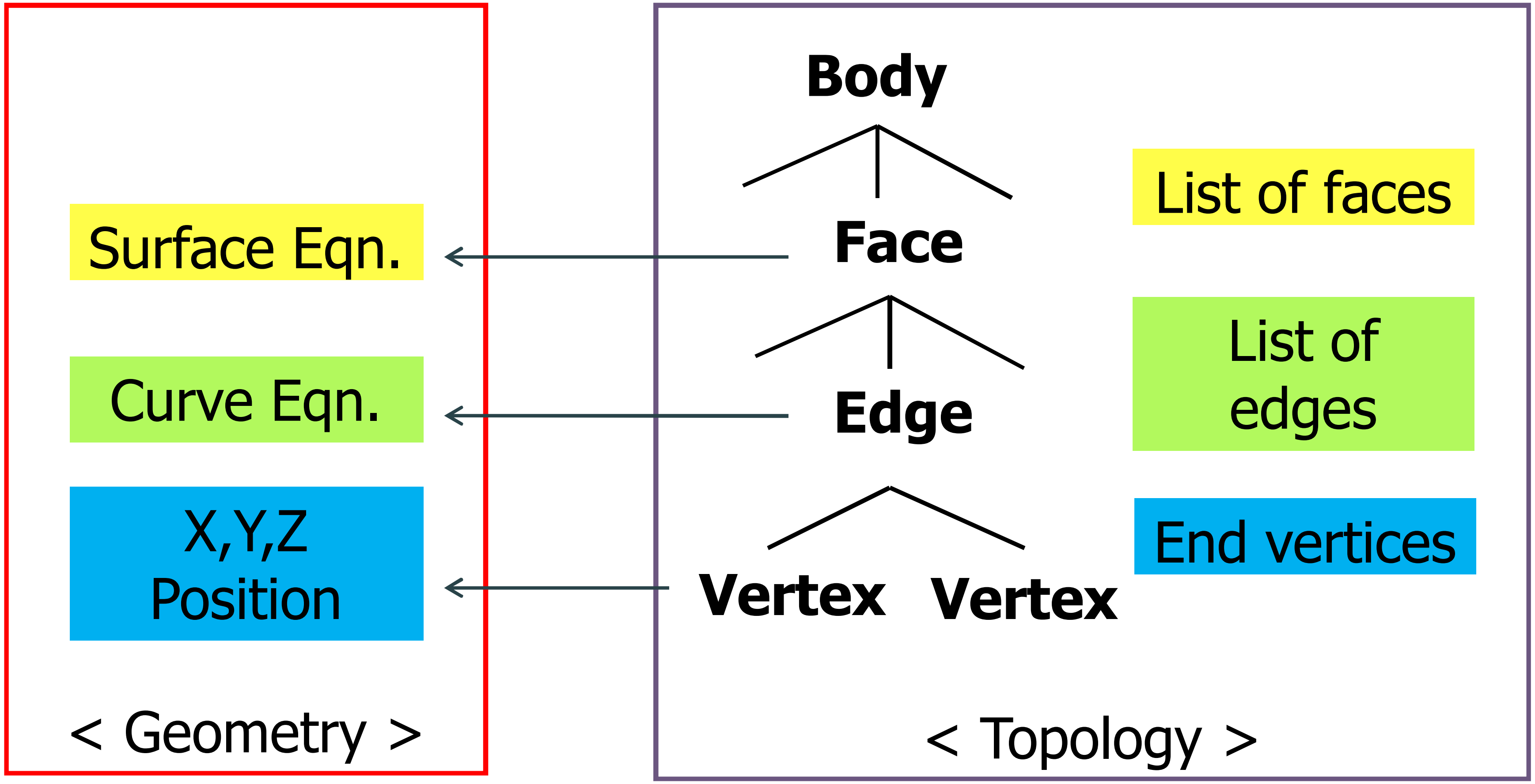
Representation and manipulation of curves

Human Centered CAD Lab.

