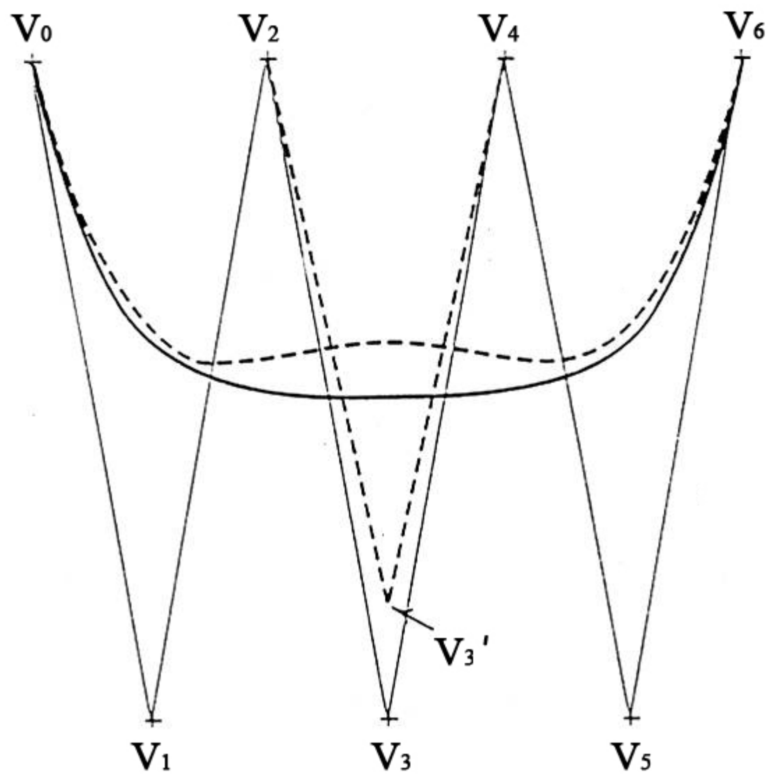
B-spline curve

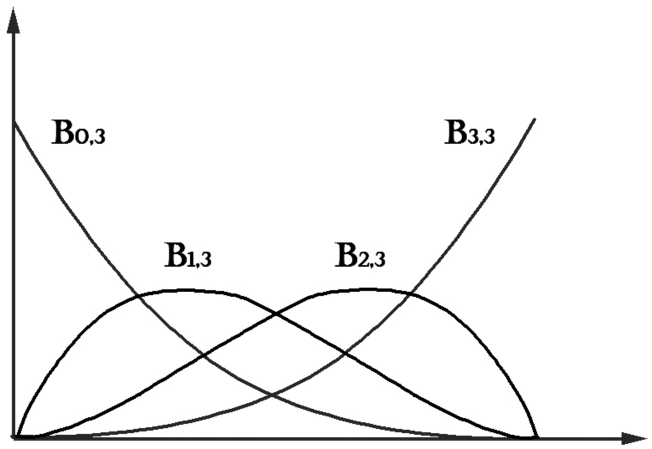
Human Centered CAD Lab.

Properties of B-spline curves

- B-spline curve:
  - Degree of curve is independent of number of control points
- Bezier curve: global modification
  - Modification of any one control point changes the curve shape everywhere
  - All the blending functions have non-zero value in the whole interval $0 \leq u \leq 1$

Bezier curve: global modification

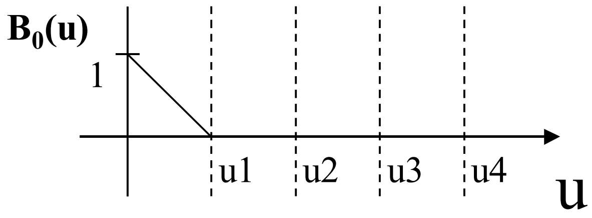




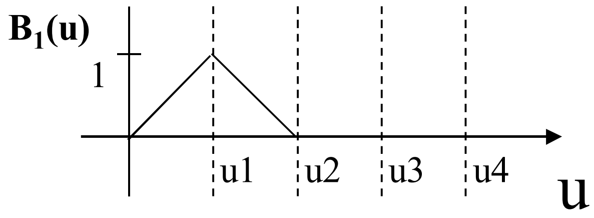
Bezier curve of degree 3

Desired Blending Function

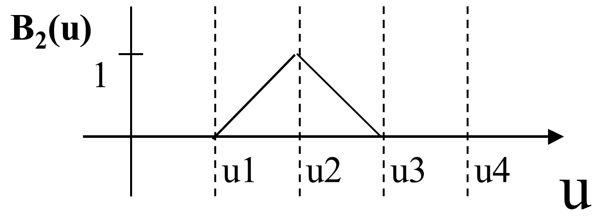
Consider degree 1 blending functions, and n=4



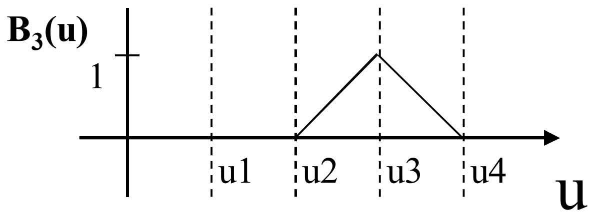
$$B_0(u) = \frac{u_1 - u}{u_1} \qquad 0 \le u \le u_1$$



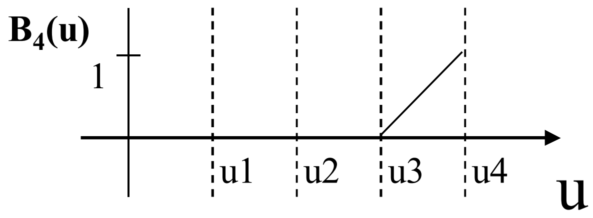
$$B_1(u) = \begin{cases} \dfrac{u}{u_1} & 0 \le u \le u_1 \\ \dfrac{u_2 - u}{u_2 - u_1} & u_1 \le u \le u_2 \end{cases}$$



$$B_2(u) = \begin{cases} \dfrac{u - u_1}{u_2 - u_1} & u_1 \le u \le u_2 \\ \dfrac{u_3 - u}{u_3 - u_2} & u_2 \le u \le u_3 \end{cases}$$



$$B_2(u) = \begin{cases} \dfrac{u - u_2}{u_3 - u_2} & u_2 \le u \le u_3 \\ \dfrac{u_4 - u}{u_4 - u_3} & u_3 \le u \le u_4 \end{cases}$$



$$B_4(u) = \frac{u - u_3}{u_4 - u_3} \qquad u_3 \le u \le u_4$$

$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i \mathbf{B}_i(u)$$

P_0 has an effect only for $0 \le u \le u_1$
P_1 has an effect only for $0 \le u \le u_2$
P_2 has an effect only for $u_1 \le u \le u_3$
P_3 has an effect only for $u_2 \le u \le u_4$
P_4 has an effect only for $u_3 \le u \le u_4$

Resulting Curve

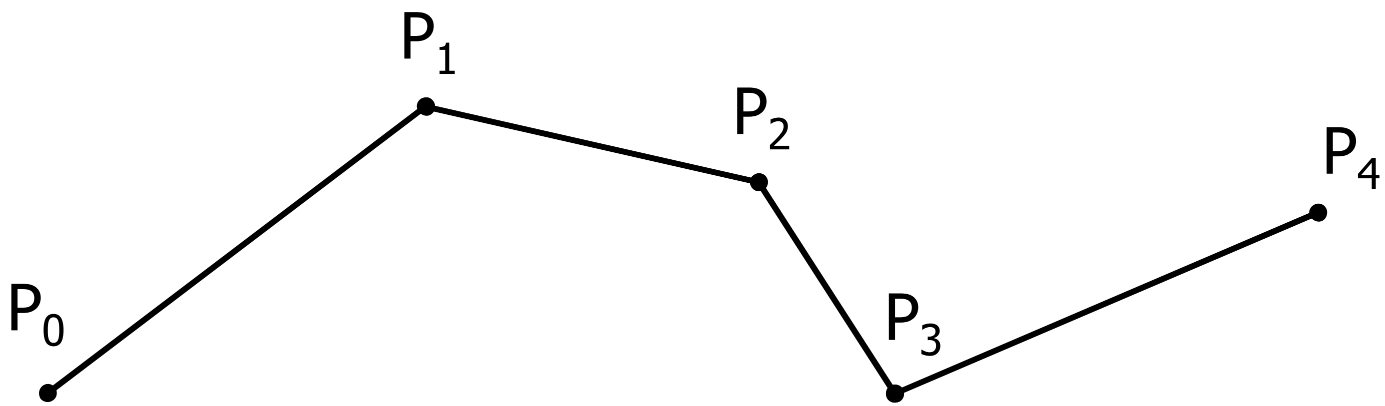
For $0 \le u \le u_1$ $\quad \mathbf{P}(u) = \mathbf{P}_0 B_0(u) + \mathbf{P}_1 \mathbf{B}_1(u) = \mathbf{P}_0(\frac{u_1 - u}{u_1}) + \mathbf{P}_1(\frac{u}{u_1})$

