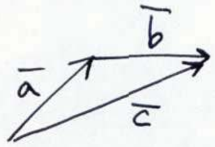
Chap. ~~8~~ 9. Vector Differential Calculus

8.1. Vector algebra in 2-D, 3-D.

· vector addition

$\bar{a} + \bar{b} = \bar{c}$



· linear combination

$\bar{b} = c_1 \bar{a}_1 + c_2 \bar{a}_2 + \cdots + c_m \bar{a}_m$

$c_1, c_2, \cdots, c_m$: scalar

· vectors : linearly independent

$c_1 \bar{a}_1 + c_2 \bar{a}_2 + \cdots + c_m \bar{a}_m = 0$

if and only if $c_1 = c_2 = \cdots = c_m = 0$

· dimension of R^n :

maximum possible number of vectors in a linearly independent set : n

8.2. Inner Product

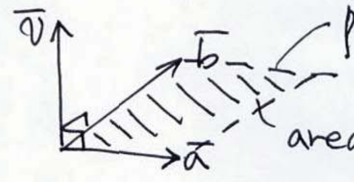
$\bar{a} \cdot \bar{b} = |\bar{a}| |\bar{b}| \cos\theta$

$|\bar{a} \cdot \bar{b}| \le |\bar{a}| |\bar{b}|$: Schwarz inequality

$|\bar{a} + \bar{b}| \le |\bar{a}| + |\bar{b}|$: triangle inequality

8.3. Vector Product (Cross product)

$$\bar{v} = \bar{a} \times \bar{b}$$



$\bar{a} = [a_1 \quad a_2 \quad a_3]$, $\quad \bar{b} = [b_1 \quad b_2 \quad b_3]$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

distributive: $\quad \bar{a} \times (\bar{b} + \bar{c}) = (\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c})$

$$(\bar{a} + \bar{b}) \times \bar{c} = \bar{a} \times \bar{c} + \bar{b} \times \bar{c}$$

not commutative: $\quad \bar{b} \times \bar{a} = -(\bar{a} \times \bar{b})$

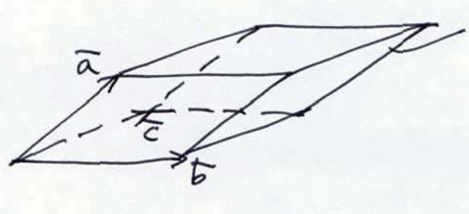
not associative: $\quad \bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$

* scalar triple product

$$(\bar{a} \ \bar{b} \ \bar{c}) = \bar{a} \cdot (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \cdot \bar{c}$$

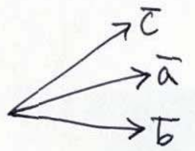
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

volume of parallelepiped



Theorem 1. Three vectors form a linearly independent set if and only if their scalar triple product is not zero.

If linearly dep.



in the same plane

$\rightarrow$ volume $= 0$.

§ 8.4. Vector and Scalar functions and fields. Derivatives

· vector function $\rightarrow$ vector field (velocity, acceleration, force, $\bar{E}$)

$$\bar{v} = \bar{v}(P) = [v_1(P), v_2(P), v_3(P)]$$

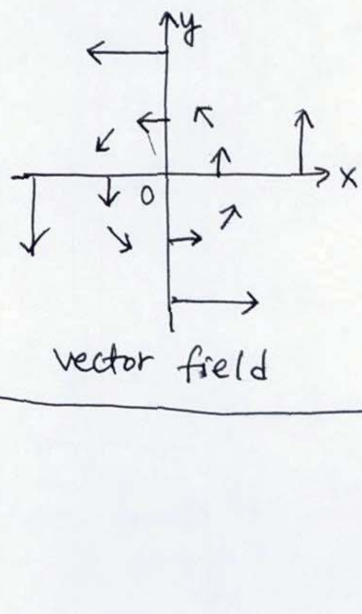
· scalar function $\rightarrow$ scalar field (temp, pressure)