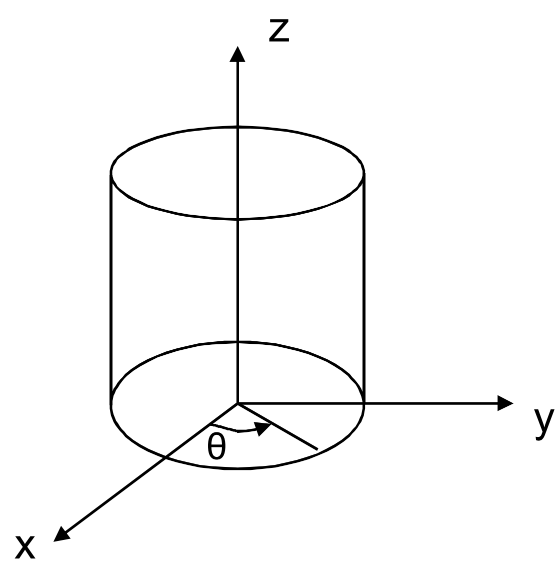
B-Rep Structure – review

Geometry vs. Topology



Types of curve equations

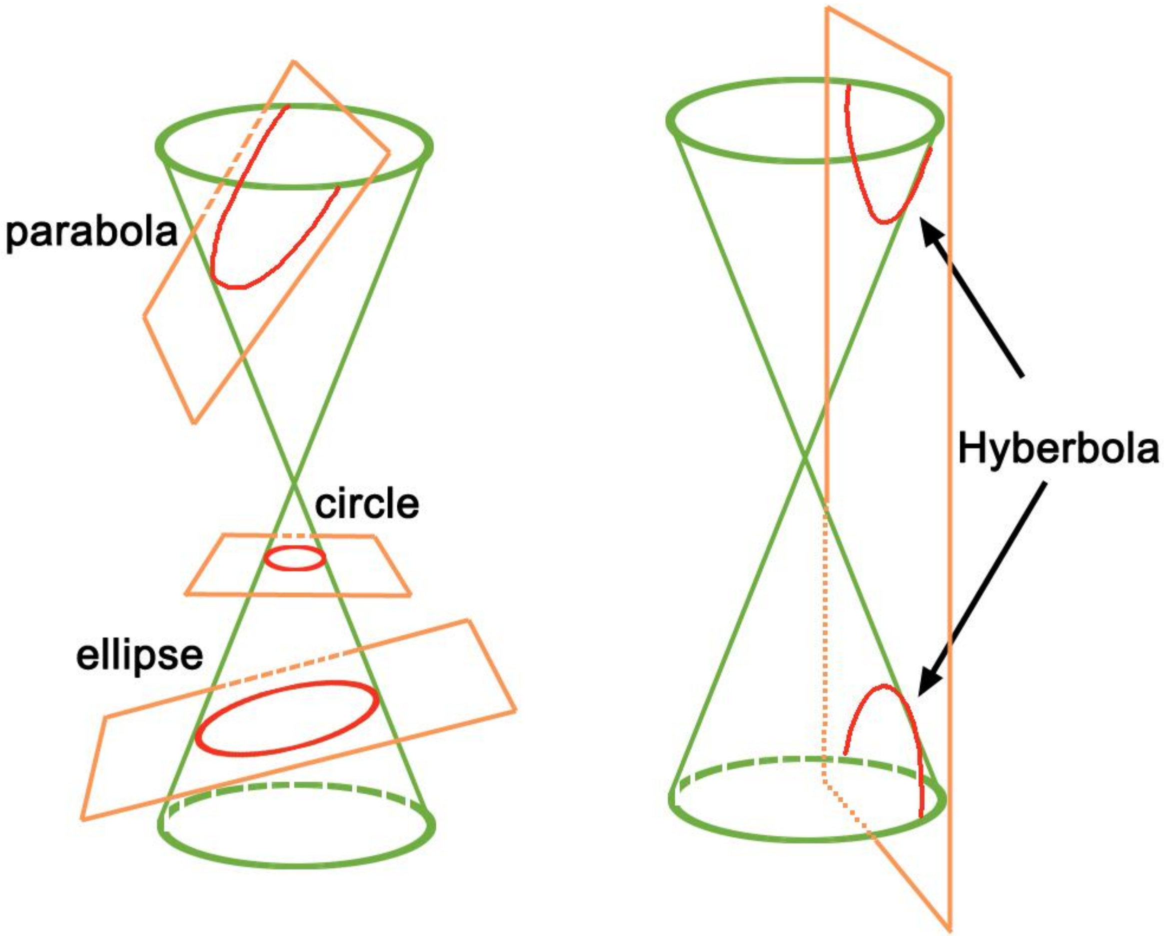
- Parametric equation
  - x=x(t), y=y(t), z=z(t)
  - Ex) $x=R\cos\theta$, $y=R\sin\theta$, $z=0$ $(0 \le \theta \le 2\pi)$
- Implicit nonparametric
  - $x^2 + y^2 - R^2 = 0, \quad z = 0$
  - F(x, y, z)=0, G(x, y, z)=0
  - Intersection of two surfaces
  - Ambiguous independent parameters
- Explicit nonparametric
  - $y = \pm\sqrt{R^2 - x^2}, \quad z = 0$
  - Should choose proper neighboring point during curve generation



Conic curves

- Curves obtained by intersecting a cone with a plane
- Circle (circular arc), ellipse, hyperbola, parabola
  - Ex) Circle (circular arc)
    - Circle in xy-plane with center (x_c, y_c) and radius R
    - $x = R\cos\theta + x_c$
    - $y = R\sin\theta + y_c$
    - $z = 0$
- Points on the circle are generated by incrementing θ by $\triangle\theta$ from 0, points are connected by line segments
- Equation of a circle lying on an arbitrary plane can be derived by transformation

