characteristic eq

$$\det(A-\lambda I)=0.$$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m}-\lambda \end{vmatrix} = \lambda^2+\frac{c}{m}\lambda+\frac{k}{m}=0.$$

§2. Basic ~~Concepts and~~ Theory of systems of ODEs

General first-order systems

$$\begin{cases} y_1' = f_1(t, y_1, \cdots, y_n) \\ y_2' = f_2(t, y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(t, y_1, \cdots, y_n) \end{cases} \qquad (1)$$

$$\bar{y}' = \bar{f}(t, \bar{y})$$

solution vector

$$\bar{y} = \bar{h}(t)$$

IVP – IC's

$$\begin{aligned} y_1(t_0) &= K_1 \\ y_2(t_0) &= K_2 \\ &\vdots \\ y_n(t_0) &= K_n \end{aligned} \qquad \bar{y}(t_0) = \bar{K}$$

Theorem 1. Existence and Uniqueness

(1) → $f_1, \cdots, f_n$: continuous. $\frac{\partial f_1}{\partial y_1}, \cdots, \frac{\partial f_1}{\partial y_n}, \cdots, \frac{\partial f_n}{\partial y_n}$: continuous

IVP $\exists$ unique sol.

※ Linear Systems

$$\begin{cases} y_1' = a_{11}(t)y_1 + \cdots + a_{1n}(t)y_n + g_1(t) \\ \quad\vdots \\ y_n' = a_{n1}(t)y_1 + \cdots + a_{nn}(t)y_n + g_n(t) \end{cases}$$

$$\bar{y}' = A\bar{y} + \bar{g}$$

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \bar{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

$\left[\begin{array}{l} \text{homogeneous : } \bar{g} = 0 : \quad \bar{y}' = A\bar{y} \\ \text{nonhomogeneous : } \bar{g} \neq 0 \end{array}\right.$

<u>Theorem 2</u>

$\frac{\partial f_1}{\partial y_1} = a_{11}, \cdots \frac{\partial f_n}{\partial y_n} = a_{nn}$ (from Thm 1)

a_{jk}, g_j : continuous $\rightarrow$ unique sol.

<u>Theorem 3</u> : Superposition principle

$\bar{y}^{(1)}, \bar{y}^{(2)}$: solutions of homogeneous linear system

$\rightarrow \bar{y} = c_1\bar{y}^{(1)} + c_2\bar{y}^{(2)}$ (linear combination) : sol.

pf.) "$\bar{y}' = A\bar{y}$"

$$[c_1\bar{y}^{(1)} + c_2\bar{y}^{(2)}]' = c_1\bar{y}^{(1)\prime} + c_2\bar{y}^{(2)\prime} = c_1 A\bar{y}^{(1)} + c_2 A\bar{y}^{(2)}$$

$$= A[c_1\bar{y}^{(1)} + c_2\bar{y}^{(2)}]$$

※ Basis / G.S / W

homogeneous system $\bar{y}' = A\bar{y}$

basis: linearly independent set of n solutions $\bar{y}^{(1)}, \dots, \bar{y}^{(n)}$

G.S.: $\bar{y} = c_1\bar{y}^{(1)} + \cdots + c_n\bar{y}^{(n)}$

$Y = [\bar{y}^{(1)} \cdots y^{(n)}]$: $n\times n$ matrix

fundamental matrix

$$\det(Y) = W(y^{(1)}, \dots, y^{(n)}) = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & \cdots & y_1^{(n)} \\ y_2^{(1)} & y_2^{(2)} & \cdots & y_2^{(n)} \\ \vdots & \vdots & & \vdots \\ y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(n)} \end{vmatrix}$$

" $W \neq 0 \rightleftarrows y^{(1)}, \dots, y^{(n)}$: basis "

2nd-order homogeneous linear diff. eq

$y'' + p(x)y' + q(x)y = 0$

basis: y, z

$$W(y, z) = \begin{vmatrix} y & z \\ y' & z' \end{vmatrix}$$

$\cdot\ y_1' = y_2$

$y_2' + py_2 + qy_1 = 0$

$\cdot\ y_2' = -py_2 - qy_1$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\bar{y} = \begin{bmatrix} y \\ y' \end{bmatrix}, \begin{bmatrix} z \\ z' \end{bmatrix}$$

∴ consistent

4.3 ~~3.3.~~ ~~Homogeneous Systems with~~ Constant-Coefficients Systems

- $\bar{y}' = A\bar{y}$. entries of A : const.

