Chapter 2.

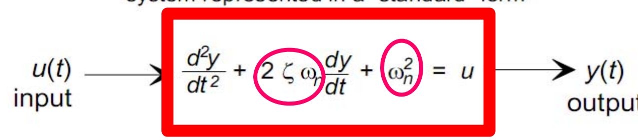
Laplace Transform

Nise Ch.2.1-2.3
MIT OCW Lec.3
Palm Ch.3

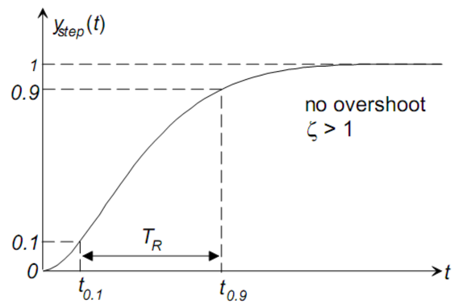
Review of the Last Lecture

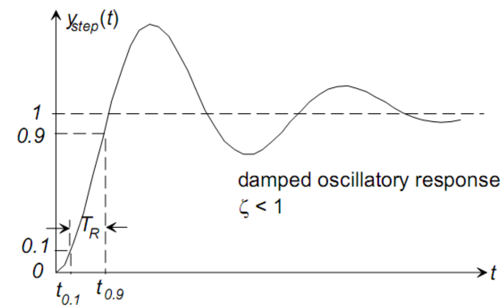
System Model

system represented in a "standard" form

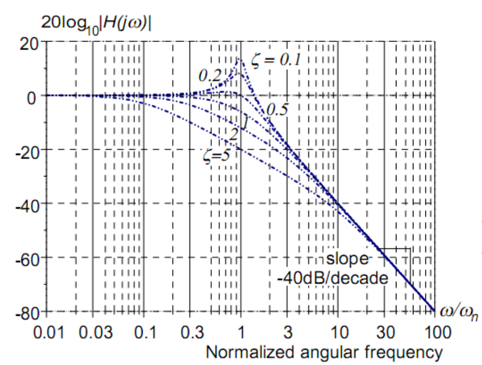


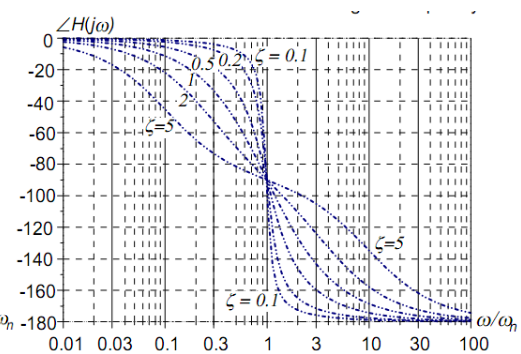
Step input response. (Transient response)



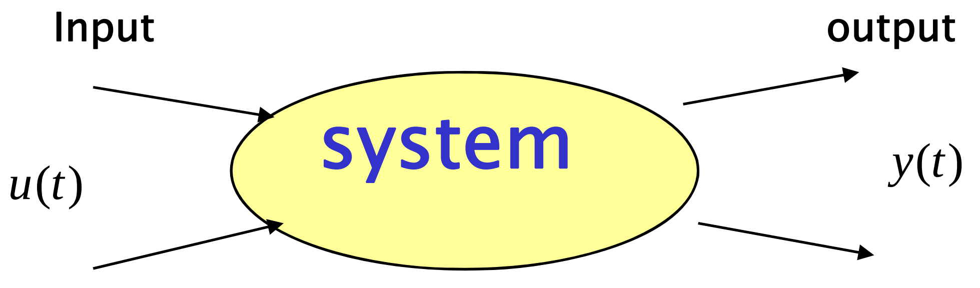


Frequency response. (Sinusoidal input response)





System, Input and Output



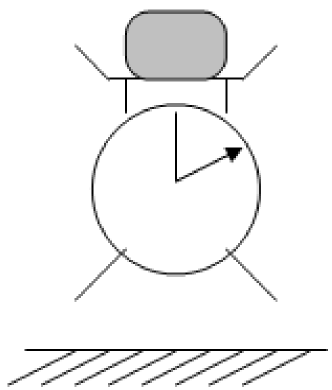
Output depends on inputs !

