Surfaces

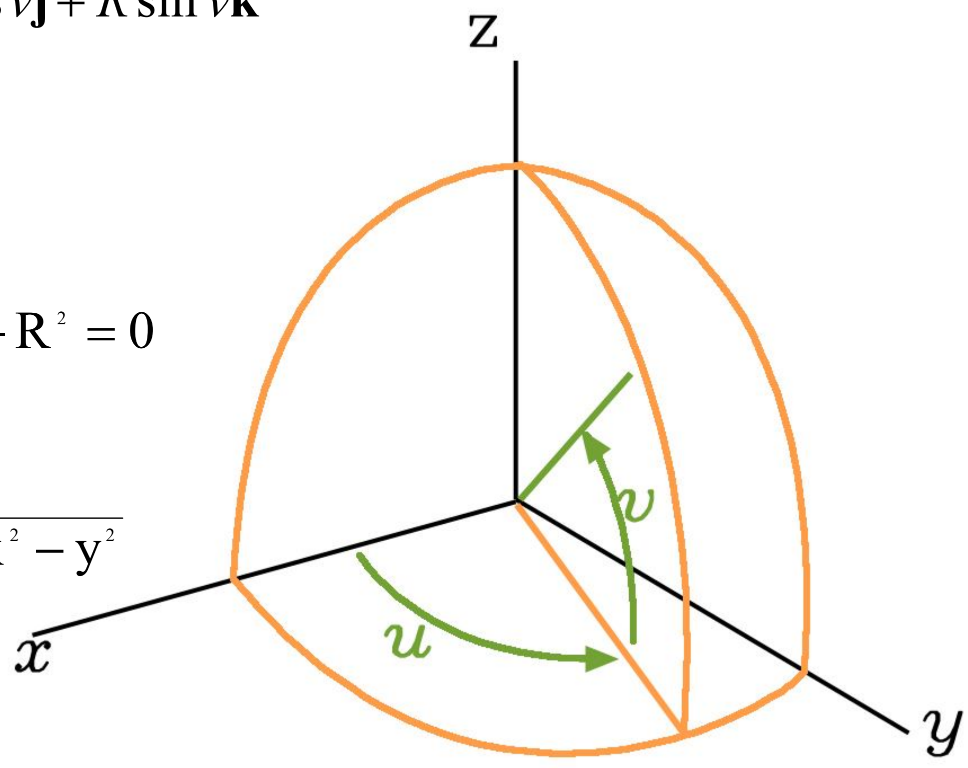
Human Centered CAD Lab.

Surfaces

- Parametric eq.

$$\mathbf{P}(u,v) = R\cos u\cos v\mathbf{i} + R\sin u\cos v\mathbf{j} + R\sin v\mathbf{k}$$

$$(0 \le u \le 2\pi,\ -\pi/2 \le v \le \pi/2)$$



- Implicit eq $\quad x^2 + y^2 + z^2 - R^2 = 0$

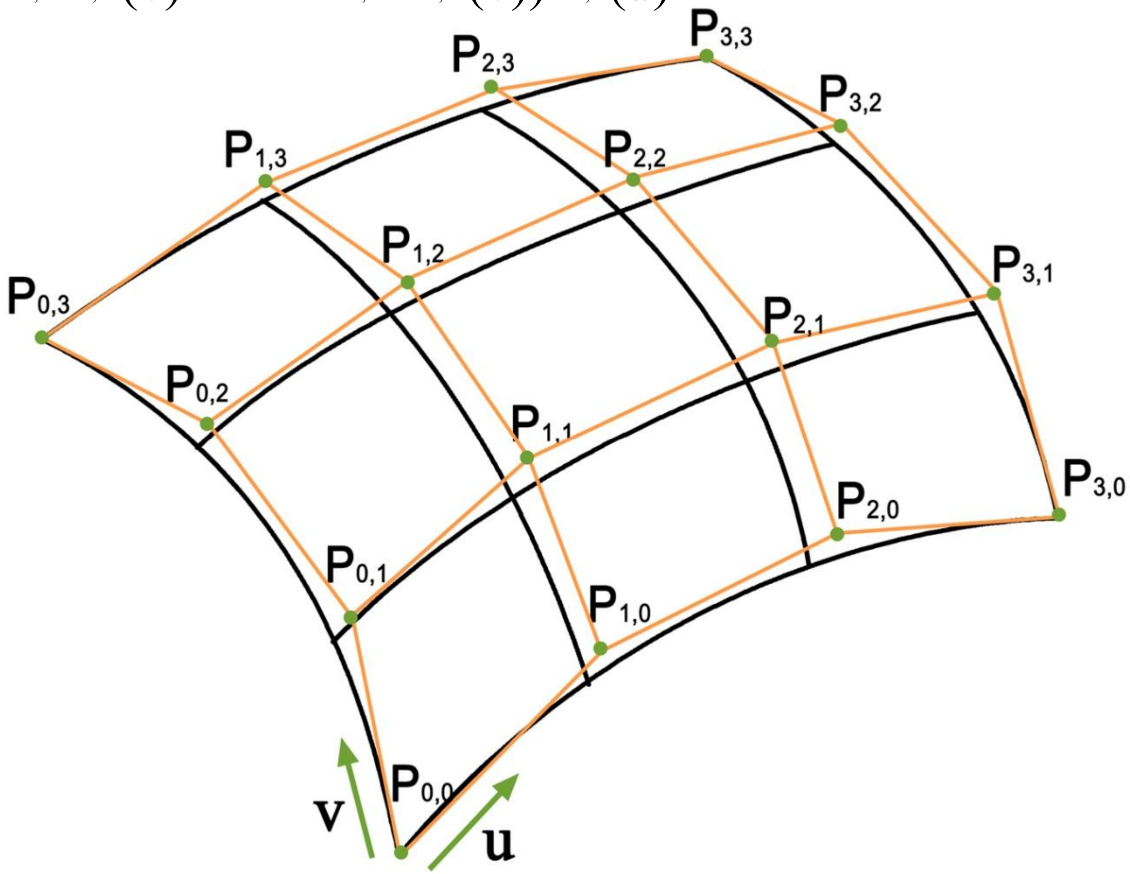
- Explicit eq $\quad z = \pm\sqrt{R^2 - x^2 - y^2}$

