Chapter 9 Wall Turbulence. Boundary-Layer Flows

9.1 Introduction

• Turbulence occurs most commonly in shear flows.

• Shear flow: spatial variation of the mean velocity

1) Wall turbulence: along solid surface → no-slip condition at surface

2) Free turbulence: at the interface between fluid zones having different velocities, and at boundaries of a jet → jet, wakes

• Turbulent motion in shear flows

- self-sustaining
- Turbulence arises as a consequence of the shear.
- Shear persists as a consequence of the turbulent fluctuations.

→ Turbulence can neither arise nor persist without shear.

9.2 Structure of a Turbulent Boundary Layer

9.2.1 Boundary layer flows

(i) Smooth boundary

Consider a fluid stream flowing past a smooth boundary.

→ A boundary-layer zone of viscous influence is developed near the boundary.

1) $\mathrm{Re} < \mathrm{Re}_{crit}$

→ The boundary-layer is initially laminar.

→ $u = u(y)$

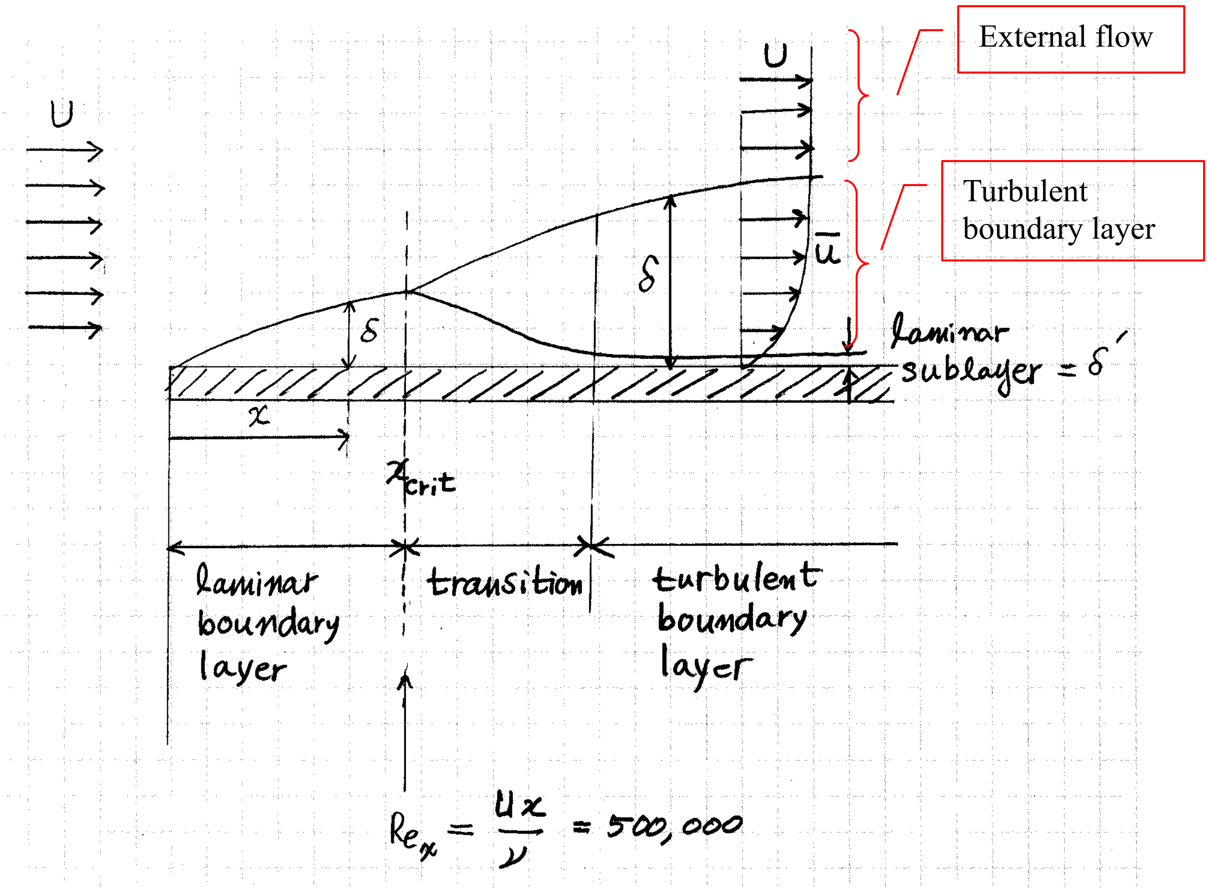
2) $\mathrm{Re} > \mathrm{Re}_{crit}$

→ The boundary-layer is turbulent.

→ $\bar{u} = \bar{u}(y)$

→ Turbulence reaches out into the free stream to entrain and mix more fluid.

→ thicker boundary layer: $\delta_{turb} \approx 4\ \delta_{lam}$

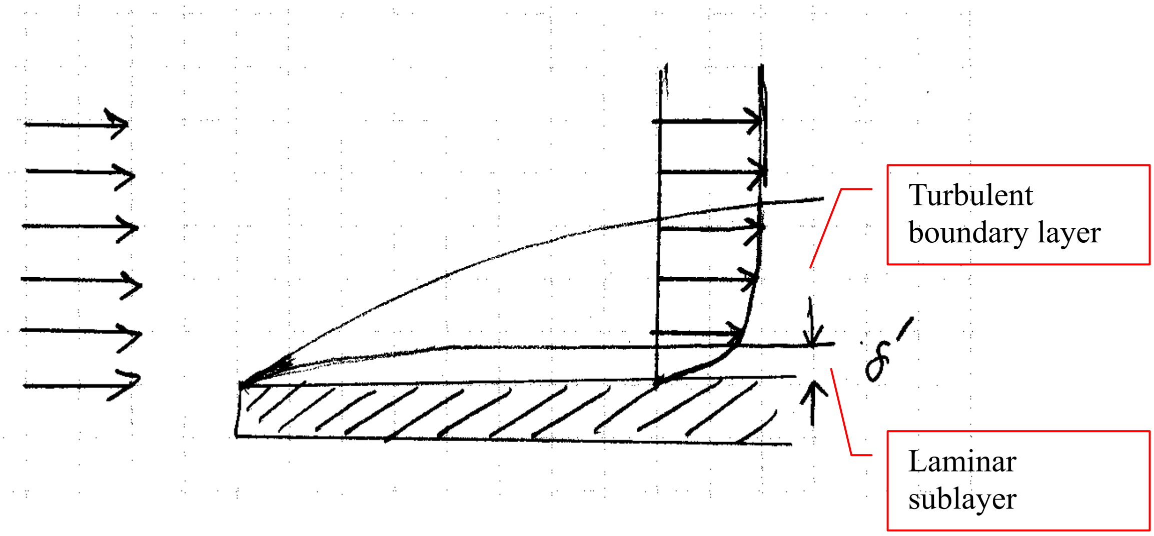


$x < x_{crit}$, total friction = laminar

$x > x_{crit}$, total friction = laminar + turbulent

(ii) Rough boundary

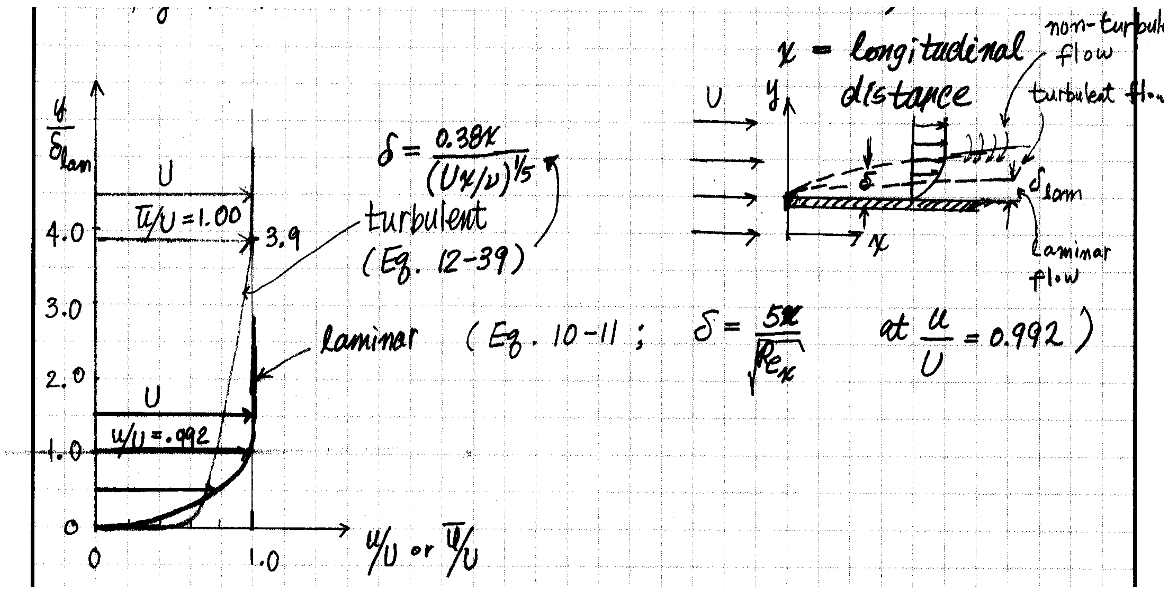
→ Turbulent boundary layer is established near the leading edge of the boundary without a preceding stretch of laminar flow.



9.2.2 Comparison of laminar and turbulent boundary-layer profiles

For the flows of the same Reynolds number $(\mathrm{Re}_x = 500{,}000)$

$$\mathrm{Re}_x = \frac{Ux\rho}{\mu}$$



1) Boundary layer thickness

$$\frac{\delta_{turb}}{\delta_{lam}} = 3.9$$

2) Mass displacement thickness, δ^*

Eq. (8.9): $$\delta^* = \int_0^h (1 - \frac{u}{U}) dy$$

$$\frac{\delta^*_{turb}}{\delta^*_{lam}} = 1.41$$

3) Momentum thickness, θ

Eq. (8.10): $$\theta = \int_0^h \frac{u}{U}(1 - \frac{u}{U}) dy$$

$$\frac{\theta_{turb}}{\theta_{lam}} = 2.84$$

→ Because of the higher flux of mass and momentum through the zone nearest the wall for turbulent flow, increases of δ^* and θ rate are not as large as δ.

9.2.3 Intermittent nature of the turbulent layer

• Outside a boundary layer

→ free-stream shearless flow (U) → potential flow (inviscid)

→ slightly turbulent flow

→ considered to be non-turbulent flow relative to higher turbulence inside a turbulent boundary layer

• Interior of the turbulent boundary layer (δ)

~ consist of regions of different types of flow (laminar, buffer, turbulent)

~ Instantaneous border between turbulent and non-turbulent fluid is irregular and changing.

~ Border consists of fingers of turbulence extending into the non-turbulent fluid and fingers of non-turbulent fluid extending deep into the turbulent region.