Conic curves – cont'



Hermite curves

- Parametric eq. is preferred in CAD systems
  - Polynomial form of degree 3 is preferred :
    - C2 continuity is guaranteed when two curves are connected

$$\therefore \mathbf{P}(u) = [x(u)\ v(u)\ z(u)] = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3 \qquad (1)$$

$$(0 \le u \le 1) : \text{algebraic eq.}$$

- Impossible to predict the shape change from change in coefficients $\Rightarrow$ not intuitive
  - Bad for interactive manipulation

Hermite curves – cont'

- Apply Boundary conditions to replace algebraic coefficients

- Use $\mathbf{P}_{(0)},\ \mathbf{P}_{(1)},\ \mathbf{P}'_{(0)},\ \mathbf{P}'_{(1)}$ $\Rightarrow$ Substitute in Eq(1)
  $\mathbf{P}_{0},\ \mathbf{P}_{1},\ \mathbf{P}'_{0},\ \mathbf{P}'_{1}$

$$\left.\begin{aligned}
\mathbf{P}_{(0)} &= \mathbf{P}_{0} = \mathbf{a}_{0} \\
\mathbf{P}_{(1)} &= \mathbf{P}_{1} = \mathbf{a}_{0} + \mathbf{a}_{1} + \mathbf{a}_{2} + \mathbf{a}_{3} \\
\mathbf{P}'_{(0)} &= \mathbf{P}'_{0} = \mathbf{a}_{1} \\
\mathbf{P}'_{(1)} &= \mathbf{P}'_{1} = \mathbf{a}_{1} + 2\mathbf{a}_{2} + 3\mathbf{a}_{3}
\end{aligned}\right\} \quad (2)$$

Hermite curves – cont'

- Solve for $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in Eq (2)

$$\left.\begin{aligned}
\mathbf{a}_0 &= \mathbf{P}_0 \\
\mathbf{a}_1 &= \mathbf{P}_0' \\
\mathbf{a}_2 &= -3\mathbf{P}_0 + 3\mathbf{P}_1 - 2\mathbf{P}_0' - \mathbf{P}_1' \\
\mathbf{a}_3 &= 2\mathbf{P}_0 - 2\mathbf{P}_1 + \mathbf{P}_0' - \mathbf{P}_1'
\end{aligned}\right\} \quad (3)$$

Hermite curves – cont'

- Substitute (3) into (1)

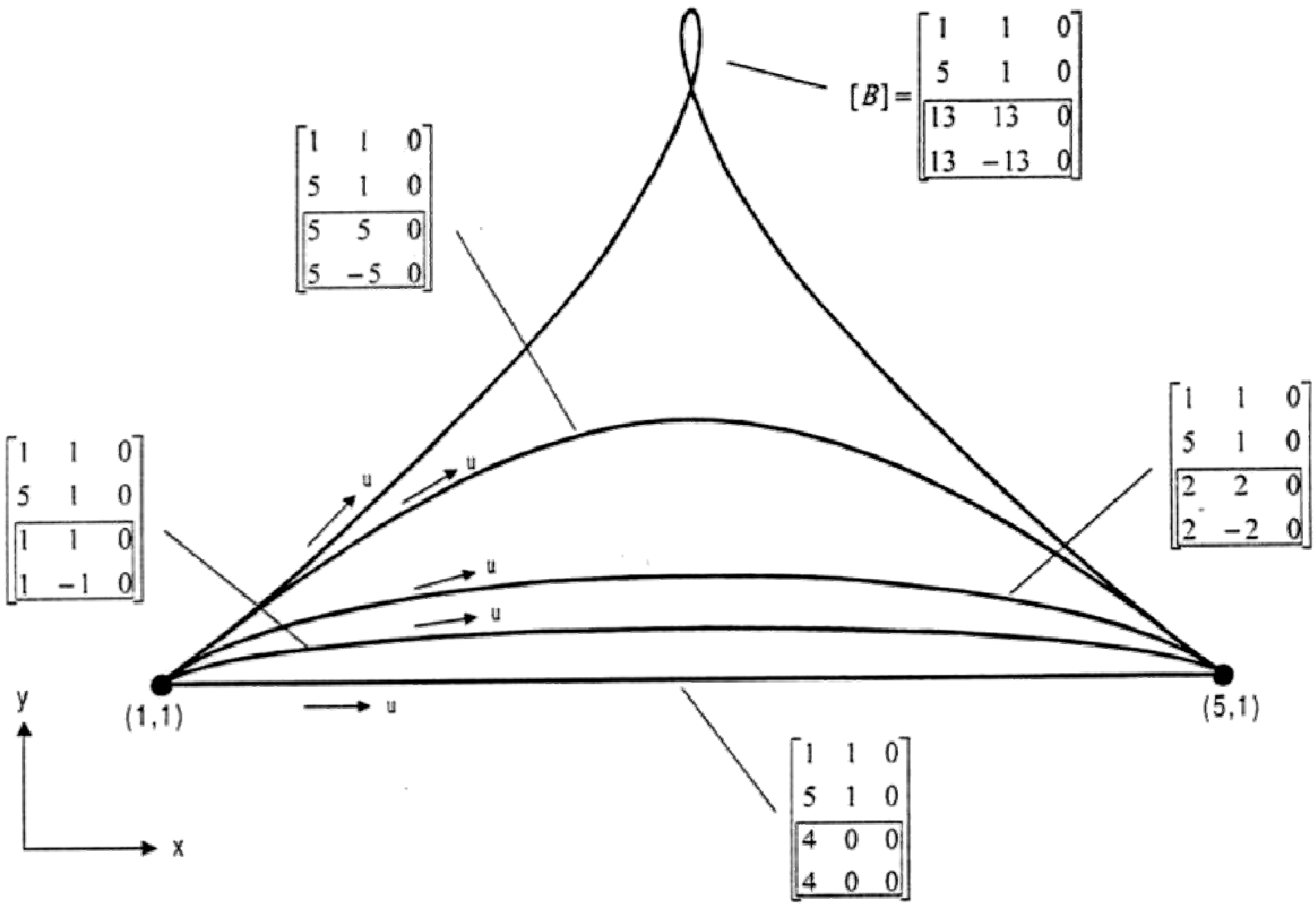
$$\mathbf{P}(u) = \begin{bmatrix} 1-3u^2+2u^3 & 3u^2-2u^3 & u-2u^2+u^3 & -u^2+u^3 \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}'_0 \\ \mathbf{P}'_1 \end{bmatrix}}_{\substack{\text{geometric} \\ \text{coefficientt}}}$$

