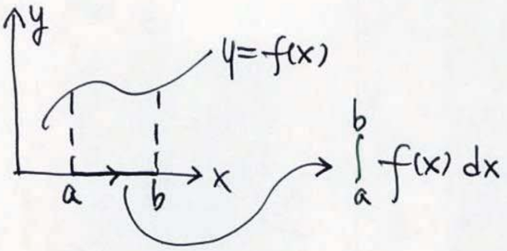
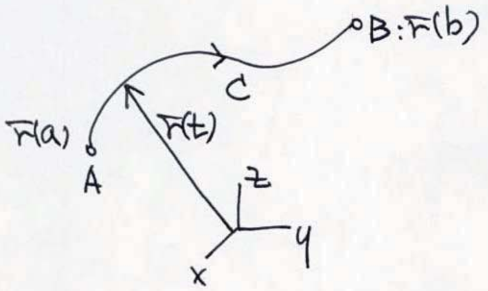
Chap. ~~9~~ 10 Vector Integral Calculus

~~9~~ 10.1. Line Integrals

simple example



In general



- Line integral of a vector function $\bar{F}(\bar{r})$ over C

$$\int_C \bar{F}(\bar{r})\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t))\cdot \frac{d\bar{r}}{dt}\,dt$$

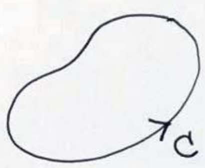
noting that
$$\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$
$$d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\int_C \bar{F}(\bar{r})\cdot d\bar{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int_a^b (F_1 x' + F_2 y' + F_3 z')\,dt$$

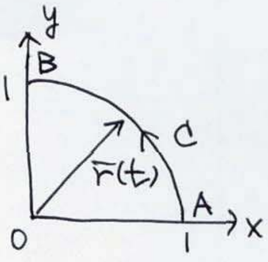
$$x' = \frac{dx}{dt}$$

- If C is closed:



$$\int_C = \oint_C$$

Ex. 1. $\overline{F}(\overline{r}) = -y\hat{i} - xy\hat{j}$. C:



$$\int_C \overline{F}\cdot d\overline{r} = ?$$

$$\overline{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} \qquad t: 0 \to \frac{\pi}{2}$$

$$= x\,\hat{i} + y\,\hat{j}$$

$$\overline{F}(\overline{r}) = -\sin t\,\hat{i} - \sin t\cos t\,\hat{j}$$

$$\int_C \overline{F}\cdot d\overline{r} = \int_0^{\frac{\pi}{2}} \overline{F}\cdot\frac{d\overline{r}}{dt}\,dt$$

$$= \int_0^{\pi/2} (-\sin t\,\hat{i} - \sin t\cos t\,\hat{j})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j})\,dt$$

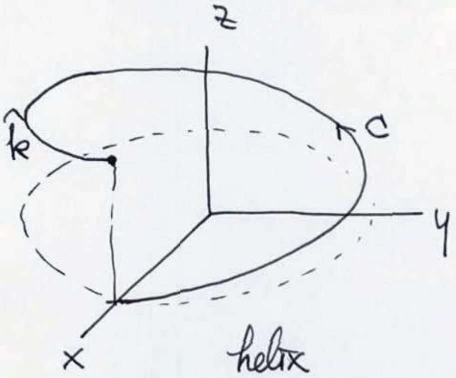
$$= \int_0^{\pi/2} (\sin^2 t - \sin t\cos^2 t)\,dt = \frac{\pi}{4} - \frac{1}{3}$$

Ex 2. $\overline{F}(\overline{r}) = z\hat{i} + x\hat{j} + y\hat{k}$

C: $\overline{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 3t\,\hat{k}$

$t: 0 \to 2\pi$

$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$



$$\int_C \overline{F}\cdot d\overline{r} = \int_0^{2\pi} \overline{F}\cdot\frac{d\overline{r}}{dt}\,dt$$

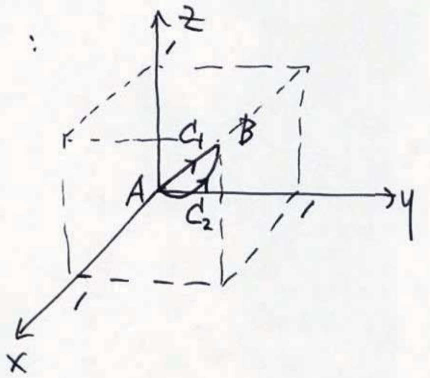
$$= \int_0^{2\pi} (3t\,\hat{i} + \cos t\,\hat{j} + \sin t\,\hat{k})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j} + 3\hat{k})\,dt$$

$$= 7\pi$$

Ex.3. Dependence of a line integral on path

$$\bar{F}(\bar{r}) = 5z\hat{i} + xy\hat{j} + x^2z\hat{k}$$

path :



