Chapter 10 Turbulence Models and Their Applications

10.1 Introduction

10.1.1 The Role of Turbulence Models

(1) Why we need turbulence models?

- Turbulent flows of practical relevance

  → highly random, unsteady, three-dimensional

  → Turbulent motion (velocity distribution), heat and mass transfer processes are extremely difficult to describe and to predict theoretically.

- Solution for turbulent flows

  - Exact equations describing the turbulent motion are known.

    → **Navier-Stokes equation**

  - Numerical procedures are available to solve N-S eqs.
  - Storage capacity and speed of present-day computers are still not sufficient to allow a solution for any practically relevant turbulent flows.

  - Average the governing equations to remove turbulent fluctuations completely

    → **Reynolds equation**

    -Describe the complete **effect of turbulence** on the average motion by using **turbulence model**

(2) Turbulence

- Scale of turbulence

  - eddying motion with a wide spectrum of eddy sizes and a corresponding spectrum of fluctuation frequencies

  i) The forms of the largest eddies (low-frequency fluctuations) are determined by the boundary conditions.

  ii) The forms of the smallest eddies (highest-frequency fluctuations) are determined by the viscous forces.

- Classification of turbulence

  i) anisotropic turbulence ~ general turbulence; it varies in intensity in direction

  ii) isotropic turbulence ~ smallest turbulence; independent of direction (orientation)

$$\overline{u_i u_j} = \begin{cases} 0, \ i \neq j \\ const. \ , i = j \end{cases}$$

  iii) nonhomogeneous turbulence

  iv) homogeneous turbulence ~ statistically independent of the location

$$\overline{u_{i_a}^{\ 2}} = \overline{u_{i_b}^{\ 2}}$$

$$\overline{\left(u_i u_j\right)_a} = \overline{\left(u_i u_j\right)_b}$$

(3) Turbulence model

~ a set of equations (algebraic or differential) which determine the turbulent transport terms in the mean-flow equations and thus close the system of equations

- Simulation of Turbulence

1)Time-averaging approaches (models)

Name	No. of turbulent transport eqns	Turbulence quantities transported
Zero equation models	0	None
One equation models	1	*k* (turbulent kinetic energy)
Two equation models	2	*k*, ε (turbulent energy, dissipation rate)
Stress/flux models	6	$\overline{u_i u_j}$ components
Algebraic stress models	2	*k*, ε used to calculate $\overline{u_i u_j}$

2) Space-averaged approaches

→ Large Eddy Simulation (LES)

- Simulate the larger and more easily-resolvable scales of the motions while accepting the smaller scales will not be properly represented

10.2 Mean Flow Equation and Closure Problem

10.2.1 Reynolds averaged basic equation

- Navier-Stokes eq.

~ Eq. of motion for turbulent motion

~ describes all the details of the turbulent fluctuating motion

~ These details cannot presently be resolved by a numerical calculation procedure.

~ Engineers are not interested in obtaining these details but interested in **average quantities**.

- Definition of mean quantities by Reynolds

$$U_i = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \widetilde{U}_i \, dt \tag{10.1a}$$

$$\Phi = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \widetilde{\Phi} \, dt \tag{10.1b}$$

where $t_2 - t_1$ = averaging time Φ = scalar quantity (temp, concentration)

- Averaging time should be long compared with the time scale of the turbulent motion but small compared with that of the mean flow in transient (unsteady) problems.

Example: in stream $t_2 - t_1$ ~ 10^1 ~ 10^2 sec

- Decomposition of instantaneous values

$$\widetilde{U}_i = U_i + u_i \tag{10.2a}$$

$$\widetilde{\Phi} = \underset{\substack{\downarrow \\ \text{mean}}}{\Phi} + \underset{\substack{\searrow \\ \text{fluctuations}}}{\phi} \tag{10.2b}$$

Substitute (10.2) into time-dependent equations of continuity and N-S eqs. and average over time as indicated by (10.1) → mean flow equations

continuity: $$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \tag{10.3}$$

x-momentum: $$\frac{\partial U}{\partial t} + \frac{\partial U^2}{\partial x} + \frac{\partial (VU)}{\partial y} + \frac{\partial (WU)}{\partial z} \tag{10.4}$$

$$= -\frac{1}{\rho}\frac{\partial P}{\partial x} + fV - \frac{\partial \overline{u^2}}{\partial x} - \frac{\partial \overline{uv}}{\partial y} - \frac{\partial \overline{uw}}{\partial z}$$

y-momentum: $$\frac{\partial V}{\partial t} + \frac{\partial UV}{\partial x} + \frac{\partial (V^2)}{\partial y} + \frac{\partial (WV)}{\partial z} \tag{10.5}$$

$$= -\frac{1}{\rho}\frac{\partial P}{\partial y} - fU - \frac{\partial \overline{uv}}{\partial x} - \frac{\partial \overline{v^2}}{\partial y} - \frac{\partial \overline{vw}}{\partial z}$$

z-momentum: $$\frac{\partial W}{\partial t} + \frac{\partial (UW)}{\partial x} + \frac{\partial (VW)}{\partial y} + \frac{\partial (W^2)}{\partial z} \tag{10.6}$$

$$= -\frac{1}{\rho}\frac{\partial P}{\partial z} - g - \frac{\partial \overline{uw}}{\partial x} - \frac{\partial \overline{vw}}{\partial y} - \frac{\partial \overline{w^2}}{\partial z}$$

scalar transport: $$\frac{\partial \Phi}{\partial t} + \frac{\partial (U\Phi)}{\partial x} + \frac{\partial (V\Phi)}{\partial y} + \frac{\partial (W\Phi)}{\partial z}$$

$$= S_{\Phi} - \frac{\partial \overline{u\phi}}{\partial x} - \frac{\partial (\overline{v\phi})}{\partial y} + \frac{\partial \overline{w\phi}}{\partial z} \tag{10.7}$$

in which P = mean static pressure

f = Coriolis parameter

ρ = fluid density

S_{Φ} = volumetric source/sink term of scalar quantity Φ

◎ Eqs. (3)-(7)