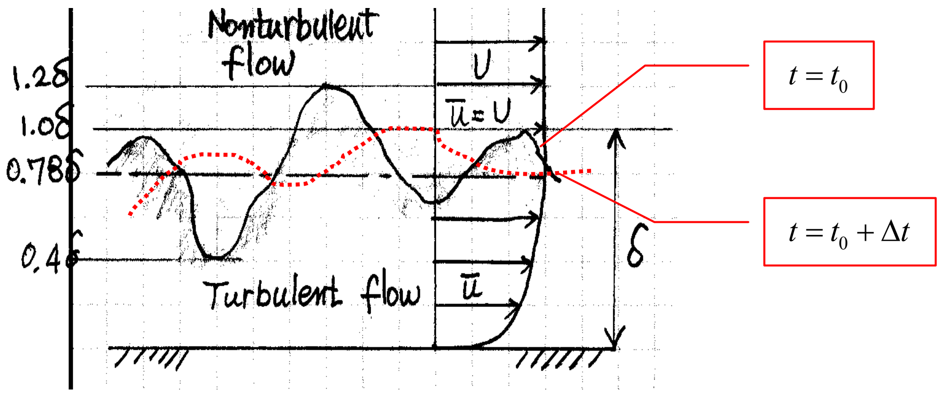
~ intermittent nature of the turbulent layer

• Intermittency factor, Ω

Ω = fraction of time during which the flow is turbulent

Ω = 1.0, deep in the boundary layer

= 0, in the free stream



①Average position of the turbulent-nonturbulent interface = 0.78 δ

②Maximum stretch of interface = 1.2 δ

③Minimum stretch of interface = 0.4 δ

162. "Typical eddy" in a turbulent boundary layer. Oil fog is illuminated by a sheet of laser light to show the lower two-thirds of a turbulent boundary layer in side view. The vortex-ring structure just below and to the right of center, which resembles a sliced mushroom leaning left, is an example of what Falcó has called a "typical eddy." It scales on wall variables (figure 161) rather than on the boundary-layer thickness. *Photograph by R. E. Falco*

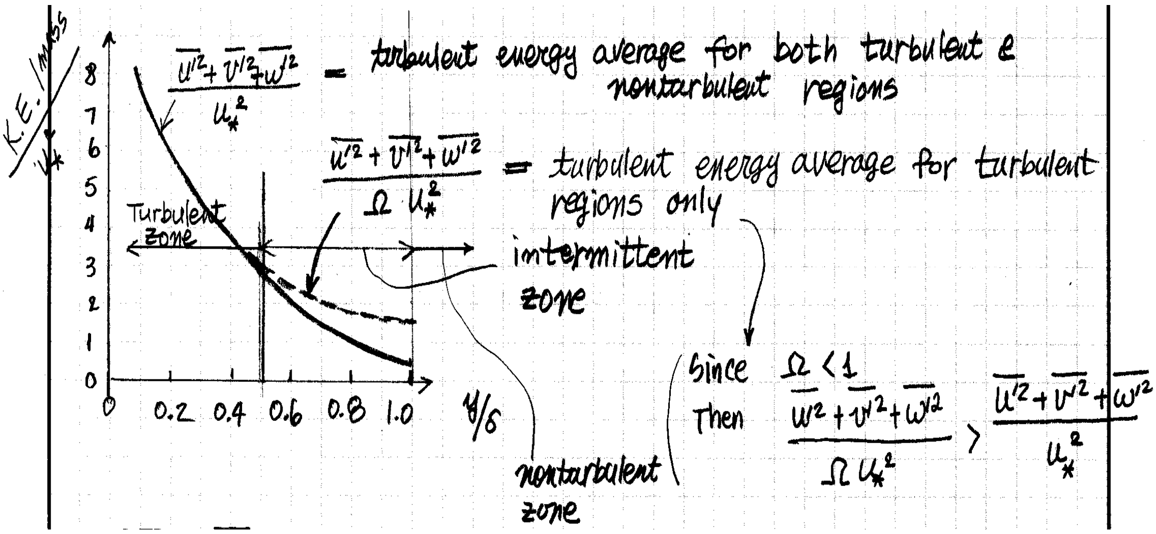
163. Oblique transverse sections of a turbulent boundary layer. The flow is viewed head-on, with smoke illuminated by a sheet of light that is inclined 45° downstream from the wall on the left and 45° upstream on the right. The Reynolds number based on momentum thickness is 600 in the upper pair of photographs and 9400 below. *Head & Bandyopadhyay 1981*

• Turbulent energy in a boundary layer, δ

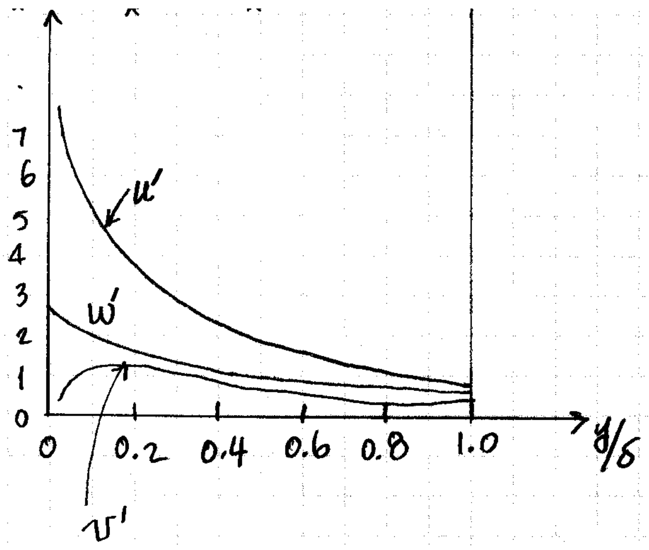
- Dimensionless energy $= \dfrac{\overline{u'^2} + \overline{v'^2}\ \overline{w'^2}}{u_*^2}$ (9.1)

where $u_* = \sqrt{\dfrac{\tau_0}{\rho}}$ = shear velocity

$$\mathrm{Re}_\delta = \frac{U\delta}{\nu} = 73{,}000 \Leftrightarrow \mathrm{Re}_x = 4\times10^6 \quad \text{for turbulent layer}$$



$$\frac{\overline{u'^2}}{\Omega u'^2_*}, \frac{\overline{v'^2}}{\Omega u'^2_*}, \frac{\overline{w'^2}}{\Omega u'^2_*},$$



- smooth wall → $v' = 0$ at wall

- rough wall → $v' \neq 0$ at wall

- smooth & rough wall

→ turbulent energy ≠ at y= δ

9.3 Mean-Flow Characteristics for turbulent boundary layer

○ Relations describing the mean-flow characteristics

→ predict velocity magnitude and relation between velocity and wall shear or pressure gradient forces

→ It is desirable that these relations should not require knowledge of the turbulence details.

○ Turbulent boundary layer

→ is composed of zones of different types of flow

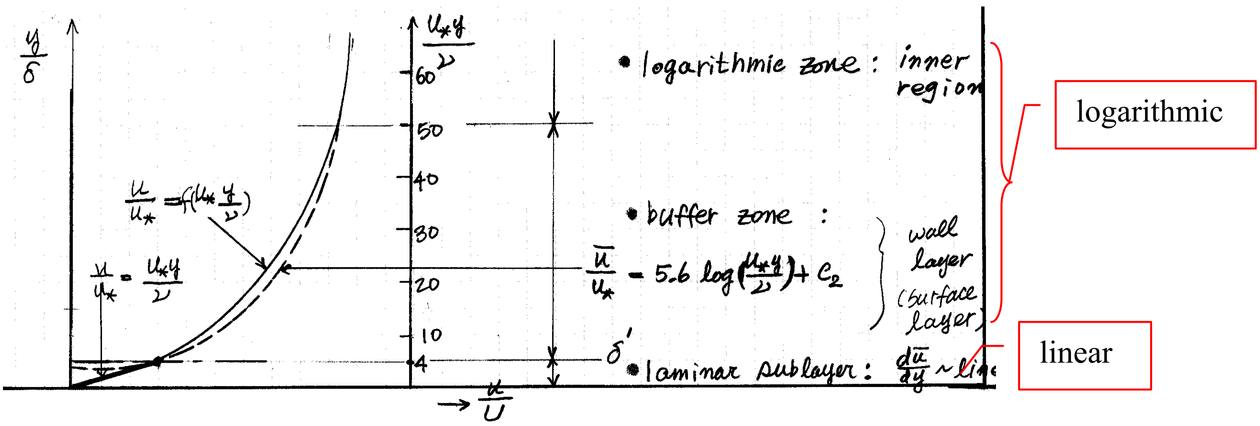
→ Effective viscosity ($\mu + \eta$) varies from wall out through the layer.

→ Theoretical solution is not practical for the general nonuniform boundary layer

→ use semiempirical procedure

9.3.1 Universal velocity and friction laws: **smooth walls**

(1) Velocity-profile regions



1) Laminar sublayer: $0 < \frac{u_* y}{\nu} \le 4$

$\frac{d\bar{u}}{dy}$ ~ linear

$$\rightarrow \frac{u}{u_*} = \frac{u_* y}{\nu} \tag{9.6}$$

~ Mean shear stress is controlled by the dynamic molecular viscosity μ .

→ Reynolds stress is negligible. → Mean flow is laminar.

~ energy of velocity fluctuation ≈0

2) Buffer zone: $4 < \frac{u_* y}{\nu} < 30 \sim 70$

~ Viscous and Reynolds stress are of the same order.

→ Both laminar flow and turbulence flow

~ Sharp peak in the turbulent energy occurs (Fig. 9.4).

3) Turbulent zone - logarithmic zone: $\frac{u_* y}{\nu} > 30 \sim 70$, *and* $y < 0.15\delta$

~ fully turbulent flow

~ inner law zone/inner region

~ Intensity of turbulence decreases.

~ velocity equation: logarithmic function

$$\frac{\bar{u}}{u_*} = 5.6 \log \frac{u_* y}{\nu} + C_2$$

4) Turbulent zone-outer region: $0.15\delta < y < 0.4\delta$

~ outer law, **velocity-defect law**

5) Intermittent zone: $0.4\delta < y < 1.2\delta$

~ Flow is intermittently turbulent and non-turbulent.

