Chap. ~~3~~ 4. Systems of ~~Differential Equations~~ ODEs, Phase Plane, Qualitative Methods

4.0. ~~Introduction~~ Basics: Vectors, Matrices, Eigenvalues

— $n \times n$ matrix

$$A = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix}$$

entries

row column

— Column vector

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

— Row vector

$$\bar{v} = [v_1 \ \cdots \ v_n]$$

* Calculations with matrices and vectors.

- Equality
- Addition
- scalar multiplication
- Matrix multiplication

$$C = AB$$

$$A = [a_{jk}], \quad B = [b_{jk}] \qquad \left\{ \begin{matrix} j = 1, \cdots, n \\ k = 1, \cdots, n \end{matrix} \right\}$$

$$c_{jk} = \sum_{m=1}^{n} a_{jm} b_{mk}$$

$AB \neq BA$: not commutative

- Differentiation

$$\bar{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}. \qquad \bar{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$

· Transposition

$$A = [a_{jk}]$$

$$A^T = [a_{kj}]$$

$$\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \bar{v}^T = [v_1 \ \ v_2]$$

· Inverse of a matrix

$n \times n$ matrix.

$$AB = BA = I \text{ ~ unit matrix}$$

A: nonsingular

B: inverse of $A = A^{-1}$.

$$AA^{-1} = A^{-1}A = I$$

if A has no inverse, it is called singular

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \qquad A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

· Linear independence

vectors $\bar{v}^{(1)}, \cdots, \bar{v}^{(r)}$ are called linearly independent

if " $c_1\bar{v}^{(1)} + \cdots + c_r\bar{v}^{(r)} = 0$.

$$\Rightarrow c_1 = c_2 = \cdots = c_r = 0. \ //$$

linearly dependent : $c_1 \neq 0$

$$\bar{v}^{(1)} = -\frac{1}{c_1}\left[c_2\bar{v}^{(2)} + \cdots + c_r\bar{v}^{(r)}\right]$$

§ ODEs. Systems of ~~Differential Equations~~ as Vector Equations

two diff. eq in two unknown fns $y_1(t)$, $y_2(t)$

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 \\ y_2' = a_{21}y_1 + a_{22}y_2 \end{cases} \qquad \bar{y}' = A\bar{y}$$

$$\bar{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = Ay = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Eigenvalues, Eigenvectors

$A = [a_{jk}]$: $n \times n$ matrix

$$A\bar{x} = \lambda \bar{x}$$

$$(n \times n)(n \times 1) = (n \times 1)$$

$\bar{x} = 0$.

$\bar{x} \neq 0$: $\begin{cases} \lambda : \text{eigenvalue of } \bar{A} \\ \bar{x} : \text{eigenvector of } A \end{cases}$

$$A\bar{x} - \lambda\bar{x} = 0$$

$$(A - \lambda I)\bar{x} = 0.$$

For $\bar{x} \neq 0$: $\underbrace{\det(A - \lambda I)}_{\text{characteristic determinant of } A} = 0$.

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} \\ &= 0 \qquad : \text{characteristic eq of } A \end{aligned}$$

Solutions: λ_1, λ_2 : eigenvalues

Ex. 1. Eigenvalues & Eigenvectors of

$$A = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -4.0-\lambda & 4.0 \\ -1.6 & 1.2-\lambda \end{vmatrix} = \lambda^2 + 2.8\lambda + 1.6 = 0$$

$$\lambda_1 = -2, \quad \lambda_2 = -0.8 \quad : \text{eigenvalues}$$

$$(A-\lambda I)\bar{x} = 0$$

$$\begin{pmatrix} -4-\lambda & 4 \\ -1.6 & 1.2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{bmatrix} (-4-\lambda)x_1 + 4x_2 \\ -1.6x_1 + (1.2-\lambda)x_2 \end{bmatrix} = 0.$$

$\lambda = \lambda_1 = -2 \rightarrow x_1 = 2, \; x_2 = 1.$

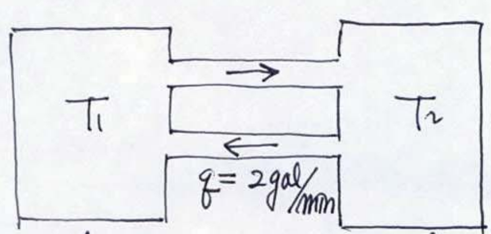
$\lambda = \lambda_2 = -0.8 \rightarrow x_1 = 1, \; x_2 = 0.8$

$$\bar{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \bar{x}^{(2)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} \quad : \text{eigenvectors}$$