$$f = f(P)$$

· $P(x, y, z)$: point in space

Ex. 2

$$\bar{v} = \omega(-y\hat{i} + x\hat{j})$$

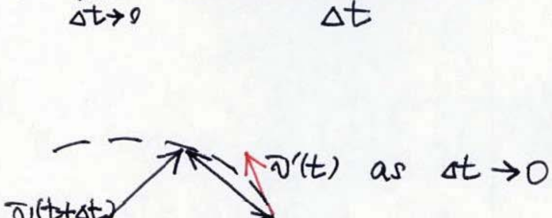


vector field

* Vector calculus

· Derivative

$$\bar{v}' = \lim_{\Delta t \to 0} \frac{\bar{v}(t+\Delta t) - \bar{v}(t)}{\Delta t}$$



$$\bar{v}'(t) = [v_1'(t), v_2'(t), v_3'(t)]$$

– rules

$$(\bar{u} \cdot \bar{v})' = \bar{u}' \cdot \bar{v} + \bar{u} \cdot \bar{v}'$$

$$(\bar{u} \times \bar{v}) = \bar{u}' \times \bar{v} + \bar{u} \times \bar{v}'$$

$$(\bar{u}\ \bar{v}\ \bar{w})' = (\bar{u}'\ \bar{v}\ \bar{w}) + (\bar{u}\ \bar{v}'\ \bar{w}) + (\bar{u}\ \bar{v}\ \bar{w}')$$

Ex. 4. vector function $\bar{v}(t)$.

of constant length $|\bar{v}| = c$

$\rightarrow$ properties of $\bar{v}' = ?$

$$c^2 = |\bar{v}|^2 = \bar{v} \cdot \bar{v}$$

$$\frac{d}{dt}(c^2) = 0 = 2\bar{v}' \cdot \bar{v}$$

$$\therefore \bar{v}' \cdot \bar{v} = 0$$

$\bar{v}'$: zero or $\perp \bar{v}$

* Partial Derivatives

$$\bar{v} = [v_1, v_2, v_3] = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

$$\frac{\partial \bar{v}}{\partial t_l} = \frac{\partial v_1}{\partial t_l}\hat{i} + \frac{\partial v_2}{\partial t_l}\hat{j} + \frac{\partial v_3}{\partial t_l}\hat{k}$$

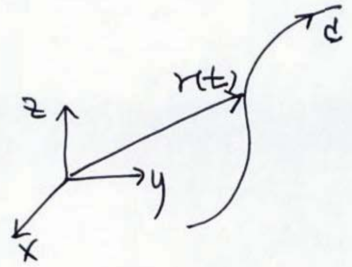
PS 8.4. # 17, 19, 21, 22

8.5. Curves, Tangents, Arc length

Differential geometry

: curves and surfaces in space

· parametric representation of Curve C



$$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

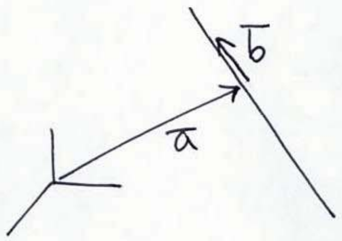
t: parameter

· other representation methods

$$\left[\begin{array}{l} y = f(x), \quad z = g(x) \\ F(x, y, z) = 0, \quad G(x, y, z) = 0. \end{array}\right.$$

Ex. 1. Straight Line

$r(t) = \bar{a} + t\bar{b}$



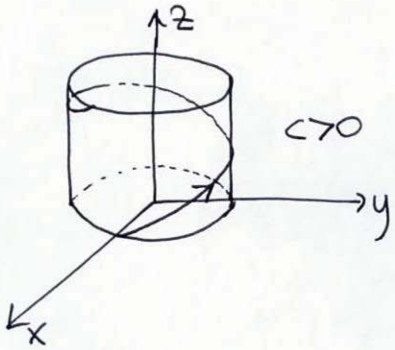
Ex. 2 Ellipse, circle

$$\bar{r}(t) = a\cos t\,\hat{i} + b\sin t\,\hat{j}$$

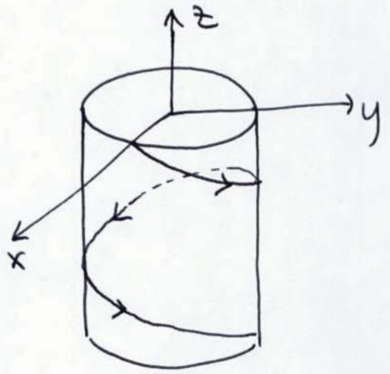
$$\frac{x^2}{a^2} + \frac{y^2}{b} = 1. \qquad z = 0.$$

