$\cancel{4}$ 5.7. Sturm-Liouville Problems, Orthogonal functions

Sturm-Liouville equation

$$[p(x)y']' + [q(x) + \lambda \overset{r}{\cancel{p}}(x)]y = 0$$

· Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$[(1-x^2)y']' + \lambda y = 0. \qquad \lambda = n(n+1)$$

$$p = 1-x^2, \quad q = 0, \quad \overset{r}{\cancel{p}} = 1$$

· Bessel's equation

$$x^2y'' + xy' + (x^2-n^2)y = 0$$

let $x = kt$

$$\frac{dy}{dt} = \frac{dy}{dx}\left(\frac{dx}{dt}\right) = k\frac{dy}{dx} = ky' = \dot{y}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right)\cdot\frac{dx}{dt} = k^2y'' = \ddot{y}$$

$$k^2t^2\frac{1}{k^2}\ddot{y} + kt\cdot\frac{1}{k}\dot{y} + (k^2t^2-n^2)y = 0$$

$$t^2\ddot{y} + t\dot{y} + (k^2t^2-n^2)y = 0 \quad /t$$

$$\frac{d}{dt}[t\dot{y}] + \left(-\frac{n^2}{t} + \lambda t\right)y = 0 \qquad \lambda = k^2$$

$$p = t \quad q = -\frac{n^2}{t}, \quad \overset{r}{\cancel{p}} = t \qquad //$$

· S.-L. problem (BVP)

$$[p(x)y']' + [q(x) + \lambda \overset{r}{\cancel{p}}(x)]y = 0 \qquad [a,b]$$

$$\text{B.C.} \begin{cases} k_1y(a) + k_2y'(a) = 0 \\ l_1y(b) + l_2y'(b) = 0 \end{cases} \qquad * \text{ assume } \overset{r}{\cancel{p}}(x) > 0.$$

↳ solution : eigenfunction $y(x)$

λ : eigenvalue for which an eigenfunction exists

Ex. 1. $y'' + \lambda y = 0$

$y(0) = 0, \quad y(\pi) = 0$ eigenvalues and eigenfns ?

S.L. ? $p=1, \ q=0, \ r=1$

(i) $\lambda < 0$. $\lambda = -\nu^2$

$y'' - \nu^2 y = 0$

$y = c_1 e^{\nu x} + c_2 e^{-\nu x}$. B.C. $\Rightarrow c_1 = c_2 = 0$. $y = 0$

(ii) $\lambda = 0$. $y'' = 0$. $y = ax + b$ B.C. $\Rightarrow y = 0$

(iii) $\lambda > 0$ $\lambda = \nu^2$: $y'' + \nu^2 y = 0$

$y = A\cos\nu x + B\sin\nu x$

$y(0) = A = 0$

$y(\pi) = B\boxed{\sin\nu\pi = 0}$

$\nu = \cancel{0}, \pm 1, \pm 2, \cdots$

$\lambda = 1, 4, 9, 16, \cdots$

$y(x) = \sin\nu x \quad (\nu = 1, 2, \cdots)$: eigenfunction

$\lambda = \nu^2$: eigenvalue

◎ p, q, r, p' : real-valued & continuous on $a \le x \le b$,

$r > 0$ throughout or $r < 0$ throughout the interval

$\Rightarrow$ all eigenvalues are real

§ ORTHOGONALITY.

· Definition

$y_1, y_2 \cdots$ orthogonal on $[a.b]$ with respect to $\overset{r}{p}(x)$

$$\boxed{\int_a^b \overset{r}{p}(x)\, y_m(x)\, y_n(x)\, dx = 0} \qquad m \neq n$$

$\underset{r}{p}(x)$: weight function > 0

norm $\quad \|y_m\| = \sqrt{\int_a^b \underset{r}{p}(x)\, y_m^2(x)\, dx}$

· orthonormal on $[a.b]$
  if orthogonal on $[a.b]$ & norm $= 1$

· $\overset{r}{p}(x) = 1$: $y_1, y_2 \cdots$ orthogonal on $[a.b]$

$$\boxed{\int_a^b y_m(x)\, y_n(x)\, dx = 0} \qquad m \neq n$$

norm $\quad \|y_m\| = \sqrt{\int_a^b y_m^2(x)\, dx}$

Ex.3 $\quad y_m(x) = \sin mx \quad (-\pi \le x \le \pi)$. $\quad m = 1, 2, \cdots$

$$\int_{-\pi}^{\pi} y_m y_n\, dx = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x\, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x\, dx$$

$$= 0 \qquad \rightarrow \text{ orthogonal set}$$

$$\|y_m\|^2 = \int_{-\pi}^{\pi} \sin^2 mx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 2mx)\, dx = \pi$$

$$\|y_m\| = \sqrt{\pi}$$

$\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \cdots \frac{\sin mx}{\sqrt{\pi}}, \cdots$: orthonormal set

- Theorem 1.

S.L. problem $\quad (py')' + (q + \lambda r)y = 0 \qquad a \le x \le b$

$\lambda_m \to y_m$

$\lambda_n \to y_n$

y_m & y_n are $\boxed{\text{orthogonal}}$ on that interval w.r.t. the weight function r

$$\int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = 0.$$

if $p(a) = 0$: no B.C. for $x = a$

$p(b) = 0$: " " $x = b$

} singular S.L.

