## Basic conservation equations in fluid mechanics (no need to repeat)

* Conservation of mass (continuity)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{u}) = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \bar{u} = 0.$$

Incompressible flow : $\nabla \cdot \bar{u} = 0.$

* Conservation of momentum

Incompressible Newtonian fluid

→ Navier-Stokes equation

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{u} + \bar{g}$$

Inertia-free flow = Very viscous flow

= Stokes flow = Creeping flow = slow flow

* Steady viscous flow

$$\rho(\bar{u} \cdot \nabla)\bar{u} = -\nabla p + \mu \nabla^2 \bar{u}$$

For steady flows,

$$\frac{\left|\frac{\partial u}{\partial t}\right|}{\left|u\frac{\partial u}{\partial x}\right|} \sim \frac{\frac{U}{\tau}}{\frac{U^2}{L}} \sim \underbrace{\frac{L}{\tau U}}_{St} \ll 1. \quad \text{(Strouhal)}$$

$$\tau \gg \frac{L}{U}$$

2 cases where $(\bar{u} \cdot \nabla)\bar{u}$ term is negligible

① Low Reynolds number

② Thin film

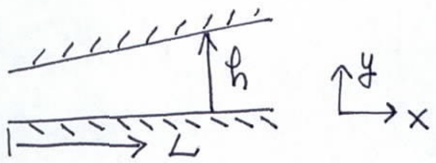
## Criteria for inertia-free flow

(1) $$\frac{O(\text{inertia})}{O(\text{viscous})} = \frac{O(\rho \frac{U^2}{L})}{O(\mu \frac{U}{L^2})} = \frac{\rho U L}{\mu}$$

$$= Re \ll 1$$

$\frac{\partial \bar{u}}{\partial t} +$ N.S. $\Rightarrow$
$\left(\bar{u}\cdot\nabla\bar{u} = \frac{1}{Re}\nabla^2\bar{u} - \nabla p\right)$

(2)




x-dir: $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ ... ①

y-dir: $\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$ ... ②

continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$ ... ③

③ $\Rightarrow O(v) \sim U\frac{h}{L}$, where $U = O(u)$

If $\boxed{\frac{h}{L} \ll 1}$ Ⓐ,

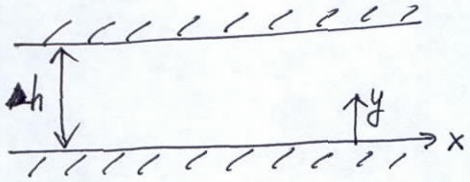
$$\frac{O\left(\frac{\partial^2 u}{\partial x^2}\right)}{O\left(\frac{\partial^2 u}{\partial y^2}\right)} \ll 1 \quad \& \quad O\left(\frac{\partial p}{\partial y}\right) \ll O\left(\frac{\partial p}{\partial x}\right)$$ : fully developed

$$\frac{O(\text{LHS of eq. ①})}{O\left(\mu\frac{\partial^2 u}{\partial y^2}\right)} = \boxed{\left(\frac{\rho U L}{\mu}\right)\left(\frac{h}{L}\right)^2 \ll 1}$$ Ⓑ

When Ⓐ & Ⓑ satisfied,

$$0 = -\frac{dP}{dx} + \mu\frac{\partial^2 u}{\partial y^2}$$

# Channel flow




Gov. Eq. $\dfrac{dp}{dx} = \mu \dfrac{d^2u}{dy^2}$

B.C. $\begin{cases} u(y=0) = \beta_1 \dfrac{du}{dy}\Big|_{y=0} \\ u(y=h) = -\beta_2 \dfrac{du}{dy}\Big|_{y=h} \end{cases}$ $\beta \geq 0.$

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} . \quad \left(\frac{dp}{dx} < 0\right)$$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx} y + c_1 .$$

$$u = \frac{1}{2\mu}\frac{dp}{dx} y^2 + c_1 y + c_2$$

B.C. $u(y=0) = c_2$ $\quad \dfrac{du}{dy}\Big|_{y=0} = c_1$

$$c_2 = \beta_1 c_1$$

$$u(y=h) = \frac{1}{2\mu}\frac{dp}{dx} h^2 + c_1 h + \beta_1 c_1 = -\beta_2 \left[\frac{1}{\mu}\frac{dp}{dx} h + c_1\right]$$

$$c_1 (h + \beta_1 + \beta_2) = \frac{1}{\mu}\frac{dp}{dx}\left(-h\beta_2 - \frac{h^2}{2}\right)$$