… straight line from P_0 to P_1

For $u_1 \le u \le u_2$ $\quad \mathbf{P}(u) = \mathbf{P}_1 B_1(u) + \mathbf{P}_2 \mathbf{B}_2(u) = \mathbf{P}_1(\frac{u_2 - u}{u_2 - u_1}) + \mathbf{P}_2(\frac{u - u_1}{u_2 - u_1})$

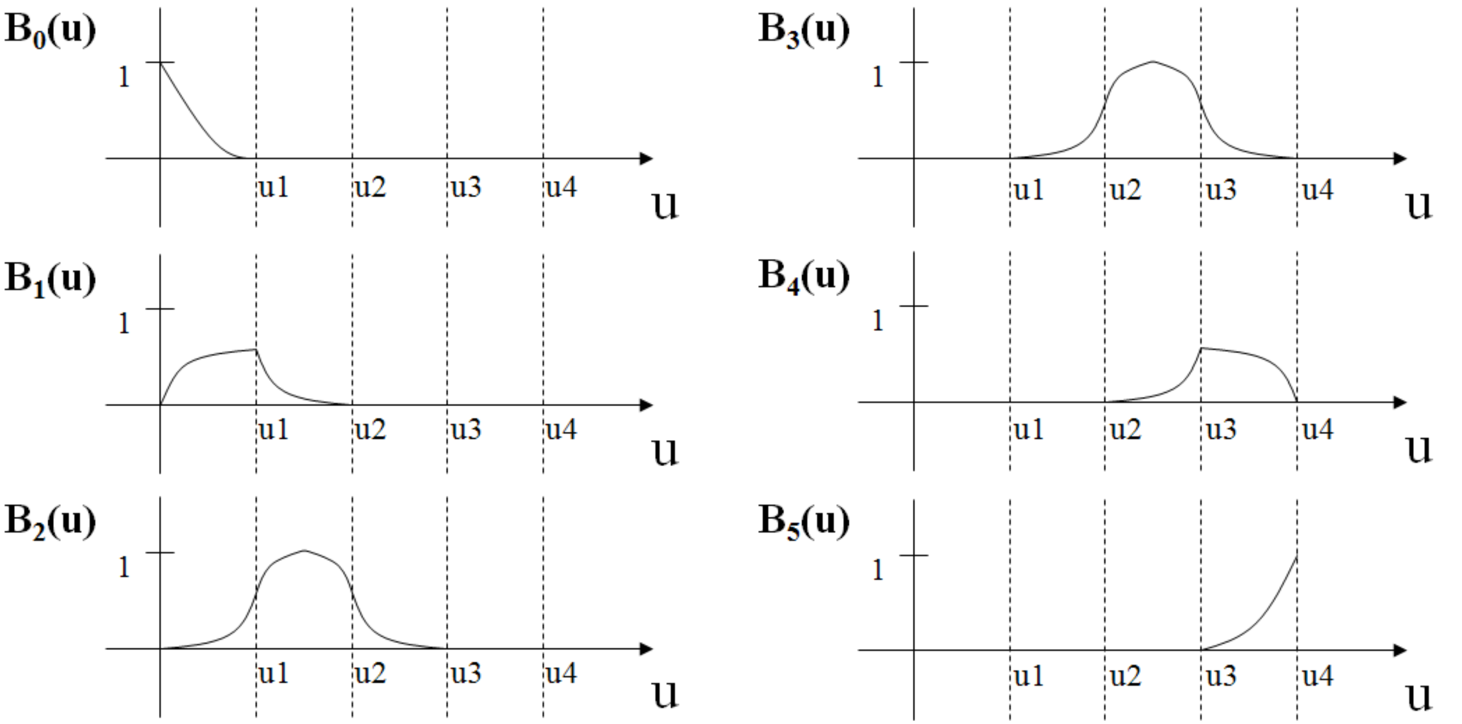
… straight line from P_1 to P_2

Similarly



Blending Function

Consider degree 2 blending functions, and n=5



P_0 has an effect only for $0 \le u \le u_1$

P_1 has an effect only for $0 \le u \le u_2$

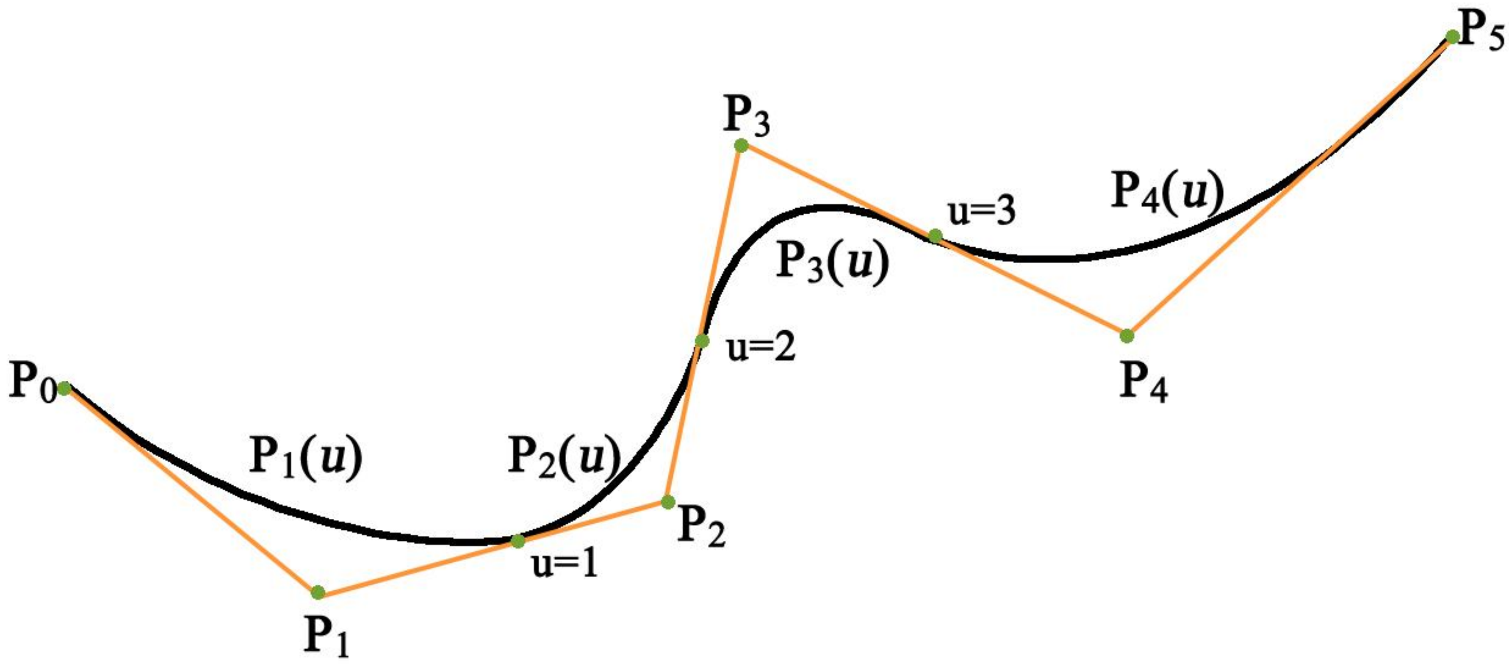
P_2 has an effect only for $0 \le u \le u_3$

P_3 has an effect only for $u_1 \le u \le u_4$

P_4 has an effect only for $u_2 \le u \le u_4$

P_5 has an effect only for $u_3 \le u \le u_4$

Resulting Curve



B-spline curve equation – cont'

$$\mathbf{P}(u) = \sum_{i=0}^{n} \mathbf{P}_i N_{i,k}(u) \qquad t_{k-1} \le u \le t_{n+1} \qquad \text{(a)}$$

$$N_{i,k}(u) = \frac{(u - t_i) N_{i,k-1}(u)}{t_{i+k-1} - t_i} + \frac{(t_{i+k} - u) N_{i+1,k-1}(u)}{t_{i+k} - t_{i+1}} \qquad \left(\frac{0}{0} = 0 \right) \qquad \text{(b)}$$

$$N_{i,1}(u) = \begin{cases} 1 & t_i \le u < t_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(c)}$$