Bezier surface

$$\mathbf{P}(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{P}_{i,j} B_{i,n}(u) B_{j,m}(v) \qquad 0 \le u \le 1, \qquad 0 \le v \le 1$$

$$= \sum_{i=0}^{n} (\mathbf{P}_{i,0} B_{0,m}(v) + \mathbf{P}_{i,1} B_{1,m}(v) + \cdots + \mathbf{P}_{i,m} B_{m,m}(v)) B_{i,n}(u)$$

Bezier curve



Bezier surface – cont’

- Surface obtained by blending (n+1) Bezier curves
  - Or by blending (m+1) Bezier curves
- Four corner points on control polyhedron lie on surface

Bezier surface – cont'

- Equation

$$\mathbf{P}(0,0) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{P}_{i,j} B_{i,n}(0) B_{j,m}(0)$$

$$= \sum_{i=0}^{n} \underbrace{\left[\sum_{j=0}^{m} \mathbf{P}_{i,j} B_{j,m}(0) \right]}_{P_{i,0}} B_{i,n}(0)$$

$$= \sum_{i=0}^{n} \mathbf{P}_{i,0} B_{i,n}(0) = \mathbf{P}_{0,0}$$

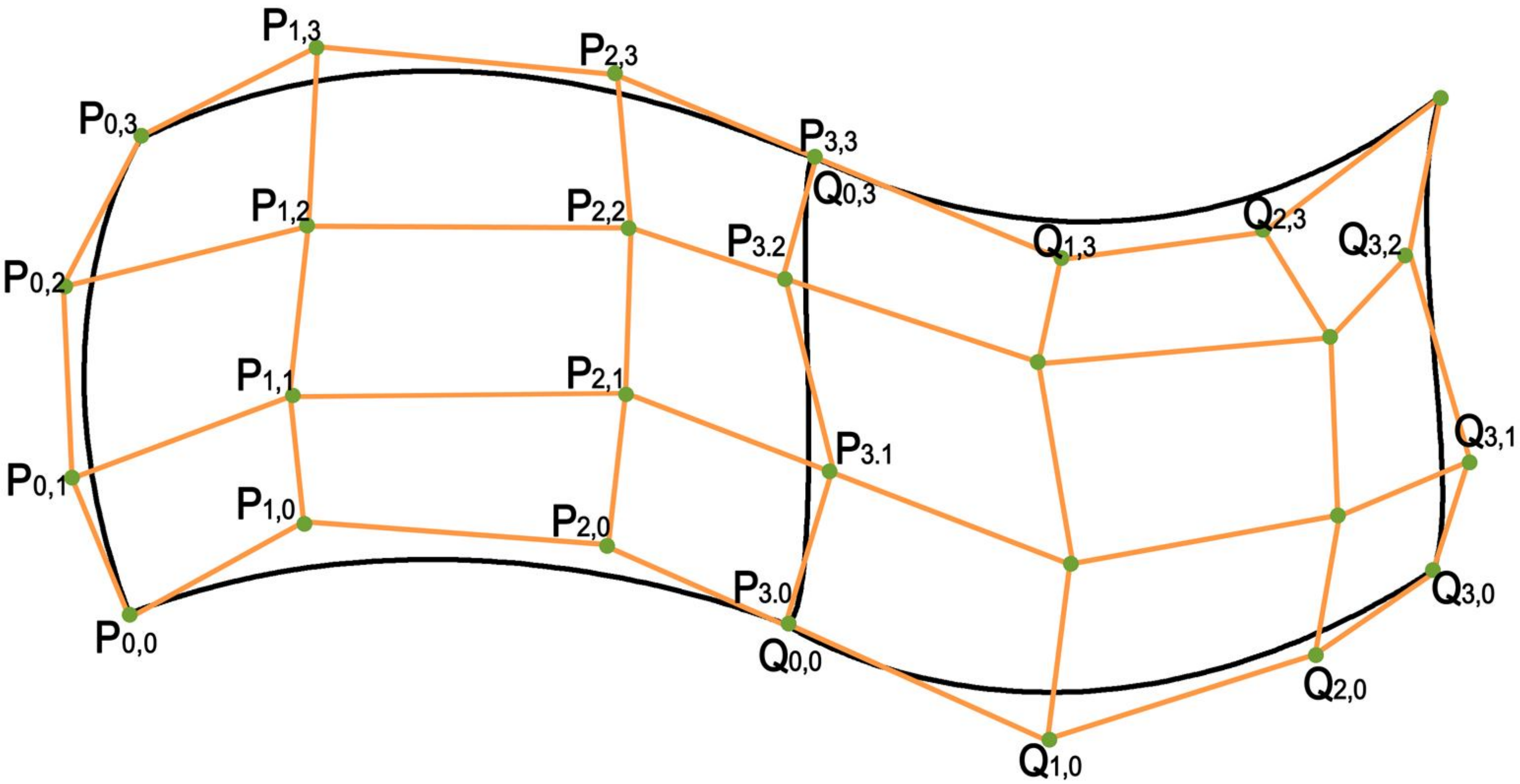
Bezier surface – cont'