→ do not form a closed set

- Non-linearity of the original N-S eq. and scalar transport eq.

$$\left(\frac{\partial u^2}{\partial x}, \frac{\partial uv}{\partial y}, \frac{\partial uw}{\partial z} \;\; ; \;\; \frac{\partial uc}{\partial x}, \frac{\partial vc}{\partial y}, \frac{\partial wc}{\partial z}, \cdots \right)$$

→ introduce unknown correlations between fluctuating velocities and between velocity and scalar fluctuations in the averaging processes

$$(\overline{u^2}, \overline{v^2}, \overline{uv}, \cdots\cdots \;\; ; \;\; \overline{u\phi} \;\; etc.,)$$

$\overline{\rho u^2} \;\; etc.$ = rate of transport of momentum

= turbulent Reynolds stresses

$\rho \overline{u\phi}$ $etc.$ = rate of transport of heat or mass

= turbulent heat or mass fluxes

- In Eqs. (3)-(7), viscous stresses and molecular heat or mass fluxes are neglected because they are much smaller than their turbulent counterparts except in the viscous sublayer very near walls.

- Eqs. (3)-(7) can be solved for average dependent variables when the turbulence correlation can be determined in some way.

  → task of the **turbulence models**

- Level of a turbulence model

~ depends on the relative importance of the turbulent transport terms

→ For the turbulent jet motion and heat and mass transport, simulation of turbulence is important.

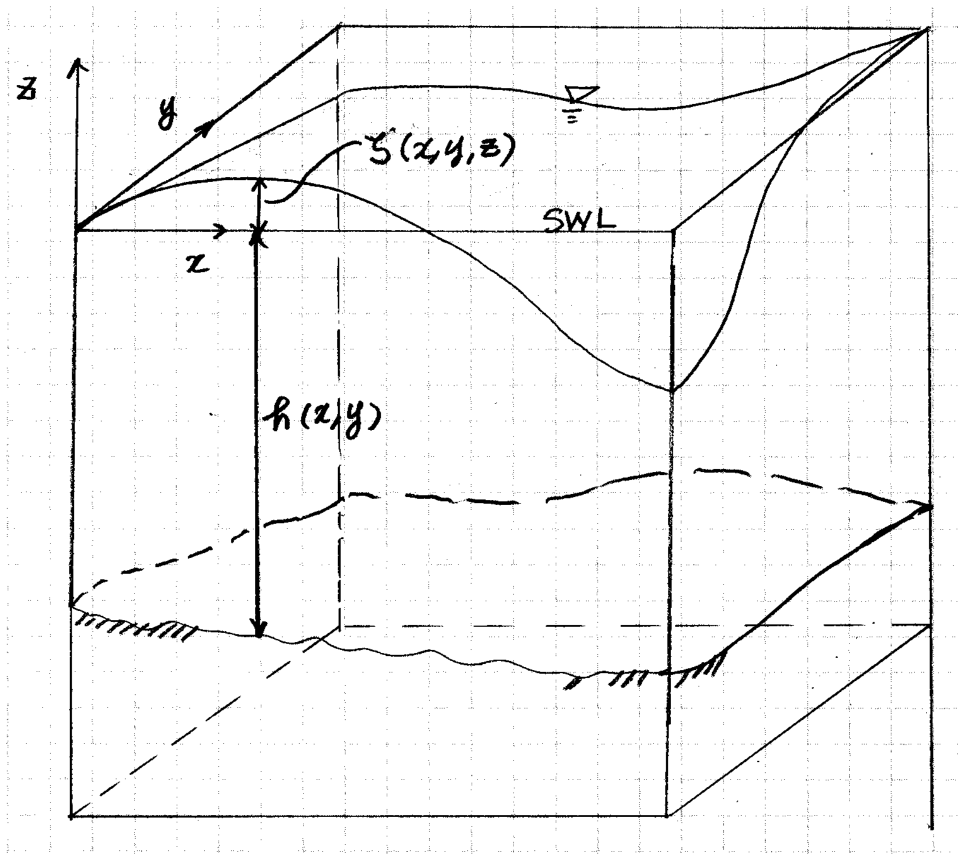
10.3 Specialized Model Equations

10.3.1 Three-Dimensional Lake Circulation and Transport Models

(1) wind-driven lake circulation / open coast transport

vertical momentum eq.: $\dfrac{\partial p}{\partial z} = -\rho g$

→ hydrostatic pressure approximation



(2) Two ways of surface approximation

1) **atmospheric pressure** at the water surface

→ calculate surface elevation ζ with kinematic boundary condition at the surface

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} + V\frac{\partial \zeta}{\partial y} - W = 0 \tag{10.8}$$

2) **rigid-lid approximation**

- assume that the surface is covered by a frictionless lid

- allows no surface deformations but permits variations of the surface pressure

→ properly accounts for the pressure-gradient terms in the momentum equations, but an error is made in the continuity equations.

