1.5 Linear Differential Eq (ODEs): Bernoulli eq.

- Linear first-order diff. eq.

$$y' + p(x)y = r(x)$$

: linear in y & y'

$$\begin{cases} r(x) \equiv 0 & : \text{homogeneous} \\ r(x) \not\equiv 0 & : \text{nonhomogeneous} \end{cases}$$

- General solution of LFODE

(1) homogeneous

$$y' + p(x)y = 0$$

$$\frac{dy}{dx} = -p(x)y \qquad \ln|y| = -\int p(x)dx + c^*$$

i) $y > 0$: $\ln y = -\int p(x)dx + c^*$

$$y = c \exp\left[-\int p(x)\,dx\right] \qquad c > 0$$

ii) $y < 0$: $\ln(-y) = -\int p(x)\,dx + c^*$

$$-y = \tilde{c} \exp\left[-\int p(x)\,dx\right]$$

$$y = c \exp\left[-\int p(x)\,dx\right] \qquad c < 0$$

* what if $c = 0$? $y(x) \equiv 0$ for all x

: trivial solution

(2) Nonhomogeneous

$$\frac{dy}{dx} + p(x)y = r(x)$$

$$\frac{dy}{dx} = r - py$$

$$(py-r)\,dx + dy = 0$$

[

$P\,dx + Q\,dy = 0$: $P = py - r$, $Q = 1$

$\xrightarrow{\text{integrating factor } F}$ $\underbrace{FP}_{=\frac{\partial u}{\partial x}} dx + \underbrace{FQ}_{=\frac{\partial u}{\partial y}} dy = du = 0.$

* always exist ? *

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$$

$$\frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = p(x)$$

$F = e^{\int p\,dx}$ $\qquad$ $F(x)$ indeed !

Following the standard method,

$$\frac{\partial u}{\partial x} = (py-r)e^{\int p\,dx}$$

$$u = \int (py-r)e^{\int p\,dx}dx + k(y)$$

$$= y\int pe^{\int p\,dx}dx - \int re^{\int p\,dx}dx + k(y)$$

$$\frac{\partial u}{\partial y} = \int \underbrace{p}_{u'}\underbrace{e^{\int p\,dx}}_{u}dx + k'(y) = e^{\int p\,dx}$$

$k'(y) = 0.$ $\qquad$ $k(y) = c$

$\int p\,dx\, e^{\int p\,dx} - \int pe^{\int p\,dx}\int p\,dx\,dx$

$he^h - \int pe^h \cdot h\,dx = e^h$

$$y\int pe^{\int p\,dx}dx - \int re^{\int p\,dx}dx = c$$

$$y\int pe^{\int p\,dx}dx = \int re^{\int p\,dx}dx + c$$

$\frac{\partial u}{\partial y} = e^{\int p\,dx}$

$u = \int e^{\int p\,dx}dy + k(x)$

$\frac{\partial u}{\partial x} = \int pe$

$e^h y - \int e^h r\,dx + c = 0$

$$\frac{\partial u}{\partial x} = FP = (py-r)e^{\int p\,dx}, \quad \frac{\partial u}{\partial y} = FQ = e^{\int p\,dx}$$

let $h = \int p\,dx$

$$\frac{\partial u}{\partial x} = (py-r)e^{h}, \quad \frac{\partial u}{\partial y} = e^{h}$$

$\hookrightarrow u = e^{h}y + k(x)$

$$\frac{\partial u}{\partial x} = h'e^{h}y + k'(x)$$

$$= pe^{h}y + k'(x) = pe^{h}y - re^{h}$$

$\therefore k'(x) = -re^{h}$

$k(x) = -\int re^{h}\,dx + \cancel{c}$

$$u = \underbrace{e^{h}y - \int re^{h}dx = c}$$

$$\boxed{y = e^{-h}\left[\int e^{h}r\,dx + c\right] \qquad h = \int p(x)\,dx}$$

Ex1. $y' - y = e^{2x}$ (1st-order! linear !)

$y' + p(x)y = r(x)$

$p = -1, \ r = e^{2x}$.

· procedure $P\,dx + Q\,dy = 0 \xrightarrow{\text{i.f. } F} FP\,dx + FQ\,dy = du = 0.$ $u = \text{const}$

Using the general solution $h = \int -1\,dx = -x$

$$y = e^{x}\left[\underbrace{\int e^{-x}e^{2x}dx}_{= e^{x}} + c\right] = e^{2x} + ce^{x}$$

(In the ~~textbook~~,
interesting rule: $e^{\int p\,dx}(y'+py) = \left(y e^{\int p\,dx}\right)'$) ee

* Reduction to Linear Form. (Bernoulli Eq)

Consider $y' + p(x)y = g(x)y^a$ (a : real)
Bernoulli eq.

if $a=0$: $y' + p(x)y = g(x)$
$a=1$: $y' + [p(x) - g(x)]y = 0$.
} linear

otherwise, "nonlinear"

then how? ↳ linear

eliminate y

set $u(x) = [y(x)]^{1-a}$

$u' = (1-a)y^{-a}y'$

$y' = gy^a - py$

$u' = (1-a)y^{-a}(gy^a - py)$

$= (1-a)(g - p\underbrace{y^{1-a}}_{=u})$

$u' = (1-a)(g - pu)$

$\boxed{u' + (1-a)pu = (1-a)g}$: linear.