C_1: $\bar{r}_1(t) = [t, t, t]$ $0 \le t \le 1$

C_2: $\bar{r}_2(t) = [t, t, t^2]$ $0 \le t \le 1$

path 1: $$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_0^1 (5t\hat{i} + t^2\hat{j} + t^3\hat{k})\cdot(\hat{i} + \hat{j} + \hat{k})\,dt$$

$$= \int_0^1 (5t + t^2 + t^3)\,dt = \frac{37}{12}$$

path 2: $$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_0^1 (5t^2\hat{i} + t^2\hat{j} + t^3\hat{k})\cdot(\hat{i} + \hat{j} + 2t\hat{k})\,dt$$

$$= \int_0^1 (5t^2 + t^2 + 2t^5) = \frac{28}{12}$$

In general, line integral is path-dependent.

* If $\bar{F}(\bar{r})$: force, c : displacement curve →

$$\text{Work} = \int_C \bar{F}\cdot d\bar{r}$$

Ex.5. Work vs. kinetic energy

$$W = \int_C \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}\cdot\frac{d\bar{r}}{dt}\,dt = \int_a^b \bar{F}\cdot\bar{v}\,dt$$

2nd law: $\bar{F} = m\bar{r}''(t) = m\bar{v}'(t)$

$$W = \int_a^b m\bar{v}'\cdot\bar{v}\,dt = \int_a^b \frac{m}{2}(|\bar{v}|^2)'\,dt = \frac{m}{2}|\bar{v}|^2\Big]_{t=a}^{t=b} = \Delta KE$$

* Other forms of line integral

$$\int_C f(\bar{r})\,dt = \int_a^b f(\bar{r}(t))\,dt$$

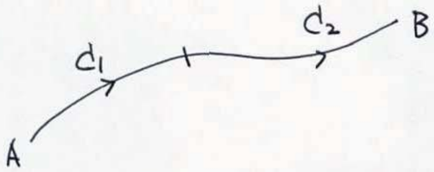
Ex. 6. $f = (x^2+y^2+z^2)^2$

$$C: \ \bar{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 3t\,\hat{k}$$
$$= x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

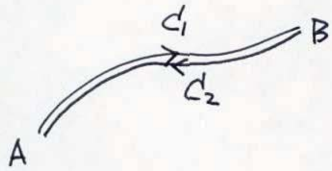
$$f = [\cos^2 t + \sin^2 t + (3t)^2]^2 = (1+9t^2)^2$$

$$\int_C f(\bar{r})\,dt = \int_0^{2\pi} (1+9t^2)^2\,dt \cong 160135.$$

* General properties of line integral



$$\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r}$$

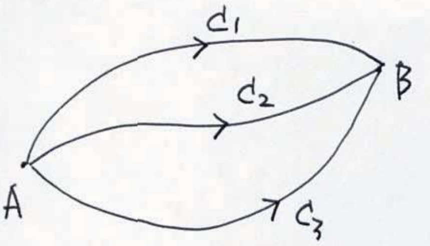


$$\int_{C_1} \bar{F}\cdot d\bar{r} = -\int_{C_2} \bar{F}\cdot d\bar{r}$$

PS 9.1 # 1, 7, 13, 19

Path independence of line integrals

10 ~~9~~.2 ~~Line integrals independent of path~~



$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_2} \bar{F}\cdot d\bar{r} = \cdots \quad : \text{path-independent}$$

Theorem 1. $\int_C \bar{F}\cdot d\bar{r}$: path-independent

iff $\bar{F} = \nabla f$ (f: potential)

OR $F_1 = \frac{\partial f}{\partial x}$, $F_2 = \frac{\partial f}{\partial y}$, $F_3 = \frac{\partial f}{\partial z}$

Proof. $\bar{F} = \nabla f \rightarrow \int_C \bar{F}\cdot d\bar{r}$: path-indep.

$$\int_C \bar{F}\cdot d\bar{r} = \int_A^B (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int_A^B \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt$$

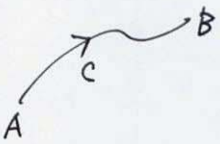
$$= \int_a^b \frac{df}{dt}\,dt = \Big[f(x(t), y(t), z(t))\Big]_{t=a}^{t=b}$$

$$= f(B) - f(A)$$

: only depends on endpoints

$\therefore$ If a line integral is path-independent,

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \int_C \nabla f\cdot d\bar{r} = f(B)-f(A)}$$



Ex. 1. $\displaystyle\int_C \bar{F}\cdot d\bar{r} = \int_C (2x\,dx + 2y\,dy + 4z\,dz)$: path-indep.

