Sol. ($\cancel{3}$ 2). $s^2Y + 3sY + 2Y = e^{-s}$

$$Y = \frac{1}{(s+1)(s+2)} e^{-s}$$

$$= \underbrace{\left(\frac{1}{s+1} - \frac{1}{s+2}\right)}_{= F(s)} e^{-s}$$

$$f(t) = \mathcal{L}^{-1}(F) = e^{-t} - e^{-2t}$$

$$y = \mathcal{L}^{-1}\{F(s) \cdot e^{-s}\}$$

$$= f(t-1)\,u(t-1) = \begin{cases} 0 & 0 \le t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & t > 1 \end{cases}$$

PS 5.3 # 5, 7, 11, 19, 23, 29 → p.160

~~5.4~~ 6.6 Differentiation and Integration of Transforms

→ p.160

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = -\int_0^\infty e^{-st}\left[t f(t)\right] dt$$

$$\therefore \boxed{\mathcal{L}\{t f(t)\} = -F'(s)} = -\frac{d}{ds}\mathcal{L}(f)$$

$$F'(s) = \mathcal{L}\{-tf\}$$

$$\mathcal{L}^{-1}\{F'(s)\} = -tf(t)$$

Ex. 1. $\mathcal{L}(t\sin\beta t) = -\frac{d}{ds}\mathcal{L}(\sin\beta t) = -\frac{d}{ds}\cdot\left(\frac{\beta}{s^2+\beta^2}\right)$

$$= \frac{\beta \cdot 2s}{(s^2+\beta^2)^2} = \frac{2\beta s}{(s^2+\beta^2)^2}.$$

$$\mathcal{L}(t\cos\beta t) = -\frac{d}{ds}\mathcal{L}(\cos\beta t)$$

$$= -\frac{d}{ds}\cdot\left(\frac{s}{s^2+\beta^2}\right)$$

$$= -\frac{s^2+\beta^2-2s^2}{(s^2+\beta^2)^2}$$

$$= \frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

$$\mathcal{L}(\sin\beta t - \beta t\cos\beta t) = \frac{\beta}{s^2+\beta^2} - \beta\frac{s^2-\beta^2}{(s^2+\beta^2)^2}$$

$$= \beta\frac{s^2+\beta^2-s^2+\beta^2}{(s^2+\beta^2)^2} = \frac{2\beta^3}{(s^2+\beta^2)^2}$$

$$\mathcal{L}(\sin\beta t + \beta t\cos\beta t) = \frac{2\beta s^2}{(s^2+\beta^2)^2}$$

* Integration of Transforms

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}}$$

$$\mathcal{L}^{-1}\left\{\int_s^\infty F(\tilde{s})\,d\tilde{s}\right\} = \frac{f(t)}{t}$$

Proof.

$$\int_s^\infty F(\tilde{s})\,d\tilde{s} = \int_s^\infty\left[\int_0^\infty e^{-\tilde{s}t}f(t)\,dt\right]d\tilde{s}$$

$$= \int_0^\infty\left[\int_s^\infty e^{-\tilde{s}t}f(t)\,d\tilde{s}\right]dt$$

$$= \int_0^\infty f(t)\Bigg[\underbrace{\int_s^\infty e^{-\tilde{s}t}\,d\tilde{s}}_{\shortparallel}\Bigg]dt$$

$$-\frac{1}{t}\left[e^{-\tilde{s}t}\right]_s^\infty = -\frac{1}{t}(0-e^{-st}) = \frac{e^{-st}}{t}$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \mathcal{L}\left\{\frac{f(t)}{t}\right\}$$