→ is valid when the relative surface elevation ζ / h is small

→ suppresses surface waves and therefore permits longer time steps in a numerical solutions

→ Bennett (1974) , J. Physical Oceanography, 4(3), 400-414

Haq and Lick (1975), J. Geophysical Res, 180, 431-437

10.3.2 Two-Dimension Depth-Averaged Models

(1) **shallow water** situations

~ vertical variation of flow quantities is small

~ horizontal distribution of vertically averaged quantities is determined

$$\overline{U} = \frac{1}{H}\int_{-h}^{\zeta} U\,dz \qquad (10.9a)$$

$$\overline{\Phi} = \frac{1}{H}\int_{-k}^{\zeta} \Phi\,dz \qquad (10.9b)$$

in which H = total water depth = $h+\zeta$

h = location of bed below still water level

ζ = surface elevation

(2) Average Eqs. (3)-(7) over depth

continuity: $$\frac{\partial \zeta}{\partial t} + \frac{\partial (H\overline{U})}{\partial x} + \frac{\partial (H\overline{V})}{\partial y} = 0 \qquad (10.10)$$

x-momentum: $$\frac{\partial (H\overline{U})}{\partial t} + \frac{\partial (H\overline{U}^2)}{\partial x} + \frac{\partial (H\overline{V}\overline{U})}{\partial y} = -gH\frac{\partial \zeta}{\partial x}$$
$$+\frac{1}{\rho}\frac{\partial (H\overline{\tau_{xx}})}{\partial x} + \frac{1}{\rho}\frac{\partial (H\overline{\tau_{xy}})}{\partial y} + \frac{\tau_{sx} = \tau_{bx}}{\rho}$$
$$+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\overline{U})^2 dz + \frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(U-\overline{U})(V-\overline{V})dz$$
(10.11)

y-momentum: $$\frac{\partial(H\bar{V})}{\partial t}+\frac{\partial(H\overline{UV})}{\partial x}+\frac{\partial(H\bar{V}^2)}{\partial y}=-gH\frac{\partial\zeta}{\partial y}$$

$$+\frac{1}{\rho}\frac{\partial(H\overline{\tau_{yx}})}{\partial x}+\frac{1}{\rho}\frac{\partial(H\overline{\tau_{yy}})}{\partial y}+\frac{\tau_{sy}=\tau_{by}}{\rho}$$

$$+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})(V-\bar{V})dz+\frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(V-\bar{V})^2dz \tag{10.12}$$

scalar transport: $$\frac{\partial(H\bar{\Phi})}{\partial t}+\frac{\partial(H\bar{U}\bar{\Phi})}{\partial x}+\frac{\partial(H\bar{V}\bar{\Phi})}{\partial y}=\frac{1}{\rho}\frac{\partial(H\bar{J}_x)}{\partial x}$$

$$+\frac{1}{\rho}\frac{\partial(H\bar{J}_y)}{\partial y}+\frac{q_s}{\rho}+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})(\Phi-\bar{\Phi})dz$$

$$+\frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(V-\bar{V})(\Phi-\bar{\Phi})dz \tag{10.13}$$

where τ_{ij} = depth-averaged stress($-\rho\overline{uv}$) acting in x_i-direction on a face perpendicular to x_j; τ_b = bottom shear stress; τ_s = surface shear stress; $\bar{J}_i$ = depth-averaged flux of $\Phi(-\rho\overline{u\phi}\ or\ -\rho\overline{v\phi})$ in direction x_i; q_s = heat flux through surface

(3) Buoyancy effects cannot be represented in a depth-averaged model.

(4) dispersion terms

~ have same physical effects as turbulent terms but do not represent turbulent transport

~ due to vertical non-uniformities (variations) of various quantities

~ consequence of the depth-averaging process

~ are very important in unsteady condition

10.3.3 Two-Dimensional Vertical Plane and Width-Averaged Models

Examples:

- long-wave-affected mixing of water masses with different densities
- salt wedges in seiche
- tide-affected estuaries
- separation regions behind obstacles, sizable vertical motion

Define width-averaged quantities

$$\overline{\overline{U}} = \frac{1}{B(x,z)} \int_{y_1(x,z)}^{y_2(x,z)} U\, dy \tag{10.14a}$$

$$\overline{\overline{\Phi}} = \frac{1}{B(x,z)} \int_{y_1(x,z)}^{y_2(x,z)} \Phi\, dy \tag{10.14b}$$

in which B = channel width (local width of the flow)

(1) Models for the vertical structure are obtained by width-averaging the original three-dimensional eqs.

continuity: $$\frac{\partial}{\partial x}(B\overline{\overline{U}}) + \frac{\partial}{\partial z}(B\overline{\overline{W}}) = 0 \tag{10.15}$$

x-momentum: $$\frac{\partial}{\partial t}(B\overline{\overline{U}}) + \frac{\partial}{\partial x}(B\overline{\overline{U^2}}) + \frac{\partial}{\partial z}(B\overline{\overline{WU}}) = -gB\frac{\partial \zeta}{\partial x} - \frac{B}{\rho_0}\frac{\partial p_d}{\partial x}$$

$$
+\frac{\tau_{wx}}{\rho_0}+\frac{1}{\rho_0}\frac{\partial}{\partial x}(B\overline{\overline{\tau_{xx}}})+\frac{1}{\rho_0}\frac{\partial}{\partial z}(B\overline{\overline{\tau_{xz}}})
$$

$$
\underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{y_1}^{y_2}\rho(U-\overline{\overline{U}})^2 dy+\frac{1}{\rho_0}\frac{\partial}{\partial z}\int_{y_1}^{y_2}(U-\overline{\overline{U}})(W-\overline{\overline{W}})dy}_{dispersion}
$$