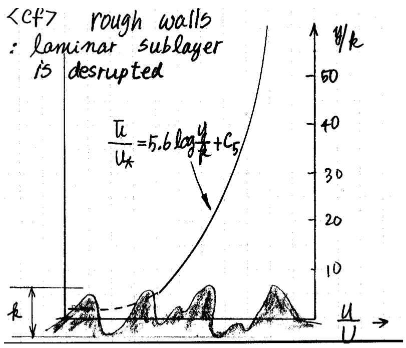
6) Non-turbulent zone

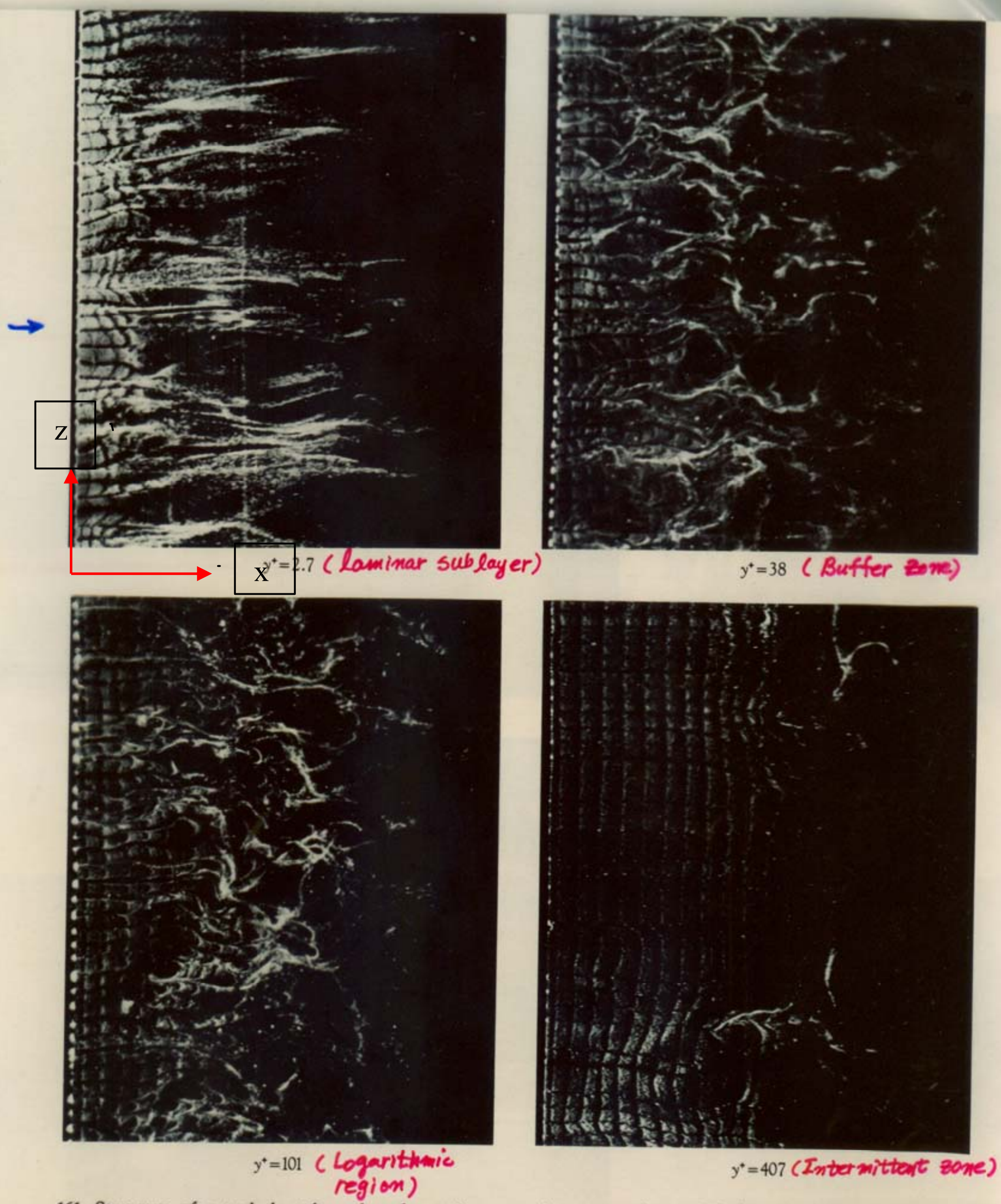
~ external flow zone

~ potential flow

[Cp] Rough wall:

→ Laminar sublayer is destroyed by the roughness elements.





161. Structure of a turbulent boundary layer. Successive layers of the flow near a flat plate in a water channel are shown by tiny hydrogen bubbles released periodically from a thin platinum wire seen at the left. The height $y^+ = y\,u_\tau/\nu$ of the wire above the plate is shown in wall variables, where $u_\tau = (\tau_w/\varrho)^{1/2}$ is the friction velocity. The characteristic low- and high-speed streaks shown in the viscous sublayer at $y^+=2.7$ become less noticeable farther away, and have disappeared in the logarithmic region at $y^+=101$. In the wake region at $y^+=407$ the turbulence is seen to be intermittent and of larger scale. *Kline, Reynolds, Schraub & Runstadler 1967*

94

(2) Wall law and velocity-defect law

Wall law $\rightarrow$ inner region

Velocity-defect law $\rightarrow$ outer region

1) Law of wall = inner law; $\frac{u_* y}{\nu} > 30 \sim 70$; $y < 0.15\delta$

~close to smooth boundaries (molecular viscosity dominant)

~ Law of wall assumes that the relation between wall shear stress and velocity $\bar{u}$ at distance y from the wall depends only on fluid density and viscosity;

$$f(\bar{u}, u_*, y, \rho, \mu) = 0$$

Dimensional analysis yields

$$\frac{\bar{u}}{u_*} = f\left(\frac{u_* y}{\nu}\right) \tag{9.3}$$

i) Laminar sublayer

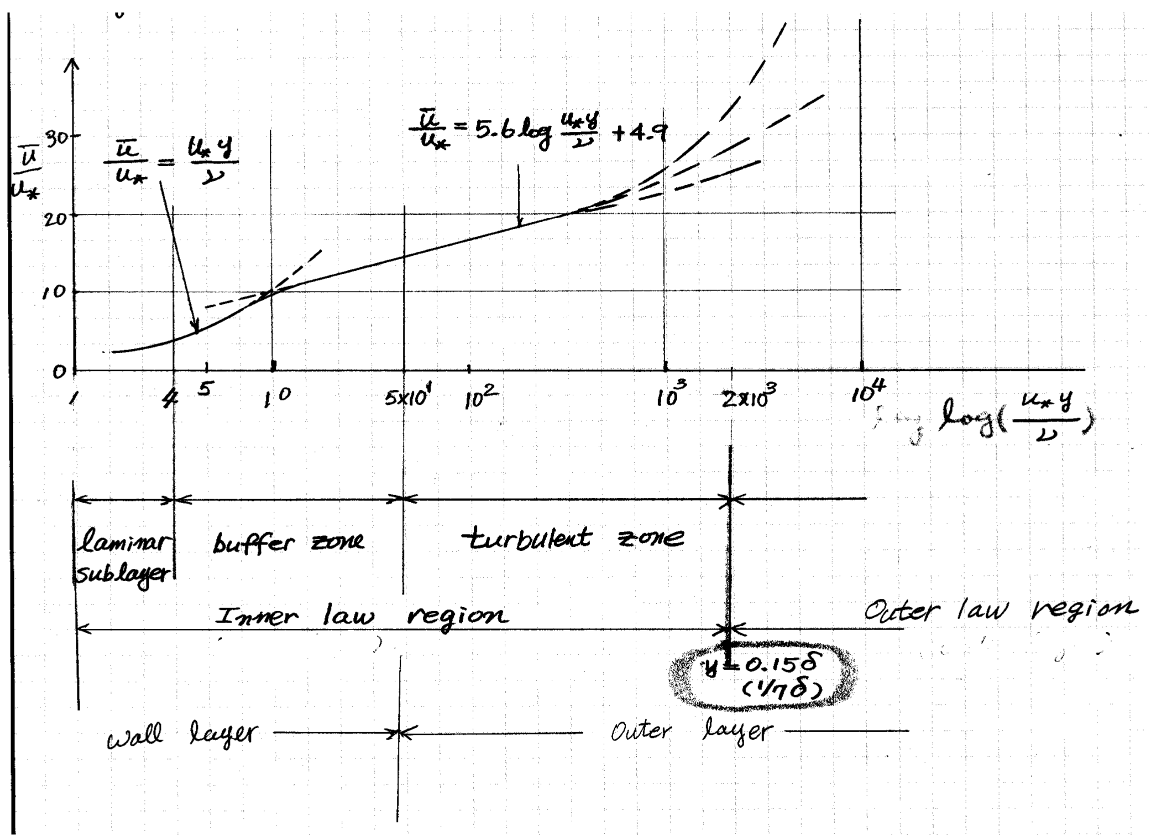
- mean velocity, $\bar{u} \equiv u$

- velcocity gradient, $\frac{\partial u}{\partial y} \sim \text{constant} \equiv \frac{u}{y}$

- shear stress, $\tau \approx \tau_0 = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \equiv \mu \frac{u}{y}$ (9.5)

- shear velocity, $u_* = \sqrt{\frac{\tau}{\rho}} \rightarrow u_*^2 = \frac{\tau}{\rho} = \frac{\mu}{\rho}\frac{u}{y}$

$$\frac{u}{u_*} = \frac{u_* y}{\nu} \tag{9.6}$$



■ Thickness of laminar sublayer

define thickness of laminar sublayer as the value of y which makes

$$\frac{u_* y}{\nu} = 4$$

$$\delta' = \frac{4\nu}{u_*} = \frac{4\nu}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{4\nu}{\left(\frac{c_f \rho U^2 / 2}{\rho}\right)^{\frac{1}{2}}} = \frac{4\nu}{U\sqrt{c_f / 2}} \tag{9.7}$$

where $\tau_0 = c_f \rho \frac{u^2}{2}$; $\quad c_f = local\ \ shear\ stress\ \ coeff.$

ii) Turbulent zone – inner region

Start with Prandtl's mixing length theory

$$\tau_0 \approx \tau = \rho l^2 \left| \frac{d\bar{u}}{dy} \right| \frac{d\bar{u}}{dy} \tag{1}$$

Near wall, $l = \kappa y$ (2)

$$\therefore \ \tau = \rho \kappa^2 y^2 \left(\frac{d\bar{u}}{dy} \right)^2 \tag{3}$$

Rearrange (3)

$$\frac{d\bar{u}}{dy} = \frac{1}{\kappa y} \sqrt{\frac{\tau}{\rho}} = \frac{u_*}{\kappa y}$$

$$d\bar{u} = \frac{u_*}{\kappa} \frac{1}{y} dy \tag{4}$$

Integrate (4)

$$\bar{u} = \frac{u_*}{\kappa} \ln y + C_1$$

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln y + C_1 \tag{9.10}$$

Substitute BC [$\bar{u} = 0$ at $y = y'$] into (9.10)

$$0 = \frac{1}{\kappa} \ln y' + C_1 \tag{5}$$

$$\therefore C_1 = -\frac{1}{\kappa}\ln y'$$

Assume $y' \propto \dfrac{\nu(m^2/s)}{u_*(m/s)} \rightarrow y' = C\dfrac{\nu}{u_*}$

Then (5) becomes

$$C_1 = -\frac{1}{\kappa}\ln y' = -\frac{1}{\kappa}\ln\left(C\frac{\nu}{u_*}\right) = C_2 - \frac{1}{\kappa}\ln\frac{\nu}{u_*} \qquad (9.11)$$

Substitute (9.11) into (9.10)

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln y + C_2 - \frac{1}{\kappa}\ln\frac{\nu}{u_*} = \frac{1}{\kappa}\ln\left(\frac{u_* y}{\nu}\right) + C_2$$

Changing the logarithm to base 10 yields

$$\frac{\bar{u}}{u_*} = \frac{2.3}{\kappa}\log_{10}\left(\frac{u_* y}{\nu}\right) + C_2 \qquad (9.12)$$

Empirical values of κ and C_2 for inner region of the boundary layer

$$\kappa = 0.41; \quad C_2 = 4.9$$

$$\frac{\bar{u}}{u_*} = 5.6\ \log\left(\frac{u_* y}{\nu}\right) + 4.9, \quad 30 \sim 70 < \frac{u_* y}{\nu}, \quad and\ \frac{y}{\delta} < 0.15 \qquad (9.13)$$

→ Prandtl's velocity distribution law; inner law; wall law

2) Velocity-defect law

～outer law

～outer reaches of the turbulent boundary layer for both smooth and rough walls

→ Reynolds stresses dominate the viscous stresses.