Ex. 3. Circular helix

$$\bar{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + ct\,\hat{k}$$



right-handed helix

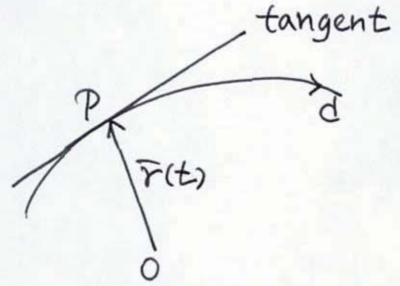


c<0

left-handed helix

· A simple curve has no multiple points

* Tangent to a curve



tangent vector $\quad \bar{r}'(t) = \lim_{\Delta t \to 0} \dfrac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t}$

tangent to C at P : $\quad \bar{q}(w) = \bar{r} + w\bar{r}'$

Ex.4. Tangent to an ellipse

$\frac{1}{4}x^2 + y^2 = 1. \qquad P : (\sqrt{2}, \frac{1}{\sqrt{2}})$

$\bar{r} = 2\cos t\,\hat{i} + \sin t\,\hat{j}$

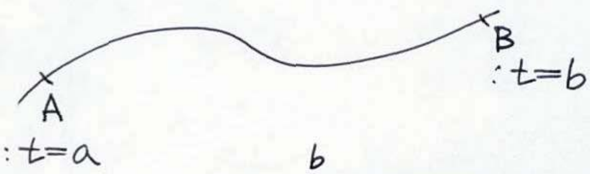
$\bar{r}'(t) = -2\sin t\,\hat{i} + \cos t\,\hat{j}$

$P : t = \frac{\pi}{4}$

$\bar{q}(w) = [\sqrt{2}, \frac{1}{\sqrt{2}}] + w\,[-\sqrt{2}, \frac{1}{\sqrt{2}}]$

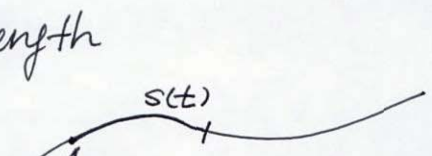
$= \sqrt{2}(1-w)\,\hat{i} + (\frac{1}{\sqrt{2}})(1+w)\,\hat{j}$

* Length of a curve



$l = \int_a^b \sqrt{\bar{r}' \cdot \bar{r}'}\, dt \qquad (\bar{r}' = \frac{d\bar{r}}{dt})$

· Arc length



$s(t) = \int_a^t \sqrt{\bar{r}' \cdot \bar{r}'}\, d\tilde{t} \qquad (\bar{r}' = \frac{d\bar{r}}{d\tilde{t}})$

$ds^2 = d\bar{r} \cdot d\bar{r}$
$= dx^2 + dy^2 + dz^2$

$C : \bar{r}(s)$

$$s(t) = \int_a^t |\bar{r}'| \, d\tilde{t}$$

$$\left(\frac{ds}{dt}\right)^2 = |\bar{r}'|^2 = \frac{d\bar{r}}{dt} \cdot \frac{d\bar{r}}{dt} = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$ds^2 = d\bar{r} \cdot d\bar{r} = dx^2 + dy^2 + dz^2$$

ds : linear element of C

* s as parameter

unit tangent vector

$$\bar{u} = \frac{1}{|\bar{r}'(t)|} \cdot \bar{r}'(t)$$

$$t = s \; : \; |\bar{r}'| = 1$$

$$\therefore \; \bar{u}(s) = \bar{r}'(s).$$