Lecture 5
Transfer Function and Block Diagram Approach to Modeling Dynamic Systems

Review of the last lecture

Item no.	$f(t)$	$F(s)$
1.	$\delta(l)$	
2.	$u(t)$	
3.	$tu(t)$	
4.	$l^n u(t)$	
5.	$e^{-at}u(t)$	
6.	$\sin \omega t u(t)$	
7.	$\cos \omega t u(t)$	

Nise Ch.2

Laplace Transform Theorems

Nise Ch.2

Item no.	Theorem		Name
1.	$\mathscr{L}[f(t)] = F(s)$	$= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)]$	$= kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)]$	$= F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)]$	$=$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)]$	$=$	Time shift theorem
6.	$\mathscr{L}[f(at)]$	$=$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right]$	$=$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right]$	$=$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right]$	$=$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{1} f(\tau)d\tau\right]$	$=$	Integration theorem
11.	$f(\infty)$	$=$	Final value theorem[1]
12.	$f(0+)$	$=$	Initial value theorem[2]

The Concept of Transfer Function

Consider the linear time-invariant system defined by the following differential

equation :

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y$$
$$= b_0 u^{(m)} + b_1 u^{(m-1)} + \cdots + b_{m-1}\dot{u} + b_m u \quad (n \geq m)$$

Where y is the output of the system, and x is the input. And the Laplace transform of

the equation is,

$$(a_0 S^n + a_1 S^{n-1} + \cdots + a_{n-1} S + a_n) Y(s)$$
$$= (b_0 S^m + b_1 S^{m-1} + \cdots + b_{m-1} S + b_m) U(s)$$

The Concept of Transfer Function

The ratio of the Laplace transform of the output (response function) to the Laplace

Transform of the input (driving function) under the assumption that all initial

conditions are Zero.

$$Transfer\ Function = \frac{Y(s)}{U(s)} = G(s) = \frac{b_0 S^m + b_1 S^{m-1} + \cdots + b_{m-1} S + b_m}{a_0 S^n + a_1 S^{n-1} + \cdots a_{n-1} S + a_n}$$

$$= \frac{P(s)}{Q(s)} \qquad (n \geq m)$$

Comments on Transfer Function

1. A mathematical model.

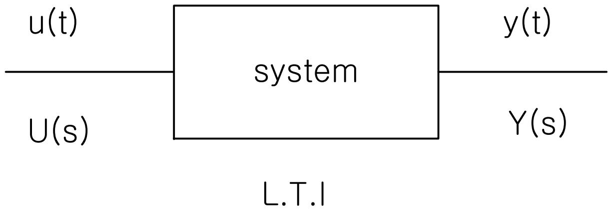
2. Property of system itself.
   - Independent of the input function and initial condition
   - Denominator of the transfer function is the characteristic polynomial,
   - TF tells us something about the intrinsic behavior of the model.

3. ODE equivalence
   - TF is equivalent to the ODE. We can reconstruct ODE from TF.

4. One TF for one input-output pair. : Single Input Single Output system.
   - If multiple inputs affect

Comments on Transfer Function

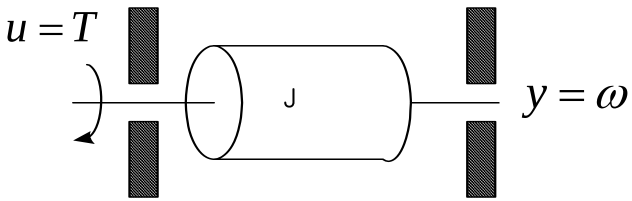
5. Analytic method and Experimental method



$$G(s)=\frac{Y(s)}{U(s)}$$

6. Different systems may have identical T.F

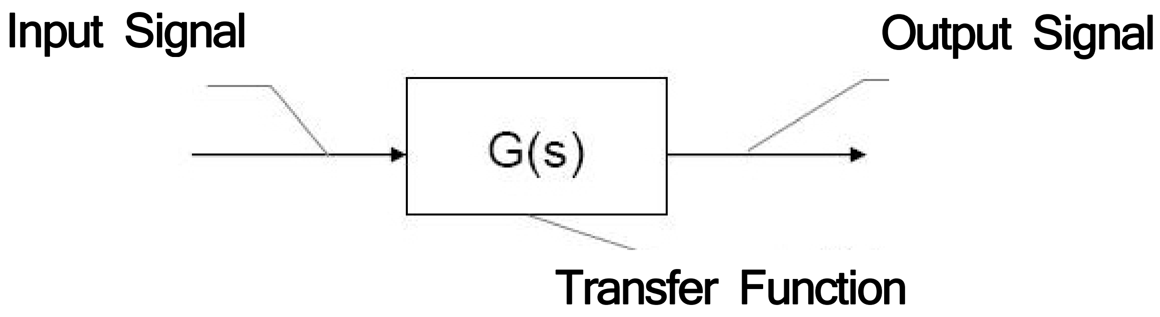
ex)

