$$LI' + RI + \frac{1}{C}\int I\,dt = E_0 \sin\omega t$$

$$LI'' + RI' + \frac{1}{C}I = E_0\omega\cos\omega t \quad \cdots \text{ electrical}$$

compare with

$$my'' + cy' + ky = F_0\cos\omega t \quad \cdots \text{ mechanical}$$

* Analogy between mechanical and electrical systems

→ p. 54

Chap 3 ~~2.~~ ~~3.~~ Higher Order Linear ~~Diff. Eq~~ ODEs

3.1 Homogeneous linear ODEs

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \quad : \text{linear}$$

$$\left(y^{(n)} = \frac{d^n y}{dx^n}\right).$$

$r(x)=0$: homogeneous

$\neq 0$: nonhomogeneous

: n th order ODE

· Homogeneous eq

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0.$$

Def. basis: $y_1, y_2, \cdots, y_n$: linearly independent.

G.S. $y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ ($c_1, \cdots, c_n$: arbitrary const.)

Def * Linear independence : " $y_1, \cdots, y_n$: linearly independent"

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \quad \text{on } I$$

$$\Rightarrow k_1 = k_2 = \cdots = k_n = 0.$$

if $k_1 \neq 0$: $y_1 = -\frac{1}{k_1}(k_2 y_2 + \cdots + k_n y_n)$: linearly dependent

Ex. 1*: $y_1 = x$, $y_2 = 3x$, $y_3 = x^2$: linearly dep.

Ex. 2: $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$: linearly indep.

§ Initial value problem.

$$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y = \overset{r(x)}{0}.$$

I.C: $y(x_0) = K_0$
$y'(x_0) = K_1$
$\vdots$
$y^{(n-1)}(x_0) = K_{n-1}$ → "n" initial conditions

→ Theorem 2 $p_0(x), \cdots, p_{n-1}(x)$: continuous, $x_0 \in I$
→ IVP : unique sol. on I.

§ Linear independence of solutions & Wronskian

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Theorem 3 $p_0, \cdots, p_{n-1}$: continuous on I

(1) $y_1, y_2, \cdots, y_n$: linearly dependent $\rightleftarrows$ $W = 0$ (for $x = x_0$).

(2) if $W = 0$ for $x = x_0$ → $W \equiv 0$ on I

(3) $\exists x_1$. $W \neq 0$ → $y_1, y_2, \cdots, y_n$: linearly independent

Thm 4 & 5 : reading assigned.

~~2.14.~~ 3.2 Higher order Homogeneous Eq with Const. Coeff

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Try $y = e^{\lambda x}$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

- Distinct real roots : $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$

  basis: $y_1 = e^{\lambda_1 x}, \ \ldots, \ y_n = e^{\lambda_n x}$: linearly independent !

  G.S: $y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}$

Ex.1. $y''' - 2y'' - y' + 2y = 0$

$y = e^{\lambda x}$

$\lambda^3 - 2\lambda - \lambda + 2 = 0$

$$\begin{array}{r|rrrr} & 1 & -2 & -1 & 2 \\ 1 & & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$(\lambda-1)(\lambda^2-\lambda-2) = 0$

$(\lambda-1)(\lambda+1)(\lambda-2) = 0$

$\lambda = -1, 1, 2$

G.S. $y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$

- Simple complex roots
  - complex conjugate roots

$\lambda = \gamma + i\omega, \quad \lambda = \gamma - i\omega$

basis: $y_1 = e^{\gamma} \cos \omega x, \quad y_2 = e^{\gamma} \sin \omega x$

Ex.2 $y''' - y'' + 100y' - 100y = 0.$ $\quad y(0) = 4$, $y'(0) = 11$, $y''(0) = -299$

try $y = e^{\lambda x}$

$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0$

$$\begin{array}{r|rrrr} & 1 & -1 & 100 & -100 \\ 1 & & 1 & 0 & 100 \\ \hline & 1 & 0 & 100 & \boxed{0} \end{array}$$