Ex. ~~4~~. $y' - Ay = -By^2$. : Logistic eq

$y' + py = gy^a$. $a=2 \Rightarrow u = y^{-1}$

$$u' = -y^{-2}y' = -y^{-2}(-By^2 + Ay) = B - A\underbrace{y^{-1}}_{u}$$

$u' + Au = B.$

$u' + pu = r$: standard linear form

recalling that $u = e^{-h}\left[\int e^{h} r\,dx + c\right]$, $h = \int p\,dx$

here $p = A$, $r = B$

$h = Ax$

$$u = e^{-Ax}\left[\int e^{Ax}\cdot B\,dx + c\right]$$

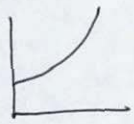
$$= e^{-Ax}\left(B\frac{1}{A}e^{Ax} + c\right)$$

$$y^{-1} = u = \frac{B}{A} + ce^{-Ax}$$

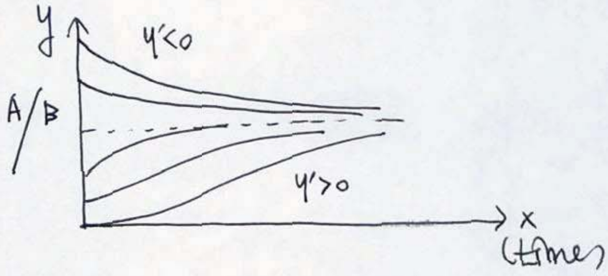
$$\therefore\ y = \frac{1}{\left(\frac{B}{A}\right) + ce^{-Ax}}$$: logistic law

§ Population dynamics $\Rightarrow$ useful in modeling human/animal populations

if $B=0$. $y' = Ay$ $\quad y = ce^{Ax}$



$-By^2$: braking term



* basic problem in nonlinear dynamics

logistic equation : $y' = f(y)$

t does not occur explicitly

$\Rightarrow$ autonomous ODE

· equilibrium solutions

$y' = 0 \Rightarrow f(y) = 0.$

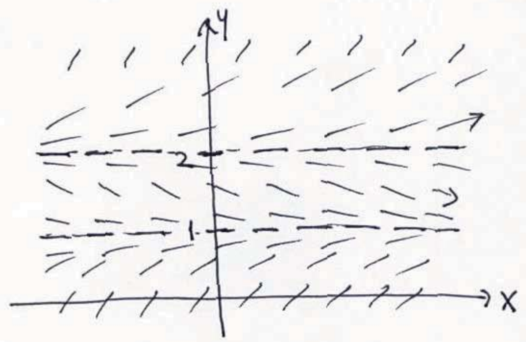
$y = \text{const.}$ ∴ zeros. : critical pts

Ex. 5. $y' = \frac{dy}{dt} = (y-1)(y-2)$

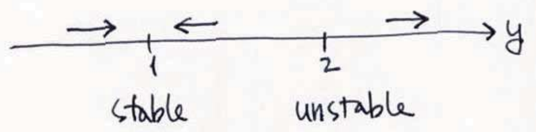
$= y^2 - 3y + 2$

$y' + 3y = y^2 + 2$ ← Bernoulli eq.

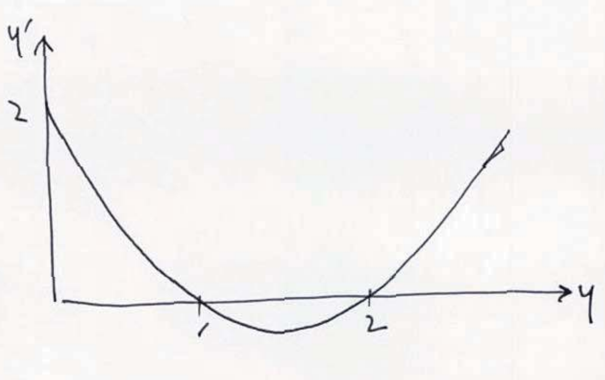
(A) Direction field



(B) Phase line

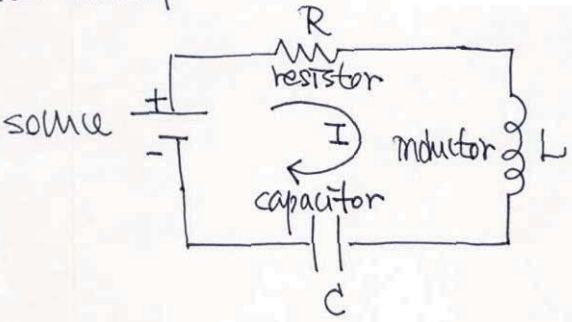


(C)



1.7. Modeling : Electric circuits

- RLC circuit

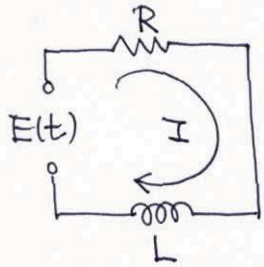


element	voltage drop
R	IR
L	$L\frac{dI}{dt}$
C	$\frac{1}{C}\int_{t_0}^{t} I\,dt$

· Kirchhoff's voltage law

in closed loop, $\sum_i (\text{voltage drop})_i = 0$.