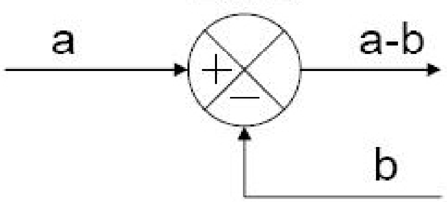


$$J\dot{\omega}+b\omega=T \qquad \frac{Y}{U}=\frac{\omega}{T}=\frac{1}{Js+b}$$

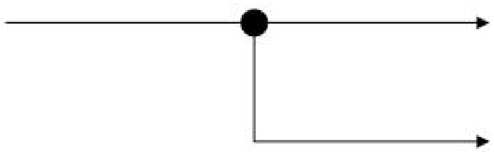
Electrical System $\frac{Y}{U}=\frac{R}{Ls+R}=\frac{1}{\frac{L}{R}s+1}$ (if b=1, J=L/R)

Block Diagram



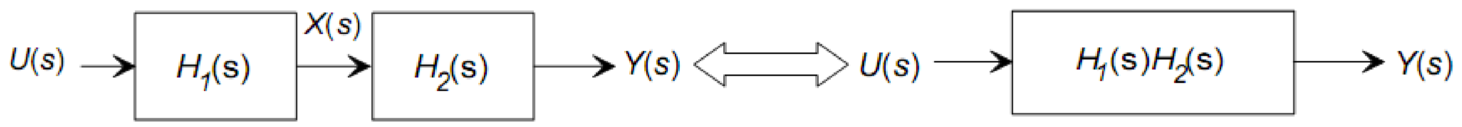


Summing Point

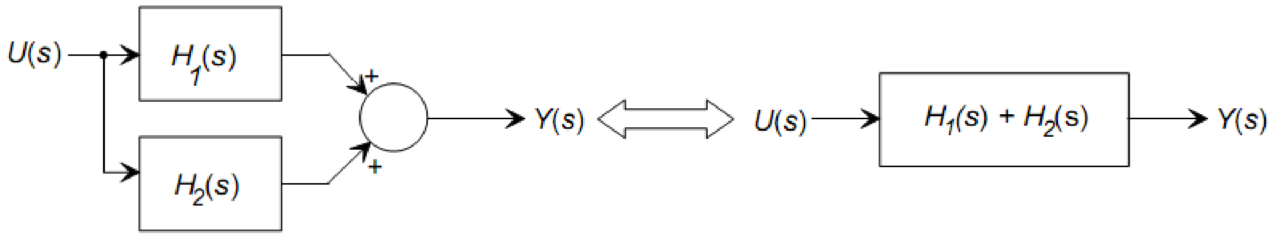


Pick Off Point

Series(Cascade) Connection



Parallel Connection



