2.2 ~~Second-Order~~ Homogeneous ~~Equations~~ linear ODEs with Constant Coefficients

$$y'' + ay' + by = 0.$$

a, b : const.

try $y = e^{\lambda x}$

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

$$\boxed{\lambda^2 + a\lambda + b = 0}$$

: characteristic eq

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$$

$$\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$$

$y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$: solutions

Discriminant : $a^2 - 4b$

i) $a^2 - 4b > 0$: two real roots

ii) $a^2 - 4b = 0$: a real double root

iii) $a^2 - 4b < 0$: complex conjugate roots

§ Case I. $a^2 - 4b > 0$: two distinct real roots λ_1 & λ_2

* basis of solutions : $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$

lin. indep.

Gen. sol.: $\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$

Ex. 1 $\quad y''-y=0 \quad : \quad y=e^{\lambda x}$

$\lambda^2-1=0 \qquad \lambda=1,\ -1$

$y=c_1e^{\lambda x}+c_2e^{-\lambda x} = c_1e^{x}+c_2e^{-x}$

Ex. 2 $\quad y''+y'-2y=0. \qquad \underbrace{y(0)=4, \quad y'(0)=-5}_{\text{IC's}} \quad :$ IVP

try $\quad y=e^{\lambda x}$

$\lambda^2+\lambda-2=0 \qquad (\lambda+2)(\lambda-1)=0. \qquad \lambda=-2,\ 1$

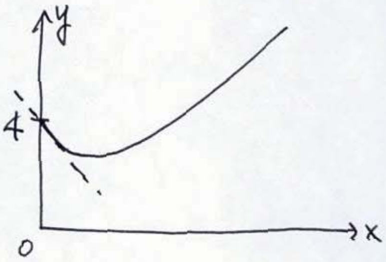
g.s. $y=c_1e^{-2x}+c_2e^{1x} \qquad y'=-2c_1e^{-2x}+c_2e^{x}$

$y(0)=c_1+c_2=4$

$y'(0)=-2c_1+c_2=-5$ $\rfloor -$

$3c_1=9. \qquad c_1=3. \quad c_2=1.$

$\therefore$ p.s. $y=3e^{-2x}+e^{x}$



§ CASE II. $\quad a^2-4b=0.$ real double root

$y''+ay'+by=0 \quad : \quad y=e^{\lambda x} \ :\ \text{“}\lambda=-\frac{a}{2}\text{''}$

basis $\left[\begin{array}{l} y_1=e^{-\frac{a}{2}x} \\ y_2=\ ? \end{array}\right.$ $\rightarrow$ reduction of order

$y_2=uy_1$

$u''y_1+u'(2y_1'+ay_1)=0.$

$$u''e^{-\frac{a}{2}x} + u'\left(\underbrace{2\cdot\left(-\frac{a}{2}\right)e^{-\frac{a}{2}x} + ae^{-\frac{a}{2}x}}_{=0.}\right) = 0$$

$\therefore\ u''e^{-\frac{a}{2}x} = 0.$

$u'' = 0$: $u' = c_1.$ $\quad u = c_1x + c_2$

" $y_2 = xy_1$ "

λ: double root $\Rightarrow$ basis: $e^{-\frac{a}{2}x}$, $xe^{-\frac{a}{2}x}$

$e^{\lambda x}$, $xe^{\lambda x}$

G.S.: $y = c_1e^{\lambda x} + c_2xe^{\lambda x}$

$= (c_1 + c_2x)e^{\lambda x}$: only for double root

Ex. 3* $\quad y'' + 8y' + 16y = 0$

$\lambda^2 + 8\lambda + 16 = 0 \qquad (\lambda + 4)^2 = 0.$

$\lambda = -4$: double root

basis: e^{-4x}, xe^{-4x}

$y = (c_1 + c_2x)e^{-4x}$

Ex. 4* $\quad y'' - 4y' + 4y = 0. \qquad y(0) = 3, \quad y'(0) = 1$

$y = e^{\lambda x}. \qquad (\lambda^2 - 4\lambda + 4) = 0. \qquad (\lambda - 2)^2 = 0.$

$\lambda = 2 \qquad y = (c_1 + c_2x)e^{2x}. \quad \Leftarrow$ I.C.