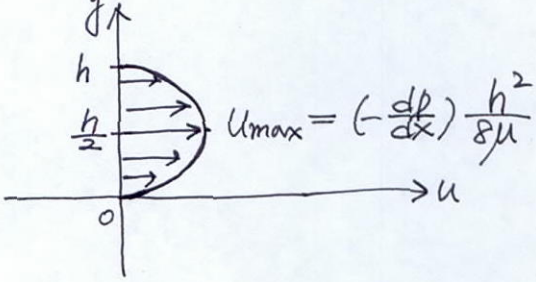
$$\begin{cases} c_1 = -\dfrac{1}{\mu}\dfrac{dp}{dx}\left(\dfrac{h\beta_2 + h^2/2}{h+\beta_1+\beta_2}\right). \\ c_2 = -\dfrac{\beta_1}{\mu}\dfrac{dp}{dx}\left(\dfrac{h\beta_2 + h^2/2}{h+\beta_1+\beta_2}\right). \end{cases}$$

(1) if $\beta_1 = \beta_2 = 0$ (no slip)

$$c_1 = -\frac{1}{\mu}\frac{dp}{dx}\left(\frac{h}{2}\right), \qquad c_2 = 0$$

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 - \frac{1}{2\mu}\frac{dp}{dx}hy$$

$$= \frac{1}{2\mu}\frac{dp}{dx}(y^2 - hy)$$




$$Q = \int_0^h \frac{1}{2\mu}\frac{dp}{dx}(y^2 - hy)\,dy$$

$$= \frac{1}{2\mu}\frac{dp}{dx}\left(\frac{h^3}{3} - \frac{h^3}{2}\right) = \left(-\frac{dp}{dx}\right)\frac{h^3}{12\mu}$$

$$u_b = \frac{Q}{h} = \left(-\frac{dp}{dx}\right)\frac{h^2}{12\mu}. \qquad \tau_w = \mu\frac{du}{dy}\Big|_{y=0} = \left(-\frac{dp}{dx}\right)\frac{h}{2}$$

$u_m = \frac{3}{2}u_b$

(2) if $\beta_1 = \beta_2 = \beta > 0$.

$$c_1 = -\frac{1}{\mu}\frac{dp}{dx}\,\frac{h\beta + h^2/2}{h+2\beta} = -\frac{1}{2\mu}\frac{dp}{dx}\,\frac{h(2\beta+h)}{h+2\beta}$$

$$= -\frac{1}{\mu}\frac{dp}{dx}\left(\frac{h}{2}\right) \quad \text{same as above}$$

$$c_2 = -\frac{\beta}{\mu}\frac{dp}{dx}\left(\frac{h\beta + h^2/2}{h+2\beta}\right)$$

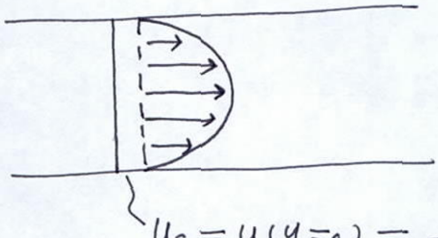
$$= -\frac{\beta}{2\mu}\frac{dp}{dx}\,\frac{h(2\beta+h)}{h+2\beta} = -\frac{1}{\mu}\frac{dp}{dx}\left(\frac{h\beta}{2}\right)$$

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 - \frac{1}{2\mu}\frac{dp}{dx}hy - \frac{1}{2\mu}\frac{dp}{dx}(h\beta)$$

$\rightarrow 0$ as $\frac{\beta}{h} \rightarrow 0$

$$= \frac{1}{2\mu}\frac{dp}{dx}(y^2 - hy - h\beta) = -\frac{1}{2\mu}\frac{dp}{dx}(-y^2 + hy + h\beta)$$

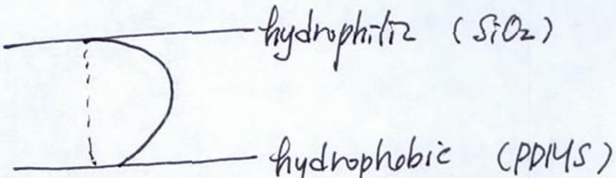
$$= -\frac{1}{2\mu}\frac{dp}{dx}h^2\left[\left(\frac{y}{h}\right)^2 + \left(\frac{y}{h}\right) + \left(\frac{\beta}{h}\right)\right]$$



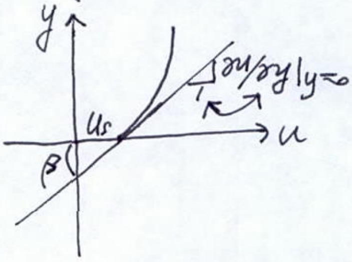
$u_s = u(y=0) = -\frac{1}{2\mu}\frac{dp}{dx} h\beta$.

