous effects $\left(\mu=0\right)\rightarrow$ inviscid fluids in motion and for all fluids at rest

$$\sigma_x=\sigma_y=\sigma_z=\bar{\sigma}=-p$$

$$\tau_{xy}=\tau_{yz}=\tau_{zx}=0$$

[Appendix 2] Bulk viscosity and thermodynamic pressure

→ Boundary-Layer Theory (Schlichting, 1979) pp. 61-63



$$\bar{\sigma} = -p + \mu'(\nabla \cdot \vec{q})$$

If fluid is compressed, expanded or made to oscillate at a finite rate, work done in a thermodynamically reversible process per unit volume is

$$W = p\nabla \cdot \vec{q} = P\frac{de}{dt}$$ ~ dissipation of energy

where μ' = bulk viscosity of fluid that represents that property which is responsible for energy dissipation in a fluid of uniform temperature during a change in volume at a finite rate = second property of a compressible, isotropic, Newtonian fluid

[Cf] μ = shear viscosity = first property

$$\mu' = 0\,, \quad p = -\bar{\sigma}$$

$$\mu' \neq 0\,, \quad p \neq -\bar{\sigma}$$

Direct measurement of bulk viscosity is very difficult.

[Appendix 3] **N**ormal stress

Normal stress = pressure + deviation from it

$$\sigma_x = -p + \sigma_x'$$

$$\sigma_y = -p + \sigma_y'$$

$$\sigma_z = -p + \sigma_z'$$

Thus, stress matrix becomes

$$\begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} \sigma_x' & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_y' & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z' \end{pmatrix}$$

Normal stresses are proportional to the volume change (compressibility) and corresponding components of linear deformation, *a*, *b*, *c*.

Thus,

$$\sigma_x = -p + \lambda(a+b+c) + 2\mu a$$

$$\sigma_y = -p + \lambda(a+b+c) + 2\mu b$$

$$\sigma_z = -p + \lambda(a+b+c) + 2\mu c$$

where λ = compressibility coefficient

Homework Assignment # 3

Due: 1 week from today

5-1. Verify Eq. (5-14)

$$G = \frac{E}{2(1+n)}.$$

5-3. Consider a fluid element under a general state of stress as illustrated in Fig. 5-1. Given that the element is in a gravity field, show that the equilibrium requirement between surface, body and inertial forces leads to the equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x = \rho a_x$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g_y = \rho a_y$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho g_x = \rho a_{\mathrm{x}}$$

5-4. Consider a fluid in two-dimensional motion. Using plane polar coordinates r, θ, and z, show that the rate of strain components are

$$\frac{\partial \varepsilon_r}{\partial t} = \frac{\partial v_r}{\partial r}, \frac{\partial \varepsilon_\theta}{\partial t} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}, \frac{\partial \gamma_{r\theta}}{\partial t} = \frac{\partial v_\theta}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}.$$