Depending on the system input output relationship,

- Static System vs. Dynamic system
- Linear System vs. Nonlinear System

Static System vs. Dynamic System

Ex) Balance



Static System Weight = K*x

Dynamic System Weight = K*x,
Settling time =

Interested in static weight measurement => Static System

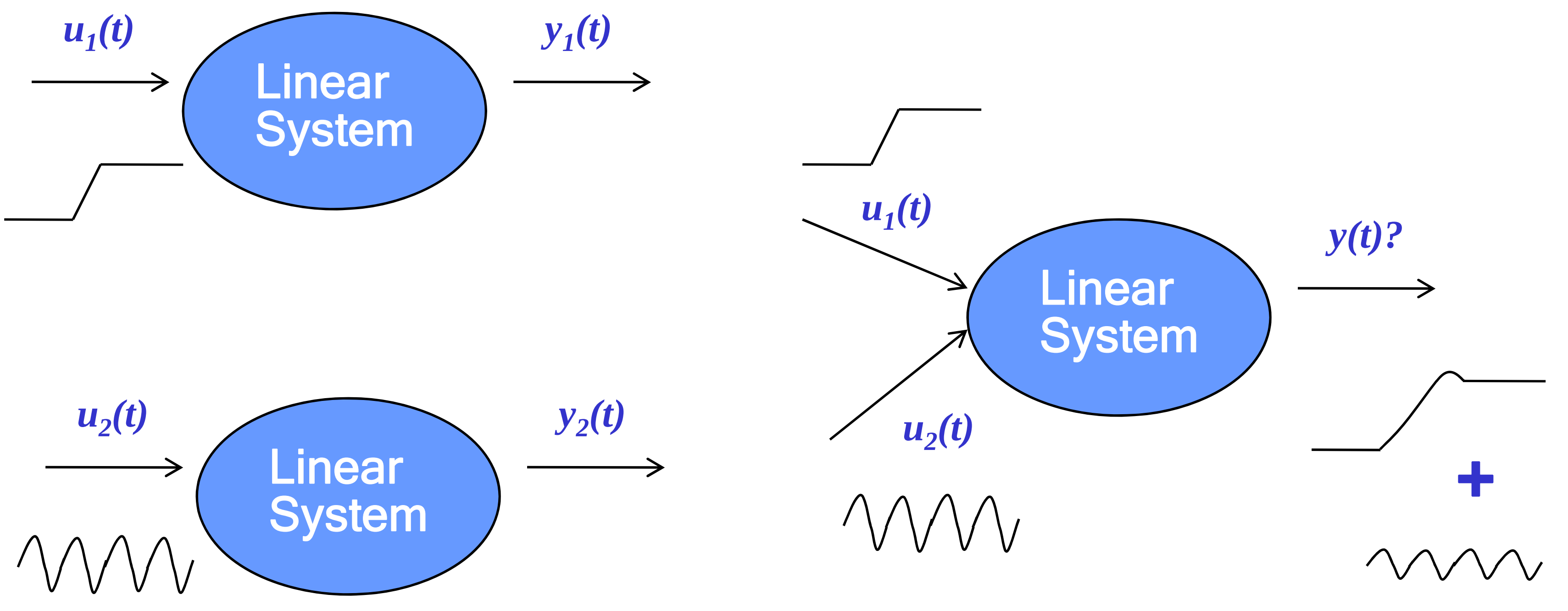
Interested in FAST weight measurement => Dynamic System