Recall : $y' = ky$

$y = ce^{kt}$

$\therefore \quad \bar{y} = \bar{x}e^{\lambda t}$

$$\begin{cases} \bar{y}' = \underline{\lambda\bar{x}e^{\lambda t}} \\ \cancel{\bar{y}'} = A\bar{y} = \underline{A\bar{x}e^{\lambda t}} \end{cases}$$

$$\boxed{A\bar{x} = \lambda\bar{x}.}$$

- eigenvalue problem

λ : eigenvalue

$\bar{x}$: eigenvector.

Assume A has a basis of n eigenvectors $\bar{x}^{(1)}, \cdots, \bar{x}^{(n)}$ corresponding to eigenvalues $\lambda_1, \cdots, \lambda_n$

sol.: $\bar{y}^{(1)} = \bar{x}^{(1)}e^{\lambda_1 t}, \cdots, \bar{y}^{(n)} = \bar{x}^{(n)}e^{\lambda_n t}$

$$W(\bar{y}^{(1)}, \cdots, \bar{y}^{(n)}) = \begin{vmatrix} x_1^{(1)}e^{\lambda_1 t} & \cdots & x_1^{(n)}e^{\lambda_n t} \\ x_2^{(1)}e^{\lambda_1 t} & \cdots & x_2^{(n)}e^{\lambda_n t} \\ \vdots & & \vdots \\ x_n^{(1)}e^{\lambda_1 t} & \cdots & x_n^{(n)}e^{\lambda_n t} \end{vmatrix}$$

$$= \underbrace{e^{\lambda_1 t + \cdots + \lambda_n t}}_{\neq 0} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & \cdots & x_n^{(n)} \end{vmatrix}^{\neq 0} \qquad \neq 0.$$

Theorem 1. $\bar{y}' = A\bar{y}$

if const matrix A has a linearly independent set of n eigenvectors $(A\bar{x} = \lambda\bar{x})$.

$\rightarrow \bar{y}^{(1)} = \bar{x}^{(1)} e^{\lambda_1 t}, \cdots, \bar{y}^{(n)} = \bar{x}^{(n)} e^{\lambda_n t}$: basis of solutions

G.S: $\bar{y} = c_1 \bar{x}^{(1)} e^{\lambda_1 t} + \cdots + c_n \bar{x}^{(n)} e^{\lambda_n t}$

4.1. Ex. 1.

$$\bar{y}' = A\bar{y} \qquad A = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$$

Try. $\bar{y} = \bar{x} e^{\lambda t}$

$$\bar{y}' = \lambda \bar{x} e^{\lambda t} = A\bar{x} e^{\lambda t}$$

$$A\bar{x} - \lambda \bar{x} = 0$$

$$(A - \lambda I)\bar{x} = 0$$

To get nontrivial solutions,

$$\det(A - \lambda I) = 0.$$

$$\det(A - \lambda I) = \begin{vmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{vmatrix}$$

$$= (0.02+\lambda)^2 - 0.02^2$$

$$= \lambda(\lambda + 0.04) = 0$$

$\lambda = 0, \ \lambda = -0.04$: eigenvalues

$$\begin{bmatrix} -0.02-\lambda & 0.02 \\ 0.02 & -0.02-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$\lambda = 0 \rightarrow -0.02x_1 + 0.02x_2 = 0$: $x_1 = x_2$

$\lambda = -0.04 \rightarrow 0.02x_1 + 0.02x_2 = 0$: $x_1 = -x_2$

take $\begin{cases} x_1=1, \; x_2=1 & (\lambda=0) \\ x_1=1, \; x_2=-1 & (\lambda=-0.04) \end{cases}$

eigenvectors $\bar{x}^{(1)}=\begin{bmatrix}1\\1\end{bmatrix}$, $\bar{x}^{(2)}=\begin{bmatrix}1\\-1\end{bmatrix}$.