Assume velocity defect (reduction) at $y \propto$ wall shear stress

$$\frac{U-\bar{u}}{u_*} = g\left(\frac{y}{\delta}\right) \tag{9.14}$$

Substituting BC [$\bar{u} = U \; at \; y = \delta$] into Eq. (9.12) leads to

$$\frac{U}{u_*} = \frac{2.3}{\kappa}\log\left(\frac{u_*\delta}{\nu}\right) + C_2' \tag{9.15}$$

Subtract (9.12) from (9.15)

$$\frac{U}{u_*} = \frac{2.3}{\kappa}\log\left(\frac{u_*\delta}{\nu}\right) + C_2'$$

$$-\left|\;\frac{\bar{u}}{u_*} = \frac{2.3}{\kappa}\log\left(\frac{u_* y}{\nu}\right) + C_2\right.$$

$$\frac{U-\bar{u}}{u_*} = \frac{2.3}{\kappa}\left\{\log\left(\frac{u_*\delta}{\nu}\right) - \log\left(\frac{u_* y}{\nu}\right)\right\} + C_2' - C_2$$

$$= \frac{2.3}{\kappa}\log\left(\frac{u_*\delta}{\nu}\frac{\nu}{u_* y}\right) + C_3$$

$$= -\frac{2.3}{\kappa}\log\left(\frac{y}{\delta}\right) + C_3$$

where x and C_3 are empirical constants

i) Inner region; $\frac{y}{\delta} \leq 0.15$

$\kappa = 0.41$, $C_3 = 2.5$

$$\frac{U-\bar{u}}{u_*} = -5.6\log\left(\frac{y}{\delta}\right) + 2.5 \tag{9.16}$$

ii) Outer region; $\frac{y}{\delta} > 0.15$

$x = 0.267$, $C_3 = 0$

$$\frac{U-\bar{u}}{u_*} = -8.6\log\left(\frac{y}{\delta}\right) \tag{9.17}$$

→ Eqs. (9.16) & (9.17) apply to both smooth and rough surfaces.

→ Eq. (9.16) = Eq. (9.13)

Fig. 9.8 → velocity-defect law is applicable for both smooth and rough walls

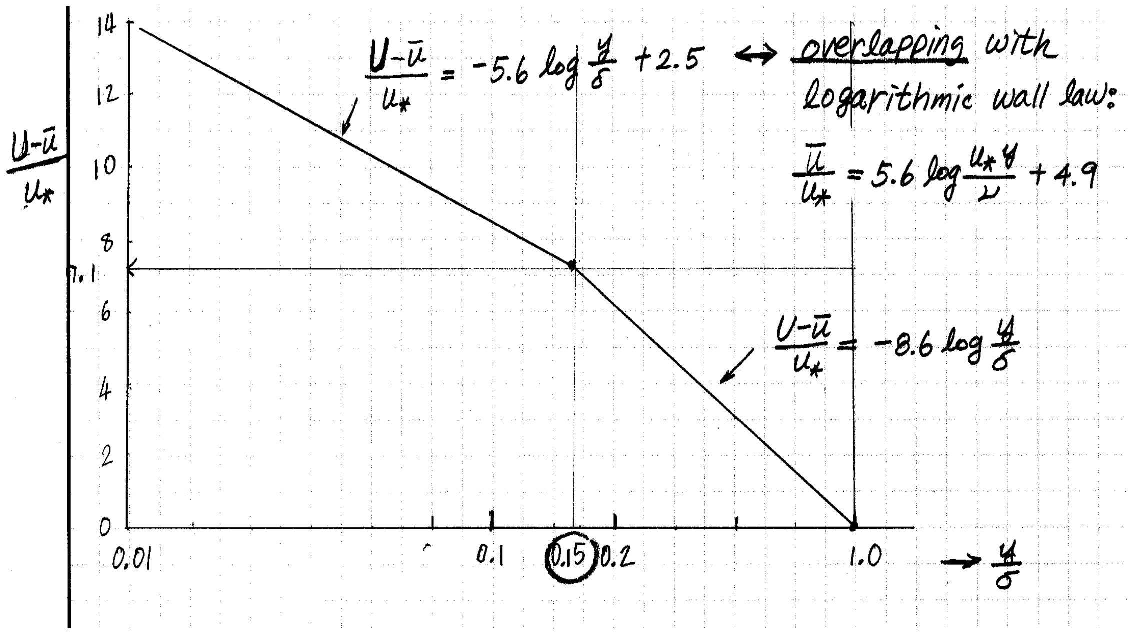
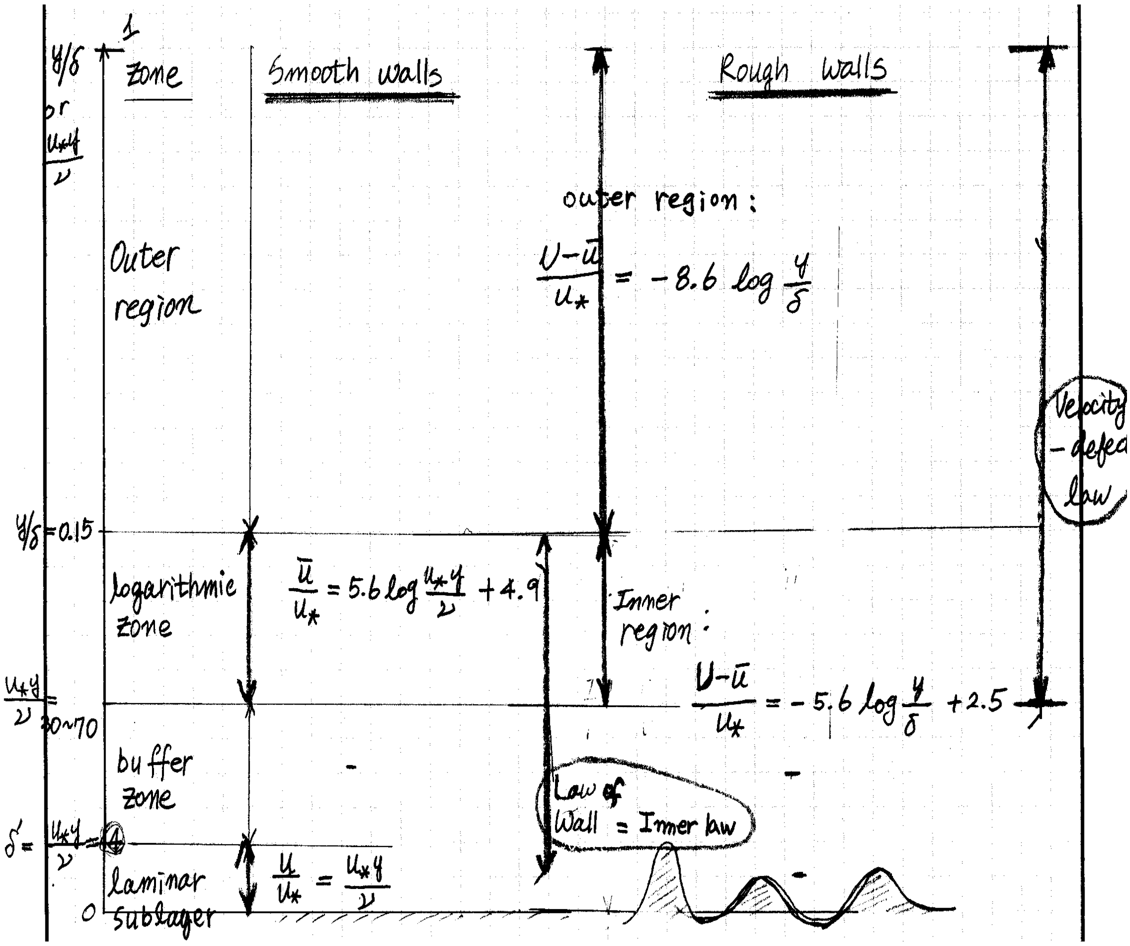


Table 9-1



■ For smooth walls

a) Laminar sublayer: $\dfrac{u}{u_*} = \dfrac{u_* y}{\nu}$

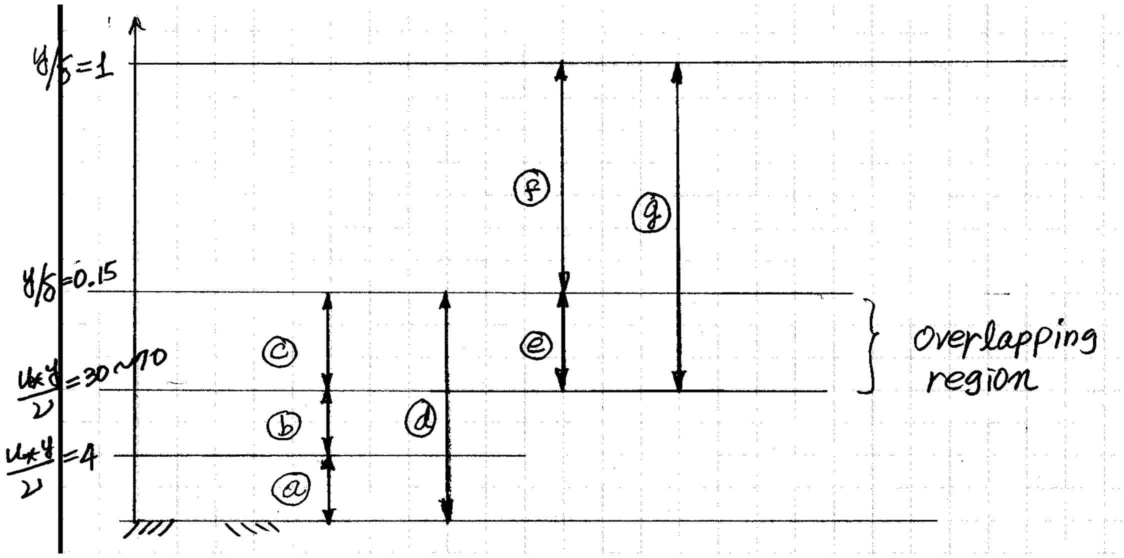
→ viscous effect dominates.