Ex. 2. $\mathcal{L}^{-1}\{\ln(1+\frac{\omega^2}{s^2})\}$

$$\frac{d}{ds}\left[\ln(1+\frac{\omega^2}{s^2})\right] = \frac{-2\omega^2 s^{-3}}{1+\frac{\omega^2}{s^2}} = \frac{-2\omega^2}{s^3+\omega^2 s}$$

$$= \frac{-2\omega^2}{s(s^2+\omega^2)}$$

$$= -2\omega^2 \cdot \left(\frac{a}{s} + \frac{bs+c}{s^2+\omega^2}\right)$$

$$\begin{cases} as^2+a\omega^2+bs^2+cs = 1 \\ (a+b)s^2+cs+a\omega^2 = 1 \\ a=\frac{1}{\omega^2},\quad b=-\frac{1}{\omega^2},\quad c=0 \end{cases}$$

$$= -2\omega^2\left(\frac{1}{\omega^2}\frac{1}{s} - \frac{1}{\omega^2}\frac{s}{s^2+\omega^2}\right)$$

$$= -2\left(\frac{1}{s} - \frac{s}{s^2+\omega^2}\right)$$

$$\int_s^\infty -2\left(\frac{1}{s}-\frac{s}{s^2+\omega^2}\right)ds = \left[\ln(1+\frac{\omega^2}{s^2})\right]_s^\infty = -\ln(1+\frac{\omega^2}{s^2})$$

$$\therefore \mathcal{L}^{-1}\{\ln(1+\frac{\omega^2}{s^2})\} = \mathcal{L}^{-1}\left\{\int_s^\infty 2\left(\frac{1}{s}-\frac{s}{s^2+\omega^2}\right)ds\right\} = \frac{f(t)}{t}.$$

$$f(t) = \mathcal{L}^{-1}\left\{2\left(\frac{1}{s}-\frac{s}{s^2+\omega^2}\right)\right\} = 2(1-\cos\omega t)$$

$$\therefore \mathcal{L}^{-1}\{\ln(1+\frac{\omega^2}{s^2})\} = \frac{2}{t}(1-\cos\omega t)$$

~~Ex.3~~ $\mathcal{L}^{-1}\{\ln(1-\frac{a^2}{s^2})\} = \frac{2}{t}(1-\cosh at)$. HW

* Differential Eq with Variable Coefficients

$$a(t)y''+b(t)y'+c(t)y=0.$$

$$\mathcal{L}(ty') = -\frac{d}{ds}\mathcal{L}(y') = -\frac{d}{ds}[sY-y(0)] = -Y - s\frac{dY}{ds}$$

$$\mathcal{L}(ty'') = -\frac{d}{ds}\mathcal{L}(y'') = -\frac{d}{ds}[s^2Y-sy(0)-y'(0)] = -2sY - s^2\frac{dY}{ds} + y(0)$$

Ex. 4. Laguerre's diff. eq..

$$ty'' + (1-t)y' + ny = 0. \qquad (n=0,1,2,\cdots)$$

L.T.

$$\left[-2sY - s^2\frac{dY}{ds} + y(0)\right] + sY - y(0) - \left(-Y - s\frac{dY}{ds}\right) + nY = 0$$

$$(s-s^2)\frac{dY}{ds} + (n+1-s)Y = 0$$

$$\frac{dY}{Y} = -\frac{n+1-s}{\underset{\underset{s(1-s)}{\parallel}}{s-s^2}}ds = \left(\frac{a}{s} + \frac{b}{s-1}\right)ds$$

$$as - a + bs = (a+b)s - a$$

$$a+b = -1$$
$$n+1 = -a$$
$$b = -1-a$$
$$= -1+n+1 = n$$

$$\frac{dY}{Y} = \left[\frac{-(n+1)}{s} + \frac{n}{s-1}\right]ds$$

$$\ln Y = n\ln(s-1) - (n+1)\ln s + c$$

$$= \ln\frac{(s-1)^n}{s^{n+1}} + c$$

$$Y = c\,\frac{(s-1)^n}{s^{n+1}}$$

Take $c=1$ for simplicity

$$Y = \frac{(s-1)^n}{s^{n+1}} \qquad \text{if } n=0:\ Y=\frac{1}{s}.\quad y=1$$

Note $\mathcal{L}(t^n e^{-t}) = \dfrac{n!}{(s+1)^{n+1}}$.