G.S. $\bar{y}=c_1\bar{x}^{(1)}e^{\lambda_1 t}+c_2\bar{x}^{(2)}e^{\lambda_2 t}$

$= c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$

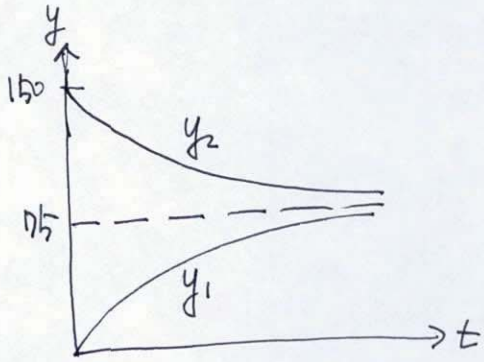
I.C. $\bar{y}(0)=\begin{bmatrix}0\\150\end{bmatrix}$ [lb] fertilizer

$\bar{y}(0)=\begin{bmatrix}c_1+c_2\\c_1-c_2\end{bmatrix}=\begin{bmatrix}0\\150\end{bmatrix}$. $c_1=75$, $c_2=-75$

$\bar{y}=75\begin{bmatrix}1\\1\end{bmatrix}-75\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.04t}$

In components

$\begin{cases} y_1(t)=75-75e^{-0.04t} \\ y_2(t)=75+75e^{-0.04t} \end{cases}$

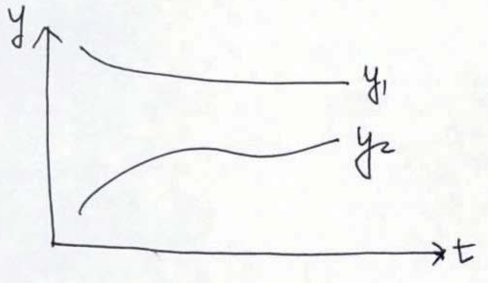


$\cancel{\text{plot}}$ How to graph solutions in the Phase Plane

Consider $\quad y_1'=a_{11}y_1+a_{12}y_2$

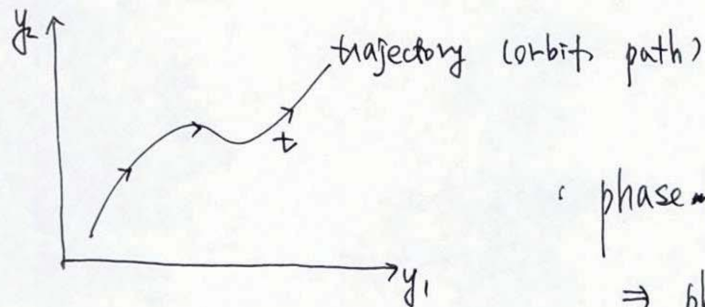
$\quad\quad\quad\quad y_2'=a_{21}y_1+a_{22}y_2$

$\begin{bmatrix}y_1\\y_2\end{bmatrix}'=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$: $\bar{y}'=A\bar{y}$



↓

$y_1 y_2$-plane : phase-plane



: phase-plane + trajectory

$\Rightarrow$ phase portrait

Ex. 1. Phase portrait

$$\bar{y}' = A\bar{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \bar{y}$$

Try $\bar{y} = \bar{x} e^{\lambda t}$. $\Rightarrow$ $A\bar{x} = \lambda \bar{x}$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

$$(\lambda+2)(\lambda+4) = 0$$

$\lambda_1 = -2$, $\lambda_2 = -4$. : eigenvalues

$$(A - \lambda I)\bar{x} = \begin{bmatrix} (-3-\lambda)x_1 + x_2 \\ x_1 - (3+\lambda)x_2 \end{bmatrix} = 0$$

$\lambda = -2$: $-x_1 + x_2 = 0$. $x_1 = x_2$ $x_1 = 1$, $x_2 = 1$

$$\bar{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -4$: $x_1 + x_2 = 0$. $x_1 = -x_2$. $x_1 = 1$, $x_2 = -1$

$$\bar{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\bar{y} = c_1 \bar{x}^{(1)} e^{\lambda_1 t} + c_2 \bar{x}^{(2)} e^{\lambda_2 t} = c_1 \bar{y}^{(1)} + c_2 \bar{y}^{(2)}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

$$\begin{cases} y_1 = c_1 e^{-2t} + c_2 e^{-4t} \\ y_2 = c_1 e^{-2t} - c_2 e^{-4t} \end{cases}$$

$c_1 = 0$: $\left.\begin{array}{l} y_1 = c_2 e^{-4t} \\ y_2 = -c_2 e^{-4t} \end{array}\right\rangle \quad y_2 = -y_1$

$c_2 = 0$: $\left.\begin{array}{l} y_1 = c_1 e^{-2t} \\ y_2 = c_1 e^{-2t} \end{array}\right\rangle \quad y_2 = y_1$