Settling time < 0.1msec

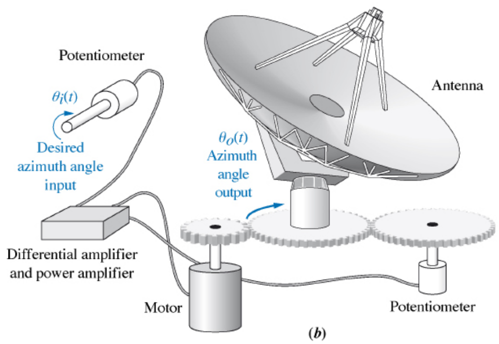
Linear vs. Nonlinear

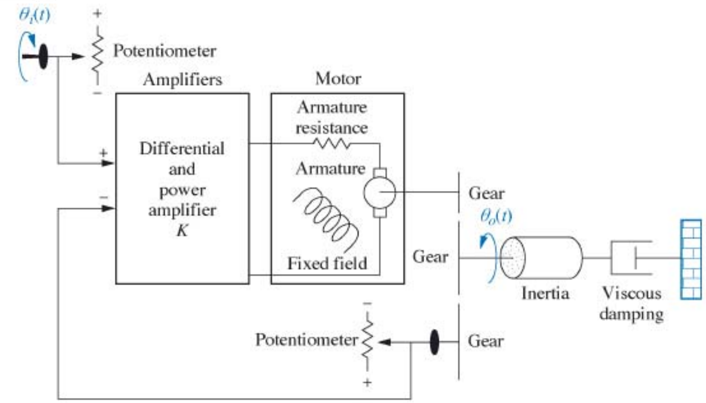
Linear systems : linear superposition principle

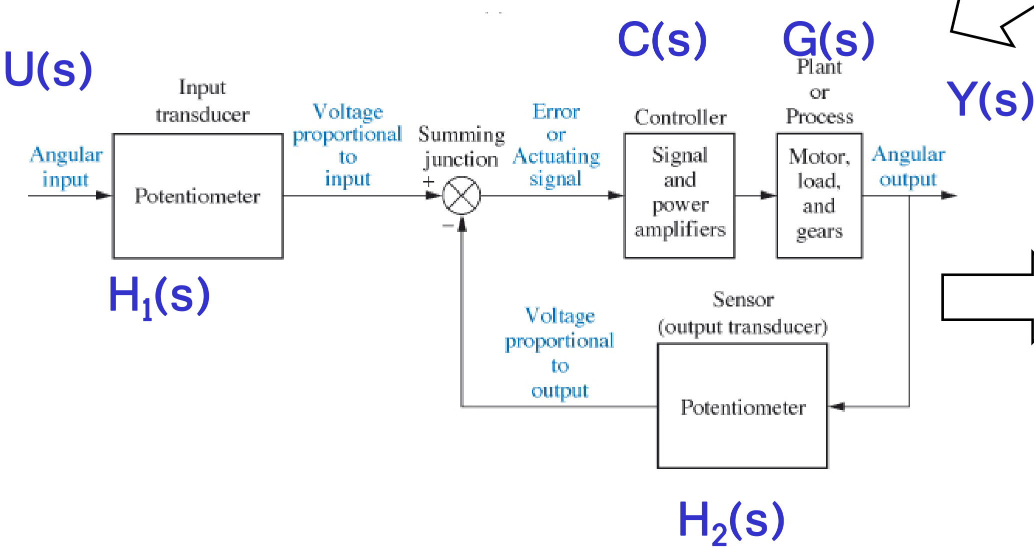
$$u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t) \longrightarrow y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$