$$\Uparrow$$

$$\text{Hermite} \quad \text{curve} \quad \text{equation}$$

- It is possible to predict the curve shape change from the change in P_0, P_1, P_0', P_1' to some extent

Hermite curves – cont’



Effect of P_0’ and P_1’ on curve shape

Hermite curves – cont'

- $1-3u^2+2u^3,\ 3u^2-2u^3,\ u-2u^2+u^3,\ -u^2+u^3$

  determine the curve shape by blending the effects of P_0, P_1, P_0', P_1' → blending function

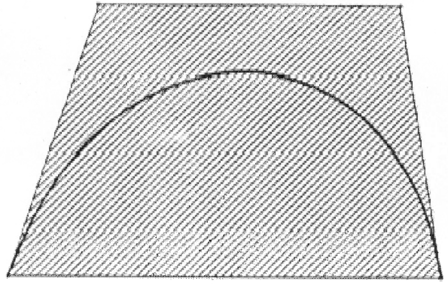
Bezier curves

- It is difficult to realize a curve in one's mind by changing size and direction of P_0', P_1' in Hermite curves
- Bezier curves
  - Invented by Bezier at Renault
  - Use polygon that enclose a curve approximately
  - Control polygon, control point

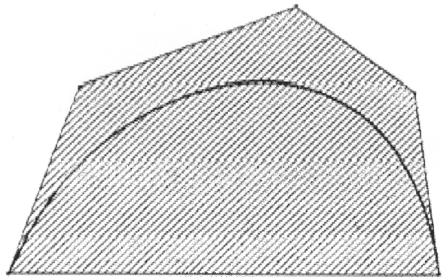
Bezier curves – cont'

- Passes through 1st and last vertex of control polygon
- Tangent vector at the starting point is in the direction of 1st segment of control polygon
- Tangent vector at the ending point is in the direction of the last segment
  - Useful feature for smooth connection of two Bezier curves
- The n-th derivative at starting or ending point is determined by the first or last (n+1) vertices of control polygon
- Bezier curve resides completely inside its convex hull
  - Useful property for efficient calculation of intersection points

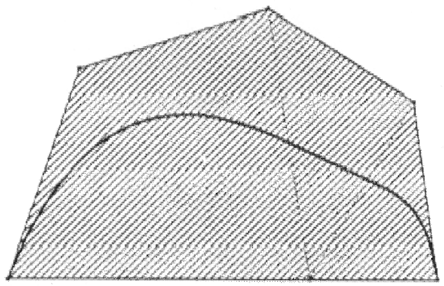
Bezier curves – cont’



(a)



(b)



(c)