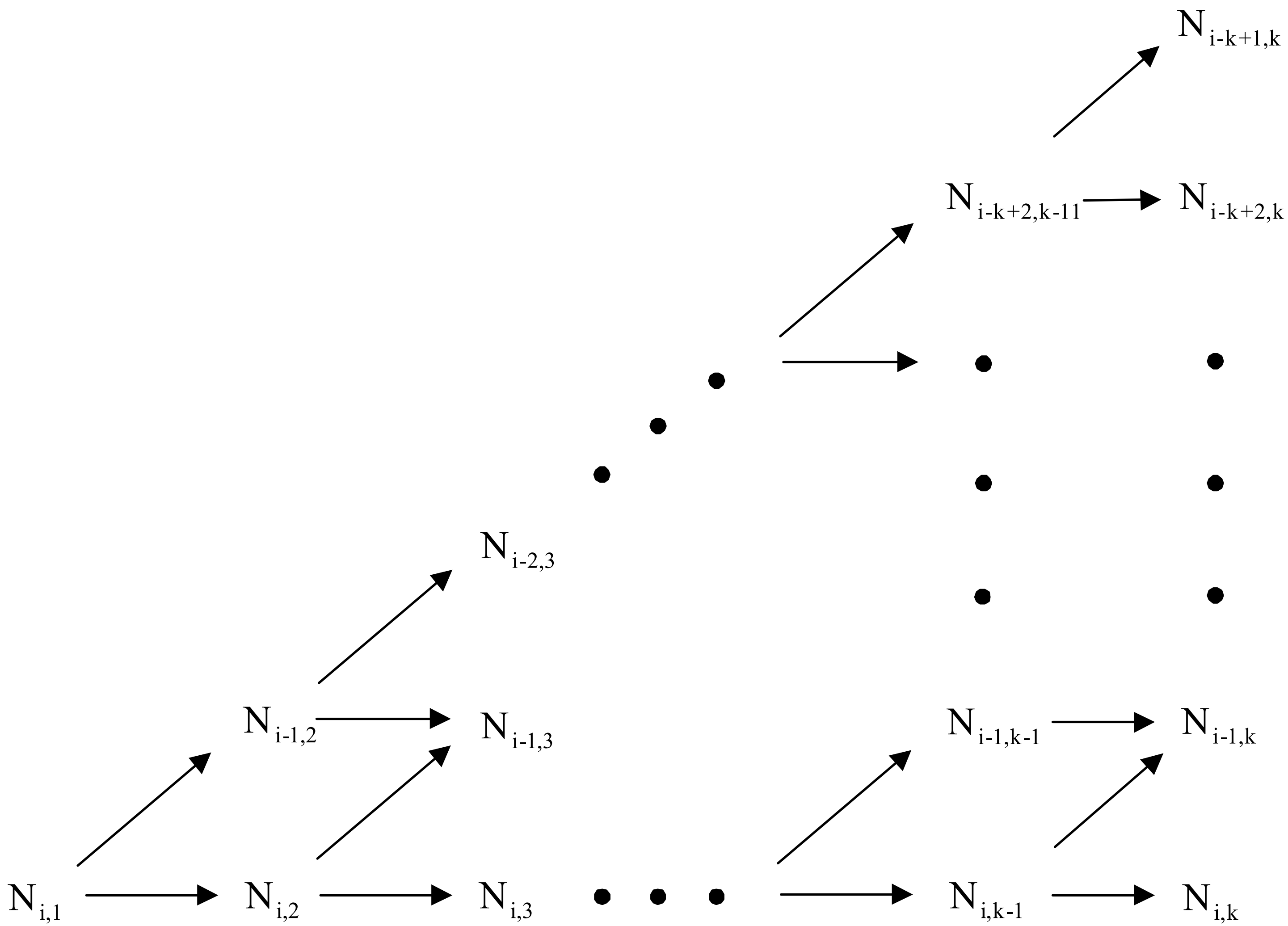
→ At any value of u, there should be only one non-zero $N_{i,1}(u)$

- $N_{i,k}$: degree (k-1) of u, k : order (independent of number of control points n)
- t_i: knot values, boundary of non-zero range of each blending function
- t_0 (for i=0)to t_{n+k} (for i=n) are needed (n+k+1 values)

B-spline curve equation – cont'

- Only the differences in t_i (i=0, ⋯, n+k) is important in (b)
- Can be shifted as a whole, parameter range should be shifted together
- A portion of B-spline curve is affected by a limited number of control points
- For u in $[t_l, t_{l+1}]$
  - Control points associated with blending functions that are non-zero in $[t_l, t_{l+1}]$ have effect
  - $N_{l,1}(u)$ is nonzero in $[t_l, t_{l+1}]$ among $N_{i,1}(u)$
  - Substitute $N_{l,1}(u)$ into the right-hand side of (b)
  - $N_{l,2}(u)$, $N_{l-1,2}(u)$ can be non-zero
  - Apply recursively
  - From $N_{l,2}(u)$, $N_{l,3}(u)$ and $N_{l-1,3}(u)$ can be non-zero
  - From $N_{l-1,2}(u)$, $N_{l-1,3}(u)$ and $N_{l-2,3}(u)$can be non-zero

B-spline curve equation – cont’



B-spline curve equation – cont'

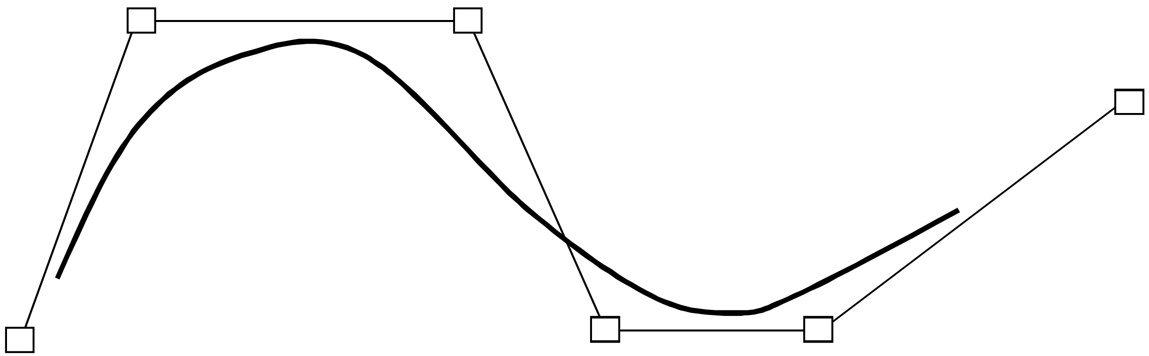
- Control points that have influence in the region $[t_l, t_{l+1}]$ are
  - $P_{l-k+1}, P_{l-k+2}, \cdots, P_l$ k control points
  - Control points modify: Example

B-spline curve - Knot

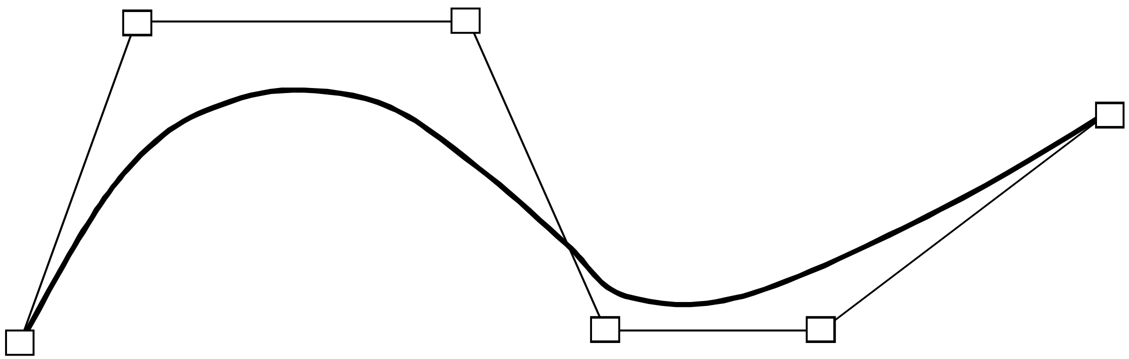
- Knot: $t_0, t_1, \cdots, t_{n+k}$
  - Parameter range is determined by knots
  - Periodic knots
    - $t_i = i-k \qquad 0 \le i \le n+k$
  - Non-periodic knots

$$t_i = \begin{cases} 0 & 0 \le i < k & \text{duplicates k times} \\ i-k+1 & k \le i \le n & \\ n-k+2 & n < i \le n+k & \text{duplicates k times} \end{cases} \qquad \text{(d)}$$