System → Schematic → Block Diagram → Transfer Functions



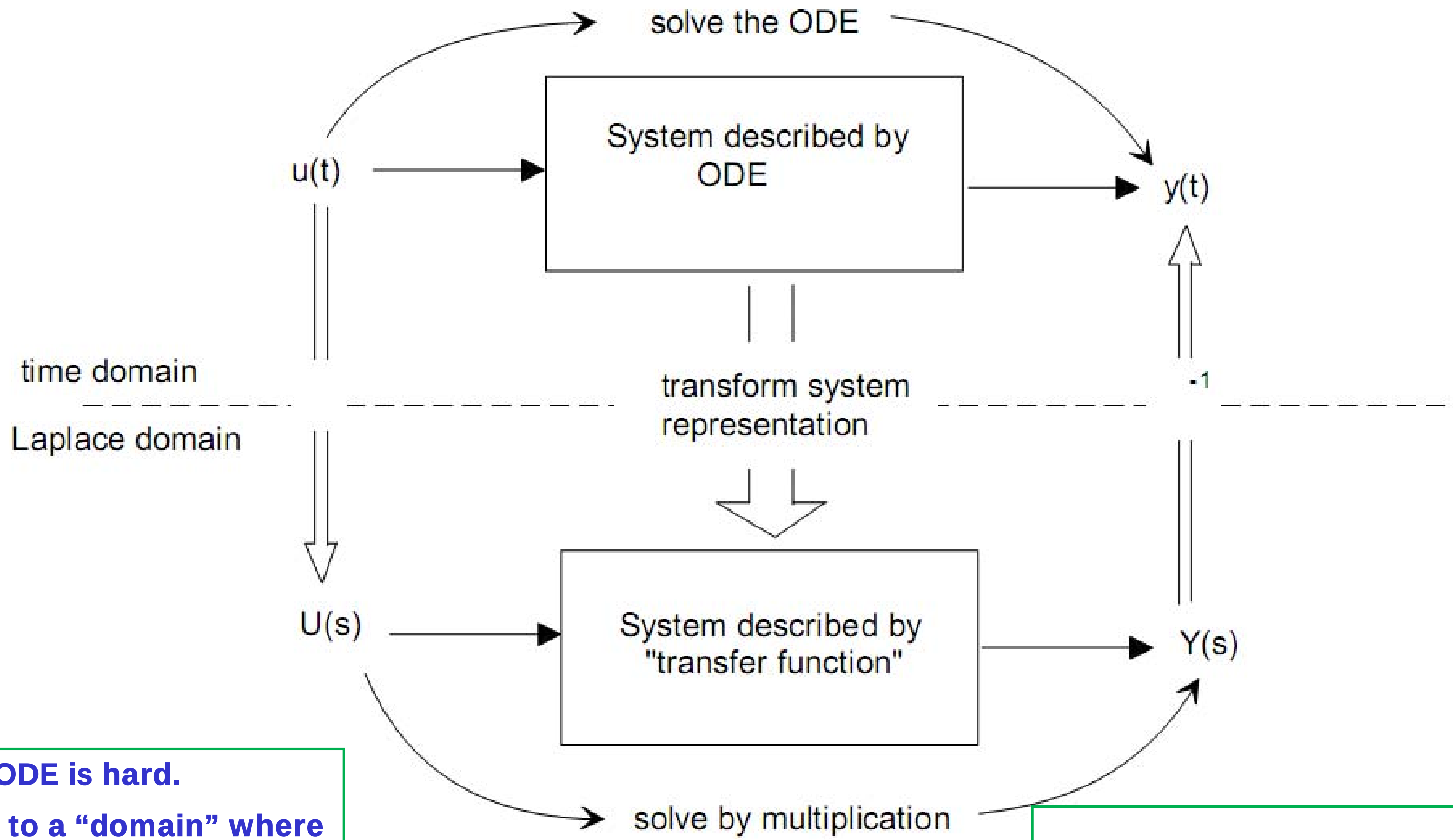




$$Y(s) = \frac{C(s)G(s)H_1(s)}{1+C(s)G(s)H_2(s)} U(s)$$

Why use Laplace Transform?

