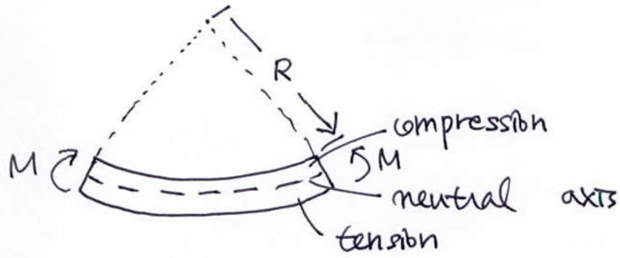
· Mechanics of slender bodies

Why slender rods? $(L \gg d)$

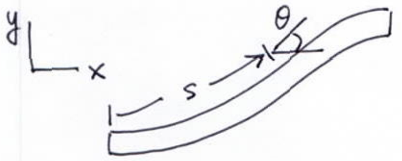
good models for (biological polymers (DNA, protein filaments)
(cilia, flagella, ...

· Beam equations



$$M = EI \frac{1}{R}$$

$I = \int y^2 dA$: second moment of inertia of the cross-section



$$\frac{1}{R} = \frac{d\theta}{ds}$$

$$\frac{d\theta}{ds} = \frac{1}{EI} M$$

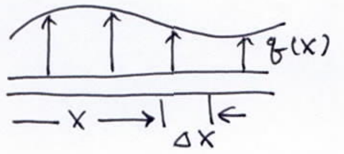
for small θ, $x \simeq s$, $\frac{dy}{dx} \simeq \frac{dy}{ds} = \sin\theta \simeq \theta$

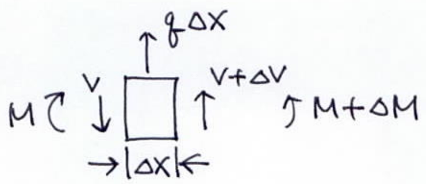
$$\frac{d^2y}{dx^2} \simeq \frac{d\theta}{ds}.$$

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}.$$

· Beam under distributed loading

q: load-intensity fn



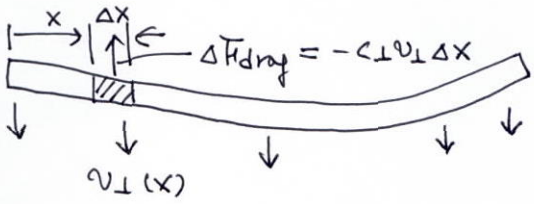


$\frac{dV}{dx} + q = 0$, $\frac{dM}{dx} + V = 0$

$$\frac{d^2M}{dx^2} = q$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2 v}{dx^2}\right) = q(x)$$

* Hydrodynamics of slender rods



drag force per unit length

$$f_\perp(x) = -c_\perp v_\perp(x) = -c_\perp \frac{\partial y}{\partial t}(x)$$

bending moment at pt x due to drag force on the RHS of pt

$$M(x) = \int_x^L f_\perp(x')(x'-x)\,dx'$$

$$\frac{\partial^2 M}{\partial x^2} = f_\perp(x) = -c_\perp \frac{\partial y}{\partial t}$$

$$EI\frac{\partial^2 y}{\partial x^2} = M$$

$$EI\frac{\partial^4 y}{\partial x^4} = \frac{\partial^2 M}{\partial x^2} = -c_\perp \frac{\partial y}{\partial t}$$

$$\frac{\partial^4 y}{\partial x^4} = -\frac{c_\perp}{EI}\frac{\partial y}{\partial t} \quad : \text{ hydrodynamic beam equation}$$

* Hydrodynamic bending modes

B.C. for an unconstrained filament

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^3 y}{\partial x^3} = 0 \quad \text{at } x=0 \ \& \ x=L$$

by separation of variables

$$y_n(x,t) = e^{-t/\tau_n}\left[\sinh\alpha_n \cos\frac{2\alpha_n}{L}\left(x-\frac{L}{2}\right) - \sin\alpha_n \cosh\frac{2\alpha_n}{L}\left(x-\frac{L}{2}\right)\right]$$

n odd

$$y_n(x,t) = e^{-t/\tau_n}\left[\cosh\alpha_n \sin\frac{2\alpha_n}{L}\left(x-\frac{L}{2}\right) + \cos\alpha_n \sinh\frac{2\alpha_n}{L}\left(x-\frac{L}{2}\right)\right]$$

n even

where

$$\tau_n = \frac{C_\perp}{EI}\left(\frac{L}{2\alpha_n}\right)^4 . \qquad \tan\alpha_n = (-1)^n \tanh\alpha_n \qquad n = 1, 2, 3, \cdots$$

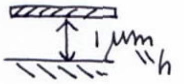
approximate values for α_n

$$\alpha_n \simeq (n + \tfrac{1}{2})\frac{\pi}{2} \qquad n = 1, 2, 3, \cdots \qquad (\alpha_1 \simeq 2.365)$$

e.g. Relaxation of microtubules and actin filaments

microtubule

$L = 50\,\mu m$, radius $= 15\ nm$, $EI = 30\times10^{-24}\ N\cdot m^2$



$\cdot\ \mu = 1\ mPa\cdot s$

relaxation time for $n=1$.