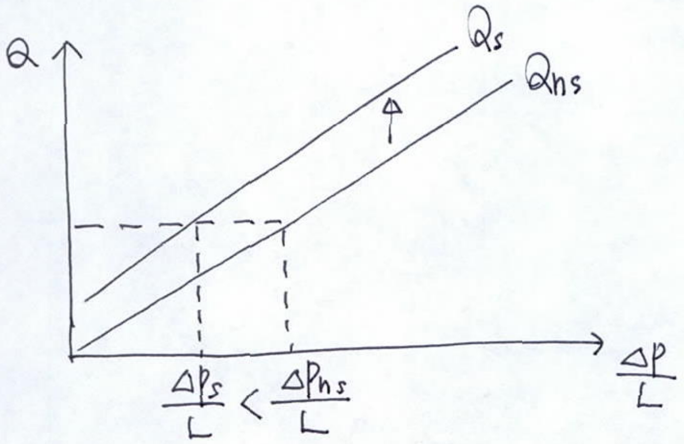
$$\begin{cases} u_{max} = u_{max,ns} + u_s \\ u_b = u_{b,ns} + u_s \\ Q = Q_{ns} + u_s h \end{cases}$$

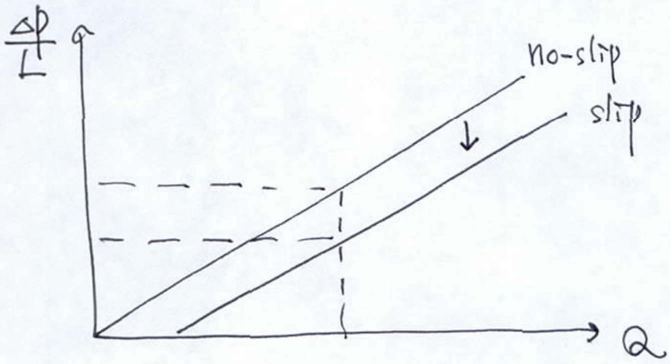
(3) if $\beta_1 \neq 0$, $\beta_2 = 0$.




* Physical significance of slip length $\beta$ [m].

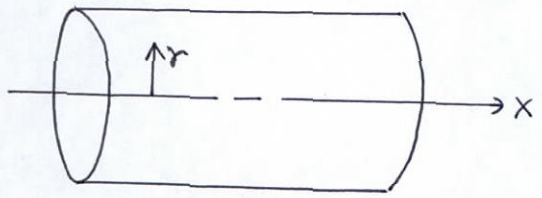



Q
Qs
Qns
ΔPs/L < ΔPns/L
ΔP/L


ΔP/L
no-slip
slip
Q

## Pipe flow

Hagen-Poiseuille flow




$$0 = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)u$$

OR

$$0 = -\frac{1}{\rho}\frac{dp}{dx} + \nu \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{1}{\mu}\frac{dp}{dx}$$

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{r}{\mu}\frac{dp}{dx}$$

$$r\frac{du}{dr} = \frac{r^2}{2\mu}\frac{dp}{dx}$$

$$\frac{du}{dr} = \frac{r}{2\mu}\frac{dp}{dx}$$

$$u = \frac{r^2}{4\mu}\frac{dp}{dx} + C$$

B.C. $u=0$ at $r=R$

$$u = -\left(\frac{dp}{dx}\right)\frac{1}{4\mu}\left(R^2 - r^2\right)$$

Volume flow rate $Q = \int u \, dA$

$$= \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dx}\right)$$

Pressure drop over $L$

$$\Delta p = \frac{8\mu L Q}{\pi R^4}$$

Darcy friction factor $f$

$$\Delta p = f \frac{L}{D} \rho \frac{U^2}{2}$$

$$f = \frac{\Delta p}{\frac{L}{D} \rho \frac{U^2}{2}} \qquad U = \frac{Q}{\pi R^2} \text{ : avg vel}$$

for a round pipe, $\quad f \equiv \left(-\frac{dp}{dx}\right) \frac{D_h}{\frac{1}{2}\rho U^2}$

$$f = \frac{\cancel{\Delta p}}{\frac{\cancel{L}}{D} \cdot \rho \frac{1}{2} U \cdot \frac{1}{\cancel{\pi} R^2} \cdot \left( \frac{\cancel{\pi} R^4}{8 \mu} \frac{\cancel{\Delta p}}{\cancel{L}} \right)}$$

$$= \frac{16}{\rho \frac{\cancel{L} U}{D R^2} \cdot \frac{R^4}{\mu}} = \frac{16}{\frac{\rho U R^2}{D \mu}}$$

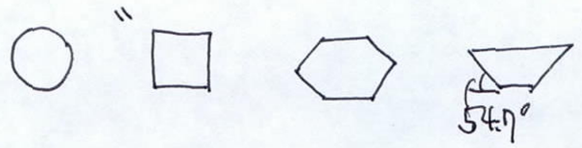
$$= \frac{16}{\rho U \left(\frac{D^2}{4}\right)} D \mu = \frac{64 \mu}{\rho U D} = \frac{64}{\frac{\rho U D}{\mu}}$$

$$\boxed{f = \frac{64}{Re_D}} \qquad Re_D = \frac{\rho U D}{\mu}$$

IN MOST CASES,

for fully developed flow through straight microchannels

$$\boxed{f \cdot Re_{D_h} = \text{const.}}$$




" " : numerical sol

"Handard"