Periodic vs. Non-periodic



Periodic knot



Non-periodic knot

B-spline curve - Knot

- By duplicating knot k times at the ends
  - Curve passes through first control point and last control point
- Periodic knot
  - First control point and last control point do not pass through curve as other control points
  - Non-periodic knots are used in most CAD systems
- knot interval is uniform in (d)
  - Uniform B-spline (vs. non-uniform B-spline)
- During manipulation of curve shape, knots are added or removed
  - Non-uniform knot → non-uniform B-spline curve

Example program

- Knot Insertion
  - Example1

  - Example2

Expansion of curve equation

- Ex)
  - $K=3, P_0, P_1, P_2$ non-periodic uniform B-spline

$$t_0=0,\ t_1=0,\ t_2=0,\ t_3=1,\ t_4=1,\ t_5=1$$

$$\underset{(=t_2)}{\overset{\uparrow}{0}} \le u \le \underset{(=t_3)}{\overset{\uparrow}{1}}$$

Expansion of curve equation – cont'

$$N_{0,1}(u) = \begin{cases} 1 & t_0 \le u \le t_1 \qquad (u = 0) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{1,1}(u) = \begin{cases} 1 & t_1 \le u \le t_2 \qquad (u = 0) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{2,1}(u) = \begin{cases} 1 & t_2 \le u \le t_3 \qquad (0 \le u \le 1) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{3,1}(u) = \begin{cases} 1 & t_3 \le u \le t_4 \qquad (u = 1) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{4,1}(u) = \begin{cases} 1 & t_4 \le u \le t_5 \qquad (u = 1) \\ 0 & \text{otherwise} \end{cases}$$

Expansion of curve equation – cont'

- At u=0, select $N_{2,1}(u)$ to be non-zero among $N_{0,1}(0)$, $N_{1,1}(0)$, $N_{2,1}(0)$
- Selection of any one is O.K.
- At u=1, select $N_{2,1}(u)$ similarly
- Only $N_{2,1}(u)$ needs to be considered among blending functions of order 1

Expansion of curve equation – cont'

$$N_{1,2}(u) = \frac{(u - t_1)N_{1,1}}{t_2 - t_1} + \frac{(t_3 - u)N_{2,1}}{t_3 - t_2} = \frac{(1-u)N_{2,1}}{1} = (1-u)$$

$$N_{2,2}(u) = \frac{(u - t_2)N_{2,1}}{t_3 - t_2} + \frac{(t_4 - u)N_{3,1}}{t_4 - t_3} = \frac{uN_{2,1}}{1} = u$$

$$N_{0,3}(u) = \frac{(u - t_0)N_{0,2}}{t_2 - t_0} + \frac{(t_3 - u)N_{1,2}}{t_3 - t_1} = \frac{(1-u)N_{1,2}}{1} = (1-u)^2$$

$$N_{1,3}(u) = \frac{(u - t_1)N_{1,2}}{t_3 - t_1} + \frac{(t_4 - u)N_{2,2}}{t_4 - t_2} = u(1-u) + (1-u)u = 2u(1-u)$$

$$N_{2,3}(u) = \frac{(u - t_2)N_{2,2}}{t_4 - t_2} + \frac{(t_5 - u)N_{3,2}}{t_5 - t_3} = u^2$$

$$\therefore \mathbf{P}(u) = (1-u)^2\mathbf{P}_0 + 2u(1-u)\mathbf{P}_1 + u^2\mathbf{P}_2$$

Expansion of curve equation – cont'

- Consider Bezier curve defined by P_0, P_1, P_2

$$\mathbf{P}(u) = \binom{2}{0} u^0 (1-u)^2 \mathbf{P}_0 + \binom{2}{1} u^1 (1-u)^1 \mathbf{P}_1 + \binom{2}{2} u^2 (1-u)^0 \mathbf{P}_2$$

- Non-periodic B-spline curve having k (order) control points ends in Bezier curve
- Bezier curve is a special case of B-spline curve

Example

- Ex)

  $K=3$, $P_0, P_1, P_2, P_3, P_4, P_5$

  $t_0=0$, $t_1=0$, $t_2=0$, $t_3=1$, $t_4=2$, $t_5=3$, $t_6=4$, $t_7=4$, $t_8=4$

  $0 \le u \le 4$

$$N_{2,1}(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{3,1}(u) = \begin{cases} 1 & 1 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{4,1}(u) = \begin{cases} 1 & 2 \le u \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$$N_{5,1}(u) = \begin{cases} 1 & 3 \le u \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Example – cont'

$$N_{1,2}(u) = \frac{(u - t_1)N_{1,1}}{t_2 - t_1} + \frac{(t_3 - u)N_{2,1}}{t_3 - t_2} = (1 - u)N_{2,1}$$

$$N_{2,2}(u) = \frac{(u - t_2)N_{2,1}}{t_3 - t_2} + \frac{(t_4 - u)N_{3,1}}{t_4 - t_3} = u \quad N_{2,1} + (2 - u)N_{3,1}$$

$$N_{3,2}(u) = \frac{(u - t_3)N_{3,1}}{t_4 - t_3} + \frac{(t_5 - u)N_{4,1}}{t_5 - t_4} = (u - 1)N_{3,1} + (3 - u)N_{4.1}$$

$$N_{4,2}(u) = \frac{(u - t_4)N_{4,1}}{t_5 - t_4} + \frac{(t_6 - u)N_{5,1}}{t_6 - t_5} = (u - 2)N_{4,1} + (4 - u)N_{5.1}$$

$$N_{5,2}(u) = \frac{(u - t_5)N_{5,1}}{t_6 - t_5} + \frac{(t_7 - u)N_{6,1}}{t_7 - t_6} = (u - 3)N_{5,1}$$

Example – cont'

$$N_{0,3}(u)=\frac{(u-t_0)N_{0,2}}{t_2-t_0}+\frac{(t_3-u)N_{1,2}}{t_3-t_1}=(1-u)N_{1,2}=(1-u)^2N_{2,1}$$

$$N_{1,3}(u)=\frac{(u-t_1)N_{1,2}}{t_3-t_1}+\frac{(t_4-u)N_{2,2}}{t_4-t_2}=u\quad N_{1,2}+\frac{2-u}{2}N_{2,2}$$

$$=\left[u(1\text{-}u)+\frac{(2\text{-}u)u}{2}\right]N_{2,1}+\frac{(2\text{-}u)^2}{2}N_{3.1}$$

$$N_{2,3}(u)=\frac{(u-t_2)N_{2,2}}{t_4-t_2}+\frac{(t_5-u)N_{3,2}}{t_5-t_3}=\frac{u}{2}N_{2,2}+\frac{3-u}{2}N_{3,2}$$

$$=\frac{u^2}{2}N_{2,1}+\left[\frac{u(2-u)}{2}+\frac{(3-u)(u-1)}{2}\right]N_{3,1}+\frac{(3-u)^2}{2}N_{4,1}$$

$$N_{3,3}(u)=\frac{(u-t_3)N_{3,2}}{t_5-t_3}+\frac{(t_6-u)N_{4,2}}{t_6-t_4}=\frac{u-1}{2}N_{3,2}+\frac{4-u}{2}N_{4,2}$$

$$=\frac{(u\text{-}1)^2}{2}N_{3,1}+\left[\frac{(u\text{-}1)(3\text{-}u)}{2}+\frac{(4\text{-}u)(u\text{-}2)}{2}\right]N_{4,1}+\frac{(4\text{-}u)^2}{2}N_{5,1}$$

Example – cont'

$$N_{4,3}(u) = \frac{(u - t_4)N_{4,2}}{t_6 - t_4} + \frac{(t_7 - u)N_{5,2}}{t_7 - t_5} = \frac{u-2}{2} N_{4,2} + (4-u)N_{5,2}$$

$$= \frac{(u-2)^2}{2} N_{4,1} + \left[\frac{(u-2)(4-u)}{2} + (4-u)(u-3) \right] N_{5,1}$$

$$N_{5,3}(u) = \frac{(u - t_5)N_{5,2}}{t_7 - t_5} + \frac{(t_8 - u)N_{6,2}}{t_8 - t_6} = (u-3)N_{5,2} = (u-3)^2 N_{5,1}$$

Example – cont'

$$\therefore \mathbf{P}(u) = (1-u)^2 N_{2,1}\mathbf{P}_0 + \left\{\left[u(1-u) + \frac{(2-u)u}{2}\right]N_{2,1} + \frac{(2-u)^2}{2}N_{3,1}\right\}\mathbf{P}_1$$