$c_1 = 3 \qquad c_2 = -5$

$\therefore\ y = (3 - 5x)e^{2x}.$

~~2.3.~~ § case Ⅲ: ~~of~~ Complex roots. $(a^2-4b<0)$

~~Ex. 1.~~ e.g. $y''+y=0$

$y=e^{\lambda x}$

$\lambda^2+1=0. \quad \lambda=i,\ -i$

$y=C_1e^{ix}+C_2e^{-ix}$

Euler formula

$$\begin{cases} e^{ix}=\cos x+i\sin x \\ e^{-ix}=\cos x-i\sin x \end{cases}$$

$$\begin{cases} \cos x=\frac{1}{2}(e^{ix}+e^{-ix}) \\ \sin x=\frac{1}{2i}(e^{ix}-e^{-ix}) \end{cases}$$

$$\begin{aligned} y &= C_1(\cos x+i\sin x)+C_2(\cos x-i\sin x) \\ &= (C_1+C_2)\cos x+i(C_1-C_2)\sin x \\ &= A\cos x+B\sin x \end{aligned}$$

Complex exponential function

$z=\gamma+it$

$$\begin{aligned} e^{z} &= e^{\gamma+it}=e^{\gamma}e^{it} \\ &= e^{\gamma}(\cos t+i\sin t) \\ &= \underbrace{e^{\gamma}\cos t}_{\text{"real"}}+i\underbrace{e^{\gamma}\sin t}_{\text{"real"}} \end{aligned}$$

~~CASE III.~~ Complex roots. $(a^2-4b<0)$

$\lambda=-\frac{1}{2}a\pm\frac{1}{2}\sqrt{a^2-4b}$

$\lambda_1=-\frac{1}{2}a+\frac{1}{2}\sqrt{a^2-4b}=-\frac{1}{2}a+\frac{i}{2}\sqrt{4b-a^2}=-\frac{1}{2}a+i\sqrt{b-\frac{a^2}{4}}$

$\omega=\sqrt{b^2-\frac{a^2}{4}}$: real

$$\lambda_1 = -\frac{a}{2} + i\omega$$
$$\lambda_2 = -\frac{a}{2} - i\omega$$

basis
$$\left[\begin{array}{l} e^{\lambda_1 x} = e^{(-\frac{a}{2}+i\omega)x} = e^{-\frac{a}{2}x}(\cos\omega x + i\sin\omega x) \\ e^{\lambda_2 x} = e^{(-\frac{a}{2}-i\omega)x} = e^{-\frac{a}{2}x}(\cos\omega x - i\sin\omega x) \end{array}\right.$$

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$
$$= e^{-\frac{a}{2}x}\left[(C_1+C_2)\cos\omega x + i(C_1-C_2)\sin\omega x\right]$$

$$\boxed{y = e^{-\frac{a}{2}x}(A\cos\omega x + B\sin\omega x)}$$

Ex. 5* $\quad y'' + 0.2y' + 4.01y = 0. \qquad y(0)=0, \quad y'(0)=2$

Try $e^{\lambda x}$

$$\lambda^2 + 0.2\lambda + 4.01 = 0$$
$$\lambda = \frac{-0.2 \pm \sqrt{0.2^2 - 4\times 4.01}}{2} = -0.1 \pm 2i$$

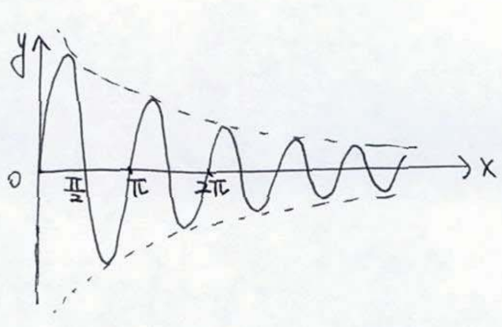
G.S. $\quad y = e^{-0.1x}(A\cos 2x + B\sin 2x)$