* $D_h$ (hydraulic diameter)

$$D_h = \frac{4A}{P}$$

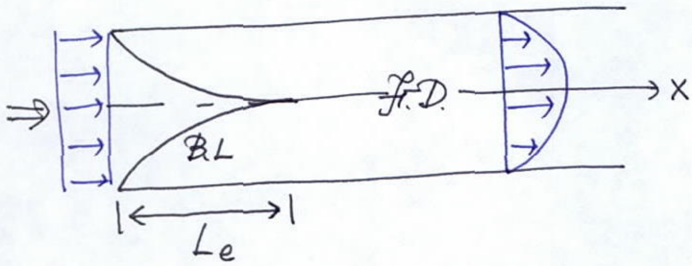
Note that

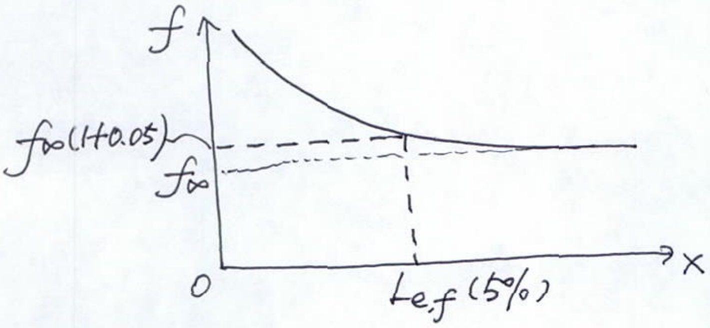
- $$f = \frac{\Delta P}{\frac{L}{\cancel{D}} \rho \cdot \frac{\cancel{U}}{\cancel{2}} \cdot \frac{Q \cdot \cancel{4}^2}{\pi R^{\cancel{2}}}} = \frac{C}{\frac{\rho U D}{\mu}}$$

$$\boxed{\frac{\Delta P}{L} \sim Q^1}$$

pressure drop

Entrance length

- 

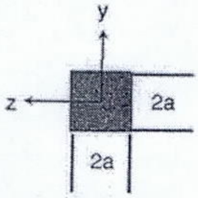
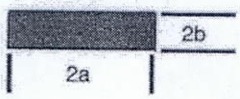
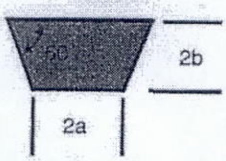
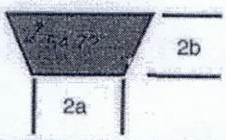




$$\frac{L_{e,f}(5\%)}{D} \simeq 0.05\, Re_D$$ : hydrodynamic entrance length

- . thermal entrance length $L_{eT}$. $Nu \rightarrow f$

$$\frac{L_{eT}(5\%)}{D} = 0.017\, Re_D Pr$$

**TABLE 6.2** Resistance to Flow in Fully Developed Flow Through Straight Microchannels of Various Cross-Sectional Geometries

| Cross Section | | $f\,Re$ | $u_{max}/u_B$ |
|---|---|---|---|
|   | | 64 | 2.000 |
|   | | 56.92 | 2.0962 |
|   α=b/a | | $96(1 - 1.3553\alpha + 1.9467\alpha^2$ $-1.7012\alpha^3 + 0.9564\alpha^4$ $-0.2537\alpha^5)$ | — |
|   | | 96 | 1.5000 |
|   | | 60 | — |
|   | 2b/2a | | |
| | 4.000 | 55.66 | 2.181 |
| | 2.000 | 55.22 | 2.162 |
| | 1.000 | 56.60 | 2.119 |
| | 0.500 | 62.77 | 1.969 |
| | 0.250 | 72.20 | 1.766 |
|   | 1.000 | 56.15 | 2.137 |

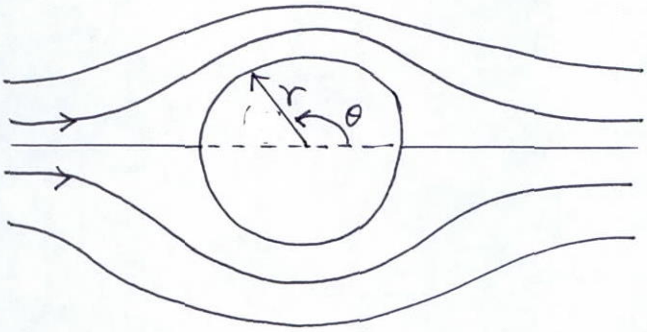
*Source:* Data from Shah, R.K., and London, A.L. (1978) *Advances in Heat Transfer*, Suppl. 1, Academic Press, New York.