Ex.1. RL- circuit



$I(t) = ?$

$$E(t) = \begin{cases} \text{const} = E_0 \\ \text{periodic} = E_0 \sin \omega t \end{cases}$$

i) modeling : $E - IR - L\frac{dI}{dt} = 0$.

$$L\frac{dI}{dt} + RI = E(t)$$

ii) solution

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E(t)}{L} \qquad \because \text{ Linear, first-order DE.}$$

Using $y' + py = r$. $\quad y_{gs} = e^{-h}\left[\int e^{h} r\,dx + c\right]$

$$h = \int p\,dx$$

$$I = e^{-\alpha t}\left[\int e^{\alpha t} \frac{E(t)}{L}\,dt + c\right].$$

$$\alpha = \frac{R}{L}$$

* special case 1

$$L\frac{dI}{dt} + RI = E_0$$

$$L\frac{dI}{dt} = -RI + E_0$$

$$\frac{dI}{-RI+E_0} = \frac{dt}{L}$$

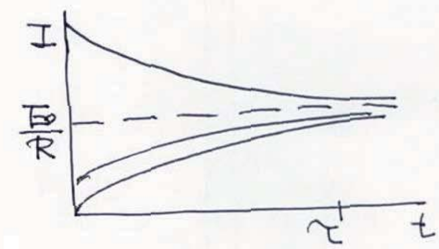
$$\frac{-R\,dI}{-RI+E_0} = -\frac{R}{L}dt$$

$$\ln|-RI+E_0| = -\frac{R}{L}t + c^*$$

$$-RI+E_0 = \tilde{c}e^{-\frac{R}{L}t}$$

$$RI = E_0 - \tilde{c}e^{-\frac{R}{L}t}$$

$$I = \frac{E_0}{R} + ce^{-\frac{R}{L}t}$$

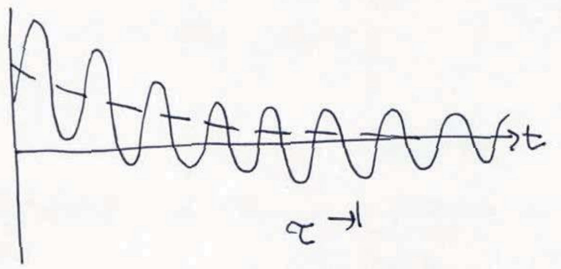


* Special case 2

$$L\frac{dI}{dt} + RI = E_0 \sin\omega t$$

G.S. $$I = ce^{-(R/L)t} + \frac{E_0}{\sqrt{R^2+\omega^2L^2}}\sin(\omega t - \delta)$$

$$\delta = \tan^{-1}\frac{\omega L}{R}$$

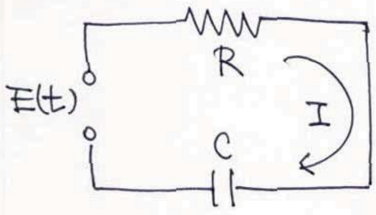


time constant

$$\tau = \frac{L}{R}$$

X

Ex. 2 RC- circuit



$I(t) = ?$

i) modeling

$$RI + \frac{1}{C}\int I\,dt = E(t)$$

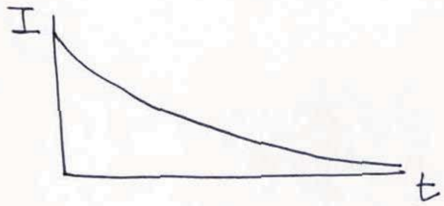
$$R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}.$$ linear first-order DE

ii) solution

G.S. $I = e^{-t/(RC)}\left[\frac{1}{R}\int e^{t/(RC)}\frac{dE}{dt}\,dt + c\right)$

* special case 1. $E = E_0$

$$I = ce^{-t/(RC)} = e^{-t/\tau_c} \qquad \tau_c = RC$$



* case 2 $E = E_0 \sin\omega t$

$$\frac{dE}{dt} = \omega E_0 \cos\omega t$$

$$I = ce^{-t/(RC)} + \frac{\omega E_0 C}{\sqrt{1+(\omega RC)^2}}\sin(\omega t - \delta)$$

$$\delta = \tan^{-1}\left(\frac{-1}{\omega RC}\right)$$