- if $p(a) = p(b)$ B.C can be replaced by

$$\begin{cases} y(a) = y(b) \\ y'(a) = y'(b) \end{cases}$$: periodic boundary conditions

→ periodic S.L. prob

pf.)

$$(py_m')' + (q + \lambda_m r)\, y_m = 0 \qquad \times\, y_n$$

$$+)\ (py_n')' + (q + \lambda_n r)\, y_n = 0 \qquad \times\, (-y_m)$$

$$y_n (py_m')' - y_m (py_n')' + (\lambda_m - \lambda_n)\, r\, y_m y_n = 0$$

$$(\lambda_m - \lambda_n)\, r\, y_m y_n = \left[(p\, y_n')\, y_m - (p y_m')\, y_n \right]'$$

- $$(\lambda_m - \lambda_n) \int_a^b r\, y_m y_n\, dx = \left[p\, (y_n' y_m - y_m' y_n) \right]_a^b$$

$$= p(b) \left[y_n'(b) y_m(b) - y_m'(b) y_n(b) \right]$$

$$- p(a) \left[y_n'(a) y_m(a) - y_m'(a) y_n(a) \right] \stackrel{?}{=} 0$$

case 1. $p(a)=0$ & $p(b)=0$

$$\int_a^b r\, y_m y_n = 0 \quad (m \neq n)$$ no need to use B.C

case 2. $p(b)=0$, $p(a)\neq 0$

$$RHS = -p(a)\left[y_n'(a)\,y_m(a) - y_m'(a)\,y_n(a)\right]$$

B.C. at $x=a$: $k_1 y_n(a) + k_2 y_n'(a) = 0$ $\quad \times\, y_m(a)$

$+)\ k_1 y_m(a) + k_2 y_m'(a) = 0$ $\quad \times\,(-y_n(a))$

$$k_2\left[y_n'(a)\,y_m(a) - y_m'(a)\,y_n(a)\right] = 0 \qquad k_2 \neq 0.$$

$RHS = 0.$

what if $k_2 = 0$ & $k_1 \neq 0$? : practice

case 3. $p(a)=0$. $p(b)\neq 0$. similar

case 4. $p(a)\neq 0$, $p(b)\neq 0$. similar

case 5. $p(a)=p(b)$

$$RHS = p(b)\left[y_n'(b)\,y_m(b) - y_m'(b)\,y_n(b) - y_n'(a)\,y_m(a) + y_m'(a)\,y_n(a)\right]$$

· original B.C $\Rightarrow$ $RHS = 0$

· $y_{\substack{m\\n}}(a) = y_{\substack{m\\n}}(b)$ $\quad y'_{\substack{m\\n}}(a) = y'_{\substack{m\\n}}(b)$

Ex. $\cancel{4}$ 5. Legendre's equation

$$[(1-x^2)y']' + \lambda y = 0. \qquad \lambda = n(n+1)$$

$p = 1-x^2$, $q=0$. $r=1$.

$p(-1) = p(1) = 0$: singular S.L prob. on $[-1,1]$

no B.C needed.

if $n = 0,1,2,\dots$: $P_n(x)$ - sol.

$$\therefore \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0 \quad (m\neq n) \quad : \text{orthogonal}$$

* Orthogonality of Bessel functions $J_n(x)$

Bessel's equation

$$\tilde{x}^2 \frac{d^2 J_n}{d\tilde{x}^2} + \tilde{x} \frac{dJ_n}{d\tilde{x}} + (\tilde{x}^2 - n^2)\, J_n(\tilde{x}) = 0 .$$

transformation: $\tilde{x} = kx$

$$[x J_n'(kx)]' + \left(-\frac{n^2}{x} + k^2 x\right) J_n(kx) = 0$$

$$[\underset{p}{r}(x) y']' + [q(x) + \lambda \underset{r}{p}(x)]\, y = 0 .$$

$\overset{p}{r}(x) = x$, $\quad q(x) = -\frac{n^2}{x}$. $\quad \overset{r}{p}(x) = x$. $\quad \lambda = k^2$

$\overset{p}{r}(0) = 0$ no need for B.C. at $x=0$

at $x = R$: $J_n(kR) = 0$. $\quad n$: fixed $\Big\}$ orthogonal

$J_n(\tilde{x}) = 0$: $\tilde{x}$ = zeros of J_n

$\tilde{x} = \alpha_{1n} < \alpha_{2n} < \cdots$: infinitely many zeros

$kR = \alpha_{mn}$. $\quad (m = 1, 2, \dots)$

$k = k_{mn} = \frac{\alpha_{mn}}{R}$ $\rightarrow$ eigenvalue $\lambda_{mn} = k_{mn}^2$

Theorem 2

$$\int_0^R x\, J_n(k_{mn} x)\, J_n(k_{jn} x)\, dx = 0 \qquad j \neq m$$

$\overset{5}{4}$.8 Orthogonal Eigenfunction Expansions

Notation:

$$(y_m, y_n) \equiv \int_a^b \overset{r}{p}(x)\, y_m(x)\, y_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

if $y_0, y_1, y_2 \cdots$: orthonormal w.r.t. $p(x)$