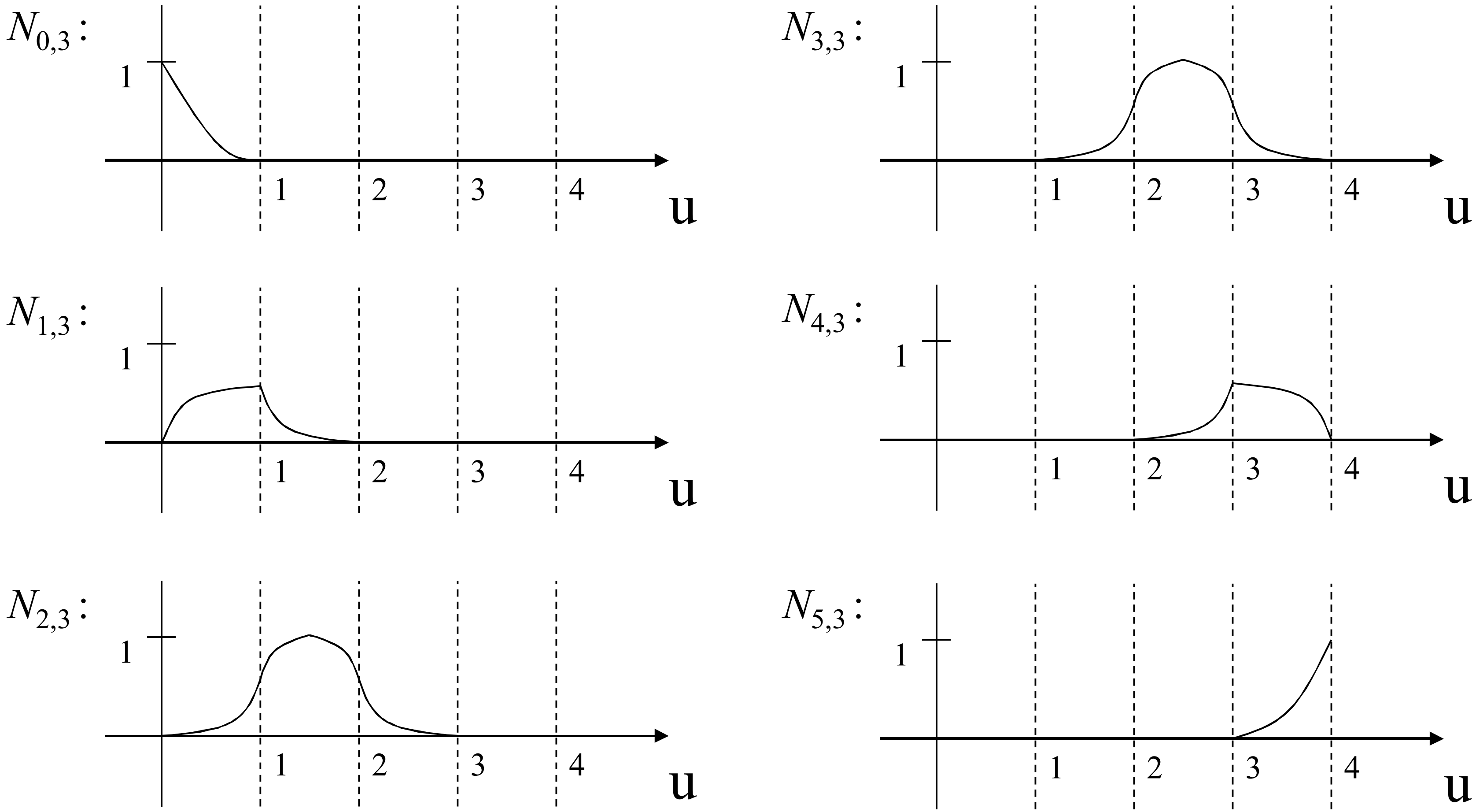
$$+\left\{\frac{u^2}{2}N_{2,1} + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2}\right]N_{3,1} + \frac{(3-u)^2}{2}N_{4,1}\right\}\mathbf{P}_2$$

$$+\left\{\frac{(u-1)^2}{2}N_{3,1} + \left[\frac{(u-1)(3-u)}{2} + \frac{(4-u)(u-2)}{2}\right]N_{4,1} + \frac{(4-u)^2}{2}N_{5,1}\right\}\mathbf{P}_3$$

$$+\left\{\frac{(u-2)^2}{2}N_{4,1} + \left[\frac{(u-2)(4-u)}{2} + (4-u)(u-3)\right]N_{5,1}\right\}\mathbf{P}_4$$

$$+(u-3)^2 N_{5,1} \quad \mathbf{P}_5$$

Example – cont'



Example – cont'

- For each knot interval, coefficients of certain control points = 0
  - Only subset of control points has influence

- For $0 \le u \le 1$, all $N_{i,1}$ except $N_{2,1}$ are 0

$$\therefore \mathbf{P}_1(u) = (1-u)^2 \mathbf{P}_0 + \left[u(1-u) + \frac{(2-u)u}{2} \right] \mathbf{P}_1 + \frac{u^2}{2} \mathbf{P}_2$$

Example – cont'

- Similarly

$$1 \le u \le 2$$

$$\mathbf{P}_2(u) = \frac{(2-u)^2}{2}\mathbf{P}_1 + \left[\frac{u(2-u)}{2} + \frac{(3-u)(u-1)}{2}\right]\mathbf{P}_2 + \frac{(u-1)^2}{2}\mathbf{P}_3$$

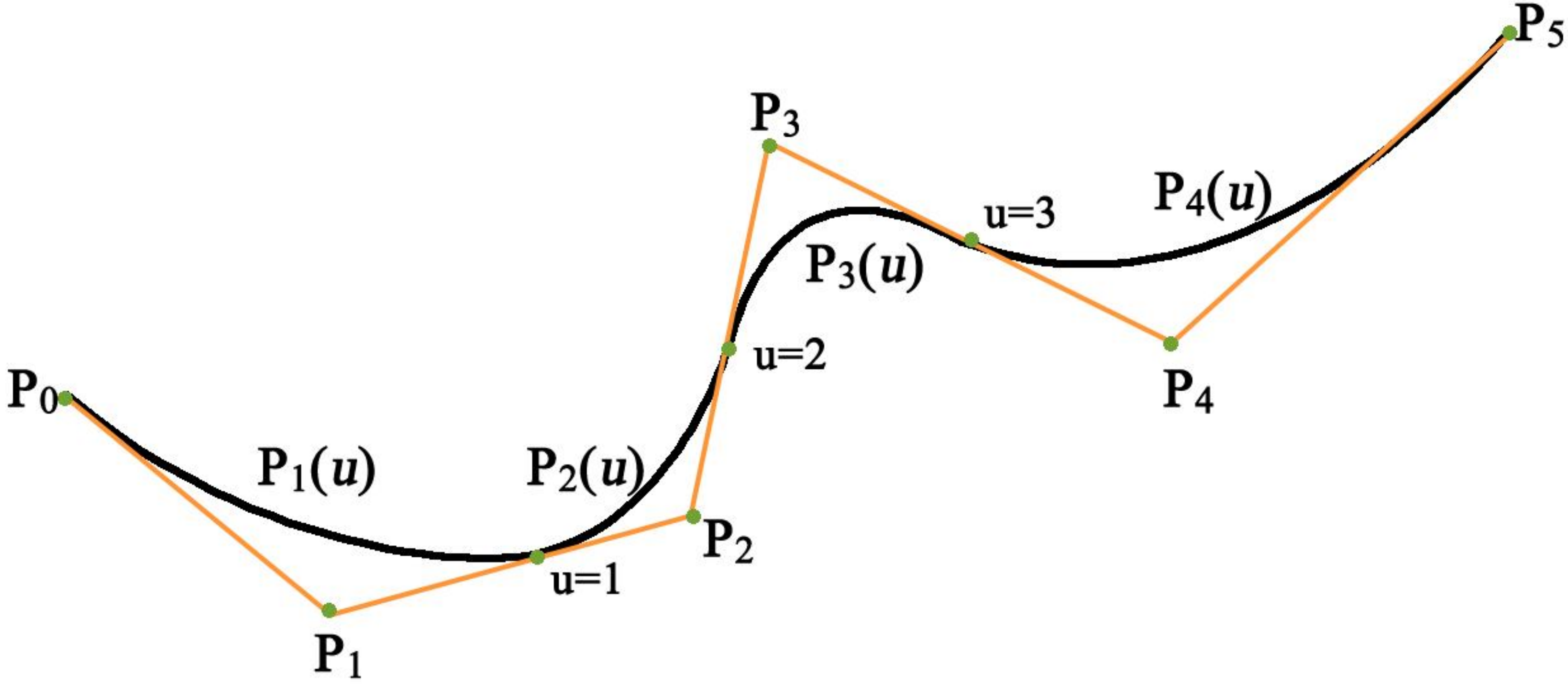
$$2 \le u \le 3$$

$$\mathbf{P}_3(u) = \frac{(3-u)^2}{2}\mathbf{P}_2 + \frac{1}{2}(-2u^2 + 10u - 11)\mathbf{P}_3 + \frac{(u-2)^2}{2}\mathbf{P}_4$$

$$3 \le u \le 4$$

$$\mathbf{P}_4(u) = \frac{(4-u)^2}{2}\mathbf{P}_3 + \frac{1}{2}(-3u^2 + 20u - 32)\mathbf{P}_4 + (u-3)^2\mathbf{P}_5$$