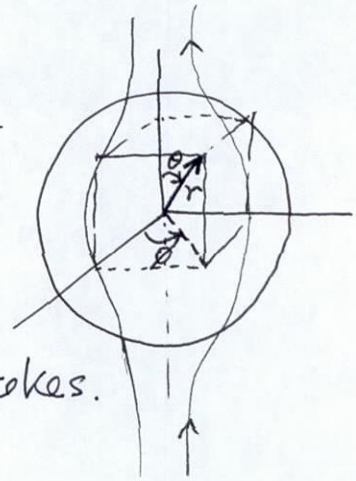
## 6.2 Experimental Studies of Flow Through Microchannels

Despite the fundamental simplicity of laminar flow in straight ducts, experimental studies of microscale flow have often failed to reveal the expected relationship between the friction factor and Reynolds number. Further, flow discrepancies are neither consistently higher nor lower than macroscale predictions. A summary of the experiments that have been conducted to investigate the behavior of fluid flow in microchannels, over a large range of Reynolds numbers, geometries and experimental conditions, is presented in Table 6.3. In reviewing these results, they will be grouped according to the results of friction factor measurements (follow macroscale predictions, higher than predictions, and lower than predictions).

The first experimental investigations of flow through microchannels in the early 1980s were motivated by the interest in high-performance heat sinking. The large surface-to-volume ratios of microchannels make them excellent candidates for efficient heat transfer devices, as discussed in the introduction to this chapter. Tuckerman and Pease (1981) studied flow through an array of microchannels with approximately rectangular cross sections (height range 50 to 56 μm, width range 287 to 320 μm). Although this study was focused primarily on heat transfer characteristics, they "confirmed that the flow rate obeyed

# Low Reynolds number flow past a sphere




Stokes.

axisymmetric flow in spherical polar coordinates

$$\bar{u} = [u_r(r,\theta),\ u_\theta(r,\theta),\ 0]$$

stream function $\psi(r,\theta)$

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$

$\nabla\cdot\bar{u} = 0$ automatically.

$$0 = -\nabla p + \mu \nabla^2 \bar{u} \qquad \cdots ①$$

Using the vector identity

$$\nabla^2\bar{u} = \nabla(\nabla\cdot\bar{u}) - \nabla\times(\nabla\times\bar{u})$$

Eq. ① is rewritten as

$$\nabla p = -\mu \nabla\times(\nabla\times\bar{u}) \qquad \cdots ②$$

$$\nabla\times\bar{u} = [0,\ 0,\ -\frac{1}{r\sin\theta}E^2\psi]$$

$E^2$ (differential operator)

$$= \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

From Eq ②

$$\begin{cases} \dfrac{\partial p}{\partial r} = \dfrac{\mu}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}E^2\psi \\ \dfrac{1}{r}\dfrac{\partial p}{\partial\theta} = \dfrac{-\mu}{r\sin\theta}\dfrac{\partial}{\partial r}E^2\psi \end{cases}$$

Eliminating pressure,

$$\left[ \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta} \left( \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \right) \right]^2 \psi = 0.$$

B.C. $\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0$ on $r = a$

$\psi \to \frac{1}{2} U r^2 \sin^2\theta$ as $r \to \infty$ $\left[ \begin{array}{l} u_r \to U\cos\theta \\ u_\theta \to U\sin\theta \end{array} \right.$

Solving, ($\psi = f(r)\sin^2\theta$) $\psi = \frac{1}{4} U \left( 2r^2 + \frac{a^3}{r} - 3ar \right) \sin^2\theta$

(symmetric fore and aft of the sphere)

What is the drag $D$?

Using $E^2\psi = \frac{3}{2} U a r^{-1} \sin^2\theta$,

$$p = p_\infty - \frac{3}{2} \frac{\mu U a}{r^2} \cos\theta$$

Stress components on the sphere:

$$\left\{ \begin{array}{l} \tau_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r} \\ \tau_{r\theta} = \mu r \frac{\partial}{\partial r}\left( \frac{u_\theta}{r} \right) + \frac{\mu}{r} \frac{\partial u_r}{\partial \theta} \\ \tau_{r\phi} = 0 \end{array} \right.$$

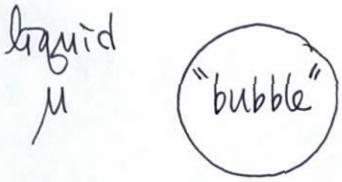
Stress component in the direction of net force $\rightarrow$.

$$\tau = \tau_r \cos\theta - \tau_\theta \sin\theta$$
$$= -p_\infty \cos\theta + \frac{3}{2} \frac{\mu U}{a}$$

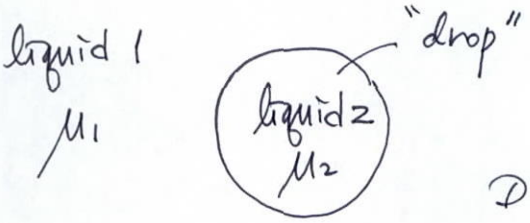
drag $D = \int_0^{2\pi} \int_0^{\pi} \tau a^2 \sin\theta \, d\theta \, d\phi = 6\pi\mu U a$

Stokes' law!