Bezier curves – cont'

$$\mathbf{P}(\mathrm{u}) = \sum_{\mathrm{i}=0}^{\mathrm{n}} \binom{\mathrm{n}}{\mathrm{i}} \mathrm{u}^{\mathrm{i}} (1-\mathrm{u})^{\mathrm{n}-\mathrm{i}} \mathbf{P}_{\mathrm{i}} \qquad (0 \le \mathrm{u} \le 1)$$

↑ Control Point

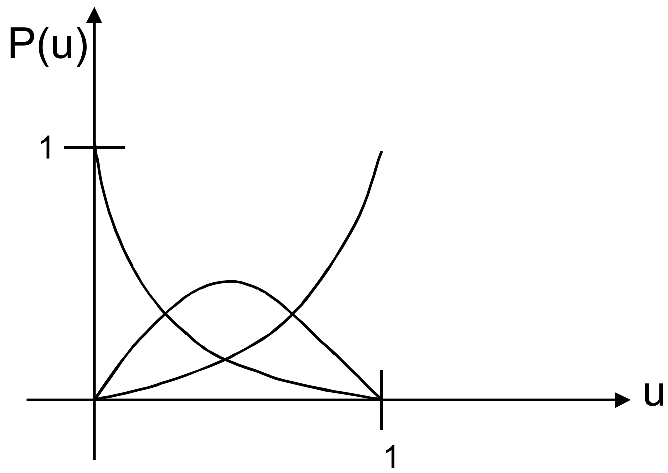
$$\mathbf{P}(u) = (1-u)\mathbf{P}_0 + u\mathbf{P}_1$$

: Straight line from P0 to P1 satisfies the desired qualities including convex hull property

$$\mathbf{P}(u) = (1-u)^2 \mathbf{P}_0 + 2(1-u)u\mathbf{P}_1 + u^2 \mathbf{P}_2$$

$$\Rightarrow (1-u)^2 + 2(1-u)u + u^2 = 1$$

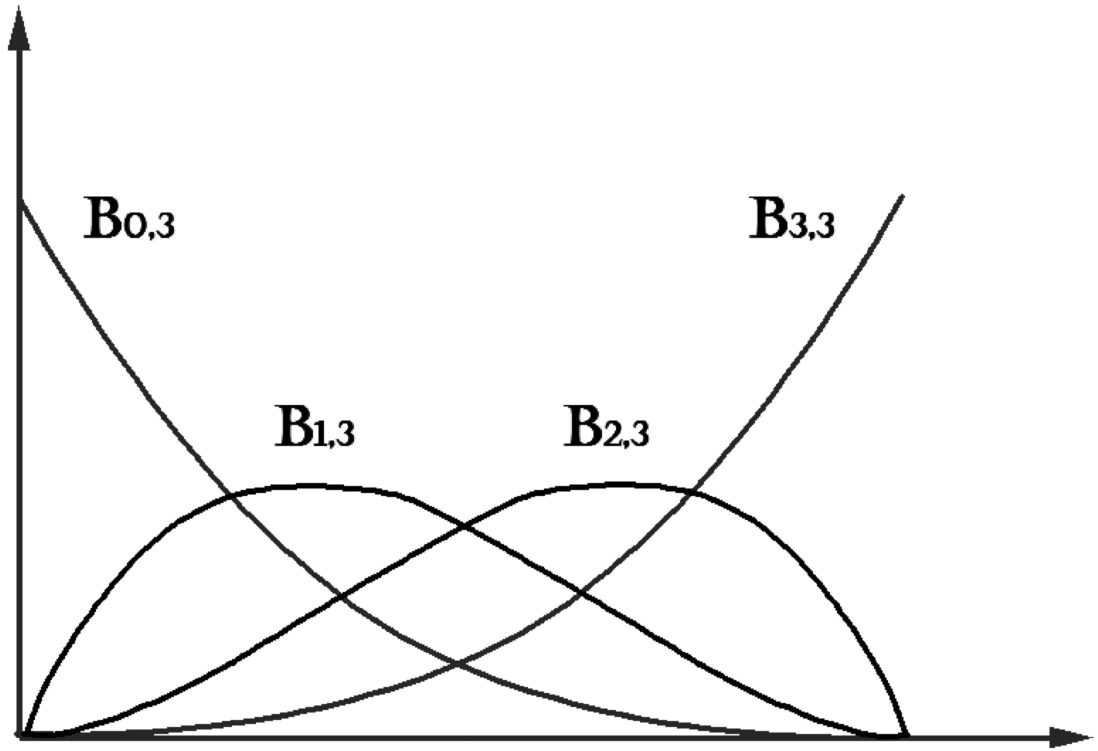
satisfies the desired qualities



Bezier curves – cont'

