Ex. ~~9~~ 6 Shifted data problems

I.C. : $t = t_0$ instead of $t = 0$

(i) G.S. (L.T.) $\rightarrow$ determine constants

(ii) Set $t = \tilde{t} + t_0$.

I.C. $t = t_0 \rightarrow \tilde{t} = 0$.

e.g. $y'' + y = 2t$. $y(\frac{\pi}{4}) = \frac{\pi}{2}$. $y'(\frac{\pi}{4}) = 2 - \sqrt{2}$

$t = \tilde{t} + \frac{\pi}{4}$ $\tilde{y}(\tilde{t}) = y(t)$

$\tilde{y}'' + \tilde{y} = 2(\tilde{t} + \frac{\pi}{4})$. $\tilde{y}(0) = \frac{\pi}{2}$, $\tilde{y}'(0) = 2 - \sqrt{2}$

step 1. L.T.

$$s^2\tilde{Y} - s(\tfrac{\pi}{2}) - (2-\sqrt{2}) + \tilde{Y} = 2\frac{1}{s^2} + \frac{\pi}{2}\frac{1}{s}$$

step 2.
$$\tilde{Y} = \frac{2}{s^2(s^2+1)} + \frac{\pi/2}{s(s^2+1)} + \frac{\pi}{2}\frac{s}{s^2+1} + \frac{2-\sqrt{2}}{s^2+1}$$

step 3.
$$\tilde{y} = 2(\tilde{t} - \sin\tilde{t}) + \frac{\pi}{2}(1 - \cos\tilde{t}) + \frac{\pi}{2}\cos\tilde{t} + (2-\sqrt{2})\sin\tilde{t}$$

$$\tilde{t} = t - \frac{\pi}{4}$$

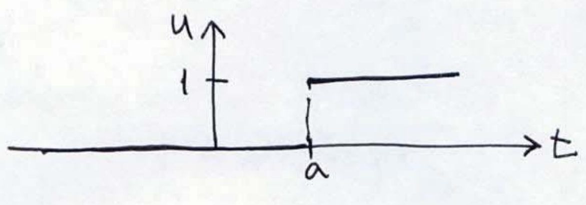
$$y = 2t - \sin t + \cos t$$

PS 5.2

\# 1, 5, 9, 15, 19

~~5~~ 6.3. Unit step function, ~~Dirac's Delta function~~

* unit step function $u(t-a)$



$$u(t-a) = \begin{cases} 0 & \text{if } t < a \quad (\text{off}) \\ 1 & \text{if } t > a \quad (\text{on}) \end{cases} \qquad a \geq 0$$

Heaviside fn Fig 112/113.

Theorem 1. Second shifting theorem

$$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$$

$$= e^{-as}\mathcal{L}\{f(t)\}$$

$$f(t-a)\,u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$$

pf.
$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau$$

$$= \int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau$$

$$\tau + a = t$$

$$e^{-as}F(s) = \int_a^\infty e^{-st}f(t-a)\,dt \quad + \int_0^a 0\,dt$$

$$= \int_0^\infty e^{-st}f(t-a)\,u(t-a)\,dt$$

$$= \mathcal{L}\{f(t-a)\,u(t-a)\}$$

L.T. of unit step fn

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \qquad (s>0)$$

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2} \quad \cdots$$

shifting theorems

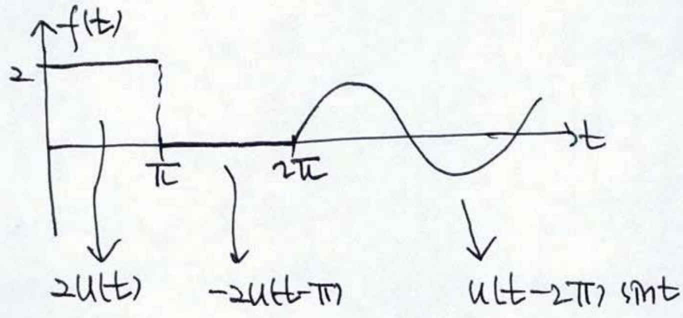
$$\left[\begin{array}{l} \mathcal{L}\{e^{at} f(t)\} = F(s-a) \\ \mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as} F(s) \end{array}\right.$$

$$\mathcal{L}(y') = sY - y(0)$$

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s} F(s)$$

Ex. 1.*

$$f(t) = \begin{cases} 2 & 0 < t < \pi \\ 0 & \pi < t < 2\pi \\ \sin t & t > 2\pi \end{cases} \qquad \mathcal{L}(f) = ?$$



$$f(t) = 2u(t) - 2u(t-\pi) + u(t-2\pi)\sin(t-2\pi)$$

$$\mathcal{L}(f) = \frac{2}{s} - 2\frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}$$