Algebraic Manipulation of ODE



Solution of ODE is hard.

Tranform in to a "domain" where it's easier to solve

Solve in the new domain

Perform "inverse" transform.

Laplace Transformation

· Definition: $\mathcal{L}\,[f(t)] = \qquad = F(s)$

$\mathcal{L}$: f(t) ⇒ F(s) , s=σ +jω (complex variable)

f(t) : a time function such that f(t)=0 for t<0

· Inverse Laplace Transformation

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

Existence of Laplace Transformation

- f(t) Laplace – transformable

  if i) f(t) piecewise-continuous

  ii) f(t) of exponential order as t approaches infinity

  – $e^{\alpha t}|f(t)|$ bounded, α exist.

  – or $e^{-\sigma t}|f(t)|$ approaches zero as t approaches infinity.

- If $\lim_{t\to\infty} e^{-\sigma t}|f(t)| = \begin{cases} 0 & for\ \sigma > \sigma_c \\ \infty & for\ \sigma < \sigma_c \end{cases}$

  the σ_c : the abscissa of convergence

Existence of Laplace Transformation

example : 1) $t, \ \sin\omega t, \ t\sin\omega t \ldots$

$$\lim_{t\to\infty} e^{-\sigma t}\left|t\sin\omega t\right| = \begin{cases} 0 & if \ \sigma > 0 \\ \infty & if \ \sigma < 0 \end{cases}$$

the abscissa of convergence $\sigma_c = 0$

2) $e^{-ct}, \ te^{-ct}, \ e^{-ct}\sin\omega t, \quad c = const.$

$$\lim_{t\to\infty} e^{-\sigma t}\left|te^{-ct}\right| = \begin{cases} 0 & if \ \sigma > -c \\ \infty & if \ \sigma < -c \end{cases}$$

the abscissa of convergence $\sigma_c = -c$

- $e^{t^2}, \quad te^{t^2}$ does not possess L. T.
- $f(t) = \begin{cases} e^{t^2} & for \quad 0 \le t \le T < \infty \\ 0 & for \quad t < 0, \ T < t \end{cases}$ L[f(t)] exists.

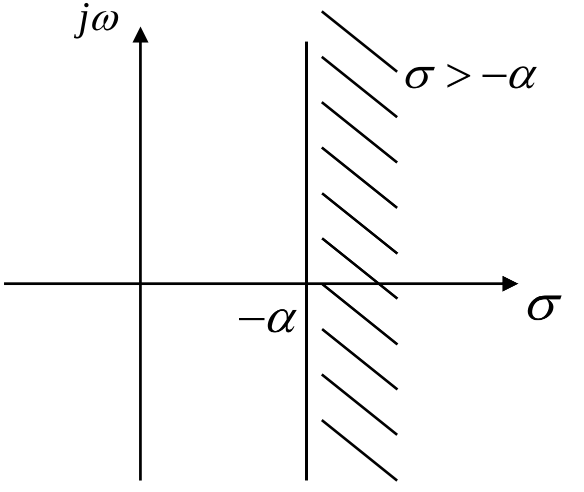
– The signals that can be physically generated always have corresponding Laplace transforms

Laplace Transformation of simple function

- Exponential function

$$f(t)=\begin{cases} 0 & for \quad t<0 \\ Ae^{-\alpha t} & for \quad t \geq 0 \end{cases}$$