(10.16)

z-momentum: $$\frac{\partial}{\partial t}(B\overline{\overline{W}})+\frac{\partial}{\partial x}(B\overline{\overline{U}}\overline{\overline{W}})+\frac{\partial}{\partial z}(B\overline{\overline{W^2}})=-\frac{B}{\rho_0}-\frac{\partial p_d}{\partial z}$$

$$
+\frac{\overline{\overline{\rho}}-\rho_0}{\rho_0}\zeta B+\frac{\tau_{wz}}{\rho_0}+\frac{1}{\rho_0}\frac{\partial}{\partial x}(B\overline{\overline{\tau_{xz}}})+\frac{1}{\rho_0}\frac{\partial}{\partial z}(B\overline{\overline{\tau_{zz}}})
$$

$$
\underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{y_1}^{y_2}\rho(U-\overline{\overline{U}})(W-\overline{\overline{W}})dy+\frac{1}{\rho_0}\frac{\partial}{\partial z}\int_{y_1}^{y_2}\rho(W-\overline{\overline{W}})^2 dy}_{dispersion}
$$

(10.17)

scalar transport : $$\frac{\partial(B\overline{\overline{\Phi}})}{\partial t}+\frac{\partial(B\overline{\overline{U}}\overline{\overline{\Phi}})}{\partial x}+\frac{\partial(B\overline{\overline{W}}\overline{\overline{\Phi}})}{\partial z}$$

$$
=\frac{Bq_s}{\rho_0}+\frac{1}{\rho_0}\frac{\partial(B\overline{\overline{J_x}})}{\partial x}+\frac{1}{\rho_0}\frac{\partial(B\overline{\overline{J_x}})}{\partial z}
$$

$$
\underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{y_1}^{y_2}\rho(U-\overline{\overline{U}})(\Phi-\overline{\overline{\Phi}})dy+\frac{1}{\rho_0}\frac{\partial}{\partial z}\int_{y_1}^{y_2}\rho(W-\overline{\overline{W}})(\Phi-\overline{\overline{\Phi}})dy}_{dispersion}
$$

(10.18)

where ρ_0 = reference density

p_d =dynamic pressure

~ pressure due to motion and buoyancy forces

= static pressure - reference hydrostatic pressure

(2) kinematic free surface condition

$$\frac{\partial \zeta}{\partial t} + \overline{\overline{U}} \frac{\partial \zeta}{\partial x} - \overline{\overline{W}} = 0 \tag{10.19}$$

(3) dispersion terms

~ due to lateral non-uniformities of the flow quantities

(4) Further simplification

Replace z-momentum Eq. by hydrostatic pressure assumption

$$\frac{\partial p_d}{\partial z} = (\overline{\overline{\rho}} - \rho_0) g \tag{10.20}$$

Replace $\dfrac{\partial p_d}{\partial x}$ in x-momentum Eq. as

$$\frac{\partial p_d}{\partial x} = g \frac{\partial}{\partial x} \int_z^{\zeta} (\overline{\overline{\rho}} - \rho_0) dz \tag{10.21}$$

Integrate continuity Eq. (10.15) over the depth and combine with Eq. (10.19)

$$\frac{\partial \zeta}{\partial t} + \frac{1}{B_s} \frac{\partial}{\partial x} \int_{-h}^{\zeta} B\overline{\overline{U}} dz = 0 \tag{10.22}$$

10.4 Turbulence-Closure Models

○ Turbulence model

~ represent the turbulence correlations $\overline{u^2}$, $\overline{uv}$, $\overline{u\phi}$ etc. in the mean-flow equations in a way which close these equations by relating the turbulence correlations to the averaged dependent variables

○ **Hypotheses** must be introduced for the behavior of these correlations which are based on empirical informations.

→ Turbulence models always contain empirical constants and functions.

→ Turbulence models do not describe the details of the turbulent fluctuations but only the average effects of these terms on the mean quantities.

○ **Parameterization of turbulence**

~ core of turbulence modeling

~ local state of turbulence and turbulence correlations are assumed to be characterized by only a few parameters.

→ Two important parameters are **velocity scale** and **length scale**.

○ Three steps of parameterization

1) choose parameters

2) establish **relation between turbulence correlation and parameters**

3) determine **distribution of these parameters** over the flow field.

10.4.1 Eddy Viscosity (Diffusivity) Concept

(1) Boussinesq (1877) introduced **eddy viscosity,** ν_t assuming that, in analogy to the viscous stresses in laminar flow, the turbulent stresses are proportional to the mean velocity gradients.

$$-\overline{u_i u_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \qquad (10.23)$$

where k = turbulent kinetic energy per unit mass; δ_{ij} = Kronecker delta, = 1 for $i = j$ and = 0 for $i \neq j$

$$k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}) \qquad (10.24)$$

(2) **Eddy viscosity**, ν_t

~ not a fluid property, and depends on state of the turbulence

~ may vary considerably over the flow field

~ is proportional to a velocity scale $\hat{V}$, and a length scale L

$$\nu_t \propto \hat{V} L \qquad (10.25)$$

(3) Turbulent heat or mass transport

~ turbulent heat or mass transport is assumed to be proportional to the gradient of the transported quantity:

$$-\overline{u_i \phi} = \Gamma_t \frac{\partial \Phi}{\partial x_i} \qquad (10.26)$$

where Γ_t = **eddy (turbulent) diffusivity** of heat or mass

(4) Turbulent Prandtl (heat) or Schmidt number (mass), σ_t

$$\sigma_t = \frac{\nu_t}{\Gamma_t} \tag{10.27}$$

10.4.2 Types of Turbulence Models

~ Classification of turbulence model according to the number of transport equations used for turbulence parameters.