4.1. Systems of ODEs

~~3.1. Introductory Examples~~

Ex. 1. Mixing problem



initial: 100 gal. pure | 100 gal water / 150 lb fertilizer

$y_1(t)$ | $y_2(t)$: amount of fertilizer

(1) Modeling

fertilizer in T_1

$$\frac{dy_1}{dt} = \text{inflow} - \text{outflow}$$

$$= y_2 \frac{2}{100} - y_1 \frac{2}{100}$$

fertilizer in T_2

$$\frac{dy_2}{dt} = y_1 \frac{2}{100} - y_2 \frac{2}{100}$$

$$\left.\begin{aligned} y_1' &= -0.02 y_1 + 0.02 y_2 \\ y_2' &= 0.02 y_1 - 0.02 y_2 \end{aligned}\right\} \text{ system of first-order diff eq}$$

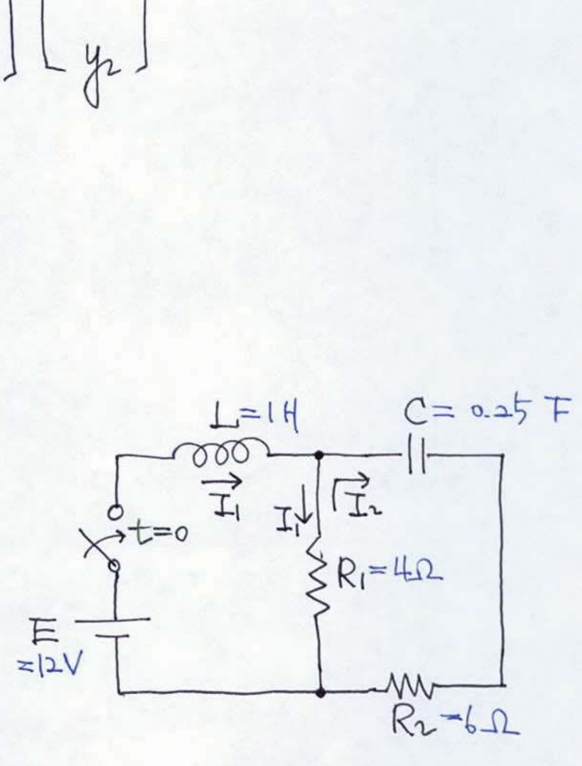
$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\bar{y}' = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\bar{y}' = A\bar{y}$$

(2) Solution : later

Ex.2 Electrical network



$$I_1' = -4I_1 + 4I_2 + 12$$

$$I_2' = -1.6 I_1 + 1.2 I_2 + 4.8$$

$$\bar{J}' = \underbrace{\begin{bmatrix} -4 & 4 \\ -1.6 & 1.2 \end{bmatrix}}_{A} \bar{J} + \underbrace{\begin{bmatrix} 12 \\ 4.8 \end{bmatrix}}_{\bar{g}}$$

§ Conversion of an nth order ~~diff. eq.~~ ODE to a system of n first-order diff. eqn's.

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$$

$$\begin{cases} y_1 = y \\ y_2 = y' \\ y_3 = y'' \\ \vdots \\ y_n = y^{(n-1)} \end{cases}$$

First-order system

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{n-1}' = y_n \\ y_n' = F(t, y_1, y_2, \cdots, y_n) \end{cases}$$

Ex. 3. $\quad y'' + \frac{c}{m} y' + \frac{k}{m} y = 0.$

- $y_1' = y_2$

$y_2' + \frac{c}{m} y_2 + \frac{k}{m} y_1 = 0$

- $y_2' = -\frac{k}{m} y_1 - \frac{c}{m} y_2$

$$\bar{y}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\bar{y}$$

characteristic eq

$$\det(A-\lambda I)=0.$$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m}-\lambda \end{vmatrix} = \lambda^2+\frac{c}{m}\lambda+\frac{k}{m}=0.$$

§2. Basic ~~Concepts and~~ Theory of systems of ODEs

General first-order systems

$$\begin{cases} y_1' = f_1(t, y_1, \cdots, y_n) \\ y_2' = f_2(t, y_1, \cdots, y_n) \\ \quad\vdots \\ y_n' = f_n(t, y_1, \cdots, y_n) \end{cases} \qquad (1)$$

$$\bar{y}' = \bar{f}(t, \bar{y})$$

solution vector

$$\bar{y} = \bar{h}(t)$$

IVP – IC's

$$y_1(t_0)=K_1$$
$$y_2(t_0)=K_2 \qquad \bar{y}(t_0)=\bar{K}$$
$$\vdots$$
$$y_n(t_0)=K_n$$

Theorem 1. Existence and Uniqueness

(1) → $f_1, \cdots, f_n$: continuous. $\frac{\partial f_1}{\partial y_1}, \cdots, \frac{\partial f_1}{\partial y_n}, \cdots, \frac{\partial f_n}{\partial y_n}$: continuous

IVP $\exists$ unique sol.