also $\mathcal{L}\{(t^n e^{-t})^{(n)}\} = s^n\mathcal{L}\{t^n e^{-t}\} - s^{n-1}\,\cancel{\text{I.C}(t=0)}^{\,0}\cdots$

$$= \frac{n!\,s^n}{(s+1)^{n+1}}$$

Using this, we get $\mathcal{L}\left\{\dfrac{e^t}{n!}\dfrac{d^n}{dt^n}(t^n e^{-t})\right\} = \dfrac{(s-1)^n}{s^{n+1}} = Y$

$$\mathcal{L}^{-1}\{Y\} = y = l_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t}) \qquad l_0 = 1.$$
$$n=1,2,\cdots$$

PS 5.4. $\rightarrow$ p.162

5, 7, 9, 13, 15

6.5. Convolution

Theorem 1. Convolution theorem

$$F(s) = \mathcal{L}(f), \quad G(s) = \mathcal{L}(g)$$

$$F(s) \cdot G(s) = H(s) = \mathcal{L}(h)$$

$$\underbrace{h(t)}_{\text{convolution of } f \text{ and } g} = (f * g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$$

$$h = \mathcal{L}^{-1}(F \cdot G)$$

Pf.

$$F(s) \cdot G(s) = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \cdot G(s)$$

$$= \int_0^\infty e^{-s\tau} G(s) f(\tau)\, d\tau$$

$$e^{-s\tau} G(s) = \mathcal{L}\{g(t-\tau) u(t-\tau)\}$$

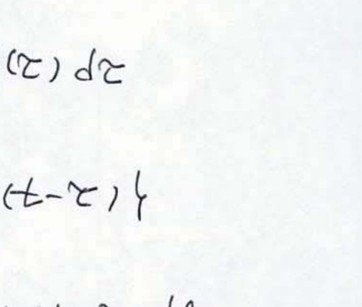
$$= \int_0^\infty e^{-st} g(t-\tau) u(t-\tau)\, dt$$

$$= \int_\tau^\infty e^{-st} g(t-\tau)\, dt$$

$$F(s) G(s) = \int_0^\infty f(\tau) \left[\int_\tau^\infty e^{-st} g(t-\tau)\, dt \right] d\tau \qquad s > k$$

$$\left[\begin{array}{l} t : \tau \to \infty \\ \tau : 0 \to \infty \end{array} \right.$$

$$\left[\begin{array}{l} \tau : 0 \to t \\ t : 0 \to \infty \end{array} \right.$$



$$F(s)\,G(s) = \int_0^\infty e^{-st} \underbrace{\int_0^t f(\tau)\,g(t-\tau)\,d\tau}_{= h(t)}\,dt$$

$$= \int_0^\infty e^{-st} h(t)\,dt = \mathcal{L}(h)$$

Ex. 1*

$$H(s) = \frac{1}{(s^2+1)^2}$$

$$h(t) = \mathcal{L}^{-1}(H) = ?$$

$$H(s) = \frac{1}{(s^2+1)} \cdot \frac{1}{(s^2+1)} = \mathcal{L}(\sin t)\cdot \mathcal{L}(\sin t)$$

$$h = \sin t * \sin t$$

$$= \int_0^t \sin\tau \sin(t-\tau)\,d\tau$$

$$= \int_0^t \frac{1}{2}\left[\cos(2\tau - t) - \cos t\right] d\tau$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin(2\tau - t) - \tau\cos t\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin t - t\cos t + \frac{1}{2}\sin t\right]$$

$$= \frac{1}{2}\left(\sin t - t\cos t\right)$$

Ex. 2*

$$H(s) = \frac{1}{s^3} = \frac{1}{s^2}\cdot\frac{1}{s} = \mathcal{L}(t)\cdot\mathcal{L}(1)$$

$$h = \mathcal{L}^{-1}(H) = t * 1$$

$$= \int_0^t \tau \cdot 1\,d\tau = \frac{t^2}{2}$$

Ex. ~~3~~. eg

$$H(s) = \frac{1}{s^2(s-a)} = \frac{1}{s^2} \cdot \frac{1}{s-a}$$

$$= \mathcal{L}(t)\,\mathcal{L}(e^{at})$$

$$h = t * e^{at} = \int_0^t \underset{u}{\tau}\, \underset{v'}{e^{a(t-\tau)}}\, d\tau$$