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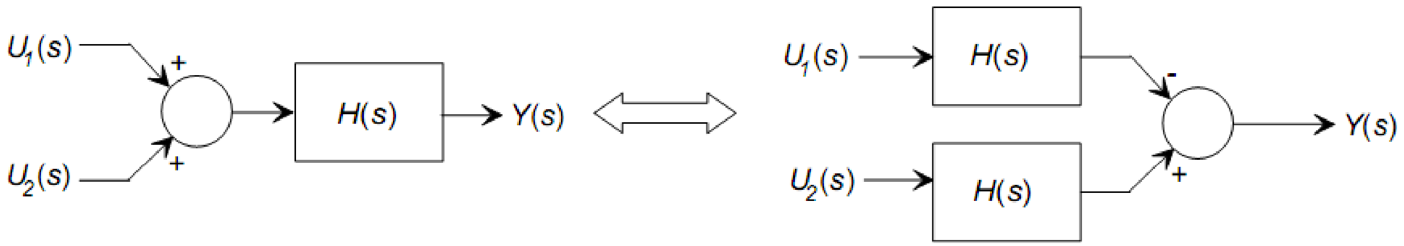
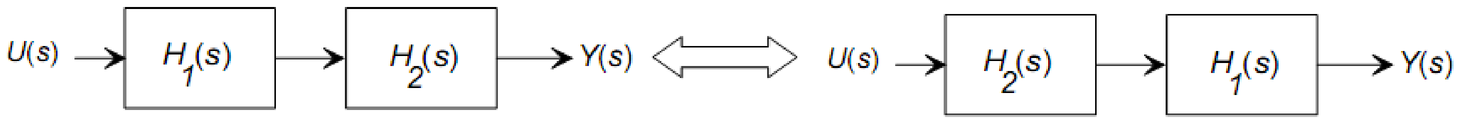
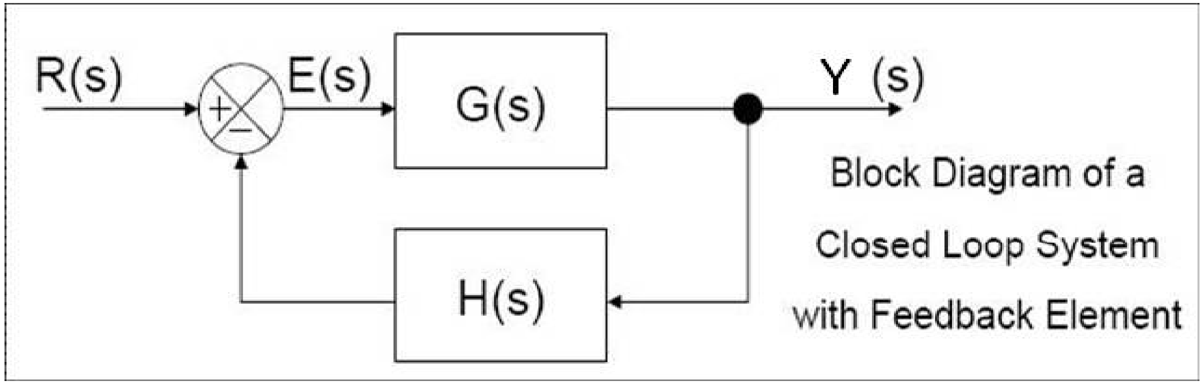
Associative Rule



Commutative Rule



Closed Loop Transfer Function



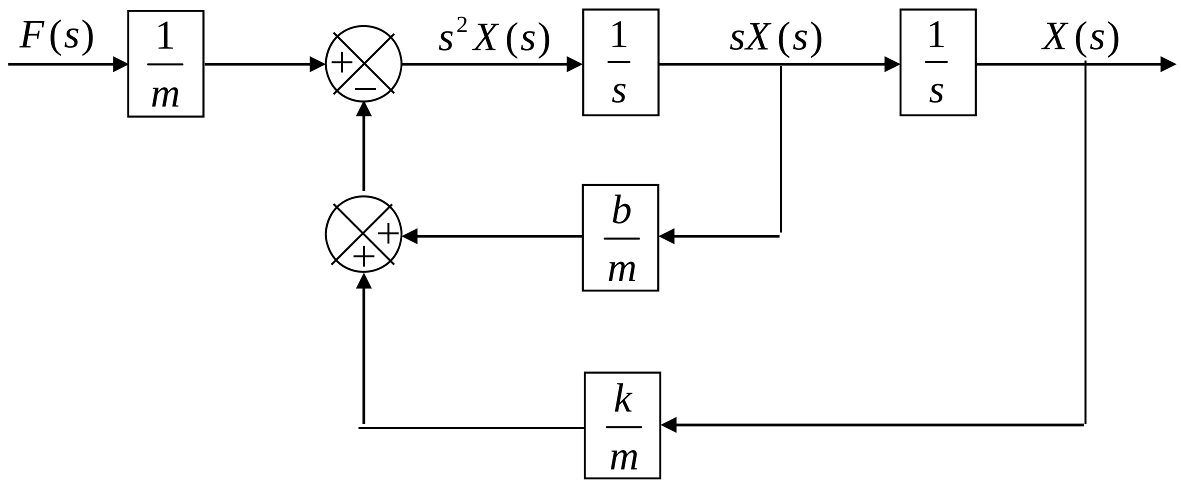
$$Y(s) = G(s)E(s) = G(s)[R(s) - H(s)Y(s)]$$

$$[1 + G(s)H(s)]Y(s) = G(s)R(s)$$

$$\therefore Transfer\ Function = \frac{Y(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)H(s)}$$