$\because\ \bar{F} = \nabla f$

$\dfrac{\partial f}{\partial x} = 2x,\quad \dfrac{\partial f}{\partial y} = 2y,\quad \dfrac{\partial f}{\partial z} = 4z$

$f = x^2+y^2+2z^2.$

$C:\ A(0,0,0) \rightarrow B(2,2,2)$

$r(t) = t\hat{i} + t\hat{j} + t\hat{k}.\qquad t: 0\rightarrow 2$

$$I = \int_0^2 (2t+2t+4t)\,dt = \int_0^2 8t\,dt = \left[4t^2\right]_0^2 = 16$$

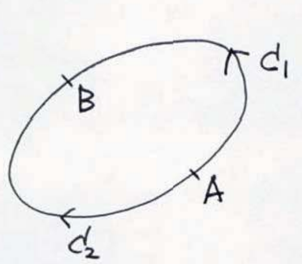
OR $I = f(2,2,2) - f(0,0,0) = 16$

Theorem 2.

$\displaystyle\int_C \bar{F}\cdot d\bar{r}$: path-independent

iff $\displaystyle\oint \bar{F}\cdot d\bar{r} = 0.$

Proof.



If $\displaystyle\int_C \bar{F}\cdot d\bar{r}$ is path-indep.

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_2} \bar{F}\cdot d\bar{r}.$$

$$\oint \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} - \int_{C_2} \bar{F}\cdot d\bar{r} = 0.$$

* Exactness and independence of path

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (\underbrace{F_1 dx + F_2 dy + F_3 dz}_{\text{exact } = df})$$

~ path-independent.

if $\bar{F} = \nabla f$

$$= \int_C \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= \int_C df$$

Theorem 3 $\int_C \bar{F}\cdot d\bar{r}$: path-independent, $\bar{F}\cdot d\bar{r}$: exact $\rightleftarrows$ curl $\bar{F} = 0$

$$\begin{cases} \dfrac{\partial F_3}{\partial y} = \dfrac{\partial F_2}{\partial z} \\ \dfrac{\partial F_1}{\partial z} = \dfrac{\partial F_3}{\partial x} \\ \dfrac{\partial F_2}{\partial x} = \dfrac{\partial F_1}{\partial y} \end{cases}$$

Proof. $\bar{F} = \nabla f$.

$\nabla\times(\nabla f) = 0$.

Ex. 3.

$$I = \int_C [2xyz^2 dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz]$$

$A(0,0,1) \to B(1, \frac{\pi}{4}, 2)$

exact? $(F_3)_y = (F_2)_z$ ...

$I = \int df$. $f = \int F_2\, dy = x^2z^2y + \sin yz + g(x, z)$

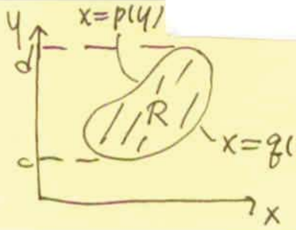
$$\frac{\partial f}{\partial x} = F_1 = 2xz^2y + \frac{\partial g}{\partial x} = 2xyz^2$$

$$\therefore \frac{\partial g}{\partial x} = 0. \quad g = g(z)$$

$$\frac{\partial f}{\partial z} = F_3 \Rightarrow \frac{\partial g}{\partial z} = 0.$$

$$f(x, y, z) = x^2yz^2 + \sin yz$$

$$I = f(B) - f(A) = \pi + 1$$



$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)\,dx \right] dy$$

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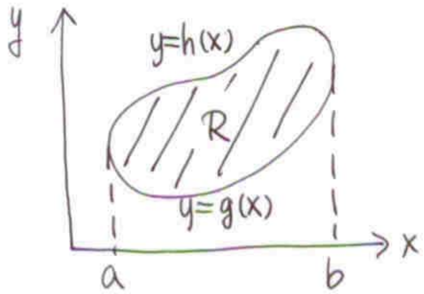
~~9~~10.3. Double Integrals Optional



$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\,dy \right] dx$$

* change of variables in double integrals

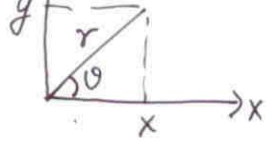
· $\displaystyle\int_a^b f(x)\,dx \overset{x\to u}{=} \int_\alpha^\beta f(x(u))\,\frac{dx}{du}\,du$

· $\displaystyle\iint_R f(x,y)\,dx\,dy \overset{(x,y)\to(u,v)}{=} \iint_{R^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

Jacobian $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

e.g. cartesian coordinates → polar coordinates

(x, y) (r, θ)



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$