- Boundary curves are Bezier curves defined by associated control points

$$
\begin{aligned}
\mathbf{P}(0, v) &= \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{P}_{i,j} B_{i,n}(0) B_{j,m}(v) \\
&= \sum_{j=0}^{m} \underbrace{\left[\sum_{i=0}^{n} \mathbf{P}_{i,j} B_{i,n} \right]_{u=0}}_{P_{0,j}} B_{j,m}(v) \\
&= \underbrace{\sum_{j=0}^{m} \mathbf{P}_{0,j} B_{j,m}(v)}
\end{aligned}
$$

- Bezier curve defined by $\mathbf{P}_{0,0}, \mathbf{P}_{0,1}, \ldots, \mathbf{P}_{0,m}$

Bezier surface – cont’



- When two Bezier surfaces are connected, control points before and after connection should form straight lines to guarantee G1 continuity

B-spline surface

$$\mathbf{P}(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{P}_{i,j} N_{i,k}(u) N_{j,l}(v) \qquad s_{k+1} \le u \le s_{n+1}, \quad t_{l-1} \le v \le t_{m+1}$$

- $N_{i,k}(u)$ is defined by $s_0, s_1, \ldots, s_{n+k}$
- $N_{j,l}(v)$ is defined by $t_0, t_1, \ldots, t_{l+m}$
- If $k=(n+1)$, $l=m+1$ and non-periodic knots are used, the resulting surface will become Bezier surface

B-spline surface – cont'

- Bezier surface is a special case of B-spline surface.
- Boundary curves are B-spline curves defined by associated control points.
- Four corner points of control polyhedron lie one the surface (when non-periodic knots are used)

NURBS surface

$$\mathbf{P}(\mathrm{u},\mathrm{v}) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} h_{i,j}\mathbf{P}_{i,j}N_{i,k}(\mathrm{u})N_{j,l}(\mathrm{v})}{\sum_{i=0}^{n}\sum_{j=0}^{m} h_{i,j}N_{i,k}(\mathrm{u})N_{j,l}(\mathrm{v})}$$

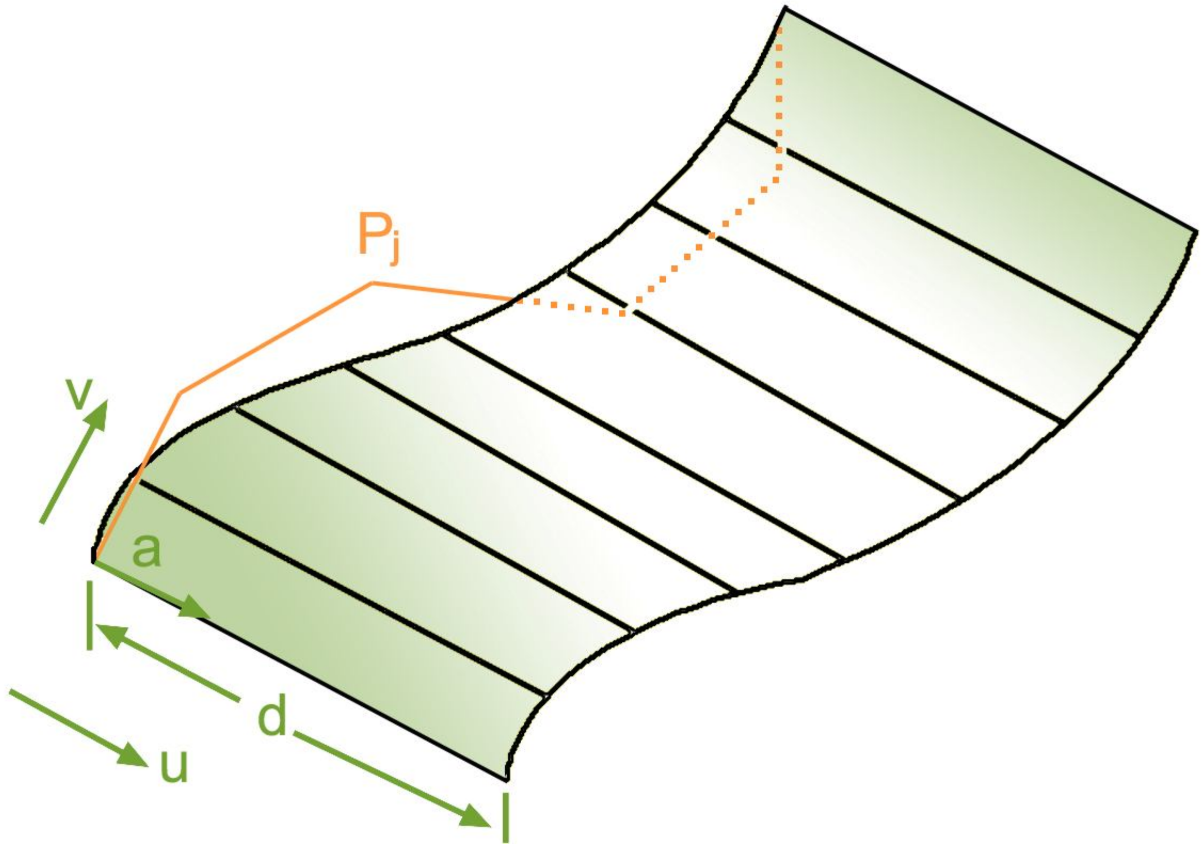
$$\mathrm{s}_{k-1} \leq \mathrm{u} \leq \mathrm{s}_{n+1}$$

$$\mathrm{t}_{\ell-1} \leq \mathrm{v} \leq \mathrm{t}_{m+1}$$

- If $\mathrm{h}_{\mathrm{i,j}} = 1$, B-spline surface is obtained
- Represent quadric surface(cylindrical, conical, spherical, paraboloidal, hyperboloidal) exactly