b) Buffer zone

c) Logarithmic zone: $\dfrac{\bar{u}}{u_*} = 5.6 \log \dfrac{u_* y}{\nu} + 4.9$

→ turbulence effect dominates.

d) Inner law (law of wall) region



■ For both smooth and rough walls

e) Inner region: $\dfrac{\bar{U}-\bar{u}}{u_*}=-5.6\log\dfrac{y}{\delta}+2.5$

f) Outer region: $\dfrac{\bar{U}-\bar{u}}{u_*}=-8.6\log\dfrac{y}{\delta}$

g) Outer law (velocity - defect law) region

(3) Surface-resistance formulas

1) Local shear-stress coefficient on smooth walls

Velocity profile ↔ shear-stress equations

$$u_* = \sqrt{\tau / \rho} = \sqrt{\frac{c_f}{2}}\, U \;\rightarrow\; \frac{U}{u_*} = \sqrt{\frac{2}{c_f}} \tag{9.18}$$

where c_f = local shear - stress coeff

[Re] $\tau_0 = \frac{1}{2}\rho c_f U^2 \rightarrow \frac{\tau_0}{\rho} = \frac{c_f}{2} U^2$

$$u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{c_f}{2}}\, U$$

i) Apply logarithmic law

Substituting into ⓐ $y = \delta$, $\bar{u} = U$ into Eq. (9.12) yields

$$\frac{U}{u_*} = \frac{2.3}{\kappa}\log\left(\frac{u_*\delta}{\nu}\right) + C_4 \tag{A}$$

Substitute (9.18) into (A)

$$\sqrt{\frac{2}{c_f}} = \frac{2.3}{\kappa}\log\left(\frac{U\delta}{\nu}\sqrt{\frac{c_f}{2}}\right) + C_4 \tag{9.19}$$

$\sim c_f$ is not given explicitly.

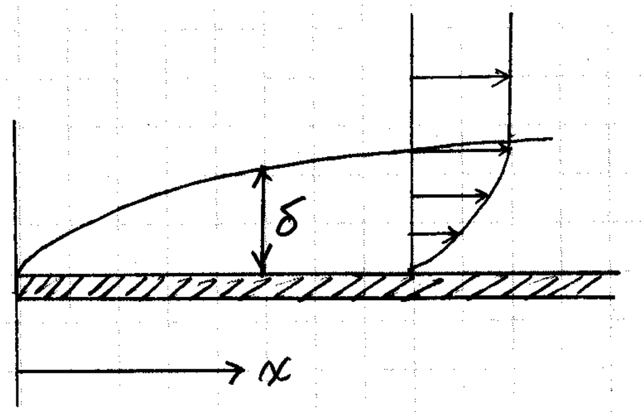
ii) For explicit expression, use displacement thickness δ^* and momentum thickness θ instead of δ

Clauser: $$\frac{1}{\sqrt{c_f}} = 3.96 \log \mathrm{Re}_{\delta^*} + 3.04 \quad (9.20)$$

Squire and Young: $$\frac{1}{\sqrt{c_f}} = 4.17 \log \mathrm{Re}_{\theta} + 2.54 \quad (9.21)$$

where $\mathrm{Re}_{\delta^*} = \dfrac{U\delta^*}{\nu}$; $\mathrm{Re}_{\theta} = \dfrac{U\theta}{\nu}$

Re_{δ^*} , $\mathrm{Re}_{\theta} = f(\mathrm{Re}_x), \mathrm{Re}_x = \dfrac{Ux}{\nu}$



iii) Karman's relation

~ assume turbulence boundary layer all the way from the leading edge

(i.e., no preceding stretch of laminar boundary layer)

$$\frac{1}{\sqrt{c_f}} = 4.15 \log(\mathrm{Re}_x c_f) + 1.7 \quad (9.23)$$

iv) Schultz-Grunow (1940)

$$c_f = \frac{0.370}{(\log \mathrm{Re}_x)^{2.58}} \tag{9.24}$$

Comparison of (9.23) and (9.24)

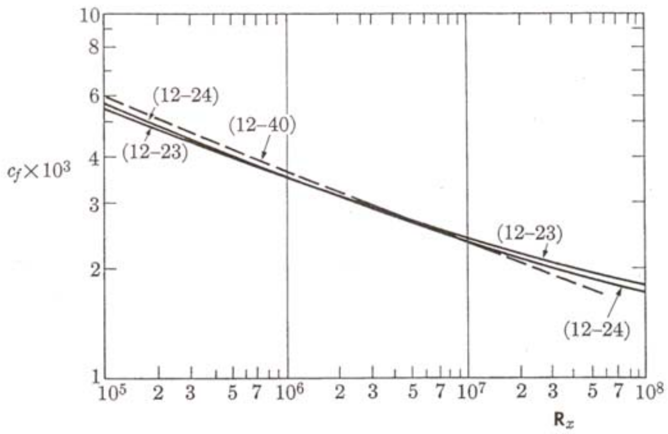


FIG. 12–9. Local coefficient of resistance.

2) Average shear-stress coefficient on smooth walls

Consider average shear-stress coefficient over a distance *l* along a flat plate of a width *b*

$$total\ drag\,(D) = \tau \times bl = \frac{1}{2} C_f \rho U^2 bl$$

$$C_f \equiv \frac{D}{bl\rho U^2/2}$$

i) Schoenherr (1932)

$$\frac{1}{\sqrt{C_f}} = 4.13 \log(\mathrm{Re}_l C_f) \tag{9.26}$$

→ implicit

where $\mathrm{Re}_l = \dfrac{Ul}{\nu}$

ii) Schultz-Grunow

$$C_f = \frac{0.427}{(\log \mathrm{Re}_l - 0.407)^{2.64}}, \quad 10^2 < \mathrm{Re}_l < 10^9 \tag{9.27}$$

Comparison of (9.26) and (9.27)

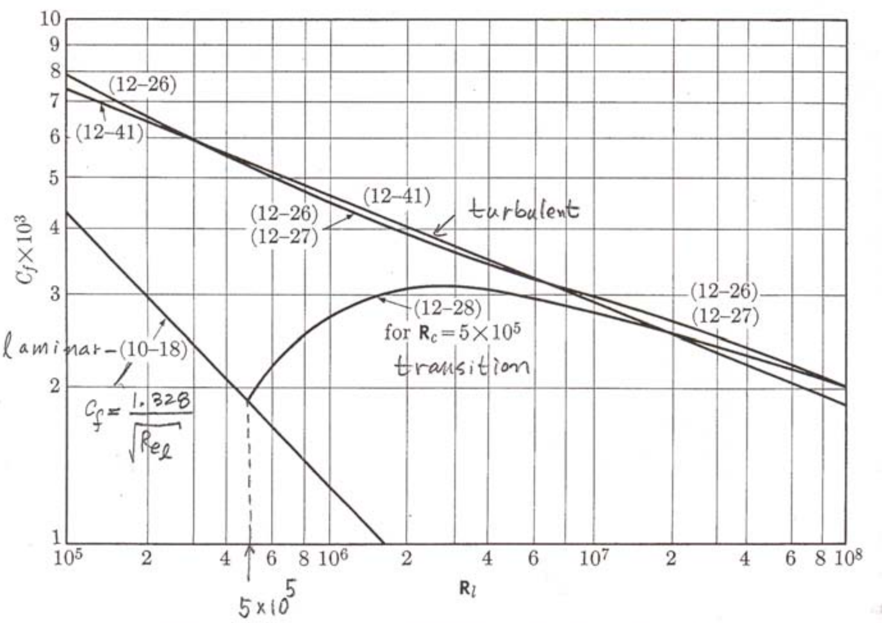
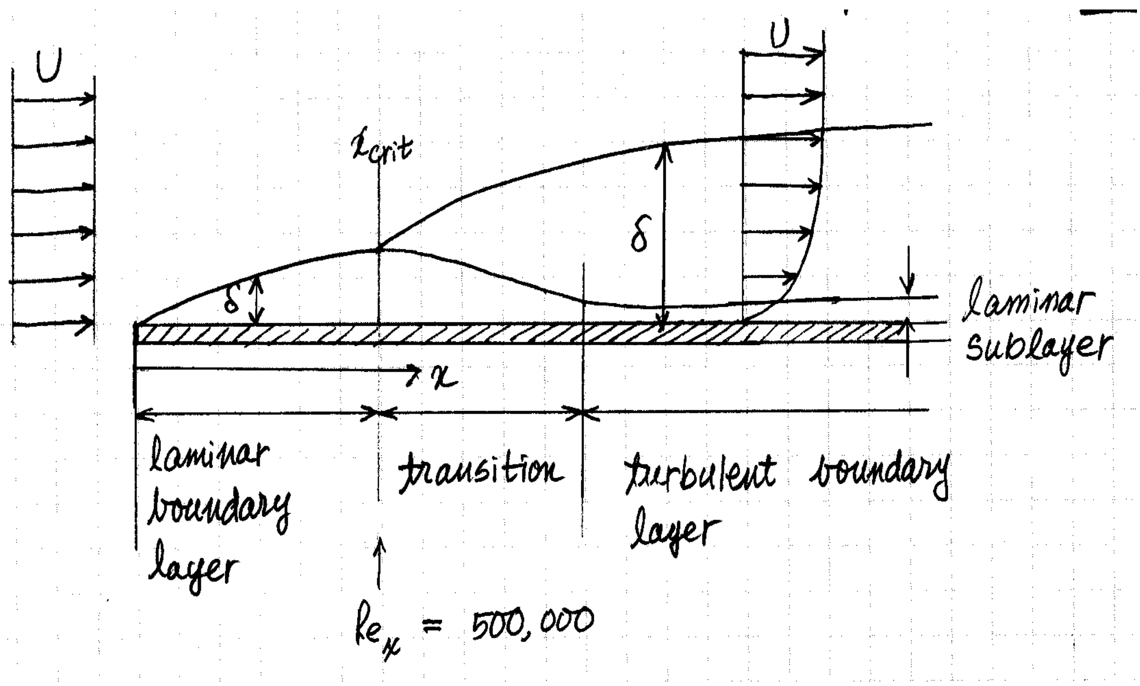


FIG. 12–10. Average coefficient of resistance for flat plates.