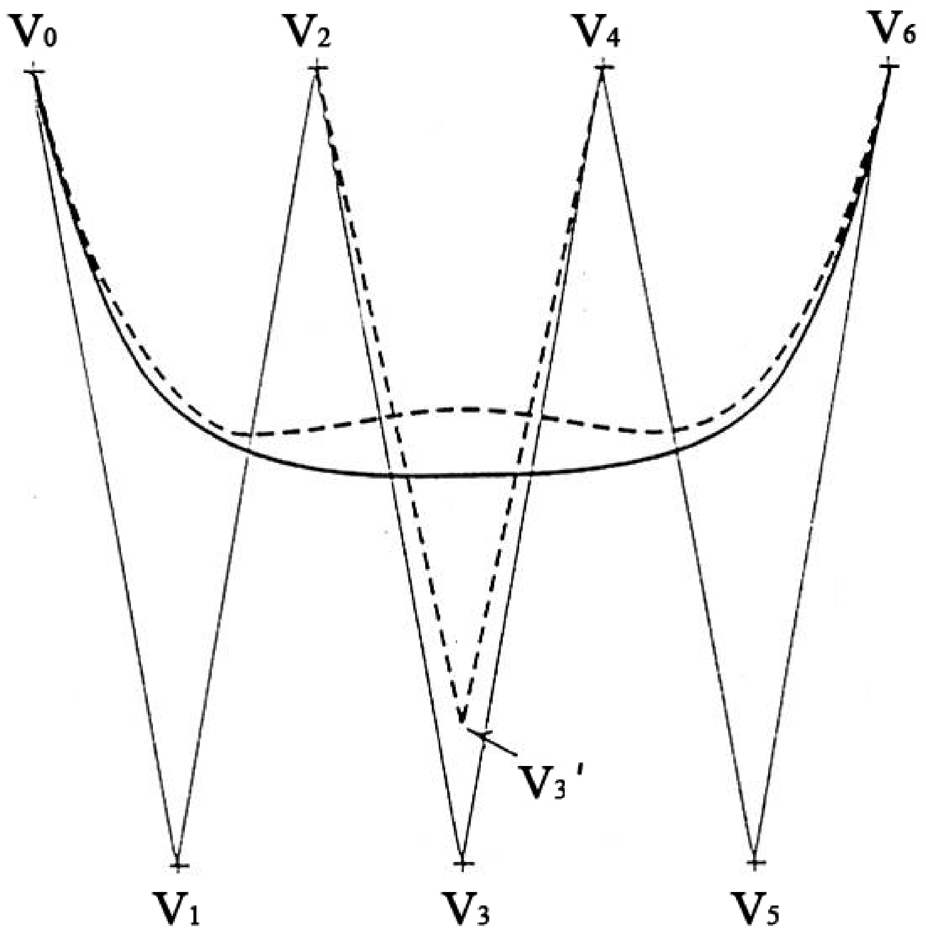
- Highest term is u^n for the curve defined by (n+1) control points
  - Polynomial of degree n
- Degree of curve is determined by number of control points
- Large number of control points are needed to represent a curve of complex shape → high degree is necessary.
  - Heavy computation, oscillation
  - Better to connect multiple Bezier curves
- Global modification property (not local modification)
  - Difficult to result a curve of desired shape by modifying portions

Blending functions in Bezier curve



for degree 3

Bezier curves – cont’



Bezier Curve does NOT have local modification property

Derivative vectors

$$\frac{d\mathbf{r}(t)}{dt} = n\sum_{i=0}^{n-1}\binom{n-1}{i}t^{i}(1-t)^{n-1-i}\mathbf{a}_{i}$$

$$where \qquad \mathbf{a}_{i} = \mathbf{P}_{i+1} - \mathbf{P}_{i} \qquad i = 0,1,\cdots,n-1$$

$pf\,)$

$$\frac{d\mathbf{r}(t)}{dt} = \sum_{i=0}^{n} i\binom{n}{i}t^{i-1}(1-t)^{n-i}\mathbf{P}_{i} - \sum_{i=0}^{n}(n-i)\binom{n}{i}t^{i}(1-t)^{n-i-1}\mathbf{P}_{i}$$

$$= \sum_{i=1}^{n} i\binom{n}{i}t^{i-1}(1-t)^{n-i}\mathbf{P}_{i} - \sum_{i=0}^{n-1}(n-i)\binom{n}{i}t^{i}(1-t)^{n-i-1}\mathbf{P}_{i}$$

$let \quad j = i-1$

$$\frac{d\mathbf{r}(t)}{dt} = \sum_{j=0}^{n-1}(j+1)\binom{n}{j+1}t^{j}(1-t)^{n-j-1}\mathbf{P}_{j+1} - \sum_{i=0}^{n-1}(n-i)\binom{n}{i}t^{i}(1-t)^{n-1-i}\mathbf{P}_{i}$$

