(2) y_p: $y_p = Ce^{0.5x} + K\cos 4x + M\sin 4x$

$y_p' = 0.5Ce^{0.5x} - 4K\sin 4x + 4M\cos 4x$

$y_p'' = 0.25Ce^{0.5x} - 16K\cos 4x - 16M\sin 4x$

$(0.25+1+5)Ce^{0.5x} + (-16K+8M+5K)\cos 4x + (-16M-8K+5M)\sin 4x$

$= 1.25e^{0.5x} + 40\cos 4x - 55\sin 4x$

$\therefore\ C = 0.2,\quad K = 0,\quad M = 5$

$y = y_h + y_p$

$= e^{-x}(A\cos 2x + B\sin 2x) + 0.2e^{0.5x} + 5\sin 4x$

→ p.56

2.10. Solution by Variation of Parameters.

· previous method (undetermined coeff.)

$y'' + \underbrace{ay' + by}_{① \text{ const.}} = \underbrace{r(x)}_{② \text{ special fn}}$

· General: $y'' + p(x)y' + q(x)y = r(x)$

p. q. r : continuous on I

$$\boxed{y_p(x) = -y_1\int \frac{y_2 r}{W}dx + y_2\int \frac{y_1 r}{W}dx}$$

$\begin{cases} y_1, y_2 : \text{basis of solutions of homo. eq} \\ W\ (\text{Wronskian}) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \end{cases}$

§ Derivation

homogeneous eq $\quad y'' + p(x)y' + q(x)y = 0$

$\rightarrow$ G.S: $\quad y_h(x) = c_1 y_1(x) + c_2 y_2(x)$

variation of para. $\rightarrow$

$$\boxed{y_p(x) = u(x)\,y_1 + v(x)\,y_2}$$

How to obtain u & v?

- condition 1) $\quad y_p'' + p y_p' + q y_p = r \quad \cdots (1)$

- condition 2) $\quad$ arbitrary $\quad$ – what?

consider $\quad y_p' = \underline{u'y_1} + uy_1' + \underline{v'y_2} + vy_2'$

$$u'y_1 + v'y_2 = 0. \quad \cdots (2)$$

$$y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$$

plugging into (1):

$$(u'y_1' + uy_1'' + v'y_2' + vy_2'') + p\,(uy_1' + vy_2') + q\,(uy_1 + vy_2) = r$$

$$u(\underbrace{y_1'' + py_1' + qy_1}_{=0}) + v(\underbrace{y_2'' + py_2' + qy_2}_{=0})$$
$$+ u'y_1' + v'y_2' = r$$

condition (1) $\rightarrow \quad u'y_1' + v'y_2' = r \quad \cdots (3)$

Solving the simultaneous eqs (2) & (3): to find u' and v':

$$u'(y_1y_2' - y_2y_1') = -y_2 r \quad : \quad u'W = -y_2 r$$
$$v'(y_1y_2' - y_2y_1') = y_1 r \quad : \quad v'W = y_1 r$$

$$u' = -\frac{y_2 r}{W}, \qquad v' = \frac{y_1 r}{W}$$

$$u = -\int \frac{y_2 r}{W}\,dx, \qquad v = \int \frac{y_1 r}{W}\,dx \quad //$$

Ex. 1. $\quad y'' + y = \sec x$

(1) homogeneous eq $\quad y_h'' + y_h = 0$

$$y_h = c_1 \cos x + c_2 \sin x$$

(2) particular ~~eq.~~ sol. y_p : $\sec x \Rightarrow$ special fn X

$$y_p = -y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx$$

$$y_1 = \cos x, \quad y_2 = \sin x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

