1.4.§ Modeling

~~Try all the examples in textbook!~~

Ex. 5 Heating problem (Newton's law of cooling)

$$\boxed{\begin{array}{l} t=0:\ T=66°F \\ t=2hr:\ T=63°F \\ \quad\downarrow \\ t=10hr:\ T=? \end{array}} \xrightarrow[Q]{} T_A = 32°F = \text{const.}$$

physical information → modeling

$$mc\frac{dT}{dt} = -Q . \qquad Q = k^*(T-T_A) \text{ : Newton's} - //$$

$$(Q, k > 0)$$

$$\frac{dT}{dt} = -\frac{k^*}{mc}(T-T_A)$$

$$\frac{dT}{dt} = -k(T-T_A).$$

$$\frac{dT}{T-T_A} = -k\,dt$$

$$\ln(T-T_A) = -kt + c^*$$

$$T = ce^{-kt} + T_A$$

2 unknowns, 2 conditions

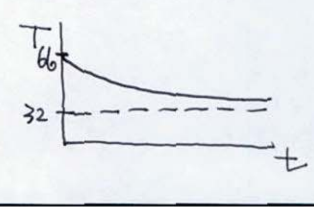
$$T(t=0) = c + T_A = 66 . \qquad c = 34$$

$$T(t=2) = 34e^{-2k} + 32 = 63$$

$$34e^{-2k} = 31 . \qquad e^{-2k} = \frac{31}{34}$$

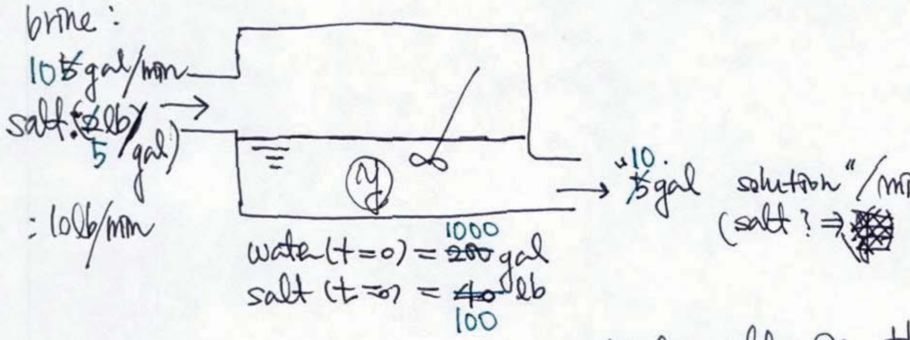
$$-2k = \ln\left(\frac{31}{34}\right). \qquad k = -\frac{1}{2}\ln\left(\frac{31}{34}\right) = 0.046$$

$$T = 34e^{-0.046t} + 32 \rightsquigarrow$$



$$T(t=10) = 53.5\ °F$$

Ex.3 Mixing problem



$y(t) = ?$: amount of salt in the tank

Essential physics

$$\boxed{\frac{dy}{dt} = \underbrace{\overbrace{(\text{influx of salt})}^{Q_{in}}}_{\text{salt inflow rate}} - \underbrace{\overbrace{(\text{outflux of salt})}^{Q_{out}}}_{\text{salt outflow rate}}}$$

$Q_{in} = \cancel{10}\ 50$ lb/min

Q_{out} ? $\quad \cancel{200}\ 1000 \text{ gal} - y$

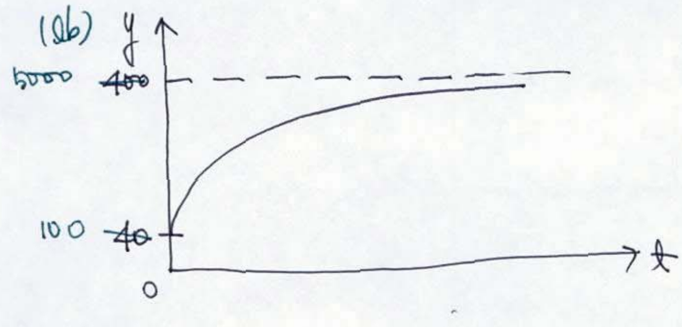
$\qquad\qquad 10\ \cancel{5} \text{ gal} - Q_{out}$.