A, α : constants



$$\mathcal{L}[Ae^{-\alpha t}]=\int_0^{\infty} Ae^{-\alpha t}\cdot e^{-st}dt$$

$$=\int_0^{\infty} Ae^{-(\alpha+s)t}dt$$

$$=A[-\frac{1}{s+\alpha}e^{-(s+\alpha)\cdot\infty}+\frac{1}{s+\alpha}e^{0}]$$

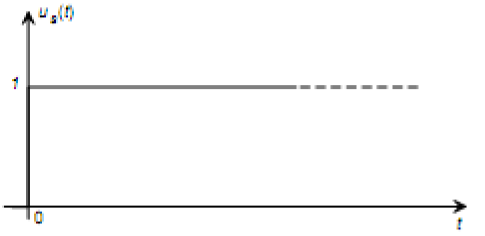
$$=A[-\frac{1}{s+\alpha}e^{-(\sigma+\alpha)\cdot\infty-j\omega\cdot\infty}+\frac{1}{s+\alpha}] \qquad (s=\sigma+j\omega)$$

$$=A[0+\frac{1}{s+\alpha}]=\boxed{}$$

Laplace Transformation of simple function

- step function

$$f(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\mathcal{L}[f(t)] = \int_0^{\infty} Ae^{-st} dt = A[-\frac{1}{s}e^{-s\cdot\infty} + \frac{1}{s}e^{-s\cdot 0}]$$

$$= A[0 + \frac{1}{s}] \qquad if \ \mathrm{Re}[s] > 0$$

- unit step input function

$$1(t - t_0) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0 \end{cases}$$

$$1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[1(t)] = \boxed{}$$

Laplace Transformation of simple function

- Sinusoidal Functions

$$f(t) = \begin{cases} A\sin\omega t & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$\cos\omega t = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin\omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

$$\mathcal{L}[A\sin\omega t] = \int_0^\infty \frac{A}{2j}(e^{j\omega t} - e^{-j\omega t})e^{-st}dt$$

$$= \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right)$$

$$= \boxed{}$$

$$\mathcal{L}[A\cos\omega t] = \boxed{}$$

- Ramp function

$$f(t) = \begin{cases} At & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$\mathcal{L}[At] = A\int_0^\infty te^{-st}dt$$

$$= A\left(t\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt \right)$$

$$= A\cdot\frac{1}{s}\int_0^\infty e^{-st}dt = \boxed{}$$

Laplace Transformation of simple function

- Pulse function

$$f(t) = \begin{cases} \dfrac{A}{t_0} & for \quad 0 \le t \le t_0 \\ 0 & for \quad t < 0,\ t_0 < t \end{cases}$$

$$f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t - t_0)$$

$$\mathcal{L}[f(t)] = \mathcal{L}\left[\frac{A}{t_0}1(t)\right] - \mathcal{L}\left[\frac{A}{t_0}1(t - t_0)\right]$$

$$= \boxed{\quad}$$

- Impulse function

$$f(t) = \begin{cases} \lim\limits_{t_0 \to \infty} \dfrac{A}{t_0} & for \quad 0 \le t < t_0 \\ 0 & for \quad t < 0, \quad t_0 < t \end{cases}$$

$$\mathcal{L}[f(t)] = \lim_{t_0 \to 0}\left[\frac{A}{t_0}\frac{1}{s}(1 - e^{-t_0 s})\right]$$

$$= \lim_{t_0 \to 0} \frac{\dfrac{d}{dt_0}\left[A(1 - e^{-t_0 s})\right]}{\dfrac{d}{dt_0}(t_0 s)} = \frac{A \cdot s}{s} = \boxed{\quad}$$

Laplace Transformation of simple function

- Unit impulse function ; impulse function of magnitude 1

$$f(t) = \begin{cases} \lim_{t_0 \to \infty} \frac{1}{t_0} & for \quad 0 \le t < t_0 \\ 0 \quad for & t < 0, \quad t_0 < t \end{cases}$$

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \ne 0 \end{cases} \qquad \mathcal{L}[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = \int_{0-}^{0+} \delta(t) dt = \boxed{}$$