Example – cont'

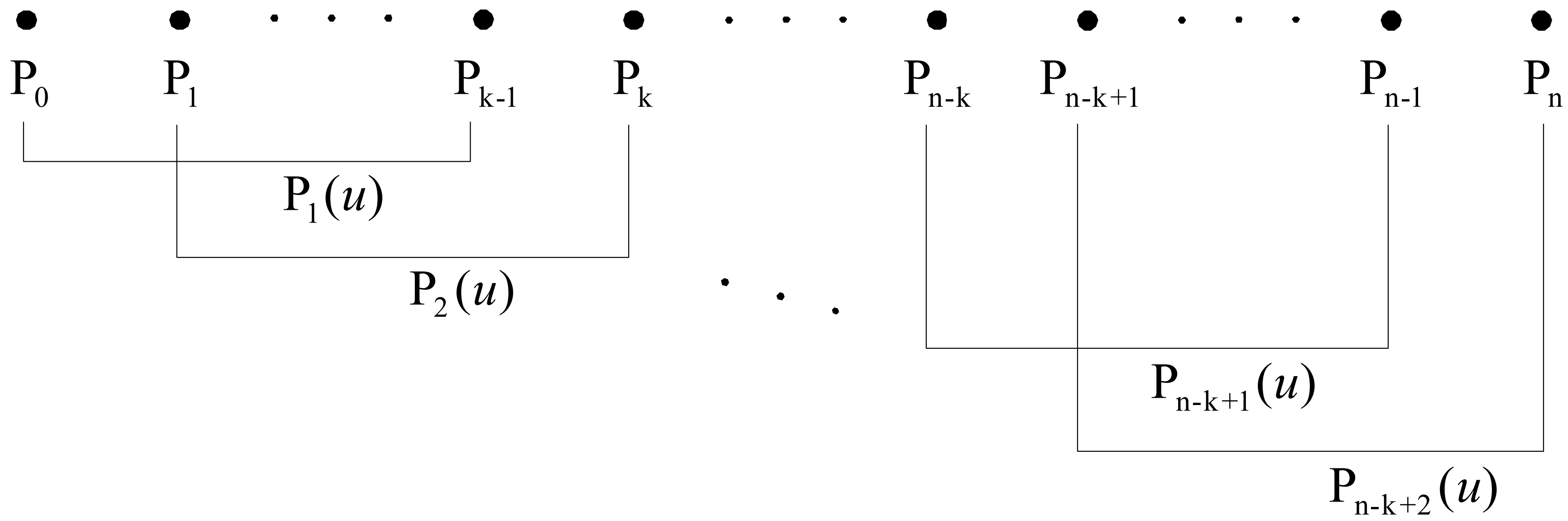


Example – cont'

- $\mathbf{P}'_1(1) = \mathbf{P}'_2(1), \quad \mathbf{P}'_2(2) = \mathbf{P}'_3(2), \quad \mathbf{P}'_3(3) = \mathbf{P}'_4(3) \quad \mathrm{C}^1\mathrm{continuiy}$
- C2 continuity is not satisfied. (∵ k=3, degree 2)
  - For curve of order k, neighboring curves have same derivatives up to (k-2)-th derivative at the common knot
- Each curve segment is defined by k control points.
- Any one control point can influence up to maximum k curve segments.

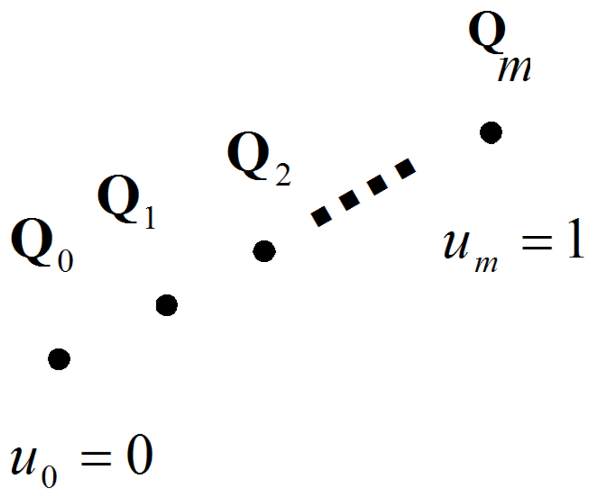
Example – cont'

count curve segment including $\mathbf{P}_{k-1}$



Least squares curve approximation (Interpolation)

- Determine parameter values corresponding to data points



$$d = \sum_{k=1}^{m} \left| \mathbf{Q}_k - \mathbf{Q}_{k-1} \right|$$

$$u_k = u_{k-1} + \frac{\left| \mathbf{Q}_k - \mathbf{Q}_{k-1} \right|}{\mathbf{d}} \quad k=1,\ldots,m-1$$

Least squares curve approximation – cont'

- Calculation of control points

$$\mathbf{C}(u) = \sum_{i=0}^{n} \mathrm{N}_{\mathrm{i}}(u)\mathbf{P}_{\mathbf{i}} \qquad u \in [0,1]$$

$$\mathbf{P}_{\mathbf{0}} = \mathbf{Q}_{\mathbf{0}},\ \mathbf{P}_{\mathbf{n}} = \mathbf{Q}_{\mathbf{m}}$$

Look for $\mathrm{P}_1 \ldots \mathrm{P}_{n-1}$ minimizing $f = \sum_{k=1}^{m-1} \left|\mathbf{Q}_k - \mathbf{C}(u_k)\right|^2$

(n is smaller than m)

Least squares curve approximation – cont’

$$f = \sum_{k=1}^{m-1} \left| \mathbf{Q}_k - \mathbf{C}(\mathrm{u}_\mathrm{k}) \right|^2$$

$$= \sum_{k=1}^{m-1} \left| \mathbf{Q}_k - \mathrm{N}_0(\mathrm{u}_\mathrm{k})\mathbf{Q}_0 - \mathrm{N}_\mathrm{n}(\mathrm{u}_\mathrm{k})\mathbf{Q}_m - \sum_{i=1}^{n-1} \mathrm{N}_i(u_k)\mathbf{P}_i \right|^2 \qquad (\because \mathbf{Q}_0 = \mathbf{P}_{0,} \quad \mathbf{Q}_m = \mathbf{P}_n)$$

$$= \sum_{k=1}^{m-1} \left| \mathbf{R}_k - \sum_{i=1}^{n-1} \mathrm{N}_i(u_k)\mathbf{P}_i \right|^2$$

$$= \sum_{k=1}^{m-1} \left(\mathbf{R}_k - \sum_{i=1}^{n-1} \mathrm{N}_i(u_k)\mathbf{P}_i \right) \bullet \left(\mathbf{R}_k - \sum_{i=1}^{n-1} \mathrm{N}_i(u_k)\mathbf{P}_i \right)$$

$$= \sum_{k=1}^{m-1} \left[\mathbf{R}_k \bullet \mathbf{R}_k - \mathbf{2} \sum_{i=1}^{n-1} \mathrm{N}_i(u_k)(\mathbf{R}_k \bullet \mathbf{P}_i) + \left(\sum_{i=1}^{n-1} \mathrm{N}_i(u_k)\mathbf{P}_i \right) \bullet \left(\sum_{i=1}^{n-1} \mathrm{N}_i(u_k)\mathbf{P}_i \right) \right]$$

Least squares curve approximation – cont'

$$\frac{\partial f}{\partial \mathbf{P}_l} = \sum_{k=1}^{m-1}\left(-2\mathrm{N}_l(u_k)\mathbf{R}_k + 2\mathrm{N}_l(u_k)\sum_{i=1}^{n-1}\mathrm{N}_i(u_k)\mathbf{P}_i\right) = 0$$

$$\therefore \qquad -\sum_{k=1}^{m-1}\mathrm{N}_l(u_k)\mathbf{R}_k + \sum_{k=1}^{m-1}\ \sum_{i=1}^{n-1}\mathrm{N}_l(u_k)\mathrm{N}_i(u_k)\mathbf{P}_i = 0$$

$$\therefore \qquad \sum_{i=1}^{n-1}\left(\sum_{k=1}^{m-1}\mathrm{N}_l(u_k)\mathrm{N}_i(u_k)\right)\mathbf{P}_i = \sum_{k=1}^{m-1}\mathrm{N}_l(u_k)\mathbf{R}_k$$