# Other external flows

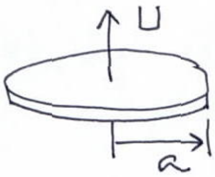
* 


$$D = 4\pi\mu U a$$

* 


$$D = 4\pi\mu_1 U a \left(\frac{\mu_1 + \frac{3}{2}\mu_2}{\mu_1 + \mu_2}\right)$$

* circular disk




$$D = 16\mu U a$$

* Elongated rod

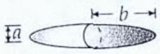
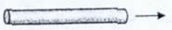
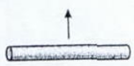
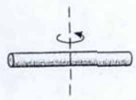

$\frac{a}{b} \gg 1.$

$$D = \frac{4\pi\mu U a}{\ln(a/b) + 0.19315}$$

J. Howard.

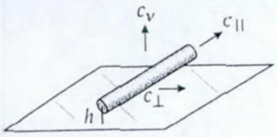
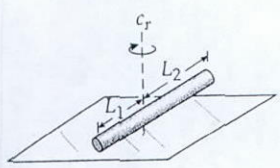


*Table 6.2 Drag coefficients in an unbounded solution*

| Parameter | Direction | Cylinder ($L>>r$) | Ellipsoid ($b>>a$) | Sphere |
|---|---|---|---|---|
| | |  |  |  |
| $\gamma_{\parallel}$ |  | $\dfrac{2\pi\eta L}{\ln(L/2r) - 0.20}$ | $\dfrac{4\pi\eta b}{\ln(2b/a) - 0.5}$ | $6\pi\eta r$ |
| $\gamma_{\perp}$ |  | $\dfrac{4\pi\eta L}{\ln(L/2r) + 0.84}$ | $\dfrac{8\pi\eta b}{\ln(2b/a) + 0.5}$ | $6\pi\eta r$ |
| $\gamma_{r}$ |  | $\dfrac{\frac{1}{3}\pi\eta L^3}{\ln(L/2r) - 0.66}$ | $\dfrac{\frac{8}{3}\pi\eta b^3}{\ln(2b/a) - 0.5}$ | $8\pi\eta r^3$ |
| $\gamma_{a}$ |  | $4\pi\eta r^2 L$ | $\frac{16}{3}\pi\eta a^2 b$ | $8\pi\eta r^3$ |

*Note:* The parallel, $\gamma_{\parallel}$, and perpendicular, $\gamma_{\perp}$, drag coefficients are defined by $F = \gamma_{\parallel}v$ and $F = \gamma_{\perp}v$, where $F$ is the force and $v$ is the velocity. The rotational drag coefficients are defined by $T = \gamma_r\omega$ and $T = \gamma_a\omega$, where $T$ is the torque and $\omega$ is the angular velocity. The values for cylinders are from Tiraldo and Garciá de la Torre, 1981, and those for prolate ellipsoids are from Perrin, 1934.

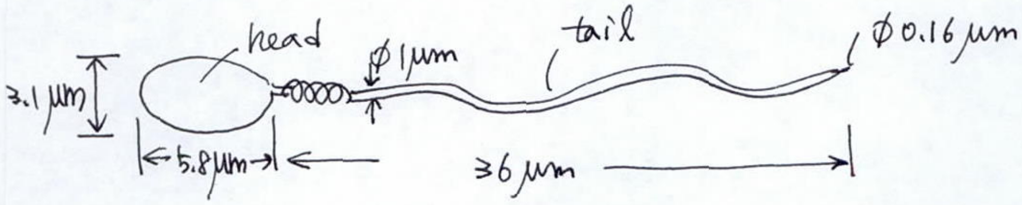
*Table 6.3 Drag coefficients (per unit length) for a cylinder near a plane surface*

| Definitions | Drag coefficient | Force or torque |
|---|---|---|
|  | $c_{\parallel} = \dfrac{2\pi\eta}{\cosh^{-1}(h/r)} \cong \dfrac{2\pi\eta}{\ln(2h/r)}$ | $F = c_{\parallel}Lv$ |
| | $c_{\perp} = 2c_{\parallel}$ | $F = c_{\perp}Lv$ |
|  | $c_v = 1/\left(c_{\perp}^{-1} - c_a^{-1}\right)$ | $F = c_v L v$ |
| | $c_a = \dfrac{4\pi\eta}{\left[1-(r/h)^2\right]^{1/2}}$ | $T = c_a L r^2 \omega$ |
| | $c_r = \dfrac{1}{3}c_{\perp}$ | $T = c_r\omega\left(L_1^3 + L_2^3\right)$ |

*Source:* From Hunt et al., 1994.