Closed Loop Transfer Function

ex)



$$\frac{1}{m}F(s)-\frac{k}{m}X(s)-\frac{b}{m}sX(s)=s^2X(s)$$

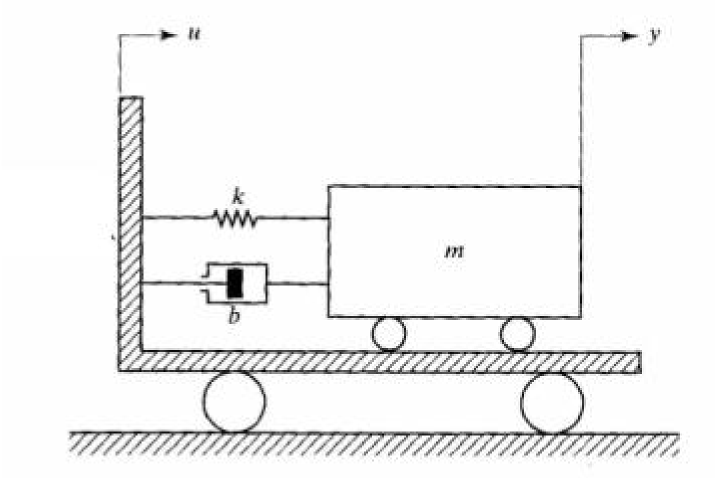
$$F(s)-\left[kX(s)+bsX(s)\right]=ms^2X(s)$$

$$(ms^2+bs)X(s)=F(s)-kX(s),\qquad (ms^2+bs+k)X(s)=F(s)$$

$$\therefore \frac{X(s)}{F(s)}=\frac{1}{ms^2+bs+k}\quad \textit{Transfer Function}$$

Example

ex)



$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

$$\text{or} \quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

Assume the initial condition is 0,

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

$$Transfer\ Function = \frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$$

If, m=10kg, b=20N-s/m, k=100N/m

$$\frac{Y(s)}{U(s)} = \frac{20s+100}{10s^2+20s+100} = \frac{2s+10}{s^2+2s+10}, \qquad U(s) = \frac{1}{s}$$

$$\therefore Y(s) = \frac{2s+10}{s^2+2s+10}\frac{1}{s} = \frac{2s+10}{s^3+2s^2+10s}$$

The law of conservation of momentum

By Newton's 2nd law,

$$F = ma = m\frac{dv}{dt} = \frac{d}{dt}(mv) \quad \rightarrow \quad F \cdot dt = d(mv)$$

$$\int_{t_1}^{t_2} F dt = \int_{v_1}^{v_2} d(mv) = mv_2 - mv_1$$

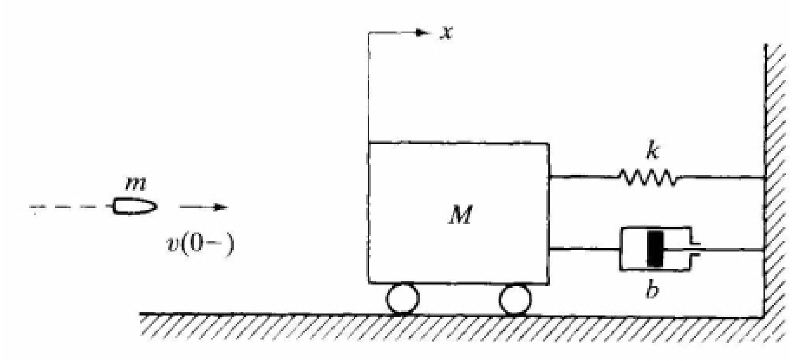
And if There is no input force, then we get