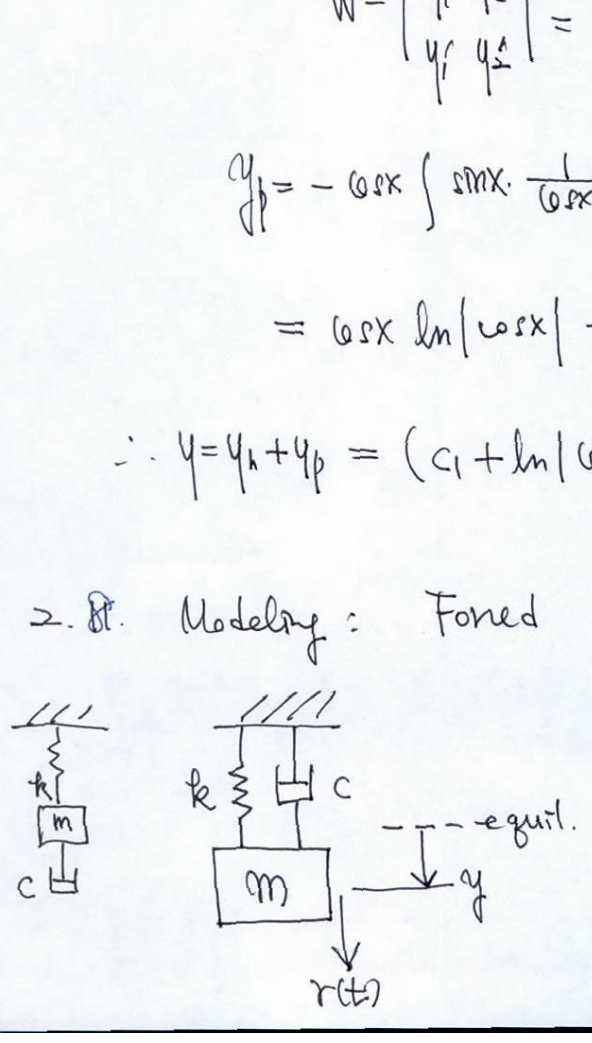
$$y_p = -\cos x \int \sin x \cdot \frac{1}{\cos x}\,dx + \sin x \int \cos x \cdot \frac{1}{\cos x}\,dx$$

$$= \cos x \ln|\cos x| + x \sin x \qquad (\text{integ. const} = 0)$$

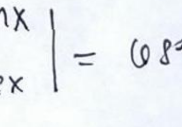
$$\therefore\ y = y_h + y_p = (c_1 + \ln|\cos x|)\cos x + (c_2 + x)\sin x$$

→ p. 61

2. 8. Modeling : Forced Oscillations



FBD.



$$\downarrow \Sigma F = -ky - cy' + r(t) = my''$$

$$\boxed{my'' + cy' + ky = r(t)}$$

We consider $\quad r(t) = F_0 \cos\omega t \quad (F_0 > 0,\ \omega > 0)$

$$\boxed{my'' + cy' + ky = F_0 \cos\omega t.}$$

(1) homogen. eq : $\quad my_h'' + cy_h' + ky_h = 0$

Try $\quad y_h = e^{\lambda t}$

$$m\lambda^2 + c\lambda + k = 0.$$

(2) y_p: $\quad my_p'' + cy_p' + ky_p = F_0 \cos\omega t.$

$$y_p = a\cos\omega t + b\sin\omega t$$

$$y_p' = \omega(-a\sin\omega t + b\cos\omega t)$$

$$y_p'' = -\omega^2(a\cos\omega t + b\sin\omega t)$$

$$[(k-m\omega^2)a + \omega c b]\cos\omega t + [-\omega c a + (k-m\omega^2)b]\sin\omega t = F_0\cos\omega t.$$

$$\left[\begin{array}{l} (k-m\omega^2)a + \omega c b = F_0 \\ -\omega c\ \ a + (k-m\omega^2)b = 0. \end{array}\right]$$

$$a = F_0 \frac{k-m\omega^2}{(k-m\omega^2)^2 + \omega^2c^2}, \qquad b = F_0 \frac{\omega c}{(k-m\omega^2)^2 + \omega^2 c^2}.$$

$$((k-m\omega^2)^2 + \omega^2c^2 \neq 0).$$

let $\omega_0 = \sqrt{\frac{k}{m}}$.

$$a = F_0 \frac{m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2 + \omega^2c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2-\omega^2)^2 + \omega^2c^2}.$$

$$y = y_h + y_p.$$

CASE 1. Undamped forced Oscillations. $(c=0)$.