Derivative vectors – cont’

$$(j+1)\binom{n}{j+1} = \frac{(j+1)n!}{(j+1)!(n-j-1)!} = \frac{n(n-1)!}{j!(n-j-1)!} = n\binom{n-1}{j}$$

$$(n-i)\binom{n}{i} = \frac{(n-i)n!}{i!(n-i)!} = \frac{n(n-1)!}{i!(n-i-1)!} = n\binom{n-1}{i}$$

$$\therefore \frac{d\mathbf{r}(t)}{dt} = n\sum_{i=0}^{n-1}\binom{n-1}{i} t^i (1-t)^{n-i-1}\mathbf{a}_i \qquad (a)$$

- Derivatives at the starting and the ending points

$$\dot{\mathbf{r}}(0) = n \times (\mathbf{P}_1 - \mathbf{P}_0)$$

$$\dot{\mathbf{r}}(1) = n \times (\mathbf{P}_n - \mathbf{P}_{n-1})$$

Derivative vectors – cont'

- Control polygon determines tangent vectors at the ends
- Differentiating (a) in the same way gives

$$\frac{d^2\mathbf{r}(t)}{dt^2} = n(n-1)\sum_{i=0}^{n-2}\binom{n-2}{i}t^i(1-t)^{n-2-i}\mathbf{b}_i$$

$$where \qquad \mathbf{b}_i = \mathbf{a}_{i+1} - \mathbf{a}_i \qquad (i = 0, \cdots, n-2)$$

Elevation of a degree

- Conversion of Bezier curve of degree n into degree (n+1), i.e. derivation of (n+1) control points into (n+2) control points

$$\mathbf{r}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \mathbf{P}_i = \{\mathbf{P}_0, \mathbf{P}_1, \cdots \mathbf{P}_n\}$$

$$\mathit{Express} \qquad \mathbf{r}(t) = t\mathbf{r}(t) + (1-t)\mathbf{r}(t)$$

Elevation of a degree – cont'

$$t\mathbf{r}(t) = \sum_{i=0}^{n} \binom{n}{i} t^{i+1}(1-t)^{n-i}\mathbf{P}_i$$

$$let \qquad i+1=k, \qquad n+1=m$$

$$\therefore\ t\mathbf{r}(t) = \sum_{k=1}^{m} \binom{m-1}{k-1} t^{k}(1-t)^{m-k}\mathbf{P}_{k-1}$$

$$\binom{m-1}{k-1} = \frac{(m-1)!}{(k-1)!(m-k)!} = \frac{k}{m}\frac{m!}{k!(m-k)!} = \frac{k}{m}\binom{m}{k}$$

$$\therefore\ t\mathbf{r}(t) = \sum_{k=0}^{m} \binom{m}{k} t^{k}(1-t)^{m-k}\frac{k}{m}\mathbf{P}_{k-1} \qquad (a)$$

Elevation of a degree – cont'

$$(1-t)\mathbf{r}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i+1} \mathbf{P}_i$$

$$let \quad k = i, \quad m = n+1$$

$$(1-t)\mathbf{r}(t) = \sum_{k=0}^{m-1} \binom{m-1}{k} t^k (1-t)^{m-k} \mathbf{P}_k$$

$$\binom{m-1}{k} = \frac{(m-1)!}{k!(m-k-1)!} = \frac{m-k}{m} \frac{m!}{k!(m-k)!} = \frac{m-k}{m} \binom{m}{k}$$

$$\therefore (1-t)\mathbf{r}(t) = \sum_{k=0}^{m-1} \binom{m}{k} t^k (1-t)^{m-k} \frac{m-k}{m} \mathbf{P}_k$$

$$= \sum_{k=0}^{m} \binom{m}{k} t^k (1-t)^{m-k} \frac{m-k}{m} \mathbf{P}_k \qquad (b)$$

Elevation of a degree – cont'

$$From\ (a)\ \&\ (b)$$

$$t\mathbf{r}(t) + (1-t)\mathbf{r}(t) = \sum_{k=0}^{n+1} \binom{n+1}{k} t^k (1-t)^{n+1-k} \left(\frac{k\mathbf{P}_{k-1} + (n+1-k)\mathbf{P}_k}{n+1} \right) \qquad (c)$$

- Control points of the Bezier curve of degree (n+1) can be derived from (c) as below

$$\left\{ \mathbf{P}_0,\ \frac{\mathbf{P}_0 + n\mathbf{P}_1}{n+1},\ \frac{2\mathbf{P}_1 + (n-1)\mathbf{P}_2}{n+1},\ \cdots,\ \frac{n\mathbf{P}_{n-1} + \mathbf{P}_n}{n+1},\ \mathbf{P}_n \right\}$$

de Casteljau algorithm

The Value of curve at t of Bezier Curve

$$\mathbf{r}(u) = \sum_{i=0}^{n} \mathbf{P}_i B_{n,i}(u) \text{ is } \mathbf{P}_0^n \text{ as calculated below.}$$