3) Transition formula

■ Boundary layer developing on a smooth flat plate

~ At upstream end (leading edge), laminar boundary layer develops because viscous effects dominate due to inhibiting effect of the smooth wall on the development of turbulence.

~ Thus, when a significant stretch of laminar boundary layer preceding the turbulent layer, total friction is the laminar portion up to x_{crit} plus the turbulent portion from x_{crit} to l.

~ Therefore, average shear-stress coefficient is lower than the prediction by Eqs. (9.26) or (9.27).

→ Use transition formula

$$C_f = \frac{0.427}{(\log \mathrm{Re}_l - 0.407)^{2.64}} - \frac{A}{\mathrm{Re}_l} \tag{9.28}$$

where A/Re_l = correction term = $f(\mathrm{Re}_{crit})$, $\mathrm{Re}_{crit} = \dfrac{Ux_{crit}}{\nu}$

⇒A = 1,060~3,340

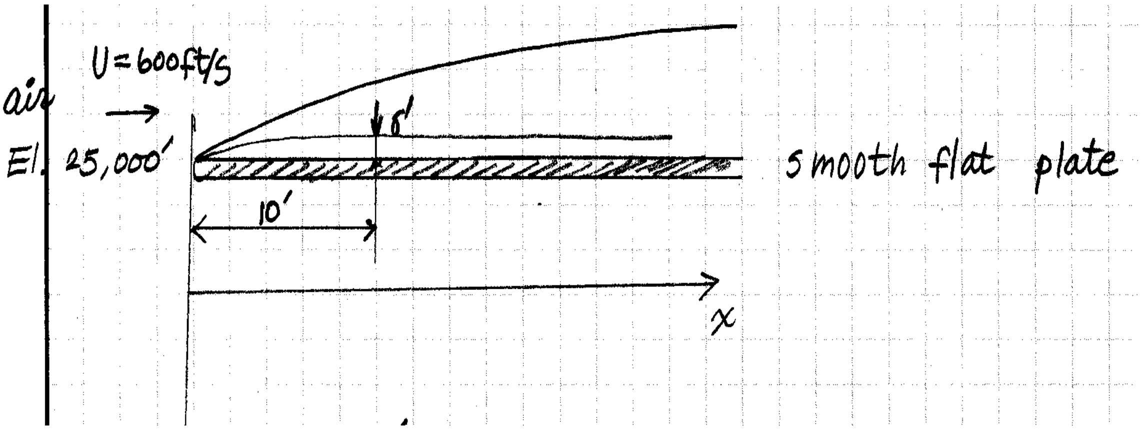
Eq. (9.28) falls between the laminar and turbulent curves.

$$Laminar\ flow: C_f = \frac{1.328}{\mathrm{Re}_l^{1/2}}$$

SURFACE RESISTANCE FORMULAS FOR BOUNDARY LAYERS WITH $d\bar{p}/dx = 0$

	Smooth walls	Rough walls
LOCAL SHEAR		
Universal equations		
Clauser (12–20)	$1/\sqrt{c_f} = 3.96 \log \mathbf{R}_{\delta^*} + 3.04$	(12–46) $\frac{1}{\sqrt{c_f}} = 3.96 \log \frac{\delta}{k} + C_8$ $C_8 = f$ (size, shape, and distribution of roughness)
Squire and Young (12–21)	$1/\sqrt{c_f} = 4.17 \log \mathbf{R}_\theta + 2.54$	
von Kármán (12–23)	$1/\sqrt{c_f} = 4.15 \log (\mathbf{R}_x c_f) + 1.7$	
Schultz-Grunow (12–24)	$c_f = \frac{0.370}{(\log \mathbf{R}_x)^{2.58}}$	
Power law (12–40)	$c_f = \frac{0.0466}{\mathbf{R}_\delta^{1/4}} = \frac{0.059}{\mathbf{R}_x^{1/5}}$	
AVERAGE SHEAR		
Universal equations		
Schoenherr (12–26)	$1/\sqrt{C_f} = 4.13 \log (\mathbf{R}_l C_f)$	
Schultz-Grunow (12–27)	$C_f = \frac{0.427}{(\log \mathbf{R}_l - 0.407)^{2.64}}$	
Power law (12–41)	$C_f = \frac{0.074}{\mathbf{R}_l^{1/5}}$	
Transition formula		
Schultz-Grunow-Prandtl (12–28)	$C_f = \frac{0.427}{(\log \mathbf{R}_l - 0.407)^{2.64}} - \frac{A}{\mathbf{R}_l}$ $A = f(\mathbf{R}_{\text{crit}})$ as given in Table 12–2	

[Ex. 9.1] Turbulent boundary-layer velocity and thickness



(a) Thickness δ' (laminar sublayer) at $x = 10\,ft$

Air ⓐ El. 25,000': $\nu = 3\times 10^{-4}\ ft^2/s$

$$\rho = 1.07\times 10^{-3}\ slug/ft^3$$

Select $\mathrm{Re}_{crit} = 5\times 10^5 = \dfrac{600(x)}{3\times 10^{-4}}$

$\therefore x_{crit} = 0.25\,ft \sim$ negligible comparaed to $l = 10\,ft$

Therefore, assume that turbulent boundary layer develops all the way from the leading edge. Use Schultz-Grunow Eq., (9.24) to compute c_f

$$\mathrm{Re}_x = \frac{Ux}{\nu} = \frac{600(10)}{(3\times 10^{-4})} = 2\times 10^7$$

$$c_f = \frac{0.370}{(\log \mathrm{Re}_x)^{2.58}} = \frac{0.370}{\left\{\log\left(2\times 10^7\right)\right\}^{2.58}} = \frac{0.370}{(7.30)^{2.58}} = 0.0022$$

$$\tau_0 = \frac{\rho}{2} c_f U^2 = \frac{1}{2}(1.07\times10^{-3})(0.0022)(600)^2 = 0.422\, lb/ft^2$$

$$u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{0.422}{1.07\times10^{-3}}} = 19.8\, ft/s$$

Eq. (9.7): $\delta' = \frac{4\nu}{u_*} = \frac{4(3\times10^{-4})}{19.8} = \underline{0.61\times10^{-4}\, ft} = 7.3\times10^{-4}\ in$

(b) Velocity $\bar{u}$ at $y = \delta'$

Eq. (9.6): $\frac{u}{u_*} = \frac{u_* y}{\nu}$

$$\therefore u = \frac{u_*^2 \delta'}{\nu} = \frac{(19.8)^2(0.61\times10^{-4})}{(3\times10^{-4})} = 79.7\ ft/s \rightarrow \underline{13\%\ \text{of}\ U}$$

$\uparrow$ $y = \delta'$

[Cf] $U = 600\ ft/s$

(c) Velocity $\bar{u}$ at $y/\delta = 0.15$

Use Eq. (9.16) – outer law

$$\frac{U - \bar{u}}{u_*} = -5.6 \log\left(\frac{y}{\delta}\right) + 2.5$$

$$\frac{600 - \bar{u}}{19.8} = -5.6\, \underline{\log(0.15)} + 2.5$$

$$\bar{u} = 600 - 91.35 - 49.5 = 459.2\, ft/s \rightarrow \underline{76\%\ \text{of}\ U}$$

[Cf] $\bar{u} = U + 5.6\, u_* \log\left(\frac{y}{\delta}\right) - 2.5 u_*$

(d) Distance y at $y/\delta = 0.15$ and thickness δ

Use Eq. (9.13) – inner law

$$\frac{\bar{u}}{u_*} = 5.6 \log\left(\frac{u_* y}{\nu}\right) + 4.9$$

$$At\,\frac{y}{\delta} = 0.15: \;\frac{459}{19.8} = 5.6 \log\left(\frac{19.8y}{3\times10^{-4}}\right) + 4.9$$

$$\log\left(\frac{19.8y}{3\times10^{-4}}\right) = 3.26; \quad \frac{19.8y}{3\times10^{-4}} = 1839$$

$$y = 0.028' = 0.33in \approx 0.8cm \qquad \text{(B)}$$

Substitute (B) into $\frac{y}{\delta} = 0.15$

$$\delta = \frac{y}{0.15} = \underline{0.186' = 2.24in} \approx 5.7cm$$

$$\text{At}\,x = 10': \;\frac{\delta}{\delta'} = \frac{0.186}{0.61\times10^{-4}} = 3049 \approx 3\times10^3$$

$$\boxed{\frac{\delta'}{\delta} = 0.0003}$$

[Ex. 9.2] Surface resistance on a smooth boundary given as Ex.9.1

(a) Displacement thickness δ^*

$$\delta^* = \int_0^h \left(1-\frac{u}{U}\right) dy$$

$$\frac{\delta^*}{\delta} = \int_0^{h/\delta} \left(1-\frac{\bar{u}}{U}\right) d\left(\frac{y}{\delta}\right), \qquad h/\delta \geq 1 \qquad \text{(A)}$$

Neglect laminar sublayer and approximate buffer zone with Eq. (9.16)

$$(i) \quad y/\delta < 0.15, \qquad \frac{U-\bar{u}}{u_*} = -5.6\log\left(\frac{y}{\delta}\right) + 2.5 \leftarrow (9.16)$$

$$\therefore\ 1-\frac{\bar{u}}{U} = -5.6\frac{u_*}{U}\log\frac{y}{\delta} + 2.5\frac{u_*}{U} \qquad \text{(B)}$$

$$(ii) \quad y/\delta > 0.15, \qquad \frac{U-\bar{u}}{u_*} = -8.6\log\left(\frac{y}{\delta}\right) \leftarrow (9.17)$$

$$\therefore\ 1-\frac{\bar{u}}{U} = -8.6\frac{u_*}{U}\log\left(\frac{y}{\delta}\right) \qquad \text{(C)}$$

$$\ln x = 2.3\log x$$

$$\int \ln x\, dx = x\ln x - x$$

Substituting (B) and (C) into (A) yields

$$\therefore\ \frac{\delta^*}{\delta} = \int_{\delta'/\delta}^{0.15}\left(-2.43\ln\frac{y}{\delta} + 2.5\right)\frac{u_*}{U} d\left(\frac{y}{\delta}\right) + \int_{0.15}^{1.0}\left(-3.74\ln\frac{y}{\delta}\right)\frac{u_*}{U} d\left(\frac{y}{\delta}\right)$$

$$= \frac{u_*}{U}\left[\left[-2.43\left\{\frac{y}{\delta}\ln\frac{y}{\delta} - \frac{y}{\delta}\right\} + 2.5\frac{y}{\delta}\right]_{0.0003}^{0.15} + \left[-3.74\left\{\frac{y}{\delta}\ln\frac{y}{\delta} - \frac{y}{\delta}\right\}\right]_{0.15}^{1}\right]$$

$$\cong 3.74\frac{u_*}{U} = 3.74\frac{(19.8)}{600} = 0.1184$$

$$\delta^* = 0.1184\delta = 0.1184\ (0.186) = 0.022\,ft$$

$$\frac{\delta^*}{\delta} = \underline{0.1184} \ \rightarrow 11.8\%$$

(b) Local surface-resistance coeff. c_f

Use Eq. (9.20) by Clauser

$$\frac{1}{\sqrt{c_f}} = 3.96 \log \mathrm{Re}_{\delta^*} + 3.04 \quad \leftarrow \quad \mathrm{Re}_{\delta^*} = \frac{600 \times 0.022}{3 \times 10^{-4}} = 44{,}000$$

$$= 3.96 \log\left(44{,}000\right) + 3.04$$

$$\therefore\ c_f = 2.18 \times 10^{-3} = \underline{0.00218}$$

[Cf] $c_f = 0.00218$ by Schultz-Grunow Eq.