(i) assume $\omega \neq \omega_0$.

$$a = F_0 \frac{1}{m(\omega_0^2-\omega^2)}. \qquad b=0.$$

$$y_p = a\cos\omega t.$$

$$y_h : \ m\lambda^2 + k = 0. \qquad \lambda = \pm i\sqrt{\frac{k}{m}}$$

$$y_h = C\cos(\omega_0 t - \delta)$$

$$\therefore y(t) = \underline{C\cos(\omega_0 t - \delta)} + \frac{F_0}{m(\omega_0^2-\omega^2)}\underline{\cos\omega t}$$

(1) superposition

(2) as $\omega \to \omega_0$. $|y| - \uparrow$: resonance

(ii) $\omega = \omega_0$, $c=0$: resonance

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega_0 t$$

$$y_h = \overset{c_1}{a}\cos\omega_0 t + \overset{c_2}{b}\sin\omega_0 t \quad //$$

$$y_p = t(a\cos\omega_0 t + b\sin\omega_0 t)$$

$$y_p' = a\cos\omega_0 t + b\sin\omega_0 t + t\omega_0(-a\sin\omega_0 t + b\cos\omega_0 t)$$

$$y_p'' = \omega_0(-a\sin\omega_0 t + b\cos\omega_0 t) + \omega_0(-a\sin\omega_0 t + b\cos\omega_0 t) + \omega_0^2 t(-a\cos\omega_0 t - b\sin\omega_0 t)$$

$$\cos\omega_0 t(2b\omega_0) + \sin\omega_0 t(-2a\omega_0)$$

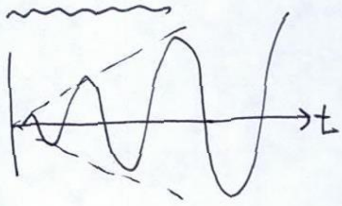
$$+ t\cos\omega_0 t(-a\omega_0^2 + a\omega_0^2) + t\sin\omega_0 t(-b\omega_0^2 + b\omega_0^2)$$

$$= \frac{F_0}{m}\cos\omega_0 t$$

$$a = 0, \quad b = \frac{F_0}{2m\omega_0}.$$

$$y_p = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

$$\therefore y = C\cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

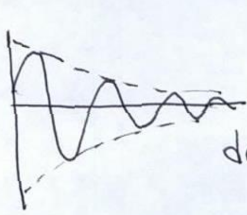


CASE 2 Damped Forced Oscillations $c > 0$.

$$my'' + cy' + ky = F_0 \cos \omega t.$$

$$y_h = e^{-\alpha t}(A\cos\omega^* t + B\sin\omega^* t)$$

$$\alpha = \frac{c}{2m}. \qquad \omega^* = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}.$$



damped out.

$$y_p = a\cos\omega t + b\sin\omega t. \qquad a, b : \text{found earlier}$$

$$= C^* \cos(\omega t - \eta)$$

$$C^* = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}}.$$

$$\tan \eta = \frac{b}{a} = \frac{\omega c}{m(\omega_0^2 - \omega^2)} = \frac{\omega c}{m\omega_0^2\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]}$$

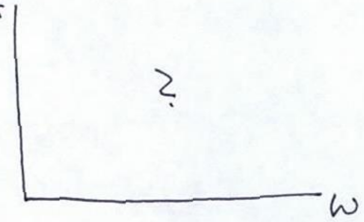
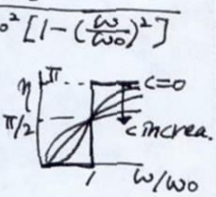


Fig. 61.