(1) Zero-Equation Models

~ do not involve transport equations for turbulence quantities

1) Constant eddy viscosity (diffusivity) model

2) Mixing-length model

3) Free-shear-layer model

(2) One-Equation Models

1) κ-equation model

2) Bradshaw et al.'s model

(3) Two-Equation Models

1) $\kappa-\varepsilon$ model

2) $\kappa-l$ model

(4) Turbulent Stress/Flux-Equation Models

1) Reynolds-stress equations

2) Algebraic stress/flux models

10.4.3 Zero-Equation Models

(1) Constant Eddy Viscosity (Diffusivity)

~ simplest turbulence model

~ used in depth-averaged model in which horizontal momentum transport is not , heat and mass transfer cannot be separated from dispersion effect due to vertical non-uniformity

~ use constant eddy viscosity (diffusivity) over the whole flow field

~ appropriate only for far-field situations where the turbulence is governed by the natural water body and not by local man-made disturbances such as discharge jets

- When turbulences are mainly bed-generated, as in the channel flow

$$\Gamma = \alpha \, du^*$$

(2) Turbulent (eddy) diffusion coeff ε (Fischer et al., 1979)

$\varepsilon_v = 0.067 \; du^*$ for uniform, straight channel

$\varepsilon_v = 0.15 \; du^*$ for uniform, straight channel

$\varepsilon_v = 0.60 \; du^*$ for meandering rivers

(3) Dispersion coeff. K

$K = 5.93 \; du^*$, due to vertical variation (Elder, 1959)

$K = 150 - 300 \; du^*$, due to transverse variation (Fischer et al., 1979)

$K = 5.71 - 11.5 \; du^*$, flow zone only separating recirculating regions in the channel (Seo and Maxwell, 1992)

(4) Mixing-Length Model

1) Near-field problems involving discharge jets, wakes, and the vicinity of banks and structures

- assumption of a constant eddy viscosity is not sufficient

- distribution of ν_t over the flow field should be determined

2) Prandtl's mixing-length hypothesis (Prandtl, 1925)

Prandtl assumed that eddy viscosity ν_t is proportional to a mean representation of the fluctuating velocity $\hat{V}$ and a mixing-length l_m.

$$\nu_t \propto \hat{V} l_m \tag{A}$$

Considering shear layers with only one significant turbulent stress ($\overline{uv}$) and velocity gradient $\partial U / \partial z$, he postulated as

$$\hat{V} = l_m \frac{\partial U}{\partial z} \tag{B}$$

Combine (A) and (B)

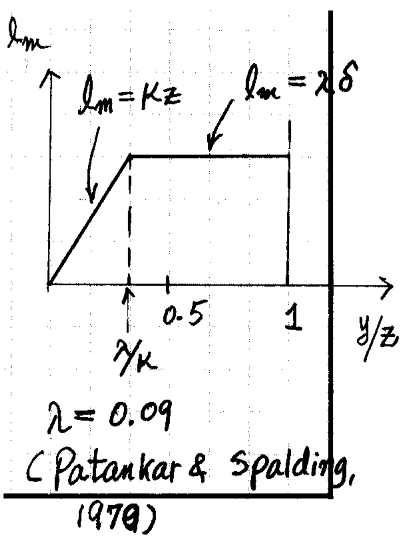
$$\nu_t = l_m^{\,2} \left| \frac{\partial U}{\partial z} \right| \tag{10.28}$$

i) Boundary-layer flows along walls:

① Near-wall region

$$l_m = \kappa z$$

in which κ = von Korman constant ($\approx$0.4)



② Outer region

$$l_m \propto \delta$$

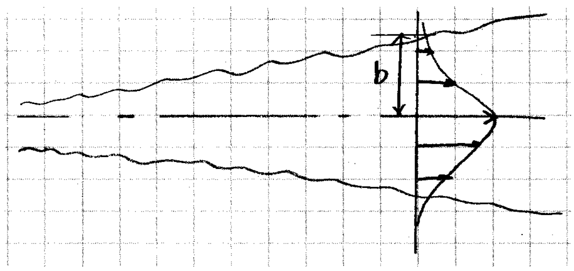
in which δ= boundary layer thickness

ii) Free shear flows … mixing layers, jets, wakes

$$l_m \propto b$$

where b = local shear-layer width

	Plane mixing layer	Plane jet	Round jet	Radial jet	Plane wake
$\frac{l_m}{b}$	0.07	0.09	0.075	0.125	0.16



(5) Effect of Buoyancy

~ Buoyancy forces acting on stratified fluid layers have a strong effect on the vertical turbulent transport of momentum and heat or mass

→ eddy viscosity relations for vertical transport must be modified by introducing a Richardson number correction

Define gradient local Richardson number R_i as

$$R_i = -\frac{g}{\rho}\frac{\partial \rho / \partial z}{\left(\dfrac{\partial U}{\partial z}\right)^2} \tag{10.29}$$

~ ratio of gravity to inertial forces

(6) Munk-Anderson (1948) relation

$$\nu_{tz} = (\nu_{tz})_0 (1 + 10R_i)^{-0.5} \tag{10.30a}$$

$$\Gamma_{tz} = \left(\Gamma_{tz}\right)_0 \left(1 + 3.3R_i\right)^{-1.5} \tag{10.30b}$$

$$\frac{l_m}{l_{m_0}} = 1 - \beta_1 R_i \ , \ R_i > 0 \quad \text{(stable stratification)} \tag{10.31a}$$

$$l_m / l_{m_0} = \left(1 - \beta_2 R_i\right)^{-1/4} \ , \ R_i < 0 \ \text{(unstable stratification)} \tag{10.31b}$$

in which $\beta_1 \approx 7$, $\beta_2 \approx 14$; subscript 0 refers to values during unstratified conditions($R_i = 0$).