$(\lambda-1)(\lambda^2+100) = 0$

$\lambda = 1, \pm 10i$

basis. $y_1 = e^x$, $y_2 = \cos 10x$, $y_3 = \sin 10x$

G.S. $y = C_1 e^x + A\cos 10x + B\sin 10x$

3 I.C's $\Rightarrow$ 3 eqns + 3 unknowns

$C_1 = 1$, $A = 3$, $B = 1$

$y = e^x + 3\cos 10x + \sin 10x$

- Multiple Real Roots

(1) double root

y_1, xy_1

(2) triple root

y_1, xy_1, $x^2 y_1$

$\vdots$

(3) root of order m : λ

basis: $e^{\lambda x}$, $xe^{\lambda x}$, $\cdots$, $x^{m-1}e^{\lambda x}$

Derivation

$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$. $\Rightarrow \mathcal{L}[y]$

sol. $y = e^{\lambda x}$

$= [\frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0]y$

$= [D^n + a_{n-1}D_{n-1} + \cdots + a_0]y$

$$\mathcal{L}[e^{\lambda x}] = [\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0]e^{\lambda x}$$
$$= (\lambda - \lambda_1)^m h(\lambda)e^{\lambda x}$$

$m = n$: $h(\lambda) = 1$

$m < n$: $h(\lambda) = (\lambda - \lambda_{m+1})\cdots(\lambda - \lambda_n)$

$$\underbrace{\frac{\partial}{\partial\lambda}\mathcal{L}[e^{\lambda x}]}_{\parallel\ \lambda, x:\ \text{independent}} = m(\lambda-\lambda_1)^{m-1}h(\lambda)e^{\lambda x} + (\lambda-\lambda_1)^m\frac{\partial}{\partial\lambda}[h(\lambda)e^{\lambda x}]$$

$\mathcal{L}[\frac{\partial}{\partial\lambda}e^{\lambda x}] = \mathcal{L}[xe^{\lambda x}]$. recall $\mathcal{L}[y] = 0$

$y = xe^{\lambda_1 x} \Rightarrow$ RHS $= 0$.

$\therefore xe^{\lambda_1 x}$ - sol.

Ex. 3 $\quad y^{v} - 3y^{iv} + 3y''' - y'' = 0$

$$\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0 \qquad \lambda^2(\lambda-1)^3 = 0.$$

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda_4 = \lambda_5 = 1$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$1, \ x \qquad e^x, \ xe^x, \ x^2e^x$$

$$y = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2)\, e^x$$

§ Multiple Complex Roots

$$(\lambda^2 + a\lambda + b)^2 = \{[\lambda - (\gamma + i\omega)][\lambda - (\gamma - i\omega)]\}^2$$

$$\left.\begin{array}{l} \lambda_1 = \lambda_2 = \gamma + i\omega \\ \lambda_3 = \lambda_4 = \gamma - i\omega \end{array}\right\} \quad \begin{array}{ll} e^{\gamma x}\cos\omega x & xe^{\gamma x}\cos\omega x \\ e^{\gamma x}\sin\omega x & xe^{\gamma x}\sin\omega x \end{array}$$

3.3. ~~2.15.~~ Higher Order Non~~linear~~ homogen ~~Eq~~ ODEs

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)$$

$$r(x) \neq 0.$$

$$y = y_h + y_p$$

y_h (G.S.) $= c_1 y_1 + \cdots + c_n y_n$.

How to obtain y_p?

— Method of Undetermined Coeff

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x)$$

$\longrightarrow$ $a_0, \ldots, a_{n-1}$: const
$r(x)$: special fn

- Basic Rule: table
- Modification Rule: $x^k y(x) \in y_h$
- Sum Rule

Ex.1. $y''' + 3y'' + 3y' + y = 30e^{-x}$.

$y(0) = 3$, $y'(0) = -3$, $y''(0) = -47$.

(1) y_h? $\quad y_h = e^{\lambda x}$

$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

$(\lambda+1)^3 = 0 \quad \lambda = -1$: triple root

basis: e^{-x}, xe^{-x}, x^2e^{-x}

$y_h = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x}$

$= (c_1 + c_2x + c_3x^2)e^{-x}$

(2) y_p? $\quad$ table $\Rightarrow$ $y_p = ce^{-x} \to cxe^{-x} \to cx^2e^{-x} \to$ " cx^3e^{-x} "

$y = y_h + y_p \to \quad y_p''' + 3y_p'' + 3y_p' + y_p = 30e^{-x}$

$y_p' = c(3x^2 - x^3)e^{-x}$

$y_p'' = c(6x \underbrace{- 3x^2 - 3x^2}_{-6x^2} + x^3)e^{-x}$

$y_p''' = c(6 - 18x + 9x^2 - x^3)e^{-x}$

$6ce^{-x} = 30e^{-x} \qquad \therefore c = 5$.

