Chap. ~~5~~ 6. Laplace Transforms

ODE → algebraic eq

PDE → ODE.

~~5~~ 6.1. Laplace Transform

$$\boxed{F(s) = \mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\, dt}$$: finite

Inverse transform

$$f(t) = \mathcal{L}^{-1}(F)$$

Ex. 1. $f(t) = 1 \quad t \geq 0.$

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty$$

$$= -\frac{1}{s}(0-1) = \frac{1}{s} \qquad (s>0)$$

* Improper integral : interval of integration infinite

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt$$

Ex. 2 $f(t) = e^{at} \quad t \geq 0$

$$\mathcal{L}(f) = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty$$

when $s-a>0$: $\mathcal{L}(f) = \frac{1}{s-a} = \mathcal{L}(e^{at})$

Theorem 1. Linearity of the Laplace Transform

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

Pf. $\mathcal{L}\{af(t)+bg(t)\} = \int_0^\infty e^{-st}(af+bg)\,dt$

$$= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt$$

$$= a\mathcal{L}(f) + b\mathcal{L}(g)$$

Ex. 3. $f(t) = \cosh at = \frac{1}{2}(e^{at}+e^{-at})$

$$\mathcal{L}(\cosh at) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} \qquad (s>a,\ a\geq 0)$$

$$= \frac{1}{2}\frac{2s}{(s-a)(s+a)} = \frac{s}{s^2-a^2}$$

$$\mathcal{L}(\sinh at) = \mathcal{L}\left\{\frac{1}{2}(e^{at}-e^{-at})\right\} = \frac{a}{s^2-a^2} \qquad (s>a\geq 0)$$

Ex. 4. $\mathcal{L}(\cos\omega t)$, $\mathcal{L}(\sin\omega t) = ?$

Use $e^{i\omega t} = \cos\omega t + i\sin\omega t$

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos\omega t) + i\mathcal{L}(\sin\omega t)$$

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s-i\omega} = \frac{s+i\omega}{(s-i\omega)(s+i\omega)} = \frac{s+i\omega}{s^2+\omega^2}$$

$$= \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$$

$$\therefore \mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2}$$

Theorem 2 First Shifting Theorem

$$\boxed{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$$

inverse: $\boxed{e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}}$

Pf. $F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$$= \int_0^\infty e^{-st}\left[e^{at}f(t)\right]dt$$

$$= \mathcal{L}\{e^{at}f(t)\}$$

Ex. 5

$\mathcal{L}(e^{at}\cos\omega t)$

$$= \frac{s-a}{(s-a)^2+\omega^2}$$

$\mathcal{L}(e^{at}\sin\omega t)$

$$= \frac{\omega}{(s-a)^2+\omega^2}$$

Table ~~5~~ 6. 1.

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}} \qquad (n = 0, 1, \cdots)$$

Induction

$n=0$: $\mathcal{L}(1) = \frac{1}{s}$.

if $n=n$. $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ is true, then

$\mathcal{L}(t^{n+1}) = \frac{(n+1)!}{s^{n+2}}$ must be true

$$\mathcal{L}(t^{n+1}) = \int_0^\infty \underbrace{e^{-st}}_{v'}\underbrace{t^{n+1}}_{u}\,dt = -\frac{1}{s}e^{-st}t^{n+1}\Big|_0^\infty + \frac{1}{s}\int_0^\infty (n+1)t^n e^{-st}\,dt$$

$$= \frac{n+1}{s}\underbrace{\int_0^\infty e^{-st}\cdot t^n\,dt}_{\mathcal{L}(t^n)} = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}}$$

$$= \frac{(n+1)!}{s^{n+2}}$$

//

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} \cdot t^a \, dt \qquad (a>0)$$

set

$st = x. \qquad dt = \frac{dx}{s}$

$$\mathcal{L}(t^a) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^a \left(\frac{dx}{s}\right) = \frac{1}{s^{a+1}} \underbrace{\int_0^\infty e^{-x} \cdot x^a \, dx}_{=\Gamma(a+1)} = \frac{\Gamma(a+1)}{s^{a+1}}$$

Ex. 6. $\mathcal{L}(t^n e^{at}) = ?$

using $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$\mathcal{L}(fe^{at}) = F(s-a)$

$\therefore \mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$

$\mathcal{L}(te^{at}) = \frac{1}{(s-a)^2}$

Existence and Uniqueness - reading assigned. $|f(t)| \le Me^{+kt}$

PS 5.1

\# 1, 5, 11, 19, 25, 33, 39

6.2. Transforms of Derivatives and Integrals

Theorem 1.

$$\boxed{\mathcal{L}(f') = s\mathcal{L}(f) - f(0).} \qquad s > k$$

Pf. $\mathcal{L}(f') = \int_0^\infty \underset{v}{e^{-st}} \underset{u'}{f'(t)} \, dt = \underbrace{\left[e^{-st} f(t)\right]_0^\infty}_{= 0 - f(0)} + s \underbrace{\int_0^\infty e^{-st} f(t) \, dt}_{= \mathcal{L}(f)}$

$= s\mathcal{L}(f) - f(0)$

Extension.