$$C_\perp = \frac{4\pi\mu}{\ln(2h/r)} = \frac{4\pi(10^{-3})}{\ln\left(\frac{2\times10^{-6}}{15\times10^{-9}}\right)} = 2.6\times10^{-3}$$

$$\tau_n = \frac{2.6\times10^{-3}}{30\times10^{-24}}\left(\frac{50\times10^{-6}}{2\times2.365}\right)^4$$

$$= 1.1\ s$$

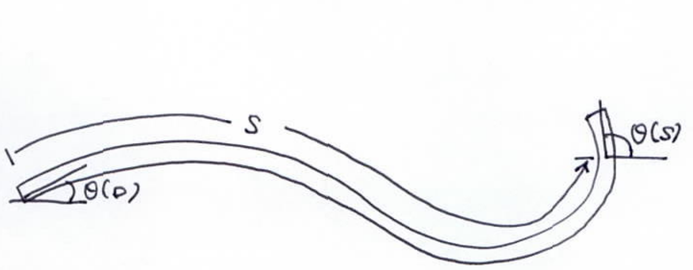
* Thermal bending of filaments

thermal forces → shape fluctuations of flexible filaments

· Persistence length L_p.

: length of filament over which thermal bending becomes appreciable

if $L \gg L_p$: tangent angles at the two ends uncorrelated.



2-D

time average of the cosine of $\theta(s) - \theta(0)$ decreases exponentially as the arc length s increases :

$$\langle \cos[\theta(s) - \theta(0)] \rangle = \exp\left(-\frac{s}{2L_p}\right)$$

Using the principle of equipartition of energy
(average energy of a molecule $\langle U \rangle = \frac{1}{2} kT$)

$$L_p = \frac{EI}{kT}$$

polymer	L_p
DNA	50 nm
actin filament	15 μm
microtubule	6 mm

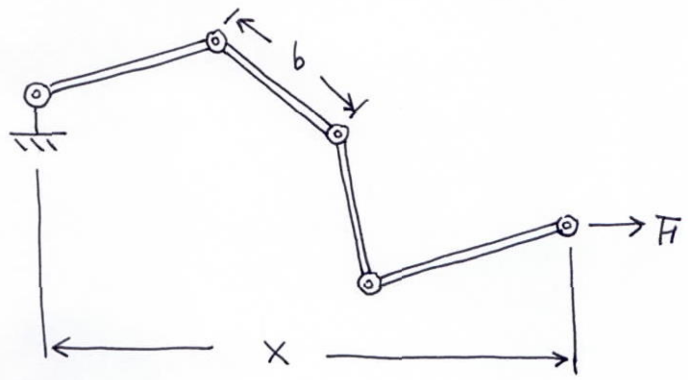
in 3-D, a filament can bend in two different directions
$\Delta\theta_1$, $\Delta\theta_2$.

$$\langle \cos[\Delta\theta_{3D}(s)] \rangle = \exp\left(-\frac{s}{L_p}\right)$$

· end-to-end length

$$\langle R^2 \rangle = 2L_p^2 \left[\exp\left(-\frac{L}{L_p}\right) - 1 + \frac{L}{L_p}\right]$$: statistical physics.

* Entropic elasticity of a freely jointed chain

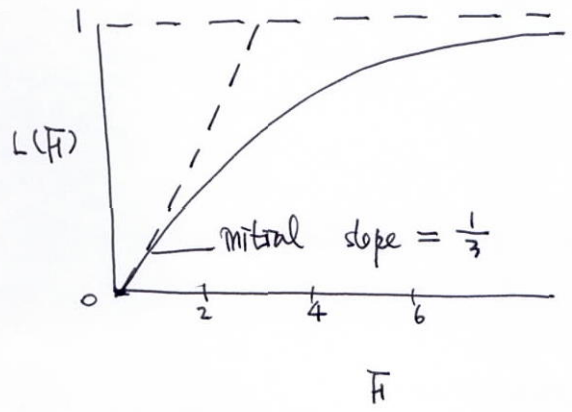


· entropic spring : { proteins that have segmental flexibility
(antibodies, motor proteins)
unfolded proteins
DNA

· resistance to extension
← extension decreases the disorder
and the corresponding decrease in entropy costs free energy

$$\langle x \rangle = nb \cdot L\left(\frac{Fb}{kT}\right)$$

$L\left(\frac{Fb}{kT}\right)$: Langevin function , $\quad L(x) \equiv \dfrac{e^x + e^{-x}}{e^x - e^{-x}} - \dfrac{1}{x}.$



for small F, $\quad L\left(\frac{Fb}{kT}\right) \simeq \dfrac{Fb}{3kT}.$

$$F = \frac{3kT}{nb^2} \langle x \rangle$$

$$kT = 4.1164 \times 10^{-21}\ \mathrm{J} \qquad (T = 25°C)$$