NURBS surface – cont'

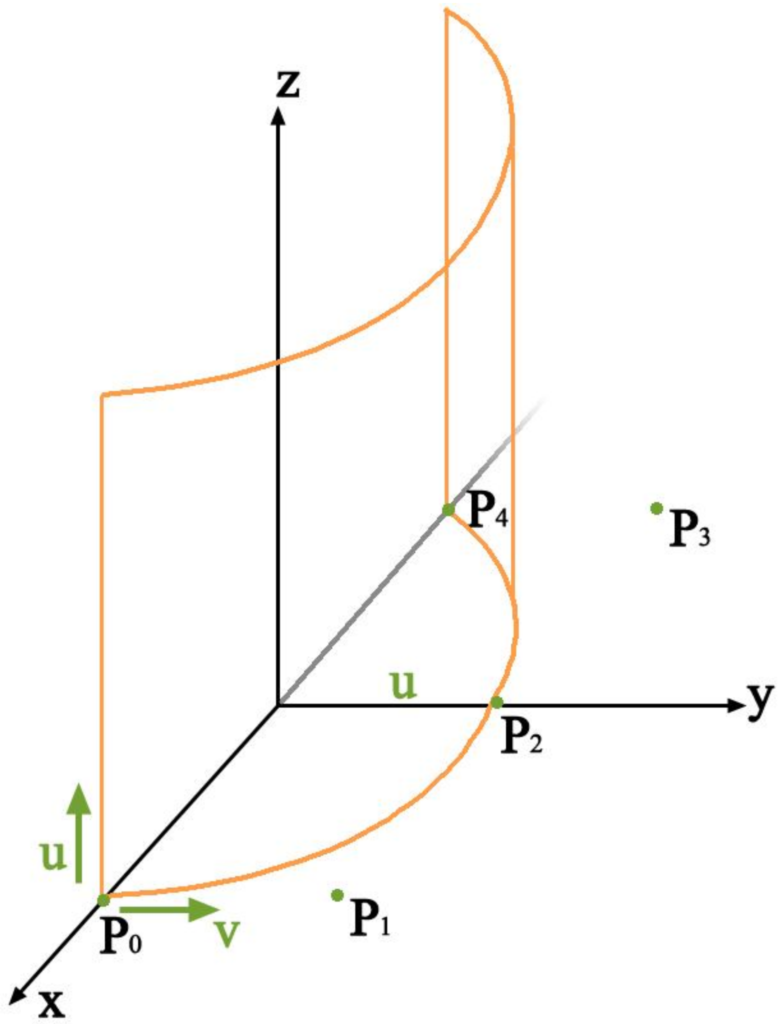
- Represent a surface obtained by sweeping a curve



NURBS surface – cont'

- Assume that v-direction of surface is a given $\mathbf{P}_j$
- v-direction knot & order is the same as the NURBS Curve's (order: l, knot: t_p)
- u-direction order is 2
- control point: 2
  - u direction knot: 0 0 1 1
  - $\mathbf{P}_{0,j} = \mathbf{P}_j$
  - $\mathbf{P}_{1,j} = \mathbf{P}_j + d\,\mathbf{a}$
  - $h_{0,j} = h_{1,j} = h_j$ from the given curve