(7) Mixing length model for general flows

$$\nu_t = l_m^{\ 2}\left[\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) + \frac{\partial U_i}{\partial x_j}\right]^{\frac{1}{2}}$$

~ very difficult to specify l_m in complex flow

~ l_m in general duct flows (Buleev, 1962)

$$l_m = \kappa \frac{1}{\pi} \int_D \frac{1}{\delta} d\Omega$$

in which δ= distance of the point at which l_m is to be determined from wall along direction Ω; D = integration domain (= cross section of the duct)

(8) **Limitation** of Mixing length model

1) Mixing-length distribution is empirical and rather problem-dependent.

→ model lacks universality

2) Close link of eddy viscosity (diffusivity) with velocity gradient, i.e. $\nu_t = 0$ when $\frac{\partial U_i}{\partial x_i} = 0$, implies that this model is based on the assumption of local equilibrium of turbulence.

<→ turbulence is locally dissipated by viscous action at the same rate as it is produced by shear.>

→ Transport and history effects are neglected (turbulence generation at previous times).

→ This model is not suitable when these effects are important as is the case in rapidly developing flows, recirculating flows and also in unsteady flows.

(9) Heat and mass transfer

The mixing-length hypothesis is also used in heat and mass transfer calculations.

$$\Gamma_t = \frac{\nu_t}{\sigma_t} = \frac{1}{\sigma_t} l_m^{\ 2} \left| \frac{\partial U}{\partial z} \right|$$

where σ_t = turbulent Prandtl (Schmidt) number

= 0.9 in near-wall flows

0.5 in plane jets and mixing layers

0.7 in round jets

1) Buoyancy effect on σ_t

→ Munk-Anderson formula

$$\frac{\sigma_t}{\sigma_{t_0}} = \frac{(1+3.33R_i)^{1.5}}{(1+10R_i)^{0.5}}$$

2) Shortcomings of mixing-length model

i) ν_t and Γ_t vanish whenever the velocity gradient is zero.

For pipes and channels,

ν_t ⓐ $\text{centerline} \approx 0.8(\nu_t)_{\max}$ in reality

$$\leftrightarrow \frac{\partial U}{\partial z} = 0 \text{ ⓐ centerline} \rightarrow \nu_t = \Gamma_t = 0$$

ii) The mixing-length model implies that turbulence is in a state of local equilibrium.

→ Thus, this model is unable to account for transport of turbulence quantities.

3) Prandtl's free-shear-layer model

Prandtl (1942) proposed a simpler model applicable only to free shear layers. (mixing layers, jets, wakes)

$$l_m \propto \delta$$

$$\hat{V} \propto \left|U_{\max} - U_{\min}\right|$$

$$\therefore \nu_t = C\delta\left|U_{\max} - U_{\min}\right|$$

	Plane mixing layers	Plane jet	Round jet	Radial jet	Plane wake
C	0.01	0.014	0.01	0.019	0.026

10.4.4 One-Equation Models

- This model accounts for transport or history effects (time-rate change) of turbulence quantities by solving differential transport equations.

- One-equation models determine the fluctuating velocity scale from a transport equation rather than the direct link between this scale and the mean velocity gradients.

(1) K-Equation Model

1) Velocity fluctuations are to be characterized by $\sqrt{k}$ where k is the **turbulent kinetic energy per unit mass** defined as

$$k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2})$$

2) Eddy viscosity ν_t

$$\nu_t \propto \hat{V} L$$

$$\nu_t = c_\mu ' \sqrt{k} L$$ … Kolmogorov-Prandtl equation

in which $c_\mu '$ = empirical constant.

3) Turbulent Kinetic Energy (TKE) equation

~ Exact form can be derived from the Navier-Stokes equation.

~ Exact equation contains certain higher-order correlations which must be approximated by models in order to achieve a closure of the equations.

~ For high Reynolds number, this equation reads

$$\underbrace{\frac{\partial k}{\partial t}}_{\text{rate of change of } k} + \underbrace{U_i \frac{\partial k}{\partial x_i}}_{\text{advective transport due to mean motion}} = \underbrace{-\frac{\partial}{\partial x_i}\left[\overline{u_i\left(\frac{u_j u_j}{2} + \frac{p}{\rho}\right)}\right]}_{\text{diffusive transport due to velocity and pressure fluctuations}} - \underbrace{\overline{u_i u_j}\frac{\partial U_i}{\partial x_j}}_{\text{production by turbulent shear stress} = P}$$

rate of change of k

advective transport due to mean motion

diffusive transport due to velocity and pressure fluctuations

production by turbulent shear stress = P

$$\underbrace{-\beta g_i \overline{u\phi}}_{\text{buoyant production / destruction} = G} - \underbrace{\nu \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}}_{\text{viscous dissipation into heat} = \varepsilon}$$

buoyant production / destruction = G due to buoyancy force

viscous dissipation into heat = ε

P = transfer of kinetic energy from the mean motion to the turbulent motion (large scale eddies)

G = exchange between the turbulent kinetic energy k and potential energy

→negative for stable stratification (k is reduced, turbulence is damped while potential energy of the system increases)

→positive for unstable stratification (k is produced at the expense of the potential energy)

ε = transfers kinetic energy into internal energy of the fluid

= negative (sink)

4) **Energy cascade**

→ Kinetic energy extracted from mean motion is first fed into large scale turbulent motion.