# Drag on a sperm

human sperm




$$\frac{V_{tail}}{V_{head}} \sim \frac{\frac{4}{3}\pi abc}{\frac{\pi}{3}\frac{L}{D}(D^3-d^3)} = \frac{\cancel{4}(5.8\times10^{-6})/\cancel{2}\cdot(3.1\times10^{-6})/\cancel{2}\cdot(3.1\times10^{-6})/2}{\frac{36\times10^{-6}}{10^{-6}}\left[10^{-18}-(0.16\times10^{-6})^3\right]}$$

$$= \frac{2.8\times10^{-17}}{3.6\times10^{-17}} = 0.78$$

in reality $\sim 0.3$

$F = \gamma v$

$\gamma_{head} \sim$ ~~...~~ ~~...~~ $\mu_w = 10^{-3}\ kg/m\cdot s$

$$\frac{4\pi\mu(2.9\times10^{-6})}{\ln\left(\frac{5.8}{1.6}\right)-0.5} = 4.6\times10^{-8}$$

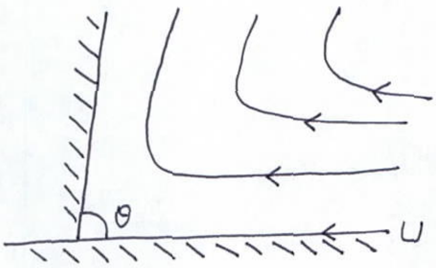
$$\gamma_{tail} \sim \frac{2\pi\mu L}{\ln\left(\frac{L}{2r}\right)-0.2} = \frac{2\pi(10^{-3})(36\times10^{-6})}{\ln\left(\frac{36}{1}\right)-0.2} = 6.7\times10^{-8}$$

$$\frac{\gamma_{tail}}{\gamma_{head}} \sim 1.5$$

if $v_{sperm} = 50\ \mu m/s$

$$f_d \sim (\gamma_{head}+\gamma_{tail})\cdot v \simeq 5.7\times10^{-12}\ N \sim 5.7\ pN$$

# Corner flow




Revisit the governing eq.

$$\nabla p = \mu \nabla^2 \bar{u}$$

$$\nabla \cdot \bar{u} = 0.$$

If the principal B.C. involve $\bar{u}$ alone, (pf. next pg)

$$\nabla^2 (\nabla \times \bar{u}) = 0 \quad , \quad \nabla \cdot \bar{u} = 0.$$

if 2-D: $\nabla \times \bar{u} = -\hat{k}\, \nabla^2 \psi$ $\qquad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \ u_\theta = -\frac{\partial \psi}{\partial r}$

$$\boxed{\nabla^2 (\nabla^2 \psi) = 0.}$$ : biharmonic eq.

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \psi = 0.$$

B.C. $\frac{\partial \psi}{\partial r} = 0 \quad , \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -U.$ at $\theta = 0$

$\frac{\partial \psi}{\partial r} = 0 \, , \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$ at $\theta = \theta_0$.

sol.) $\psi(r, \theta) = r f(\theta)$ :

$$f(\theta) = A \sin\theta + B \cos\theta + C\theta \sin\theta + D\theta \cos\theta$$

$$\text{B.C.} \rightarrow f(0) = f(\theta_0) = f'(\theta_0) = 0$$

$$f'(0) = -U$$

$$A, B, C, D = (-\theta_0^2, 0, \theta_0 - \sin\theta_0 \cos\theta_0, \sin^2\theta_0) \cdot \frac{U}{\theta_0^2 - \sin^2\theta_0}.$$

For proving, we take $\nabla \times$ on both sides.

$$\nabla \times \nabla p = \nabla \times (\mu \nabla^2 \bar{u})$$

$\nabla \times \nabla p = 0$ by vector identity

$$\therefore \nabla \times (\nabla^2 \bar{u}) = 0$$

We want to show that $\nabla \times (\nabla^2 \bar{u})$
$= \nabla^2 (\nabla \times \bar{u})$

By vector identity $\nabla^2 \bar{u} = \nabla(\overset{0}{\cancel{\nabla \cdot \bar{u}}}) - \nabla \times (\nabla \times \bar{u})$

taking curl : $\nabla \times \nabla^2 \bar{u} = - \nabla \times \nabla \times (\nabla \times \bar{u})$ ... (1)

~~We use another vector identity :~~

$$\cancel{\nabla \times (\nabla \times \bar{u}) = \nabla (\underset{0}{\nabla \cdot \bar{u}}) - \nabla \cdot \nabla \bar{u}}$$