$$y' = -0.1e^{-0.1x}(A\cos 2x + B\sin 2x) + e^{-0.1x}(-2A\sin 2x + 2B\cos 2x)$$

I.C. $\quad y(0) = A = 0.$

$$y'(0) = -0.1\underset{0}{\cancel{A}} + 2B = 2.$$
$$B = 1.$$

$$\therefore\ y = e^{-0.1x}\sin 2x$$



: damped vibration

Ex. 6. Complex roots

$$y'' + \omega^2 y = 0. \qquad \omega \neq 0. \quad \text{const.}$$

$$y = A\cos\omega x + B\sin\omega x.$$

SUMMARY

$$y'' + ay' + by = 0. \qquad a.b = \text{const.} \qquad e^{\lambda x}$$

Case	Roots	Basis	Gen. sol.
$a^2-4b>0$	distinct real λ_1, λ_2	$e^{\lambda_1 x}$, $e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
$a^2-4b=0$	real double $\lambda = -\frac{a}{2}$	$e^{-\frac{a}{2}x}$, $xe^{-\frac{a}{2}x}$	$y = (c_1 + c_2 x)e^{-\frac{a}{2}x}$
$a^2-4b<0$	complex conjugate $\lambda_1 = -\frac{a}{2} + i\omega$, $\lambda_2 = -\frac{a}{2} - i\omega$	$e^{-\frac{a}{2}x}\cos\omega x$, $e^{-\frac{a}{2}x}\sin\omega x$	$y = e^{-\frac{a}{2}x}(A\cos\omega x + B\sin\omega x)$

↓ * Boundary Value Problems
postpone

$$y'' + ay' + by = 0.$$

P_1 interval P_2 $\qquad y(P_1) = k_1, \quad y(P_2) = k_2$: boundary conditions

$\Rightarrow$ BVP.

~~Ex. 4~~ e.g. $\qquad y'' + y = 0. \qquad y(0) = 3. \qquad y(\pi) = -3$

$$y = c_1\cos x + c_2\sin x$$

$$\text{B.C.} \Rightarrow y(0) = c_1 = 3. \qquad c_1 = 3.$$

$$\cancel{y' = -c_1 \sin}$$

$$y(\pi) = c_1(-1) + c_2 \cdot 0 = -3$$

$$-3 + 0 \cdot c_2 = -3. \qquad c_2 : \text{arbitrary const.}$$

$y = 3\cos x + c_2 \cdot \sin x \quad \rightarrow \quad$ not unique.

- Uniqueness of BVP

PS 23 (p. 81)

$$y'' + ay' + by = 0. \qquad P_1 \;\text{———}\; P_2$$

If $y = c_1 y_1 + c_2 y_2$

y_1, y_2: linearly independent

y_2 is such that $\quad y_2(P_1) = 0$

$\qquad\qquad\qquad\qquad y_2(P_2) = 0.$

B.C. $\quad y(P_1) = c_1\, y_1(P_1) + c_2\, \underbrace{y_2(P_1)}_{0} = k_1$

$$c_1 = \frac{k_1}{y_1(P_1)}$$

- $$y(P_2) = \frac{k_1}{y_1(P_1)}\, y_1(P_2) + c_2 \overbrace{y_2(P_2)}^{0} = k_2$$

no info. for c_2

$$\therefore\; y = \frac{k_1}{y_1(P_1)}\, y_1 + c_2 \cdot y_2 \qquad : \text{not unique.}$$

BVP's solution unique	$\rightleftarrows$	no solution satisfies $y(P_1) = y(P_2) = 0$.

$p \qquad \rightleftarrows \qquad q$

$\sim q \rightarrow \sim p \quad$ OR $\quad p \rightarrow q$

if BVP's sol. not unique. --> $y(p_1)=y(p_2)=0$ such y exists

$Y_1 \neq Y_2$ 

But we know $Y_1(P_1)=k_1$ $\quad Y_1(P_2)=k_2$

$Y_2(P_1)=k_1$ $\quad Y_2(P_2)=k_2$

$Y_1(P_1)-Y_2(P_1)=0.$ $\quad Y_1(P_2)-Y_2(P_2)=0.$

G.S. to diff-eq

$y=c_1Y_1+c_2Y_2$, $\quad c_1=1,\ c_2=-1$

$y=Y_1-Y_2$ also satisfies $y''+ay'+by=0$

$y(p_1)=0$ and $y(p_2)=0$: such y exists!

therefore, $\sim p \to \sim q$ proven.

$\equiv q \to p$

$\therefore p \rightleftarrows q$