$$d(mv) = 0, \quad mv = const.$$ The law of conservation of momentum

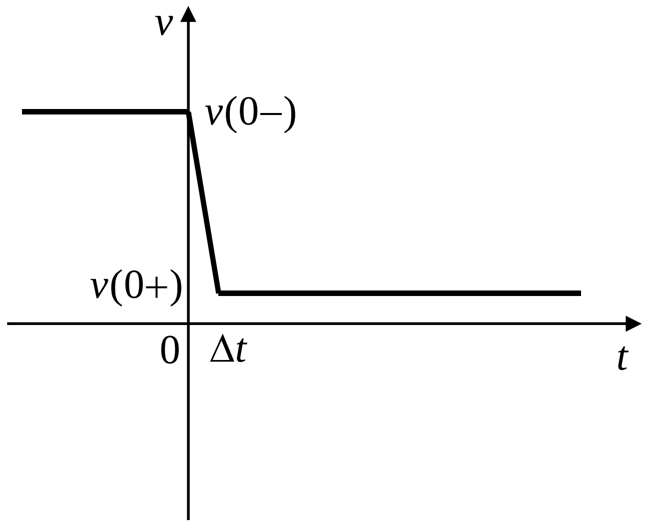
And also,

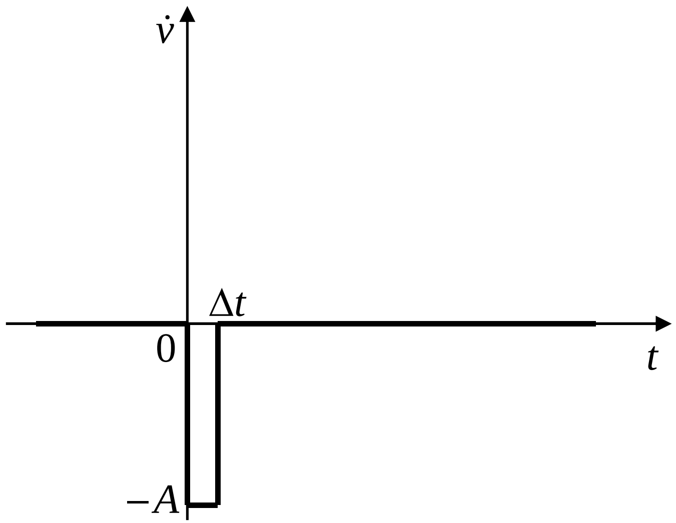
$$j\omega = const.$$ The law of conservation of angular momentum

Impulse Input



A bullet stuck in mass M





Sudden change in the velocity and acceleration of bullet

Impulse Input

System equation : $(M+m)\ddot{x}+b\dot{x}+kx=F(t)$

Impulse force : $F(t)=-m\dot{v}=A\Delta t\ \delta(t),\quad A\Delta t$: Magnitude of impulse force

$$\int_{0-}^{0+} A\Delta t\ \delta(t)\ dt=-m\int_{0-}^{0+}\dot{v}\ dt,\quad A\Delta t=mv(0-)-mv(0+)$$

Noting that, Magnitude of impulse force is equal to change of momentum!!

$v(0+)=\dot{x}(0+)=$ Initial velocity of M+m

Thus the system equation becomes

$$(M+m)\ddot{x}+b\dot{x}+kx=F(t)=\left[mv(0-)-m\dot{x}(0+)\right]\delta(t)$$

Impulse Input

By taking the Laplace Transform,

$$(M+m)\left[s^2X(s)-sx(0-)-\dot{x}(0-)\right]+b\left[sX(s)-x(0-)\right]+kX(s)=mv(0-)-m\dot{x}(0+)$$

Noting that $\dot{x}(0-)=x(0-)=0$ we obtain $X(s)=\dfrac{mv(0-)-m\dot{x}(0+)}{(M+m)s^2+bs+k}$

$$\dot{x}(0+)=\lim_{t\to 0+}\dot{x}(t)=\lim_{s\to\infty}s\left[sX(s)\right]=\lim_{s\to\infty}\frac{s^2\left[mv(0-)-m\dot{x}(0+)\right]}{(M+m)s^2+bs+k}=\frac{mv(0-)-m\dot{x}(0+)}{M+m}$$

$$\rightarrow \quad \dot{x}(0+)=\frac{m}{M+2m}v(0-)$$

$$\therefore X(s)=\frac{(M+m)\dot{x}(0+)}{(M+m)s^2+bs+k}=\frac{1}{(M+m)s^2+bs+k}\frac{(M+m)mv(0-)}{M+2m}$$