$$\therefore Q_{out} = \frac{10\ \cancel{5}}{\cancel{200}\ 1000} y = \cancel{0.025}\ 0.01\, y$$

$$\therefore y' = \cancel{10}\ 50 - \cancel{0.025}\ 0.01\, y \qquad y(0) = 100.$$

$$\cancel{\frac{dy}{dt} = 10 - 0.025y}$$

$$\rightsquigarrow y = \cancel{400}\ 5000 - \cancel{360}\ 4900\, e^{-0.\cancel{025}01\, t} \quad (\text{lb})$$



1.4. Exact Differential Equations (ODEs), Integrating Factors

- If a function $u(x,y)$ has continuous partial derivatives, its (total) differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

From this, if $u(x,y) = c$
then $du = 0$.

Ex). If $u = x + x^2y^3 = c$. ← solution

$$du = (1+2xy^3)\,dx + (3x^2y^2)\,dy = 0$$

$$\boxed{y' = \frac{dy}{dx} = -\frac{1+2xy^3}{3x^2y^2}} \quad \text{ODE}$$

- Consider

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \quad \cdots (1)$$

This is called an exact diff. eq.
if "$M\,dx + N\,dy$" is exact,
that is,

$$M\,dx + N\,dy \Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du$$

$(1) \Rightarrow du = 0.$

gen. sol. : $u(x,y) = c$.

- Problem solving technique for Eq (1)

※ → ①

※② find $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M, \quad \frac{\partial u}{\partial y} = N.$$

① First check whether (1) is exact DE

How?

remembering " $\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$, $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$ "

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \iff \boxed{\text{(1) is exact D.E.}}$$

$u = \int M dx + k(y)$. $\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int M dx + k' = N(x, y)$

OR $u = \int N dy + l(x)$. $\quad$ "$k = $ ___ "

L4 ↓

Ex. 1. $\underbrace{(x^3 + 3xy^2)}_{= M} dx + \underbrace{(3x^2y + y^3)}_{= N} dy = 0$

diff. from 9th ed

① Test for exactness

$\frac{\partial M}{\partial y} = 6xy$. $\quad \frac{\partial N}{\partial x} = 6xy \quad \therefore$ exact.

② $\frac{\partial u}{\partial x} = M = x^3 + 3xy^2$. $\quad \frac{\partial u}{\partial y} = N$

$\therefore u = \int (x^3 + 3xy^2) dx + k(y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + k(y)$

$\frac{\partial u}{\partial y} = 3x^2y + k'(y) = 3x^2y + y^3$. $\quad k'(y) = y^3$.

$$k(y) = \frac{1}{4}y^4 + C$$

$$\therefore \ u(x,y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + C$$

ans.) $u = \text{const}$: $\frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 = c.$

Ex. 3.

$$\underbrace{-y}_{\substack{\| \\ M \\ (=\frac{\partial u}{\partial x}?)}} dx + \underbrace{x}_{\substack{\| \\ N \\ (=\frac{\partial u}{\partial y}?)}} dy = 0$$

① Test: $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$. $\therefore$ not exact!

Then how?

(1st) Separation: $x\,dy = y\,dx$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln|y| = \ln|x| + c$$

(i) $x>0,\ y>0$: $\ln y = \ln x + c$. $\quad y = cx$. $\quad c>0$

(ii) $x>0,\ y<0$: $\ln(-y) = \ln x + c$. $\quad -y = \overset{*}{c}x \rightarrow$ "$y = cx$". $\quad c<0$.

$\vdots$

$$\therefore \ y = cx$$

Alternatively, make the form exact!