Ex) Translate half circle to make cylinder

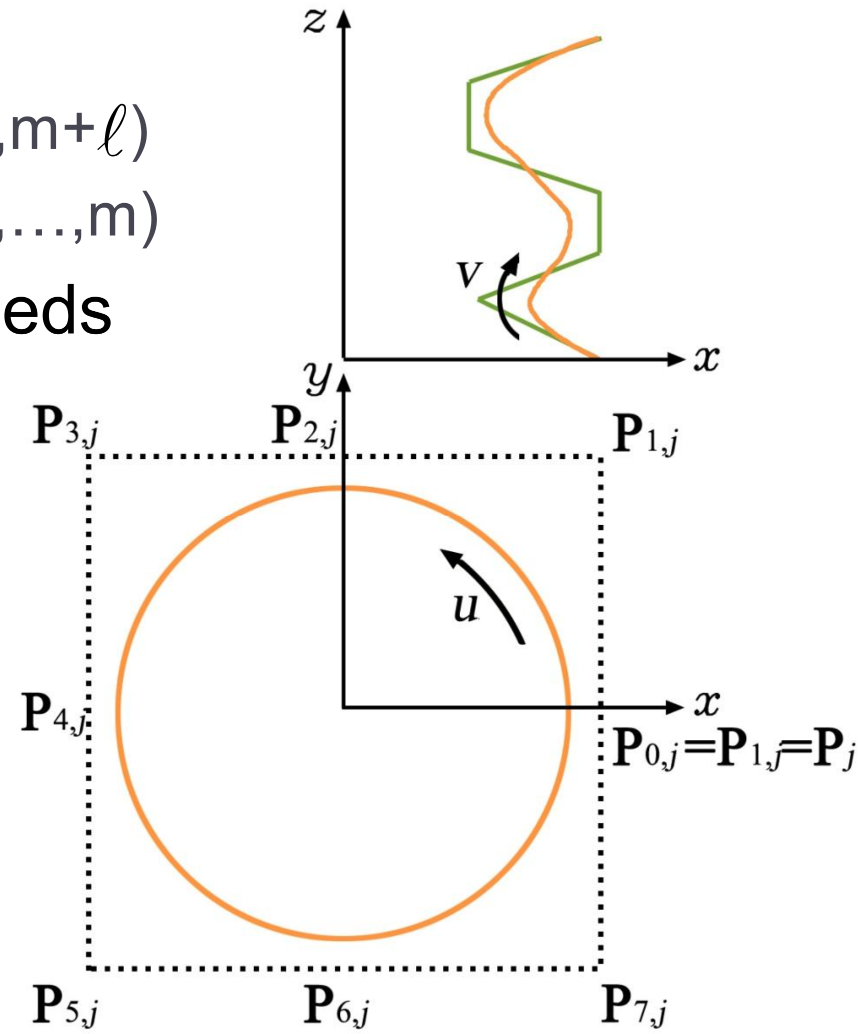


Ex) Translate half circle to make cylinder

- $\mathbf{P}_0=(1, 0, 0)$ $h_0=1$ $\mathbf{P}_1=(1, 1, 0)$ $\mathrm{h}_1=1/\sqrt{2}$
- $\mathbf{P}_2=(0, 1, 0)$ $h_2=1$ $\mathbf{P}_3=(-1, 1, 0)$ $\mathrm{h}_3= 1/\sqrt{2}$
- $\mathbf{P}_4=(-1, 0, 0)$ $h_4=1$
- $\mathbf{P}_{0,0} = \mathrm{P}_0,$ $\mathbf{P}_{1,0} = \mathbf{P}0+H\mathbf{k}$ $h_{0,0} = h_{1,0}=1$
- $\mathbf{P}_{0,1} = \mathrm{P}_1,$ $\mathbf{P}_{1,1} = \mathbf{P}1+H\mathbf{k}$ $h_{0,1} = h_{1,1}= 1/\sqrt{2}$
- $\mathbf{P}_{0,2} = \mathrm{P}_2,$ $\mathbf{P}_{1,2} = \mathbf{P}_2+H\mathbf{k}$ $h_{0,2} = \mathrm{h}_{1,2}=1$
- $\mathbf{P}_{0,3} = \mathrm{P}_3,$ $\mathbf{P}_{1,3} = \mathbf{P}_3+H\mathbf{k}$ $h_{0,3} = \mathrm{h}_{1,3}= 1/\sqrt{2}$
- $\mathbf{P}_{0,4} = \mathrm{P}_4,$ $\mathbf{P}_{1,4} = \mathbf{P}_4+H\mathbf{k}$ $h_{0,4} = h_{1,4}=1$
- Knots for v: 0 0 0 1 1 2 2 2
- Knots for u: 0 0 1 1

Ex) Surface obtained by revolution

- Curve
  - Order ℓ, knot t_p (p=0,1,…,m+ℓ)
  - Control points P_j, h_j (j=0,1,…,m)
- Original control points needs to be split into 9.



Ex) Surface obtained by revolution – cont'

- $P_{0,j} = P_j$ $\quad$ $h_{0,j} = h_j$
- $P_{1,j} = P_{0,j} + x_j\,\mathbf{j} = P_j + x_j\,\mathbf{j}$ $\quad$ $h_{1,j} = h_j \cdot 1/\sqrt{2}$
- $P_{2,j} = P_{1,j} - x_j\,\mathbf{i} = P_j - x_j\,(\mathbf{i}\text{-}\mathbf{j})$ $\quad$ $h_{2,j} = h_j$
- $P_{3,j} = P_{2,j} - x_j\,\mathbf{i} = P_j - x_j\,(2\,\mathbf{i}\text{-}\mathbf{j})$ $\quad$ $h_{3,j} = h_j \cdot 1/\sqrt{2}$
- $P_{4,j} = P_{3,j} - x_j\,\mathbf{j} = P_j - 2x_j\,\mathbf{i}$ $\quad$ $h_{4,j} = h_j$
- $P_{5,j} = P_{4,j} - x_j\,\mathbf{j} = P_j - x_j\,(2\,\mathbf{i}+\mathbf{j})$ $\quad$ $h_{5,j} = h_j \cdot 1/\sqrt{2}$
- $P_{6,j} = P_{5,j} + x_j\,\mathbf{i} = P_j - x_j\,(\mathbf{i}+\mathbf{j})$ $\quad$ $h_{6,j} = h_j$
- $P_{7,j} = P_{6,j} + x_j\,\mathbf{i} = P_j - x_j\,\mathbf{j}$ $\quad$ $h_{7,j} = h_j \cdot 1/\sqrt{2}$
- $P_{8,j} = P_{0,j} = P_j$ $\quad$ $h_{8,j} = h_j$
- u-direction order: 3
- u-direction knot: 0 0 0 1 1 2 2 3 3 4 4 4

Synthesize four quarter circles

Interpolation surface

- Generate a surface from data points
  - Essential in reverse engineering
- Derive B-spline surface passing through data points $Q_{p,q}(p=0,\ldots n,\ \ q=0,\ldots m)$.
  - Constraints: $(n+1)\times(m+1)$
  - B-spline's control points:

    minimum $(n+1)\times(m+1)$ control points are needed.