$$= \left[\tau \cdot \frac{1}{-a} e^{a(t-\tau)}\right]_0^t + \frac{1}{a}\int_0^t e^{a(t-\tau)}\, d\tau$$

$$= -\frac{t}{a} - \frac{1}{a^2}\left[e^{a(t-\tau)}\right]_0^t$$

$$= -\frac{t}{a} - \frac{1}{a^2}\left(1 - e^{at}\right)$$

$$= \frac{1}{a^2}\left(e^{at} - at - 1\right)$$

* Properties of the convolution

$f * g = g * f$: commutative law

$f * (g_1 + g_2) = f * g_1 + f * g_2$: distributive law

$(f * g) * v = f * (g * v)$: associative law

$f * 0 = 0 * f = 0.$

$f * 1 \neq f$

Application to nonhomogeneous

* Differential Eq

$$y'' + ay' + by = r(t) \qquad y(0) = y'(0) = 0$$

$$(s^2 + as + b)Y = R$$

$$Y = QR \quad . \quad Q = \frac{1}{s^2+as+b} \text{ : transfer fn.}$$

$$y = q * r \qquad q = \mathcal{L}^{-1}(Q)$$

$$= \int_0^t q(t-\tau)\, r(\tau)\, d\tau .$$

Ex. ~~4~~ 5. $y'' + 3y' + 2y = r(t) \quad . \qquad y(0) = y'(0) = 0$

$$r(t) = \begin{cases} 0 & 0 < t < 1 \\ 1 & 1 < t < 2 \\ 0 & t > 2 \end{cases}$$

$$Y = QR. \qquad Q = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$y = \int_0^t q(t-\tau)\, \underbrace{r(\tau)}\, d\tau. \qquad q(t) = \mathcal{L}^{-1}(Q) = e^{-t} - e^{-2t}$$

$0 < t < 1$: $y = 0$

$1 < t < 2$: $y = \int_0^t q(t-\tau)\, r(\tau)\, d\tau$

$$= \int_1^t \left[e^{-(t-\tau)} - e^{-2(t-\tau)} \right] d\tau$$

$$= \left[e^{-(t-\tau)} - \frac{1}{2} e^{-2(t-\tau)} \right]_1^t$$

$$= 1 - \frac{1}{2} - \left(e^{-(t-1)} - \frac{1}{2} e^{-2(t-1)} \right)$$

$$= \frac{1}{2} - e^{-(t-1)} + \frac{1}{2} e^{-2(t-1)}$$

$t > 2 \qquad y = \int_0^t \sim dt = \int_1^2 g(t-\tau)\, r(\tau)\, d\tau$

$$= e^{-(t-2)} - e^{-(t-1)} - \frac{1}{2}\left[e^{-2(t-2)} - e^{-2(t-1)}\right]$$

* Integral Eq.

Ex. ~~5~~ 6. $\quad y(t) = t + \int_0^t y(\tau)\sin(t-\tau)\, d\tau$

$$y = t + (y * \sin t)$$

$$Y = \frac{1}{s^2} + Y \cdot \frac{1}{s^2+1}$$

$$Y\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2}$$

$$Y = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$= \frac{1}{s^2} + \frac{3!}{s^4} \cdot \frac{1}{3!}$$

$$y = \mathcal{L}^{-1}(Y) = t + \frac{1}{6}t^3$$

→ p.153

PS 5.5 # 7, 17, 25, 31

p.153

§ ~~5.6.~~ Partial Fractions

Case 1. unrepeated factor $(s-a)$

Ex. 1. $\quad y'' + y' - 6y = 1. \qquad y(0) = 0. \quad y'(0) = 1$

$$Y(s) = \frac{s+1}{s(s-2)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$$

$$A_1 = -\frac{1}{6}, \quad A_2 = \frac{3}{10}, \quad A_3 = -\frac{2}{15}$$