Least squares curve approximation – cont'

- One equation with n variables $P_1 \ldots\ldots, P_{n-1}$.
  - l can be $1, \ldots\ldots, n-1$
  - $n-1$ equations with $n-1$ variables can be can be generated
- $(N^T N)\, P = R$

Least squares curve approximation – cont'

$$\mathbf{N} = \begin{bmatrix} N_1(u_1) & N_2(u_1) & \cdots & N_{n-1}(u_1) \\ N_1(u_2) & N_2(u_2) & \ddots & \vdots \\ \vdots & & & \vdots \\ N_1(u_{m-1}) & N_2(u_{m-1}) & \cdots & N_{n-1}(u_{m-1}) \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} N_1(u_1)\mathbf{R}_1 + N_1(u_2)\mathbf{R}_2 + \cdots + N_1(u_{m-1})\mathbf{R}_{m-1} \\ \vdots \\ N_{n-1}(u_1)\mathbf{R}_1 + N_{n-1}(u_2)\mathbf{R}_2 + \cdots + N_{n-1}(u_{m-1})\mathbf{R}_{m-1} \end{bmatrix}$$

Least squares curve approximation – cont'

- Determination of Knot vector
  - (n+k+1) knots are needed for order k
  - (n-k+1) internal knots need to be determined since k multiple knots at 0, 0, … , 0 and 1,1, … ,1 already exist

Least squares curve approximation – cont'

$$q = \frac{m+1}{n-k+2}$$

$for \quad j = 1 \quad to \quad n-k+1$

$\{$

$\quad i = \text{int}(jq) \qquad // \qquad i$ is the highest integer satisfying $i \le jq$

$\quad \alpha = jq - i$

$\quad t_{k-1+j} = (1-\alpha)u_{i-1} + \alpha u_i \qquad // \qquad$ *Internal knots*

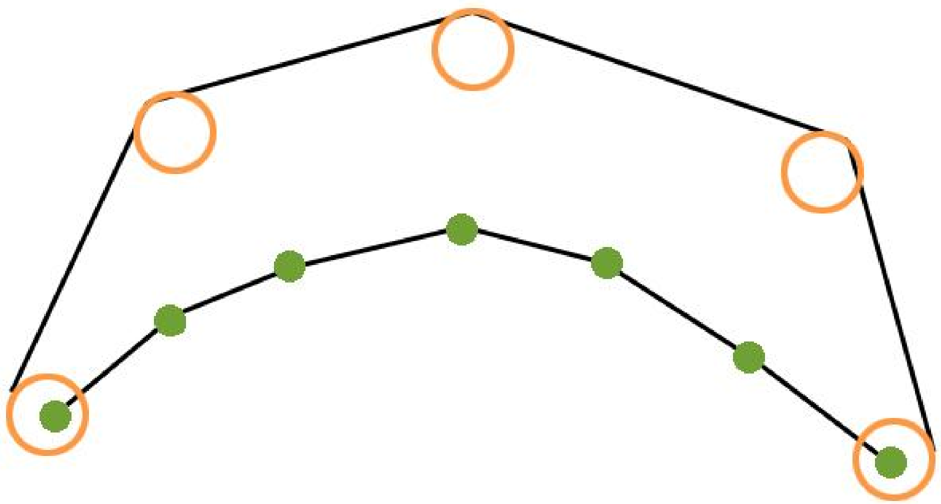
$\}$

Least squares curve approximation – cont’

Example 1

When $\begin{pmatrix} n = 4 \\ k = 4 \\ m = 6 \end{pmatrix}$, $q = \dfrac{6+1}{4-4+2} = 3.5$

For $j = 1$



$i = \mathrm{int}(1 \cdot 3.5) = 3$

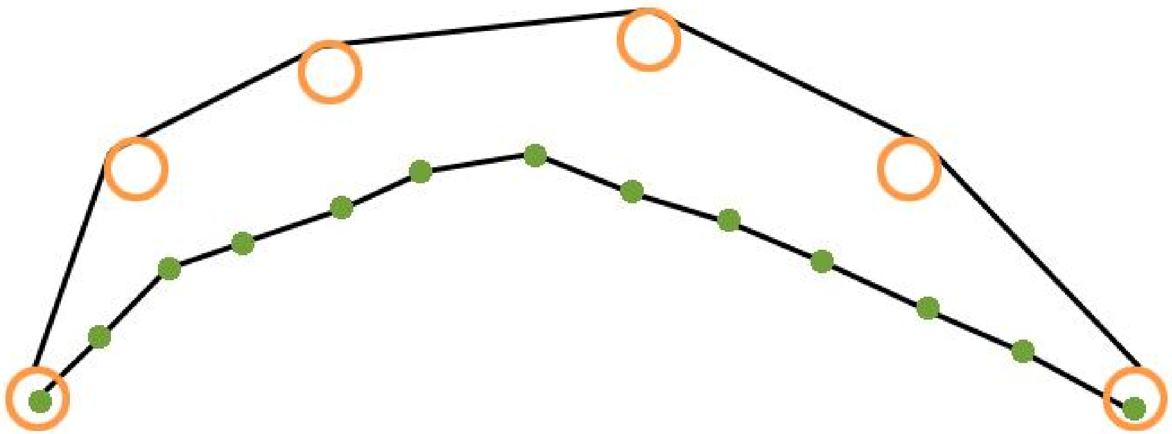
$\alpha = 1 \cdot 3.5 - 3 = 0.5$

$T_{4-1+1} = t_4 = (1-0.5)u_{3-1} + 0.5u_3$
$= 0.5(u_2 + u_3)$

Knot vector = $\{0,0,0,0,t_4,1,1,1,1\}$

Least squares curve approximation – cont'

Example 2



When $\begin{pmatrix} n = 5 \\ k = 4 \\ m = 12 \end{pmatrix}$,

$$q = \frac{12+1}{5-4+2} = \frac{13}{3}$$

Least squares curve approximation – cont'

for $j = 1$

$$i = \text{int}(1 \cdot \frac{13}{3}) = 4$$

$$\alpha = 1 \cdot \frac{13}{3} - 4 = \frac{1}{3}$$

$$t_{4-1+1} = t_4 = (1 - \frac{1}{3}) u_{4-1} + \frac{1}{3} u_4$$

$$= \frac{1}{3}(2u_3 + u_4)$$

for $j = 2$

$$i = \text{int}(2 \cdot \frac{13}{3}) = 8$$

$$\alpha = 2 \cdot \frac{13}{3} - 8 = \frac{2}{3}$$

$$t_{4-1+2} = t_5 = (1 - \frac{2}{3}) u_{8-1} + \frac{2}{3} u_8$$

$$= \frac{1}{3}(u_7 + 2u_8)$$

$\therefore$ *knot vector* $=$

$$\{0,0,0,0,t_4,t_5,1,1,1,1\}$$

Intersection between curves

- $\mathbf{P}(u) - \mathbf{Q}(v) = 0$
- 3 scalar equations, two unknowns
  - $P_x(u) - Q_x(v) = 0$
  - $P_y(u) - Q_y(v) = 0$
- Use Newton Raphson method
  - Derivative of P_x, Q_x, P_y, Q_y need to be calculated
  - $$\begin{aligned} f_1(x_1,\ldots,x_n) &= 0 \\ f_2(x_1,\ldots,x_n) &= 0 \\ &\vdots \\ f_n(x_1,\ldots,x_n) &= 0 \end{aligned}$$

Intersection between curves – cont'

$$f_1(x_1+\Delta x_1, x_2+\Delta x_2, \cdots, x_n+\Delta x_n) = f_1(x_1, \cdots, x_n) + \frac{\partial f_1}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial f_1}{\partial x_n}\Delta x_n$$

$$\vdots$$

$$f_n(x_1+\Delta x_1, x_2+\Delta x_2, \cdots, x_n+\Delta x_n) = f_n(x_1, \cdots, x_n) + \frac{\partial f_n}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial f_n}{\partial x_n}\Delta x_n$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} \cdots \frac{\partial f_2}{\partial x_n} \\ \vdots \\ \frac{\partial f_n}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_n \end{bmatrix} = \begin{bmatrix} -f_1 \\ -f_2 \\ \vdots \\ -f_n \end{bmatrix}$$

Intersection between curves – cont'

- If initial values of u, v are too far from real solution, the iteration diverges.
- Hard to find all the intersection points.
- Cannot handle the case of overlapping curves.
- Two curves are regarded to intersect each other if they lie within numerical tolerance.
- Control polygons are approximated to the curve by subdivision and initial values of u,v can be provided closely by intersecting control polygons
- Better to detect special situation in advance before resorting to numerical solution.
- Tuning tolerance values is necessary