9.3.2. Power-law formulas: Smooth walls

● **Logarithmic equations** for velocity profile and shear-stress coeff .

~ universal

~ applicable over almost entire range of Reynolds numbers

● **Power-law equations**

~ applicable over only limited range of Reynolds numbers

~ simpler

~ explicit relations for $\bar{u}/U$ and c_f

~ explicit relations for δ in terms of *Re* and distance x

(1) Assumptions of power-law formulas

1) Except very near the wall, mean velocity is closely proportional to a root of the distance y from the wall.

$$\bar{u} \propto y^{\frac{1}{n}} \qquad \text{(A)}$$

2) Shear stress coeff. c_f is inversely proportional to a root of Re_δ

$$c_f \propto \frac{1}{\mathrm{Re}_\delta^m} \quad , \quad \mathrm{Re}_\delta = \frac{U\delta}{\nu}$$

$$c_f = \frac{A}{\left(\frac{U\delta}{\nu}\right)^m} \qquad (9.29)$$

where A, m = constants

[Cf] Eq. (9.29) is similar to equation for laminar boundary layer, $c_f = \dfrac{3.32}{\mathrm{Re}_\delta}$

(2) Derivation of power equation

Combine Eqs. (9.18) and (9.29)

(9.18): $u_* = U\sqrt{\frac{c_f}{2}}$

$$c_f = \frac{A}{\left(\frac{U\delta}{\nu}\right)^m}$$

$$\therefore \frac{U}{u_*} = \sqrt{\frac{2}{c_f}} = \frac{\sqrt{2}}{\sqrt{A}}\left(\frac{U\delta}{\nu}\right)^{\frac{m}{2}}$$

$$\frac{U^{1-\frac{m}{2}}}{u_*} = \frac{\sqrt{2}}{\sqrt{A}}\left(\frac{\delta}{\nu}\right)^{\frac{m}{2}}$$

$$\left(\frac{U}{u^*}\right)^{1-\frac{m}{2}} = \sqrt{\frac{2}{A}}\left(\frac{u_*\delta}{\nu}\right)^{\frac{m}{2}}$$

$$\therefore \frac{U}{u_*} = B\left(\frac{u_*\delta}{\nu}\right)^{\frac{m}{2-m}} \tag{9.30}$$

Substitute Assumption (1) into Eq. (9.30), replace δ with y

$$\frac{\bar{u}}{u_*} = B\left(\frac{u_* y}{\nu}\right)^{\frac{m}{2-m}} \tag{9.31}$$

Divide (9.31) by (9.30)

$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{\frac{m}{2-m}} \tag{9.32}$$

For **3,000 <** Re_δ **< 70,000**; $m = \frac{1}{4}$, $A = 0.0466$, $B = 8.74$

$$\frac{\bar{u}}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \tag{9.33}$$

$$\frac{U}{u_*} = 8.74\left(\frac{u_*\delta}{\nu}\right)^{\frac{1}{7}} \tag{9.34}$$

$$\frac{\bar{u}}{u_*} = 8.74\left(\frac{u_* y}{\nu}\right)^{\frac{1}{7}} \tag{9.35}$$

$$c_f = \frac{0.0466}{\left(Re_\delta\right)^{\frac{1}{4}}} \tag{9.36}$$

(3) Relation for δ

Adopt integral-momentum eq. for steady motion with $\frac{\partial p}{\partial x} = 0$

~ integration of Prandtl's 2-D boundary-layer equations

$$\frac{\tau_0}{\rho} = U^2 \frac{\partial \theta}{\partial x} = c_f \frac{U^2}{2} \leftarrow \tau_0 = \frac{\rho}{2} c_f U^2 \tag{9.37}$$

where θ = momentum thickness

$$\theta = \int_0^h \frac{\bar{u}}{U}\left(1 - \frac{\bar{u}}{U}\right) dy \tag{A}$$

Substitute Eq. (9.33) into (A) and integrate

$$\theta = \int_0^h \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \left\{1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right\} dy$$
$$= \frac{7}{72}\delta \approx 0.1\delta \tag{9.38}$$

Substitute Eqs. (9.36) and (9.38) into (9.37) and integrate w.r.t. *x*

$$\delta = \frac{0.318x}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} = \frac{0.318x}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}} \quad , \ \mathrm{Re}_x < 10^7 \tag{9.39}$$

$$c_f = \frac{0.059}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}} \quad , \ \mathrm{Re}_x < 10^7 \tag{9.40}$$

Integrate (9.40) over *l* to get average coefficient

$$c_f = \frac{0.074}{\left(\mathrm{Re}_l\right)^{\frac{1}{5}}} \quad , \ \mathrm{Re}_l < 10^7 \tag{9.41}$$

[Re] Derivation of (9.39) and (9.40)

$$U^2 \frac{\partial \theta}{\partial x} = c_f \frac{U^2}{2}$$

$$\frac{\partial \theta}{\partial x} = \frac{c_f}{2} \tag{B}$$

Substitute (9.38) and (9.36) into (B)

$$\frac{\partial}{\partial x}\left(\frac{7}{72}\delta\right)=\frac{1}{2}\left(0.0466\,/\,(\mathrm{Re}_{\delta})^{\frac{1}{4}}\right)$$

$$\frac{7}{72}\;\frac{\partial \delta}{\partial x}=\frac{0.0233}{\left(\mathrm{Re}_{\delta}\right)^{\frac{1}{4}}}=\frac{0.0233}{\left(\dfrac{U\delta}{\nu}\right)^{\frac{1}{4}}}$$

$$\frac{\partial \delta}{\partial x}=\frac{0.2397}{\left(\dfrac{U\delta}{\nu}\right)^{\frac{1}{4}}}$$

Integrate once w.r.t. *x*

$$\delta=\frac{0.2397}{\left(\dfrac{U\delta}{\nu}\right)^{\frac{1}{4}}}x+C$$

B.C.: $\delta \cong 0$ at $x=0$

$$0=\frac{0.2397}{\left(\dfrac{U\delta}{\nu}\right)^{\frac{1}{4}}}0+C \qquad \rightarrow\; C=0$$

$$\delta=\frac{0.2397}{\left(\dfrac{U\delta}{\nu}\right)^{\frac{1}{4}}}x$$

$$\delta^{\frac{5}{4}}=\frac{0.2397}{\left(\dfrac{Ux}{\nu}\right)^{\frac{1}{4}}}x^{\frac{5}{4}}$$

$$\therefore \delta = \left\{ \frac{0.2397}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{4}}} x^{\frac{5}{4}} \right\}^{\frac{4}{5}} = \frac{(0.2397)^{\frac{4}{5}}}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} x$$

$$\therefore \delta = \frac{0.318}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} x = \frac{0.318}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}} x \qquad \rightarrow \text{ Eq. (9.39)}$$

(9-36): $c_f = \dfrac{0.0466}{\left(\mathrm{Re}_\delta\right)^{\frac{1}{4}}} = \dfrac{0.0466}{\left(\frac{U\delta}{\nu}\right)^{\frac{1}{4}}}$ (C)

Substitute (9.39) into (C)

$$\therefore c_f = \frac{0.0466}{\left(\frac{U}{\nu}\right)^{\frac{1}{4}} \left\{ \frac{0.318}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} \right\}^{\frac{1}{4}}} = \frac{0.062}{\left(\frac{U}{\nu}\right)^{\frac{1}{4}} \left\{ \frac{x^{\frac{4}{5}}}{\left(\frac{U}{\nu}\right)^{\frac{1}{5}}} \right\}^{\frac{1}{4}}}$$

$$= \frac{0.062}{\left(\frac{U}{\nu}\right)^{\frac{1}{4}} \frac{x^{\frac{1}{5}}}{\left(\frac{U}{\nu}\right)^{\frac{1}{20}}}} = \frac{0.062}{\left(\frac{U}{\nu}\right)^{\frac{1}{5}} x^{\frac{1}{5}}} = \frac{0.062}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} = \frac{0.062}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}} \qquad \rightarrow \text{ (9.40)}$$

Integrate (9.40) over l

$$\overline{C}_f = \frac{1}{l} \int_0^l \frac{0.062}{\left(\mathrm{Re}_x\right)^{\frac{1}{5}}} dx = \frac{0.062}{l} \int_0^l \frac{1}{\left(\frac{Ux}{\nu}\right)^{\frac{1}{5}}} dx \, \frac{0.062}{l\left(\frac{U}{\nu}\right)^{\frac{1}{5}}} \int_0^l \frac{1}{x^{\frac{1}{5}}} dx = \frac{0.076}{\left(\mathrm{Re}_l\right)^{\frac{1}{5}}}$$

9.3.3. Laws for rough walls

(1) Effects of roughness

rough walls: velocity distribution and resistance = f (Reynolds number , roughness)

smooth walls: velocity distribution and resistance = f (Reynolds number)

- For natural roughness, k is random, and statistical quantity

$\rightarrow k = k_s$ = uniform sand grain

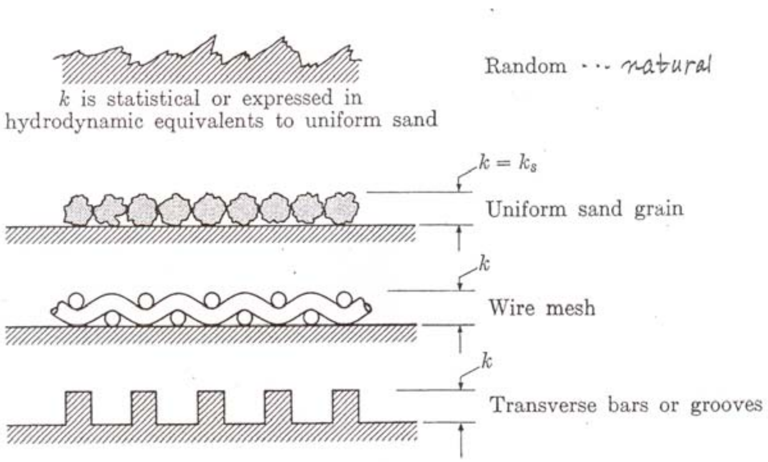


FIG. 12–11. Example of roughness types and definitions of roughness magnitude *k*.