Ex. 2*

$$F(s) = \frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{se^{-\pi s}}{s^2+1}$$

$$f(t) = ?$$

$$f(t) = 2t - 2(t-2)u(t-2) - 4u(t-2) + \cos(t-\pi)u(t-\pi)$$

$$= 2t - 2tu(t-2) - \cos t\, u(t-\pi)$$

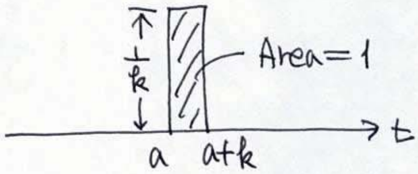
$$= \begin{cases} 2t & (0 < t < 2) \\ 0 & (2 < t < \pi) \\ -\cos t & (t > \pi) \end{cases}$$

Ex. 3. Practice

6.4

* Short impulses, Dirac's delta fn

$$f_k(t-a) = \begin{cases} \frac{1}{k} & a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$



· Impulse $f_k = \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1$

· $f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]$

$$\mathcal{L}\{f_k(t-a)\} = \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right] = e^{-as} \cdot \frac{1-e^{-ks}}{ks}$$

Dirac delta function (unit impulse fn)

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a)$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

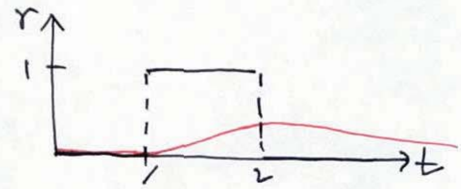
$$\delta(t-a) = \begin{cases} \infty & \text{if } t=a \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^\infty \delta(t-a)\,dt = 1.$$

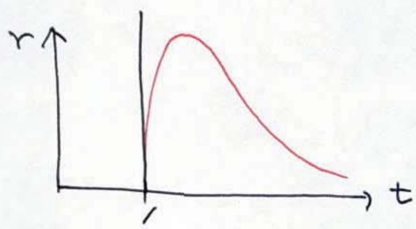
spring-mass-damper system.

Ex. ~~4~~. 1&2 $y''+3y'+2y=r(t)$. $y(0)=0$, $y'(0)=0$.

1. ~~(A)~~ $r(t)=u(t-1)-u(t-2)$



2 ~~(B)~~ $r(t)=\delta(t-1)$



Sol. ~~(A)~~ 1 $s^2Y+3sY+2Y=\frac{e^{-s}}{s}-\frac{e^{-2s}}{s}$

$$Y=\frac{1}{s(s+1)(s+2)}(e^{-s}-e^{-2s})$$

$$=\underbrace{\left(\frac{1/2}{s}-\frac{1}{s+1}+\frac{1/2}{s+2}\right)}_{=F(s)}(e^{-s}-e^{-2s})$$

$$f(t)=\mathcal{L}^{-1}(F)=\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}$$

$$y=\mathcal{L}^{-1}\{F(s)e^{-s}-F(s)e^{-2s}\}$$

$$=f(t-1)u(t-1)-f(t-2)u(t-2)$$

$$=\begin{cases}0 & (0<t<1)\\ \frac{1}{2}-e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)} & (1<t<2)\\ -e^{-(t-1)}+\frac{1}{2}e^{-2(t-1)}+e^{-(t-2)}-\frac{1}{2}e^{-2(t-2)}. & (t>2)\end{cases}$$

Sol. (~~3~~ 2). $s^2Y + 3sY + 2Y = e^{-s}$

$$Y = \frac{1}{(s+1)(s+2)} e^{-s}$$

$$= \underbrace{\left(\frac{1}{s+1} - \frac{1}{s+2}\right)}_{= F(s)} e^{-s}$$

$$f(t) = \mathcal{L}^{-1}(F) = e^{-t} - e^{-2t}$$

$$y = \mathcal{L}^{-1}\{F(s) \cdot e^{-s}\}$$

$$= f(t-1)\,u(t-1) = \begin{cases} 0 & 0 \le t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & t > 1 \end{cases}$$

ps 5.3 # 5, 7, 11, 19, 23, 29 $\rightarrow$ p.160

p.160 $\rightarrow$ ~~5.4~~ 6.6 Differentiation and Integration of Transforms

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$$

$$F'(s) = -\int_0^\infty e^{-st}\left[t f(t)\right] dt$$

$$\therefore \boxed{\mathcal{L}\{t f(t)\} = -F'(s)} = -\frac{d}{ds}\mathcal{L}(f)$$

$$F'(s) = \mathcal{L}\{-tf\}$$

$$\mathcal{L}^{-1}\{F'(s)\} = -tf(t)$$

Ex. 1. $\mathcal{L}(t \sin\beta t) = -\frac{d}{ds}\mathcal{L}(\sin\beta t) = -\frac{d}{ds} \cdot \left(\frac{\beta}{s^2+\beta^2}\right)$

$$= \frac{\beta \cdot 2s}{(s^2+\beta^2)^2} = \frac{2\beta s}{(s^2+\beta^2)^2}.$$