→ This energy is passed on to smaller and smaller eddies by vortex stretching (vortex trail, vortex street) until viscous force become active and dissipate the energy.

5) **Local isotropy**

- Large-scale turbulences are anisotropic, whereas small-scale turbulences are isotropic.
- Because of interaction between large-scale turbulent motion and mean flow, the large-scale turbulent motion depends strongly on the boundary conditions.

→ large-scale turbulence = anisotropic

- During the energy cascade process, energy is passed on to smaller eddies by vortex stretching.

→ The direction sensitivity is diminished.

→ small-scale turbulence = isotropic

6) Modeled form of the k-equation

~ The exact k-equation contains new unknown correlations.

→ To obtain a closed set of equations, model assumptions must be introduced for these terms.

i) diffusion term

~ In analogy to the diffusion expression for the scalar quantity ϕ, the diffusion flux of k is assumed proportional to the gradient k.

$$\overline{-u_i\left(\frac{u_j u_j}{2}+\frac{p}{\rho}\right)}=\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}$$

in which σ_k = empirical diffusion constant.

ii) Reynolds stress

$$-\overline{u_i u_j}=\nu_t\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)$$

iii) heat (mass) flux

$$-\overline{u_i\phi}=\frac{\nu_t}{\sigma_t}\frac{\partial \Phi}{\partial x_i}$$

in which σ_t = turbulent Prandtl or Schmidt number

iv) viscous dissipation

$$\varepsilon = c_D \frac{k^{3/2}}{L}$$

in which c_D = empirical constant.

Substitute i) ~ iv) into exact *k*-equation

$$\frac{\partial k}{\partial t} + U_i \frac{\partial k}{\partial x_i}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + \nu_t\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} + \beta g_i \frac{\nu_t}{\sigma t}\frac{\partial \Phi}{\partial x_i} - c_D \frac{k^{3/2}}{L}$$

~ This model is restricted to high Reynolds number flows;

$$c_\mu{}' c_D \approx 0.08 \;\; and \;\; \sigma_k \approx 1$$

~ For low Reynolds number flows, a viscous diffusion term should be accounted for and empirical constants are functions of the turbulent Reynolds number, $\mathrm{Re}_f = \sqrt{k}L / \nu$.

7) Special case of **local equilibrium**

~ rate of change, advection and diffusion are negligible.

→ production of *k* is equal to dissipation.

For non-buoyant shear layers

$$\nu_t \left(\frac{\partial U}{\partial z}\right)^2 = c_D \frac{k^{3/2}}{L}$$

Substitute this into Kolmogorov-Prandtl expression

$$\nu_t = c_u' \sqrt{k} L$$

$$\nu_t = \left(\frac{c_\mu'^3}{c_D} \right)^{\frac{1}{2}} L^2 \left| \frac{\partial U}{\partial z} \right|$$

Set l_m = mixing length = $\left(\frac{c'\mu^3}{c_D} \right)^{\frac{1}{4}} L$,

then $\nu_t = l_m^{\ 2} \left| \frac{\partial U}{\partial z} \right|$ → mixing-length model

8) Length-scale determination

~ Because the length scale *L* appears both in Kolmogorov-Prandtl equation and in dissipation term of the *k*-equation, this must be specified empirically.

~ In most models, *L* is determined from simple empirical relations similar to those for the mixing length l_m.

→ Launder and Spalding (1972) for estuary

Smith and Takhar (1977) for open-channel

9) Bobyleva et al. (1965) 's length scale formula

~ similar to von Kaman's formula

$$L = \kappa \frac{\Psi}{\partial \Psi / \partial z}$$

where κ = von Karman's const.

$$\Psi = \frac{k^{\frac{1}{2}}}{L} = \text{ turbulence parameter}$$

~ applicable to flows where turbulent transport is mainly in vertical direction

~ When the turbulence is in local equilibrium in the shear layer,

$$\frac{k^{\frac{1}{2}}}{L} \propto \frac{\partial u}{\partial z}$$

$$\therefore L = l_m = \kappa \left| \frac{\partial u / \partial z}{\partial^2 u / \partial z^2} \right|$$

→ von Karman's formula

(2) Bradshaw et al.'s Model ($\overline{uv}$ -equation model)

~ This model does not employ the eddy-viscosity concept.

~ It solves a transport equation for the shear stress $\overline{uv}$.

For wall boundary layers (2-D)

$$\frac{\overline{uv}}{k} = a_1 \approx const \approx 0.3 \text{ (experiment)}$$

Convert k-equation into $\overline{uv}$ -equation for steady flows

$$\underbrace{U \frac{\partial \frac{\overline{uv}}{a_1}}{\partial x} + V \frac{\partial \frac{\overline{uv}}{a_1}}{\partial y}}_{\text{advection}} = \underbrace{-\frac{\partial}{\partial y}\left[G\overline{uv}(\overline{uv}_{\max})^{\frac{1}{2}} \right]}_{\text{diffusion}} - \underbrace{\overline{uv}\frac{\partial U}{\partial y}}_{\text{production}} - \underbrace{\frac{\overline{uv}^{3/2}}{L}}_{\text{dissipation}}$$

in which $G=\left(\dfrac{\overline{uv}_{\max}}{U_\infty^2}\right)^{\frac{1}{2}} f_1\left(\dfrac{y}{\delta}\right)$

$$L=f_2\left(\frac{y}{\delta}\right)\delta \ldots \text{ empirical}$$

1) Heat and Mass Transfer

i) Find eddy viscosity (ν_t) or the shear stress ($\overline{u_i u_j}$) using one-equation model.

ii) Use gradient-diffusion concept to calculate heat and mass transfer

$$\Gamma_t=\frac{\nu_t}{\sigma_t}$$

$$-\overline{u_i\phi}=\Gamma_t\frac{\partial\Phi}{\partial x_i}$$

iii) Solve scalar transport equation

2) Assessment of one-equation models

i) Avdvantages :

① One-equation models can account for advective and diffusive transport and for history effects on the turbulent velocity scale.