$$\frac{dC^*}{d\omega} = \frac{-F_0 \frac{1}{2}\left[m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2\right]^{-1/2} \cdot \left[m^2 \cdot 2(\omega_0^2 - \omega^2)(-2\omega) + 2c^2\omega\right]}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

$$\frac{dC^*}{d\omega} = 0 \quad : \quad -2m^2(\omega_0^2 - \omega^2) + c^2 = 0.$$

$$2m^2\omega^2 = 2m^2\omega_0^2 - c^2$$

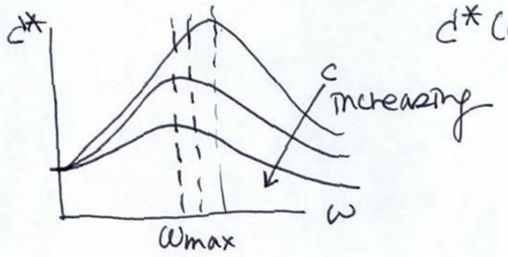
$$\omega^2 = \omega_0^2 - \frac{c^2}{2m^2}.$$

i) $\omega_0^2 > \frac{c^2}{2m^2}$: $\quad c^2 < 2m^2\underbrace{\omega_0^2}_{=\frac{k}{m}}$. $\quad c < \sqrt{2}\, m\omega_0$

$$c^2 < 2mk$$

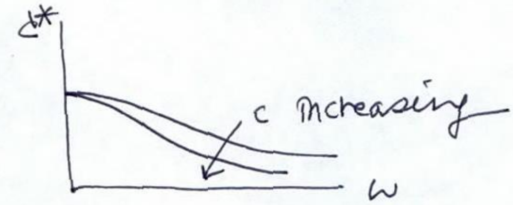
then $\omega = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}} = \omega_{max}$.

$$\alpha^*(\omega_{max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2 - c^2}}$$



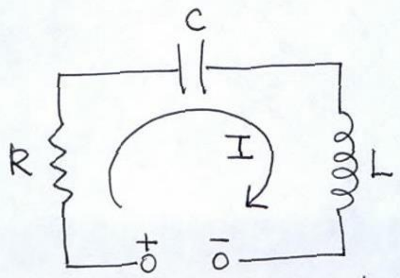
ii) $\omega_0^2 < \frac{c^2}{2m^2}$.

$\omega^2 < 0$: no real ω.



2. ~~12~~ 9. Modeling of Electric Circuits

RLC - circuit



$E(t) = E_0 \sin\omega t$

Kirchhoff's voltage law

$$E - RI - \frac{1}{C}\int I\,dt - L\frac{dI}{dt} = 0.$$

$$LI' + RI + \frac{1}{C}\int I\,dt = E_0 \sin \omega t$$

$$LI'' + RI' + \frac{1}{C} I = E_0 \omega \cos \omega t \quad \cdots \text{ electrical}$$

compare with

$$my'' + cy' + ky = F_0 \cos \omega t \quad \cdots \text{ mechanical}$$

* Analogy between mechanical and electrical systems

→ p. 54

Chap 3 ~~2.13.~~ Higher Order Linear ODEs ~~Diff. Eq~~

3.1. Homogeneous linear ODEs

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = r(x) \quad : \text{linear}$$

$$\left(y^{(n)} = \frac{d^n y}{dx^n}\right). \qquad r(x) = 0 \ : \text{homogeneous}, \quad \neq 0 \ : \text{nonhomogeneous}$$

: n th order ODE

· Homogeneous eq

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = 0.$$

Def. basis: $y_1, y_2, \cdots, y_n$: linearly independent.

G.S. $y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ ($c_1, \cdots, c_n$: arbitrary const.)

Def * Linear independence : " $y_1, \cdots, y_n$: linearly independent"

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \quad \text{on } I$$

$$\Rightarrow k_1 = k_2 = \cdots = k_n = 0.$$

if $k_1 \neq 0$: $y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n)$: linearly dependent

Ex. 1*: $y_1 = x$, $y_2 = 3x$, $y_3 = x^2$: linearly dep.

Ex. 2: $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$: linearly indep.