$\therefore y = (c_1 + c_2x + c_3x^2)e^{-x} + 5x^3e^{-x}$.

I.C.'s

$c_1 = 3$, $c_2 = 0$, $c_3 = -25$

$y = (3 - 25x^2)e^{-x} + 5x^3e^{-x}$

§ Method of Variation of Parameters

$$y_h = c_1y_1 + \cdots + c_ny_n$$

$$y_p = y_1\int \frac{W_1(x)}{W(x)} r(x)\,dx + y_2\int \frac{W_2(x)}{W(x)} r(x)\,dx + \cdots + y_n\int \frac{W_n(x)}{W(x)} r(x)\,dx$$

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}, \quad W_j = \begin{vmatrix} y_1 & \cdots & \cancel{y_j}\ 0 & \cdots & y_n \\ y_1' & & 0 & & y_n' \\ \vdots & & \vdots & & \\ y_1^{(n-1)} & & 1 & & y_n^{(n-1)} \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2 \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

Ex. 2. $x^3y''' - 3x^2y'' + 6xy' - 6y = x^4\ln x$

(1) $y_h \ni x^m$

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0.$$

$$m^3 - 3m^2 + 2m - 3m^2 + 3m + 6m - 6 = 0$$

$$m^3 - 6m^2 + 11m - 6 = 0$$

1	1	−6	11	−6
		1	−5	6
	1	−5	6	0

$$(m-1)(m^2-5m+6) = 0$$

$$(m-1)(m-3)(m-2) = 0$$

$$m = 1, 2, 3$$

$$y_h = c_1x + c_2x^2 + c_3x^3$$

from standard form: $y''' \cdots = \underbrace{x\ln x}_{=r(x)}$

(2) $y_p = y_1\int \frac{W_1}{W} r\,dx + y_2\int \frac{W_2}{W} r\,dx + y_3\int \frac{W_3}{W} r\,dx$

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3. \quad W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$W_2 = -2x^3, \quad W_3 = x^2$$

$$y_p = x\int \frac{x^4}{2x^3} x \ln x \, dx + x^2 \int \frac{-2x^3}{2x^3} x \cdot \ln x \, dx + x^3 \int \frac{x^2}{2x^3} x \cdot \ln x \, dx$$

$$= x \int \tfrac{1}{2} x^2 \ln x \, dx - x^2 \int \tfrac{1}{2} x \ln x \, dx + x^3 \int \tfrac{1}{2} \ln x \, dx$$

$$\int \underbrace{x^2}_{v'} \underbrace{\ln x}_{u} \, dx = \tfrac{1}{3} x^3 \ln x - \int \tfrac{1}{x} \cdot \tfrac{1}{3} x^3 \, dx$$

$$= \tfrac{1}{3} x^3 \ln x - \tfrac{1}{3}\tfrac{1}{3} x^3 = \tfrac{1}{3}(x^3 \ln x) - \tfrac{1}{9} x^3$$

$$\int \underbrace{x}_{v'} \underbrace{\ln x}_{u} \, dx = \tfrac{1}{2} x^2 \ln x - \int \tfrac{1}{2} x^2 \cdot \tfrac{1}{x} \, dx$$

$$= \tfrac{1}{2} x^2 \ln x - \tfrac{1}{4} x$$

$$\int \ln x \, dx = x \ln x - x$$

$$y_p = \frac{x^4}{6}\left(\ln x - \frac{11}{6}\right)$$

$$\therefore y = c_1 x + c_2 x^2 + c_3 x^3 + \frac{x^4}{6}\left(\ln x - \frac{11}{6}\right)$$

<u>Homework</u>

PS 2.15.

1, 6, 7, 11, 13

$\hookrightarrow y = [c_1 + c_2 x + c_3 x^2 + \frac{8}{105} x^{7/2}] e^{2x}$