→ It is superior to the mixing-length model when these effects are important (Examples: nonequilibrium shear layers with rapidly changing free stream conditions, abrupt changes in the boundary conditions, shear layers in estuary with velocity reversal, heat and mass exchange in area with vanishing velocity gradients)

② Buoyancy term appears automatically in the *k*-equation model.

ii) Disadvantages;

①The application is restricted to shear-layer situation not applicable to more complex flows.

②The empirical formulas for calculating length scale in general flows so far been tested insufficiently.

10.4.5 Two-Equation Models

○ Length scale *L* is also subject to transport processes in a similar manner to the kinetic energy *k*.

Examples:

① Eddies generated by a grid are advected downstream so that their size at any station depends on their initial size. → history effect

② Dissipation destroys the small eddies and thus effectively increases the eddy size.

③ Vortex stretching connected with the energy cascade reduces the eddy size.

→ The balance of all these processes can be expressed in a transport model for L.

(1) Length scale equations

$Z = k^m l^n$ ← general form

1) Energy dissipation rate:

$\varepsilon \propto k^{3/2} / L$ by Chou (1945), Davidov (1961), Jones & Launder (1972)

$\varepsilon \propto kL$ by Rotta (1968)

2) Frequency: $k^{\frac{1}{2}}L$ by Kolmogorov (1941)

3) Turbulence vorticity: k / L^2 by Spalding (1971), Saffman (1970)

(2) **Length scale transport equation**

~ exact form can be derived from Navier-Stokes eq.

$$\underbrace{\frac{\partial Z}{\partial t}}_{\text{rate of change}} + \underbrace{U_i \frac{\partial Z}{\partial x_i}}_{\text{advection}} = \underbrace{\frac{\partial}{\partial x_i}\left(\frac{\sqrt{k}L}{\sigma_z}\frac{\partial Z}{\partial x_i}\right)}_{\text{diffusion}} + \underbrace{c_{z1}\frac{Z}{k}P}_{\text{production}} - \underbrace{c_{z2}Z\frac{\sqrt{k}}{L} + S}_{\text{destruction}}$$

where σ_z, σ_{z1}, σ_{z2} = empirical constants

P = production of kinetic energy $\left(= -\overline{u_i u_j}\frac{\partial u_i}{\partial x_j}\right)$

S = secondary source term which is important near walls

(3) The $k-\varepsilon$ model

- $Z=\varepsilon$ model works better near walls than other equations.
- The ε-equation does not require a near-wall correction term S.

○ At high Reynolds numbers where local isotropy prevails,

$$\varepsilon \propto \nu \overline{\left(\frac{\partial u_i}{\partial x_j} \right)^2}$$

where ν = molecular kinematic viscosity

- Exact ε-equation can be derived from N-S equations for **fluctuating vorticity**.

→ rate of change + advection = diffusion + generation of vorticity due to vortex stretching + viscous destruction of vorticity

→ need model assumptions for diffusion, generation, and destruction terms (diffusion is modelled with gradient assumption).

(4) **Modeled ε-equation**

$$\underbrace{\frac{\partial \varepsilon}{\partial t}}_{\text{rate of change}} + \underbrace{U_i \frac{\partial \varepsilon}{\partial x_i}}_{\text{advection}} = \underbrace{\frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_i} \right)}_{\text{diffusion}} + \underbrace{c_{1\varepsilon} \frac{\varepsilon}{k}(P + c_{3\varepsilon} G) - c_{2\varepsilon} \frac{\varepsilon^2}{k}}_{\text{generation-destruction}}$$

where P = stress production of kinetic energy k;

G = buoyancy production of kinetic energy k

(5) **Complete $k-\varepsilon$ model**

$$\nu_t = c_\mu{}' \sqrt{k} L \qquad \text{(a)}$$

$$\varepsilon = c_D \frac{k^{3/2}}{L} \rightarrow L = c_D \frac{k^{3/2}}{\varepsilon} \qquad \text{(b)}$$

Substitute (b) into (a)

$$\nu_t = c_\mu{}'\sqrt{k}\,\frac{c_D k^{3/2}}{\varepsilon} = c_\mu{}' c_D \frac{k^2}{\varepsilon}$$

Set $c_\mu = c_\mu{}' c_D$

$$\nu_t = c_\mu \frac{k^2}{\varepsilon} \qquad ①$$

$$\Gamma_t = \frac{\nu_t}{\sigma_t} \qquad ②$$

$$k-eq: \frac{\partial k}{\partial t} + U_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + \nu_t\left(\frac{\partial U_i}{\partial x_i} + \frac{\partial U_j}{\partial x_j}\right)\frac{\partial U_i}{\partial x_j} + \beta g_i \frac{\nu_t}{\sigma_t}\frac{\partial \Phi}{\partial x_i} - \varepsilon \qquad ③$$

$$\varepsilon-eq: \frac{\partial \varepsilon}{\partial t} + U_i \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + c_{1\varepsilon}\frac{\varepsilon}{k}\left(P + c_{3\varepsilon} G\right) - c_{2\varepsilon}\frac{\varepsilon^2}{k}$$

④