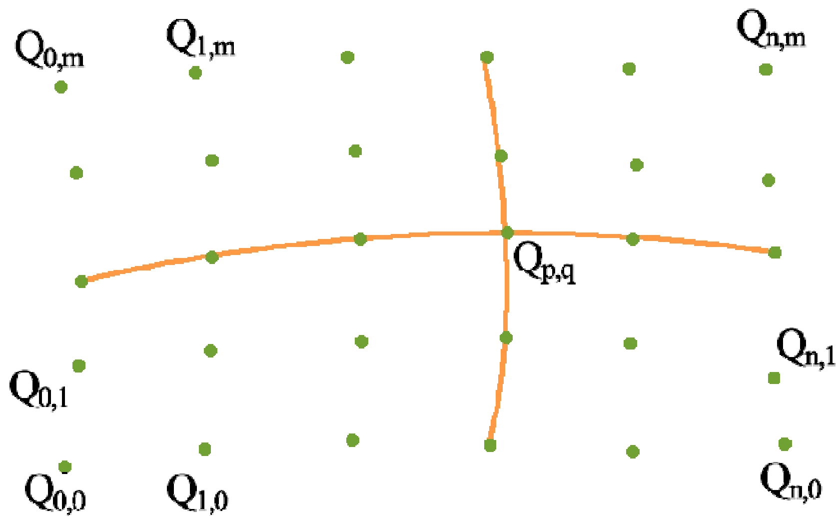
Interpolation surface – cont'

$$\mathbf{P}(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{P}_{i,j} N_{i,k}(u) N_{j,l}(v)$$

- $P_{i,j}$ should be obtained
  - Let u_p, v_q to be parameter value of $Q_{p,q}$
    - u_p is obtained while interpolating $Q_{0,q}, Q_{1,q}, \ldots, Q_{n,q}$
    - v_q is obtained while interpolating $Q_{p,0}, Q_{p,1}, \ldots, Q_{p,m}$

$$\mathbf{Q}_{p,q} = \sum_{i=0}^{n} \sum_{j=0}^{m} \mathbf{P}_{i,j} N_{i,k}(u_p) N_{j,l}(v_q)$$

$$= \sum_{i=0}^{n} \left(\underbrace{\sum_{j=0}^{m} \mathbf{P}_{i,j} N_{j,l}(v_q)}_{\mathbf{C}_i(v_q)} \right) N_{i,k}(u_p)$$



Interpolation surface – cont'

$$\mathbf{Q}_{p,q} = \sum_{i=0}^{n} \mathbf{C}_i(v_q) N_{i,k}(u_p) \qquad (a)$$

substitute 0 *to m for q in* (*a*)

$$\mathbf{Q}_{p,0} = \sum_{i=0}^{n} \mathbf{C}_i(v_0) N_{i,k}(u_p)$$

$$\mathbf{Q}_{p,1} = \sum_{i=0}^{n} \mathbf{C}_i(v_1) N_{i,k}(u_p)$$

$$\vdots$$

$$\mathbf{Q}_{p,m} = \sum_{i=0}^{n} \mathbf{C}_i(v_m) N_{i,k}(u_p)$$

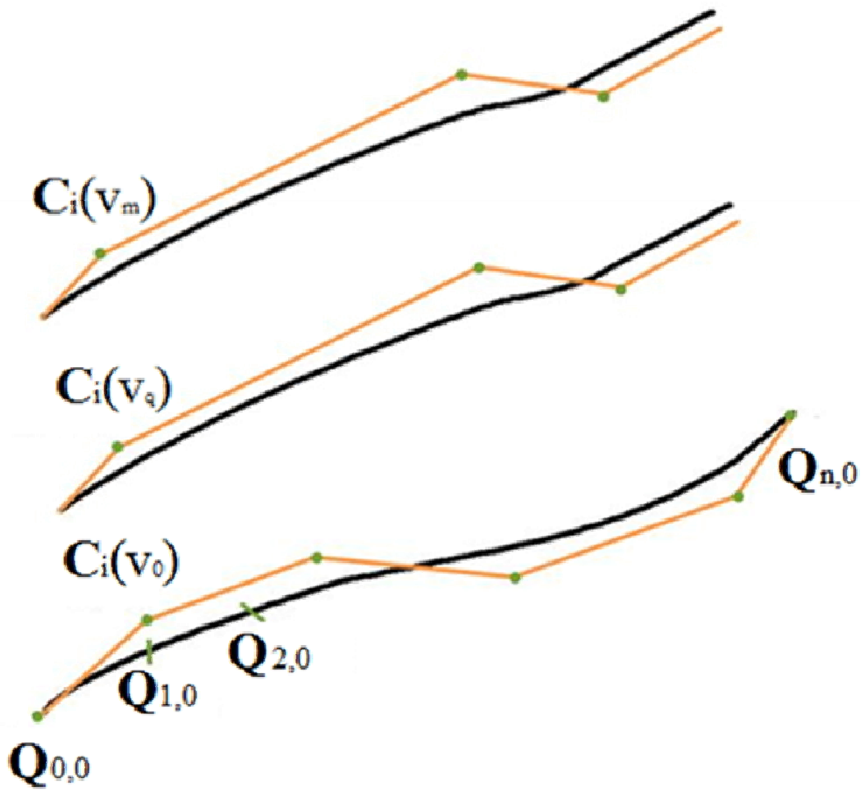
- $C_i(v_0)$ $(i=0,\ldots,n)$ are control points of B-spline interpolating $Q_{0,0}$, $Q_{1,0}$, …, $Q_{n,0}$
- Similarly $C_i(v_1)$ $(i=0,\ldots,n)$ are control points of B-spline interpolating $Q_{0,1}$, $Q_{1,1}$, $Q2,1$, … , $Q_{n,1}$

Interpolation surface – cont'

- Derive $P_{i,j}$ from $C_i(v_q)$ (q=0,…,m)

$$\mathbf{C}_i(v_q) = \sum_{j=0}^{m} \mathbf{P}_{i,j} N_{j,l}(v_q) \qquad \text{(b)}$$