Straight line vs. curve

- $\mathbf{P(u) = P_0 + u(P_1 - P_0)}$
- $\mathbf{Q(v) = P_0 + u(P_1 - P_0)}$ (a)
- Apply dot product $(P_0 \times P_1)$ to both sides of eq(a) gives

$$(P_0 \times P_1) \cdot Q(v) = 0$$

non-linear equation of v

Non-uniform Rational B-spline (NURBS) curve

- Use same Blending functions as B-spline
- Control points are given in homogeneous coordinates

$$(x_i, \quad y_i, \quad z_i) \Rightarrow (x_i \bullet h_i, \quad y_i \bullet h_i, \quad z_i \bullet h_i, \quad h_i)$$

$$x \cdot h = \sum_{i=0}^{n} (h_i \cdot x_i) \quad N_{i,k}(u)$$

$$y \cdot h = \sum_{i=0}^{n} (h_i \cdot y_i) \quad N_{i,k}(u)$$

$$z \cdot h = \sum_{i=0}^{n} (h_i \cdot z_i) \quad N_{i,k}(u)$$

$$h = \sum_{i=0}^{n} h_i \quad N_{i,k}(u)$$

Non-uniform Rational B-spline (NURBS) curve – cont'

$$\mathrm{P}(u) = \frac{\sum_{i=0}^{n} h_i \mathrm{P}_i N_{i,k}(u)}{\sum_{i=0}^{n} h_i \quad N_{i,k}(u)}$$

Passes through the 1st and the last control points

(When non-periodic knots are used)

Numerator is B-spline with $h_i\mathrm{P}_i$ as control points

$\rightarrow$ $h_0\mathrm{P}_0$, $h_n\mathrm{P}_n$ at parameter boundary values

Similarly denominator has values of h_0, h_n at parameter boundary values

Non-uniform Rational B-spline (NURBS) curve – cont'

Directions of tangent vectors are P_1-P_0, P_n-P_{n-1} at starting and ending points

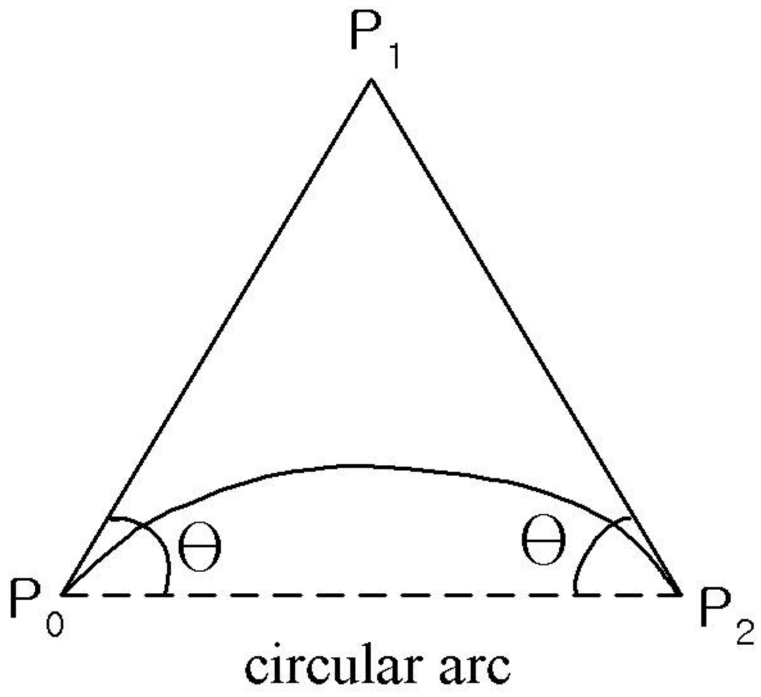
$$h_i = 1 \quad \sum_{i=0}^{n} N_{i,k}(u) = 1 \quad \Rightarrow \quad B - spline$$

B-spline curve is a special case of NURBS

Non-uniform Rational B-spline (NURBS) curve – cont'

- Curve shape can be changed by changing weight(h_i)
- Increasing weight has an effect of pulling curve toward associated control point
  - Example program
- Conic curve can be represented exactly
  - Reducing program coding effort

Control points of NURBS curve equivalent to a circular arc



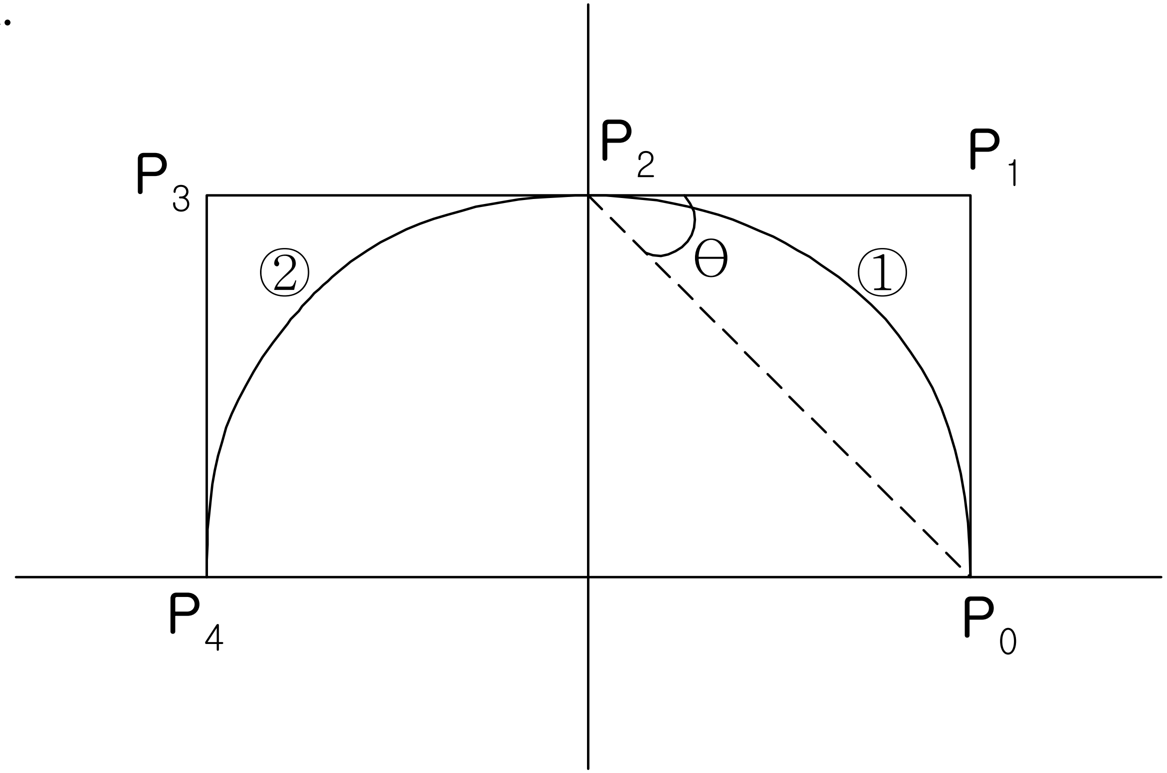
$h_0=h_2=1$
$h_1=\cos\Theta$

Can be used when center angle is less than 180°.

Arc with a center angle bigger than 180° is split into two and combined later

Example

Ex.



Example – cont'

$$P_0 = (1,\ \ 0),\qquad P_1 = (1,\ \ 1),\qquad P_2 = (0,\ \ 1)$$

$$h_0 = 1 \qquad h_1 = \cos 45^\circ = \frac{1}{\sqrt{2}} \qquad h_2 = 1$$

$$knot\quad 0\quad 0\quad 0\quad 1\quad 1\quad 1\quad (n = 2,\ \ k = 3)$$

Similarly,

$$P_2 = (0,\ \ 1),\quad P_3 = (-1,\ \ 1),\quad P_4 = (-1,\ \ 0)$$

$$h_2 = 1, \qquad h_3 = \frac{1}{\sqrt{2}}, \qquad h_4 = 1$$

$$knot\quad 0\quad 0\quad 0\quad 1\quad 1\quad 1\quad \Rightarrow\quad 1\quad 1\quad 1\quad 2\quad 2\quad 2$$

Example – cont'

Composition

$$P_0 = (1,\ \ 0),\quad P_1 = (1,\ \ 1),\quad P_2 = (0,\ \ 1),$$

$$P_3 = (-1,\ \ 1),\quad P_4 = (-1,\ \ 0)$$

$$h_0 = 1 \qquad h_1 = \frac{1}{\sqrt{2}} \qquad h_2 = 1$$

$$h_3 = \frac{1}{\sqrt{2}}, \qquad h_4 = 1$$

$$knot \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 2 \quad 2 \quad 2$$