§ Reduction to Exact form (Integrating factors)

ex) multiply $1/x^2$

$$\underbrace{-\frac{y}{x^2}}_{\substack{\| \\ M}} dx + \underbrace{\frac{1}{x}}_{\substack{\| \\ N}} dy = 0 \quad \Leftarrow \quad d\left(\frac{y}{x}\right) = 0.$$

Test : $\frac{\partial M}{\partial y} = -\frac{1}{x^2}$. $\frac{\partial N}{\partial x} = -\frac{1}{x^2}$. $\therefore$ exact

Solving $\frac{y}{x} = c$

* $P(x,y)\,dx + Q(x,y)\,dy = 0$: non exact

$\Downarrow \times F(x,y)$ integrating factor

$F P\,dx + F Q\,dy = 0$: exact

$(\frac{1}{x^2})$ others $(\frac{1}{y^2})$ $(\frac{1}{x^2+y^2})$ too.

§ How to find Integrating factors

If $FP\,dx + FQ\,dy = 0$ is exact,

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$$

Then what about solving the DE to get F? $F(x,y)$

$$F_y P + F P_y + F_x Q + F Q_y = 0$$

too difficult to solve !

→ Golden rule :

Try to find an integrating factor "depending only on one variable".

i.e. let $F = F(x)$

or $F = F(y)$.

if $F = F(x)$

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$$

$$\Rightarrow \quad FP_y = F'Q + FQ_x$$

$$Q\frac{dF}{dx} = F\frac{\partial P}{\partial y} - F\frac{\partial Q}{\partial x}$$

$$\frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$\underbrace{\frac{dF}{F}}_{x \text{ only}} = \underbrace{\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}_{=R} dx$$

if R depends on "x" only

$$\Rightarrow \quad \frac{dF}{F} = R(x)\,dx$$

not necessary

$$\ln F(x) = \int R(x)\,dx + C$$

$$\boxed{F(x) = \exp\left[\int R(x)\,dx\right].}$$

if $F = F(y)$. similar result

Ex. 5. diff. from 9th ed.

$$\underbrace{2\sin y^2}_{P}\,dx + \underbrace{xy\cos y^2}_{Q}\,dy = 0. \quad y(2) = \sqrt{\frac{\pi}{2}}$$

① check $\frac{\partial P}{\partial y} = 4y\cos y^2$, $\frac{\partial Q}{\partial x} = y\cos y^2$: not exact

② $FP\,dx + FQ\,dy = 0.$ find F to make it exact.

$$\boxed{\begin{aligned} F(x) &= \exp\left[\int R(x)\,dx\right] \\ R(x) &= \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \end{aligned}}$$

$$R(x) = \frac{1}{xy\cos y^2} \cdot 3y\cos y^2 = \frac{3}{x}$$

o.K. !

i.f. $F(x) = \exp\left[\int \frac{3}{x}dx\right] = \exp[3\ln x] = e^{\ln x^3} = x^3$

③ $FP\,dx + FQ\,dy = 0$

$$\Rightarrow \underbrace{2x^3\sin y^2}_{=\frac{\partial u}{\partial x}} dx + \underbrace{x^4 y\cos y^2}_{=\frac{\partial u}{\partial y}} dy = 0$$

: $du = 0$

$u = \text{const.}$

$$\frac{\partial u}{\partial x} = 2x^3\sin y^2$$

$$u = \frac{1}{2}x^4\sin y^2 + k(y)$$

$$\frac{\partial u}{\partial y} = x^4 y\cos y^2 + k'(y) = x^4 y\cos y^2.$$

: $k' = 0$

$k = c.$

$$\therefore u = \frac{1}{2}x^4\sin y^2 + c^* = \text{const.}$$

$$\therefore \frac{1}{2}x^4\sin y^2 = c \qquad : \text{g.s.}$$

④ P.S $x = 2,\ y = \sqrt{\frac{\pi}{2}}$

$$\frac{1}{2}\cdot 16\cdot \sin\frac{\pi}{2} = 8 = c$$

$$\therefore x^4\sin y^2 = 16. \qquad : \text{P.S.}$$