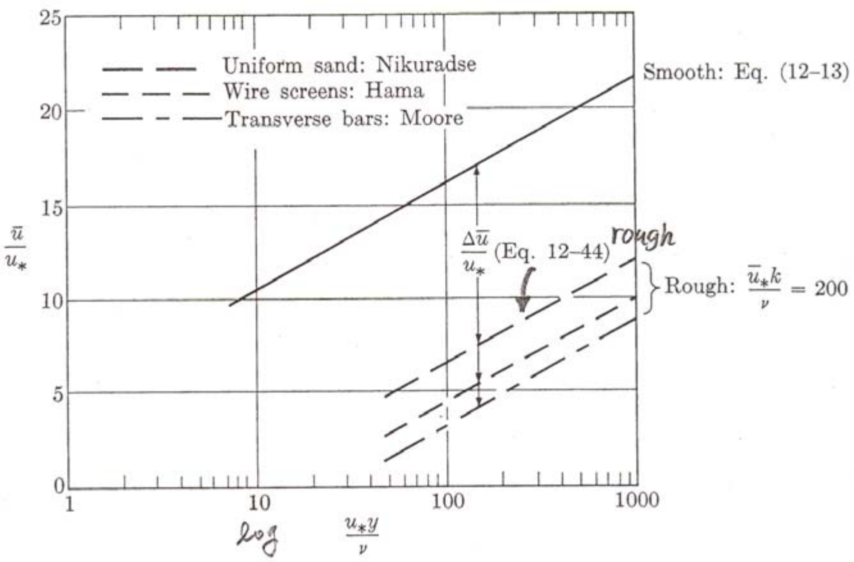


FIG. 12–12. Boundary-layer velocity-profile data illustrating effect of roughness.

- Measurement of roughness effects

  a) experiments with sand grains cemented to smooth surfaces

  b) evaluate roughness value ≡ height k_s

  c) compare hydrodynamic behavior with other types and magnitude of roughness

- Effects of roughness

i) $\frac{k_s}{\delta'} < 1$

~ roughness has negligible effect on the wall shear

→ hydrodynamically smooth

$$\delta' = \frac{4\nu}{u_*} = \text{laminar sublayer thickness}$$

ii) $\frac{k_s}{\delta'} > 1$

~ roughness effects appear

~ roughness disrupts the laminar sublayer

~ smooth-wall relations for velocity and c_f no longer hold

→ hydrodynamically rough

iii) $\frac{k_s}{\delta'} > 15 \sim 25$

~ friction and velocity distribution depend only on roughness rather than Reynolds number

→ fully rough flow condition

▪ Critical roughness, k_{crit}

$$k_{crit} = \delta'$$

$$= \frac{4\nu}{u_*} = \frac{4\nu}{U\sqrt{c_f / 2}} \quad \propto \quad \mathrm{Re}_x \quad \propto \quad x$$

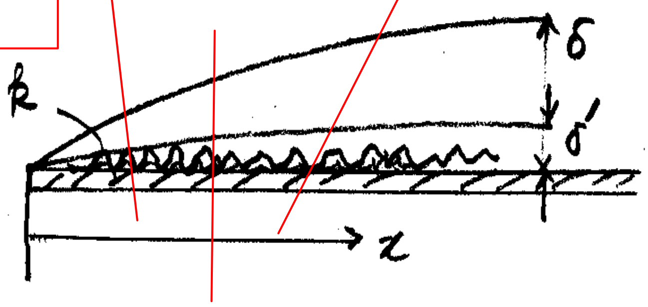
$$c_f \quad \propto \quad \frac{1}{\mathrm{Re}_x}$$

If x increases, then c_f decreases, and δ' increases.

Therefore, for a surface of uniform roughness, it is possible to be hydrodynamically rough upstream, and hydrodynamically smooth downstream.

$k < \delta'$, downstream

$k > \delta'$, upstream



(2) Rough-wall velocity profiles

Assume roughness height k accounts for magnitude, form, and distribution of the roughness. Then

$$\frac{\bar{u}}{u_*} = f\left(\frac{y}{k}\right) \tag{9.42}$$

Make f in Eq. (9.42) be a logarithmic function to overlap the velocity-defect law, Eq. (9.16), which is applicable for both rough and smooth boundaries.

(9.16): $$\frac{U-\bar{u}}{u_*} = 5.6\log\left(\frac{y}{\delta}\right) + 2.5, \quad \frac{y}{\delta} < 0.15$$

i) For rough walls, in the wall region

$$\frac{\bar{u}_{rough}}{u_*} = -5.6\log\left(\frac{k}{y}\right) + C_5, \quad \frac{u_* y}{\nu} > 50 \sim 100, \quad \frac{y}{\delta} < 0.15 \qquad (9.43)$$

where C_5 = const = f (size, shape, distribution of the roughness)

ii) For smooth walls, in the wall region

(9.13): $$\frac{\bar{u}_{smooth}}{u_*} = 5.6\log\left(\frac{u_* y}{\nu}\right) + C_2, \quad \frac{u_* y}{\nu} > 30 \sim 70, \quad \frac{y}{\delta} < 0.15$$

where $C_2 = 4.9$

Subtract Eq. (9.43) from Eq. (9.13)

$$\frac{\Delta\bar{u}}{u_*} = \frac{\bar{u}_{smooth} - \bar{u}_{rough}}{u_*} = 5.6\log\left(\frac{u_* k}{\nu}\right) + C_6 \qquad (9.44)$$

→ Roughness reduces the local mean velocity $\bar{u}$ in the wall region

where C_5 and C_6 → Table 9-4

TABLE 12–4

VALUES OF CONSTANTS IN ROUGH-WALL EQUATIONS FOR THE WALL REGION

($y/\delta < 0.15$; $u_* k/\nu > 50$ to 100)

Roughness type	Source of data	C_5, Eq. (12–43)	C_6, Eq. (12–44)	C_8, Eq. (12–46)
Uniform sand grains	Nikuradse [11] (pipes)	8.2	−3.3	7.55
Wire screens	Hama [12] (plates)	6.1	−1.2	6.1
Transverse bars	Moore [7] (plates)	4.9	0	5.25

Eq. (12–43): $\bar{u}/u_* = -5.6 \log (k/y) + C_5$,

Eq. (12–44): $\Delta\bar{u}/u_* = 5.6 \log (u_* k/\nu) + C_6$,

Eq. (12–46): $1/\sqrt{c_f} = 3.96 \log (\delta/k) + C_8$.

(Constants in this table were evaluated graphically from Fig. 12–12.)

(6) Surface-resistance formulas: rough walls

Combine Eqs. (9.43) and (9.16)

$$\frac{U-\bar{u}}{u_*} = -5.6 \log\left(\frac{y}{\delta}\right) + 2.5 \quad , \qquad \frac{y}{\delta} < 0.15$$

$$+ \left| \quad \frac{\bar{u}}{u_*} = -5.6 \log\left(\frac{k}{y}\right) + C_5 \right.$$

$$\rightarrow \quad \frac{U}{u_*} = 5.6 \log\left(\frac{\delta}{k}\right) + C_7 \tag{9.45}$$

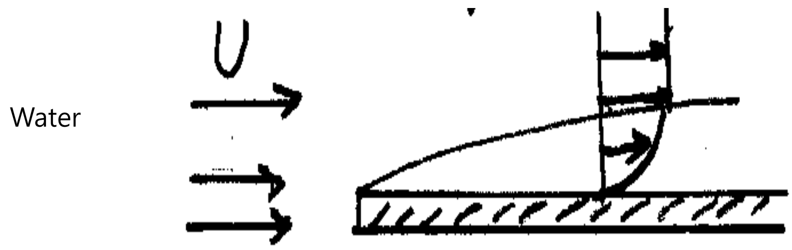
$$\frac{U}{u_*} = \sqrt{\frac{2}{c_f}} = 5.6 \log\left(\frac{\delta}{k}\right) + C_7$$

$$\therefore \frac{1}{\sqrt{c_f}} = 3.96 \log\left(\frac{\delta}{k}\right) + C_8 \tag{9.46}$$

[Ex. 9.3]

Rough wall velocity distribution and local skin friction coefficient

- Comparison of the boundary layers on a smooth plate and a plate roughened by sand grains



▪ Given: $\tau_0 = 0.485\ lb / ft^2$ on both plates

U = 10 ft / sec past the rough plate

k_s = 0.001 ft

Water temp. = 58 °F on both plates

(a) Velocity reduction Δu due to roughness

From Table 1-3:

$$\rho = 1.938\ \ slug / ft^3; \quad \nu = 1.25 \times 10^{-5} ft^2 / \sec$$

Eq. (9.18)

$$\therefore u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{0.485}{1.938}} = 0.5 ft / \sec$$

$$c_f = 2\left(\frac{u_*}{U}\right)^2 = 2\left(\frac{0.5}{10}\right)^2 = 0.005$$

$$\frac{u_* k_s}{\nu} = \frac{0.5(0.001)}{1.25 \times 10^{-5}} = 40$$

Eq. (9.44): $\frac{\Delta u}{u_*} = 5.6 \log\left(\frac{u_* k_s}{\nu}\right) - 3.3$

$$\therefore \Delta u = 0.5\{5.6 \log 40 - 3.3\} = 2.83 ft / \sec$$

(b) Velocity $\overline{u}$ on each plate at $y = 0.007 \ ft$

i) For rough plate

Eq. (9.43): $\frac{\overline{u}}{u_*} = 5.6 \log \frac{y}{k} + 8.2$

$$\therefore \overline{u} = 0.5(5.6 \log \frac{0.007}{0.001} + 8.2) = 6.47 ft / \sec$$

ii)For smooth plate,

Eq. (9.13): $\frac{\overline{u}}{u_*} = 5.6 \log \frac{u_* y}{\nu} + 4.9$

$$\frac{u_* y}{\nu} = \frac{0.5(0.007)}{1.25 \times 10^{-5}} = 280$$

$$\therefore \overline{u} = 0.5\{5.6 \log(280) + 4.9\} = 9.3 ft / \sec$$

Check $\Delta \overline{u} = 9.3 - 6.47 = 2.83 \rightarrow$ same result as (a)

(c) Boundary layer thickness δ on the rough plate

Eq. (9.46):

$$\frac{1}{\sqrt{c_f}} = 3.96 \log \frac{\delta}{k_s} + 7.55$$

$$\therefore \log \frac{\delta}{k_s} = \frac{1}{3.96} \left\{ \frac{1}{\sqrt{0.005}} - 7.55 \right\} = 1.66$$

$$\therefore \frac{\delta}{k_s} = 46 \rightarrow \delta = 0.046 ft = 0.52 in = 1.4 cm$$