$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0)$

$= s[s\mathcal{L}(f) - f(0)] - f'(0) \Rightarrow$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\vdots$$

Theorem 2

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Ex. 1.* $f(t) = t^2$. $\mathcal{L}(f) = ?$ $\Leftarrow \mathcal{L}(1) = \frac{1}{s}$

$f' = 2t$, $f'' = 2$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s\underbrace{f(0)}_{=0} - \underbrace{f'(0)}_{=0}$$

$$\mathcal{L}(t^2) = \frac{1}{s^2}\mathcal{L}(f'') = \frac{1}{s^2}\cdot\frac{2}{s} = \frac{2}{s^3}$$

Ex. 2* $\mathcal{L}(\cos \omega t) = ?$

$f = \cos\omega t$. $f' = -\omega\sin\omega t$, $f'' = -\omega^2\cos\omega t = -\omega^2 f$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s\underbrace{f(0)}_{=1} - \underbrace{f'(0)}_{=0} = s^2\mathcal{L}(f) - s$$

$$\mathcal{L}(f'') = -\omega^2\mathcal{L}(f)$$

$$\therefore\ s^2\mathcal{L}(f) - s = -\omega^2\mathcal{L}(f)$$

$$\mathcal{L}(f) = \frac{s}{s^2+\omega^2}$$

Ex. 3* $\mathcal{L}(\sin^2 t) = ?$

$f = \sin^2 t$. $f' = 2\sin t\cos t = \sin 2t$

$$\mathcal{L}(f') = s\mathcal{L}(f) - \underbrace{f(0)}_{=0} = s\mathcal{L}(f)$$

$$\mathcal{L}(f) = \frac{1}{s}\mathcal{L}(f') = \frac{1}{s}\mathcal{L}(\sin 2t) = \frac{1}{s}\cdot\frac{2}{s^2+4} = \frac{2}{s(s^2+4)}$$

Ex.4. $f(t) = t\sin\omega t$

$$f' = \sin\omega t + \omega t\cos\omega t$$

$$f'' = \omega\cos\omega t + \omega\cos\omega t - \omega^2 t\sin\omega t$$

$$= 2\omega\cos\omega t - \omega^2 f$$

$$\mathcal{L}(f'') = 2\omega\mathcal{L}(\cos\omega t) - \omega^2\mathcal{L}(f)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s\underbrace{f(0)}_{=0} - \underbrace{f'(0)}_{=0}$$

$$s^2\mathcal{L}(f) = -\omega^2\mathcal{L}(f) + 2\omega\cdot\frac{s}{s^2+\omega^2}$$

$$\mathcal{L}(t\sin\omega t) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

* Differential Equations

IVP

$$y'' + ay' + by = r(t). \qquad y(0) = K_0, \; y'(0) = K_1$$

$$r(t): \text{input} \longrightarrow y(t): \text{output}$$

step 1. $\mathcal{L}(y''+ay'+by) = \mathcal{L}(r)$

$$s^2Y - sy(0) - y'(0) + a[sY - y(0)] + bY = R(s)$$

$$(s^2+as+b)Y = ((s+a)y(0) + y'(0) + R(s)$$

step 2. $Q(s) = \dfrac{1}{s^2+as+b}$: transfer function

$$Y = [(s+a)y(0) + y'(0)]Q(s) + R(s)Q(s)$$

If $y(0) = y'(0) = 0$. $Y = RQ$

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

Step 3. $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

- Ex. ~~5~~ 4 $\quad y'' - y = t. \quad y(0) = 1. \quad y'(0) = 1 \quad$: t-space