~~taking curl :~~ $\cancel{\nabla \times (\nabla \times (\nabla \times \bar{u})) = - \nabla \times \nabla^2 \bar{u}}$

by the same vector identity

$$\nabla^2 \bar{F} = \nabla(\nabla \cdot \bar{F}) - \nabla \times (\nabla \times \bar{F}) \qquad . \quad \bar{F} = \nabla \times \bar{u}$$

$$\nabla^2 (\nabla \times \bar{u}) = \nabla (\underbrace{\nabla \cdot (\nabla \times \bar{u})}_{\substack{\| \\ 0}}) - \nabla \times [\nabla \times (\nabla \times \bar{u})] \quad \cdots (2)$$

by vector identity

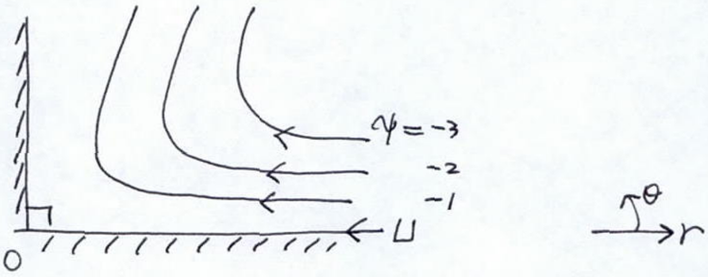
Comparing (1) & (2)

We get $\nabla \times (\nabla^2 \bar{u}) = \nabla^2 (\nabla \times \bar{u})$

note. this is true only when $\nabla \cdot \bar{u} = 0$.

If $\theta_0 = \frac{\pi}{2}$

$$\psi = \frac{rU}{\frac{1}{4}\pi^2 - 1}\left(-\frac{\pi^2}{4}\sin\theta + \frac{\pi}{2}\theta\sin\theta + \theta\cos\theta\right)$$




tangential stress

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$

$$r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = -r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) \qquad \psi = rf(\theta)$$

$$\propto r\frac{\partial}{\partial r}\left(\frac{f}{r}\right)$$

$$\sim \frac{r}{r^2}f \sim \frac{1}{r}$$

$$\frac{1}{r}\frac{\partial u_r}{\partial \theta} \sim \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)$$

$$\sim \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{\cancel{r}}\cancel{r}f(\theta)\right)$$

$$\sim \frac{1}{r}$$

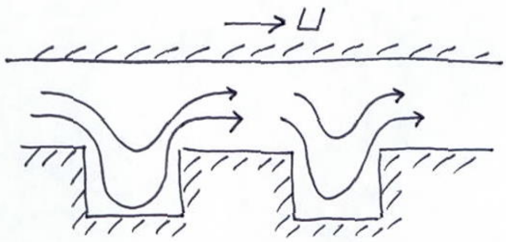
$\tau_{r\theta} \to \infty$ as $r \to 0$.

: related to contact line singularity

Batchelor pp. 224–226.

Another example

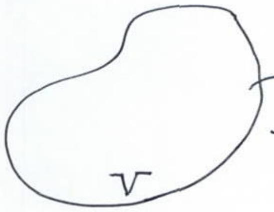
CMP ( Chemical Mechanical Planarization )



## Uniqueness and reversibility of slow flows

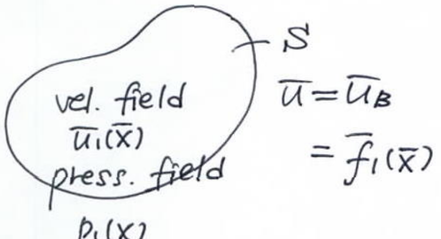
$$\left.\begin{aligned} 0 &= -\nabla p + \mu \nabla^2 \bar{u} \\ \nabla \cdot \bar{u} &= 0. \end{aligned}\right\} \quad (*)$$

· Uniqueness.




There is at most one solution of the eq. (*) which satisfies the B.C.

· Reversibility




$\xrightarrow[\text{B.C}]{\text{change}}$ $\bar{u}_B = -\bar{f}_1(\bar{x})$

Sol. of (*)

: $-\bar{u}_1(x)$

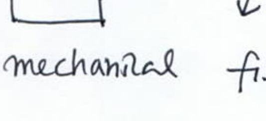
pressure $= c - p_1(x)$

→ only solution

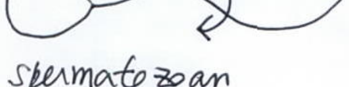
∴ Inasmuch as the slow flow eqns hold, 'reversed' boundary conditions lead to reversed flow.

## Swimming at low Reynolds number

(do something which is not time-reversible)



mechanical fish : make no progress when $Re \ll 1$.



spermatozoan : rotating helical coil → can swim

2. How to drive liquid flow in microchannel with no interfaces

(1) Syringe pump

(2) Vacuum pump

(3) Centrifugal system

(4) Active actuation with microdevices (PZT...)

(5) Electroosmosis

(6) Thermo/electro capillary action

(7) etc