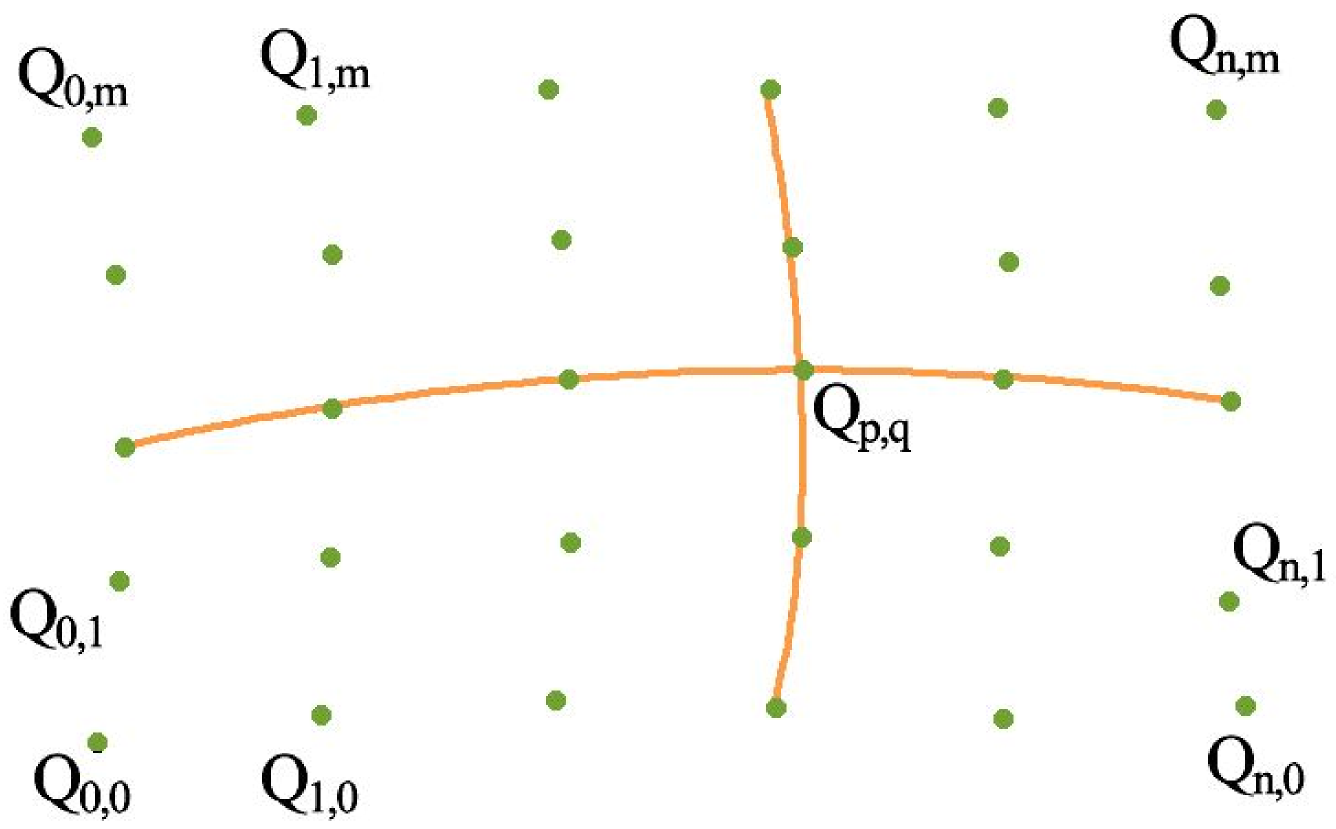
- Substitute 0 to m for q in (b)



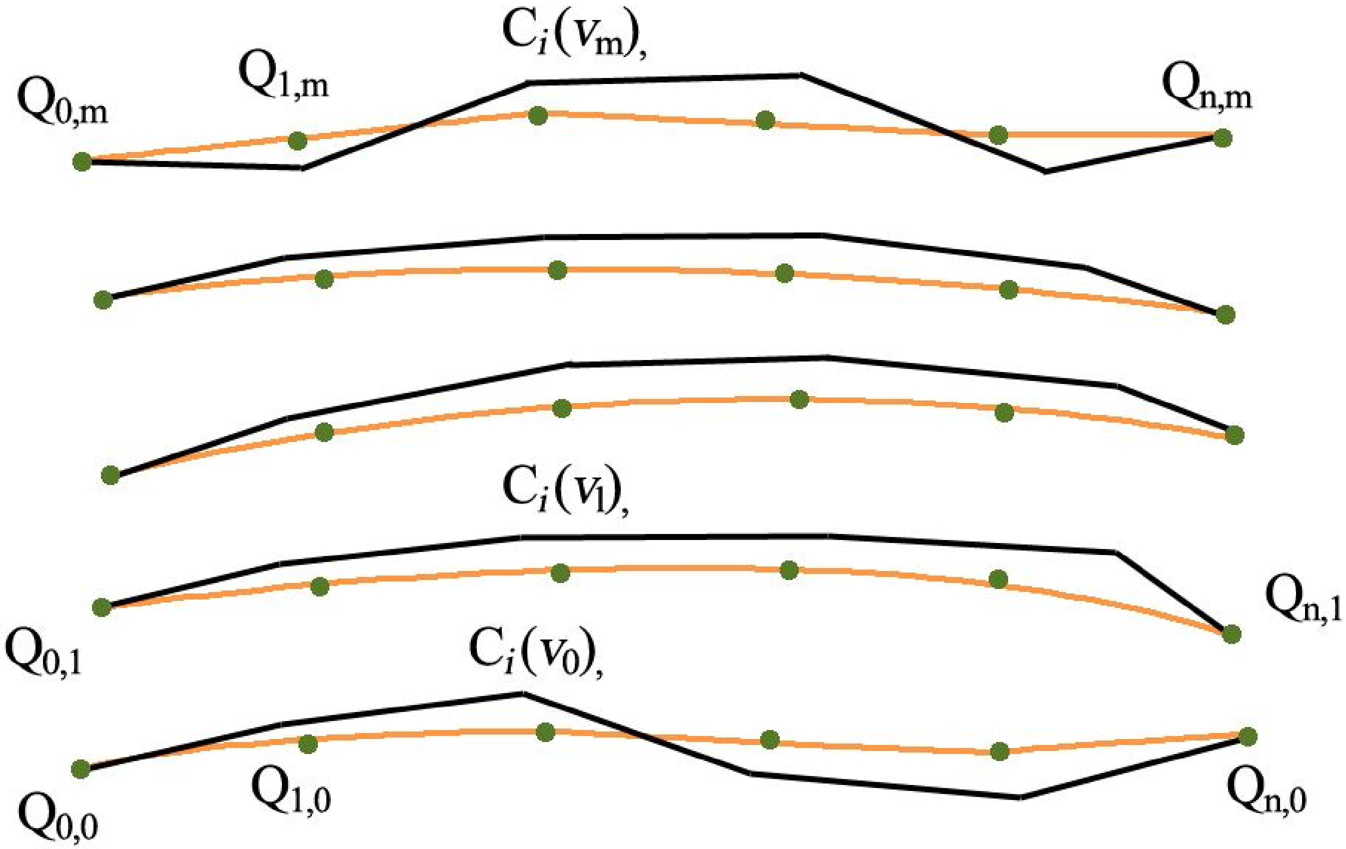
Interpolation surface – cont'