Step 1 $\quad s^2Y - sy(0) - y'(0) - Y = \frac{1}{s^2} \quad$: s-space

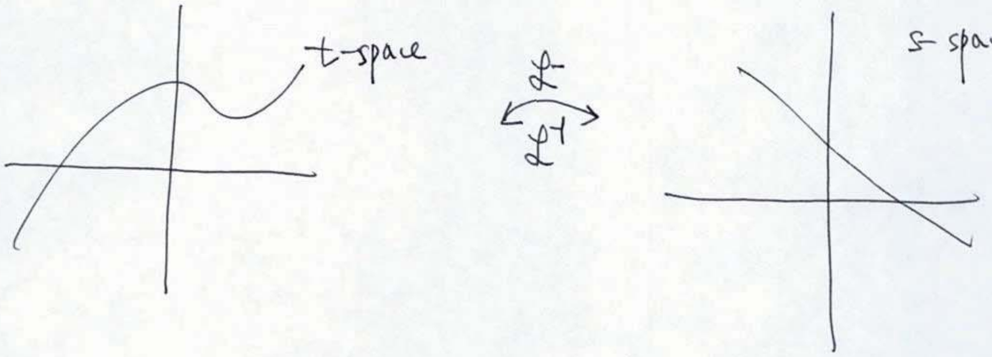
$$(s^2-1)Y = s + 1 + \frac{1}{s^2}$$

Step 2 $\quad Y = \frac{1}{s-1} + \frac{1}{s^2(s^2-1)}$

$$= \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$

Step 3. $\quad y(t) = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$

$$= e^t + \sinh t - t \quad : \text{t-space}$$

- 

~~Ex. 6.~~ HW

method in Chap. 2 $\Leftrightarrow$ Laplace Transform

* Laplace Transform of the Integral of a function

- Theorem 3.

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$$

Let $g(t) = \int_0^t f(\tau)\,d\tau$

$$g' = f$$

$$\mathcal{L}(f) = \mathcal{L}(g') = s\mathcal{L}(g) - \underbrace{g(0)}_{=0}$$

$$\therefore \mathcal{L}(g) = \frac{1}{s}\mathcal{L}(f) \quad : \quad \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s)$$

Ex. ~~7~~ 3(a) $\mathcal{L}(f) = \dfrac{1}{s(s^2+\omega^2)}$. $\quad f(t) = ?$

$$= \frac{1}{s}\cdot\frac{\omega}{s^2+\omega^2}\cdot\frac{1}{\omega}$$

$$f = \frac{1}{\omega}\mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{\omega}{s^2+\omega^2}\right)$$

$$= \frac{1}{\omega}\cdot\int_0^t \sin\omega\tau\,d\tau$$

$$= \frac{1}{\omega}\cdot\left(-\frac{1}{\omega}\right)\left[\cos\omega\tau\right]_0^t$$

$$f = \frac{1}{\omega^2}(1-\cos\omega t)$$

Ex. ~~8~~ 3(b) Prove $\quad \mathcal{L}\left\{\frac{1}{\omega^3}(\omega t - \sin\omega t)\right\} \overset{\mathcal{L}^{-1}}{=} \dfrac{1}{s^2(s^2+\omega^2)}$

i) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\dfrac{1}{(s^2+\omega^2)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{\omega^2}\left(\dfrac{1}{s^2} - \dfrac{1}{s^2+\omega^2}\right)\right\}$

$$= \frac{1}{\omega^2}\mathcal{L}^{-1}\left(\frac{1}{s^2} - \frac{1}{s^2+\omega^2}\right) = \frac{1}{\omega^2}\left(t - \frac{1}{\omega}\sin\omega t\right)$$

ii) $\mathcal{L}^{-1}\left\{\dfrac{1}{s}\cdot\dfrac{1}{s(s^2+\omega^2)}\right\} = \dfrac{1}{\omega^2}\displaystyle\int_0^t (1-\cos\omega\tau)\,d\tau$

$$= \frac{1}{\omega^2}\left(t - \frac{1}{\omega}\sin\omega t\right)$$

Ex. ~~9~~ 6 Shifted data problems

I.C. : $t = t_0$ instead of $t=0$

(i) G.S. (L.T.) → determine constants

(ii) Set $t = \tilde{t} + t_0$.

I.C. $t=t_0 \to \tilde{t} = 0$.

e.g. $y'' + y = 2t$. $y(\frac{\pi}{4}) = \frac{\pi}{2}$. $y'(\frac{\pi}{4}) = 2-\sqrt{2}$

$t = \tilde{t} + \frac{\pi}{4}$. $\tilde{y}(\tilde{t}) = y(t)$

$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \frac{\pi}{4})$. $\tilde{y}(0) = \frac{\pi}{2}$, $\tilde{y}'(0) = 2-\sqrt{2}$

Step 1. L.T.

$$s^2\tilde{Y} - s(\tfrac{\pi}{2}) - (2-\sqrt{2}) + \tilde{Y} = 2\frac{1}{s^2} + \frac{\pi}{2}\frac{1}{s}$$

Step 2. $$\tilde{Y} = \frac{2}{s^2(s^2+1)} + \frac{\pi/2}{s(s^2+1)} + \frac{\pi}{2}\frac{s}{s^2+1} + \frac{2-\sqrt{2}}{s^2+1}$$

Step 3. $$\tilde{y} = 2(\tilde{t} - \sin\tilde{t}) + \frac{\pi}{2}(1-\cos\tilde{t}) + \frac{\pi}{2}\cos\tilde{t} + (2-\sqrt{2})\sin\tilde{t}$$

$$\tilde{t} = t - \frac{\pi}{4}$$

$$y = 2t - \sin t + \cos t$$

PS 5.2

1, 5, 9, 15, 19

~~5~~ 6. 3. Unit step function, ~~Dirac's Delta function~~

* unit step function $u(t-a)$