The unit-impulse function occuring at $t = t_0$

$$\delta(t) = \begin{cases} \infty & t = t_0 \\ 0 & t \ne t_0 \end{cases} \quad \mathcal{L}[\delta(t - t_0)] = \int_0^\infty \delta(t - t_0) e^{-st} dt = \int_{t_0-}^{t_0+} \delta(t - t_0) e^{-st_0} dt = \boxed{}$$

$$\delta(t - t_0) = \frac{d}{dt} 1(t - t_0)$$

Useful Theorems

Theorem 1. Linearity

$$\mathcal{L}[af(t)] = aF(s)$$

Theorem 2. Superposition

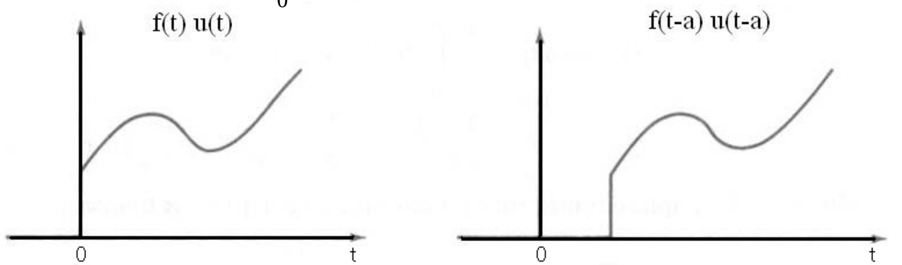
$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

Theorem 3. Translation in time.

$$\mathcal{L}[f(t-a)1(t-a)] = \int_0^\infty f(t-a)1(t-a)e^{-st}dt \qquad (a>0)$$

$$= \int_{-a}^\infty f(\tau)1(\tau)e^{-s(\tau+a)}d\tau \qquad (let.\, t-a=\tau)$$

$$= \int_0^\infty f(\tau)1(\tau)e^{-s\tau}e^{-sa}d\tau = \boxed{} \qquad (\because f(\tau)1(\tau)=0 \quad for\ \tau<0)$$



Useful Theorems

Theorem 4. Complex differentiations

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[t \cdot 1] = -\frac{d}{ds}F(s) = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \mathcal{L}[t \cdot t] = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3} \qquad \mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

$$proof) \quad let. \;\; F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$\frac{d}{ds}F(s) = \int_0^\infty \frac{\partial}{\partial s}\left[f(t)e^{-st} \right]dt = -\int_0^\infty tf(t)e^{-st}dt = -\mathcal{L}[tf(t)]$$

$$similarly, \quad \mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}F(s) \qquad \mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$$

Theorem 5. Translation in the s-domain

$$\mathcal{L}[e^{at} f(t)] = F(s-a) \qquad \mathcal{L}[e^{at}\cos\omega t] = \boxed{}$$

Theorem 6. Real Differentiation

$$Df(t) = \frac{d}{dt} f(t)$$

$$proof)\quad let.\quad \mathcal{L}[\frac{d}{dt} f(t)] = \int_0^\infty \frac{d}{dt} f(t) e^{-st} dt$$

$$= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)e^{-st} dt(-s)$$

$$= -f(0) + s \cdot F(s)$$

$$=$$

$$similarly,\quad \mathcal{L}[D^2 f(t)] = \mathcal{L}[D \cdot Df(t)] = \mathcal{L}[Df'(t)]$$

$$= s\{s \cdot F(s) - f(0)\} = s^2 \cdot F(s) - s \cdot f(0) - f'(0)$$

$$\mathcal{L}[\frac{d^n}{dt^n} f(t)] = s^n \cdot F(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0)$$

$$\ldots - f^{(n-1)}(0)$$

Useful Theorems

Theorem 7. Real Integration

$$\int_0^t f(t)dt = D^{-1}f(t) - D^{-1}f(0)$$

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \int_0^\infty \int_0^t f(\tau)d\tau \cdot e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\int_0^t f(\tau)d\tau\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}f(t)dt$$

$$= \frac{1}{s}\int_0^t f(\tau)d\tau\Big|_{t=0} + \frac{1}{s}\int_0^\infty e^{-st}f(t)dt = \boxed{}$$