$$\mathrm{P}_i^r = (1-t)\,\mathbf{P}_i^{r-1} + t \quad \mathbf{P}_{i+1}^{r-1}$$

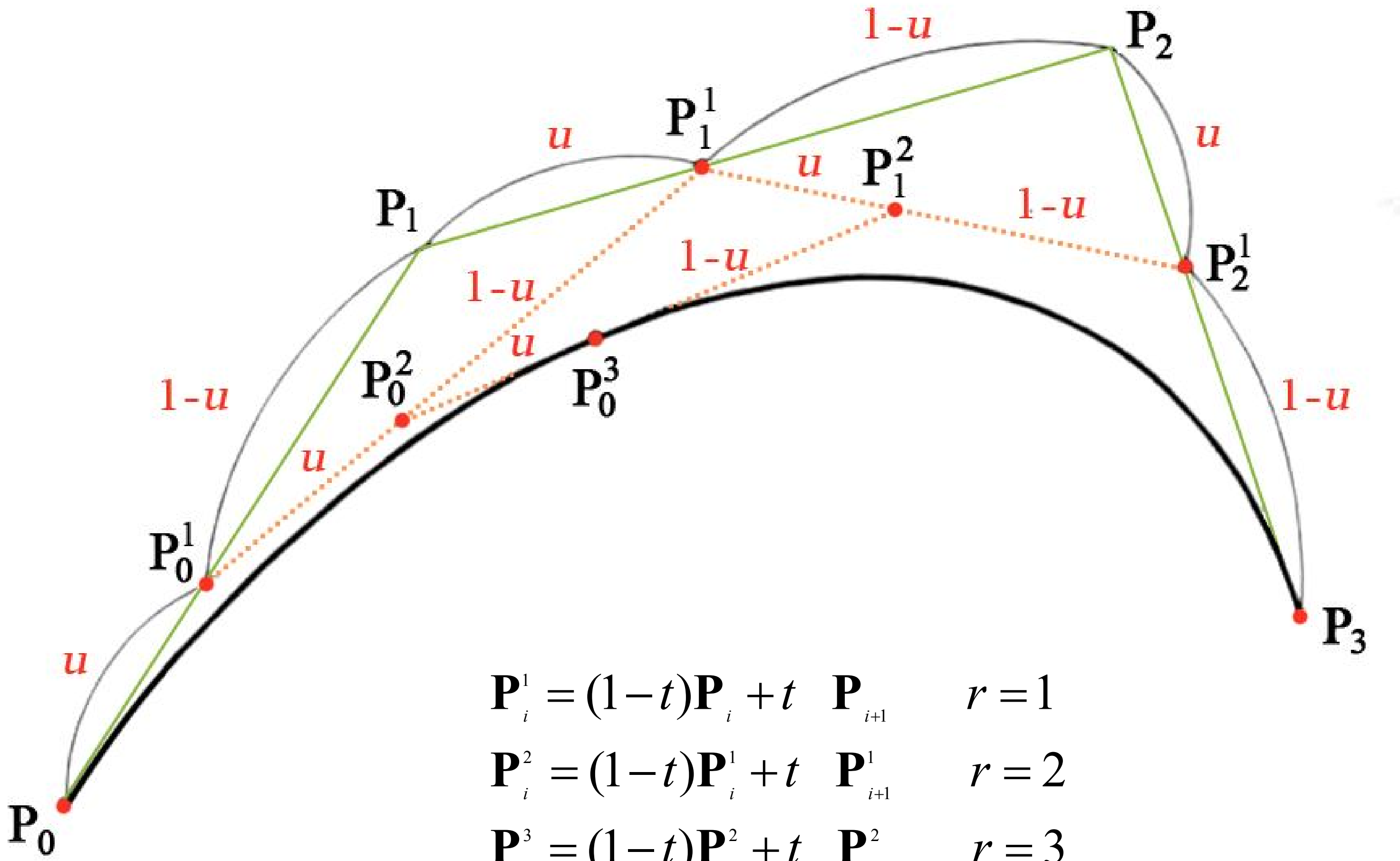
$$r = 1, \cdots, n$$

$$i = 0, \cdots, n-r$$

$$\mathbf{P}_i^0 = \mathbf{P}_i$$

$$For \quad Cubic \quad Bezier$$

de Casteljau algorithm – cont’



$$\mathbf{P}_i^1 = (1-t)\mathbf{P}_i + t\ \mathbf{P}_{i+1} \qquad r=1$$

$$\mathbf{P}_i^2 = (1-t)\mathbf{P}_i^1 + t\ \mathbf{P}_{i+1}^1 \qquad r=2$$

$$\mathbf{P}_i^3 = (1-t)\mathbf{P}_i^2 + t\ \mathbf{P}_{i+1}^2 \qquad r=3$$

de Casteljau algorithm – cont’