- $P_{i,j}$ are control points of B-spline interpolating $C_i(v_0)$, $C_i(v_1),\ldots,C_i(v_m)$
- $P_{0,j}$ are from $C_0(v_0)$, $C_0(v_1)$, …, $C_0(v_m)$
- $P_{1,j}$ are from $C_1(v_0)$, $C_1(v_1)$, …, $C_1(v_m)$
- $P_{n,j}$ are from $C_n(v_0)$, $C_n(v_1)$, …, $C_n(v_m)$

Interpolation surface – cont’



Interpolation surface – cont'



Interpolation surface – cont'

- Order k, ℓ are usually set to be 4.
- Knot values
- (m+1) set of knot values are obtained during interpolation along u direction
- (m+1) set are averaged to result knot values in u direction
  - Knot values in v direction are derived in the same way

Intersection between surfaces

- Many points on the intersection curves are obtained by numerical solution
- Intersection curve should be calculated in Boolean operation, surface modeling, etc.
- $P(u,v) - Q(s,t) = 0$
- Four unknowns u, v, s, t with three scalar eqs.
- Set one parameter value and determine remaining parameter values, repeat with many values
- Dependent on Initial value.
- Cannot guarantee all intersection curves

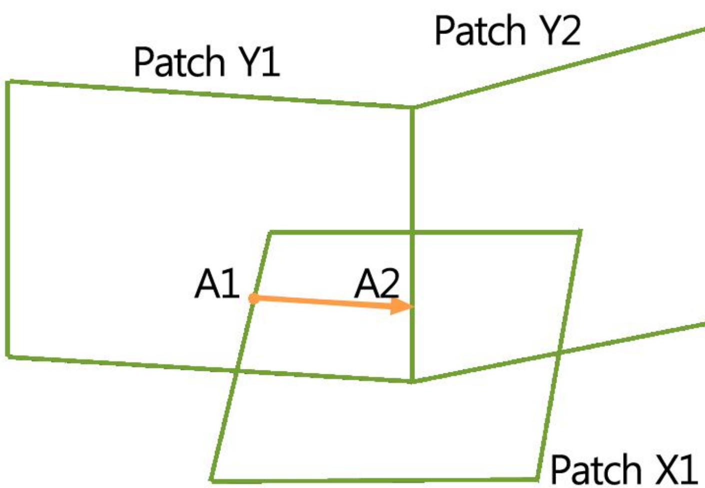
Subdivision method

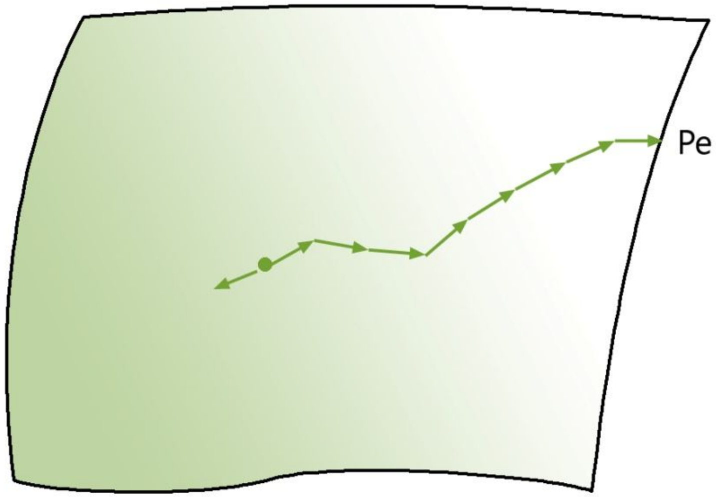
- Subdivide both surfaces until control polyhedron approximates the surface
- Look for intersecting pairs of planar quadrilaterals and calculate the intersection
  - Initial values are obtained from the intersection of planar quadrilaterals

Subdivision method – cont’

- Start from one end of intersection and look for the next pair to be considered and calculate intersection
  - Until surface boundary is reached
  - Traverse in the opposite direction starting from the other end
- Start from different pair until all the intersection curves are obtained

Tracing intersection segments





Searching for the next pair

Tracing in reverse