Theorem 8. Complex Integration

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$$

Useful Theorems

Theorem 9. Final value Theorem

$$\lim_{t\to\infty} f(t) = \boxed{}$$

Theorem 10. Initial value Theorem

$$\lim_{t\to 0} f(t) = \boxed{}$$

Inverse Laplace Transformation

$$\mathcal{L} : f(t) \to F(s)$$
$$\mathcal{L}^{-1} : F(s) \to f(t)$$

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi j}\int_{c-j\omega}^{c+j\omega} F(s)e^{st}ds$$ (c : real constant)

* Inverse Laplace Transformation by Partial Fraction Method

$$F(s) = \frac{P(s)}{Q(s)} = \frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_0}{s^n + b_{n-1}s^{n-1} + \cdots b_1 s + b_0} \qquad (n \geq m)$$

$s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0$ $\Rightarrow$ real complex conjugate, a , b : real num.

$$= (s - c_1)(s - c_2)\cdots(s^2 + d_1 s + d_2)$$

$$\Rightarrow F(s) = \frac{\alpha_1}{s - c_1} + \frac{\alpha_2}{s - c_2} + \cdots + \frac{\beta_1 s + \beta_2}{s^2 + d_1 s + d_2} + \cdots$$

Examples of Inverse Laplace Transformation

ex) $$F(s)=\frac{1}{(s+2)^2(s+3)}=\frac{a}{(s+2)}+\frac{b}{(s+2)^2}+\frac{c}{(s+3)}$$

$$a(s+2)(s+3)+b(s+3)+c(s+2)^2=1$$

$$let\ s=-2,\ than\ b=1,\ \ let\ s=-3,\ than\ c=1$$

$$a(s+2)+b+c\cdot\frac{s+2}{s+3}=\frac{1}{s+3}\ \xrightarrow{\frac{d}{ds}}\ a+c\cdot\frac{(s+3)-(s+2)}{s+3}=\frac{-1}{(s+3)^2}$$

$$\therefore\ a=-1,\ b=1,\ c=1,\qquad F(s)=\frac{-1}{(s+2)}+\frac{1}{(s+2)^2}+\frac{1}{(s+3)}$$

=> Inverse Laplace Transformation

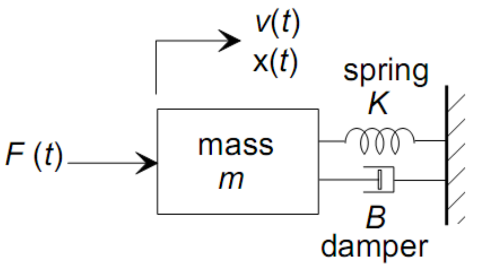
$$f(t)=-e^{-2t}+te^{-2t}+e^{-3t}$$

(partial fraction method)

ex) $$F(s)=\frac{10}{s^2+6s+25}=\frac{10\times\frac{1}{4}\times 4}{(s+3)^2+4^2}=\frac{10}{4}\frac{4}{(s+3)^2+4^2}$$

$$\therefore\ f(t)=\frac{10}{4}\sin 4te^{-3t}$$

Solution of Differential Equation by Laplace Transformation



$$m\ddot{x} + B\dot{x} + Kx = F$$

$$y'' + 2y' + 4y = 1 \quad y(0) = 0\ ,\ y'(0) = 2$$

L.T:

$$Y(s) = \frac{2s+1}{s(s^2+2s+4)} = \frac{1}{4s} - \frac{1}{4}\frac{s+1-1}{(s+1)^2 + \left(\sqrt{3}\right)^2}$$

$\therefore y(t) =$

Laplace Transform Table

Item no.	$f(t)$	$F(s)$
1.	$\delta(l)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$l^n u(t)$	$\frac{n1}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Nise Ch.2

Laplace Transform Theorems

Nise Ch.2

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{1} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]