Partial Fraction Expansion with MATLAB

$$\frac{B(s)}{A(s)} = \frac{num}{den} = \frac{b(1)S^{h} + b(2)S^{h-1} + \cdots + b(h+1)}{a(1)S^{n} + a(2)S^{n-1} + \cdots + a(n+1)}$$

$$num = [b(1) \quad b(2) \quad \cdots \quad b(h)], \quad den = [a(1) \quad a(2) \quad \cdots \quad a(h)]$$

[r, p, k] = residue (num, den)

$$\frac{B(s)}{A(s)} = k(s) + \frac{r(1)}{s-p(1)} + \frac{r(2)}{s-p(2)} + \cdots + \frac{r(n)}{s-p(n)}$$

ex) $$\frac{B(s)}{A(s)} = \frac{s^4 + 8s^3 + 16s^2 + 9s + 6}{s^3 + 6s^2 + 11s + 6}$$

```
>>num=[1   8   16   9   6];
>>den=[1   6   11   6];
>>[r,p,k]=residue(num,den)
r=
        -6.0000
        -4.0000
         3.0000
p=
        -3.0000
        -2.0000
        -1.0000
k=
        1         2
```

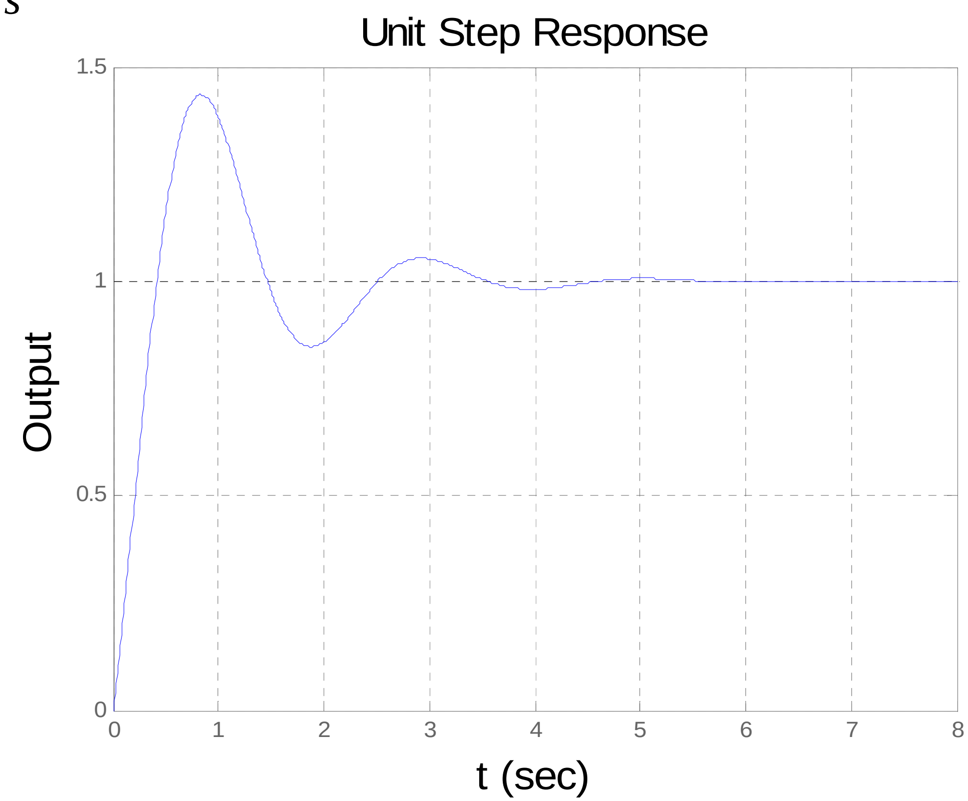
$$= s + 2 + \frac{-6}{s+3} + \frac{-4}{s+2} + \frac{3}{s+1}$$

Transient Response Analysis with MATLAB

$$\frac{Y(s)}{U(s)} = \frac{20s+100}{10s^2+20s+100} = \frac{2s+10}{s^2+2s+10}, \qquad U(s) = \frac{1}{s}$$

$$\therefore Y(s) = \frac{2s+10}{s^2+2s+10}\frac{1}{s} = \frac{2s+10}{s^3+2s^2+10s}$$

```
t=0:0.01:8;
num=[2 10];
den=[1 2 10];
sys=tf(num,den);
step(sys,t)
grid
title('Unit Step Response','Fontsize',15)
xlabel('t(sec)','Fontsize',15)
ylabel('Output','Fontsize',15)
```



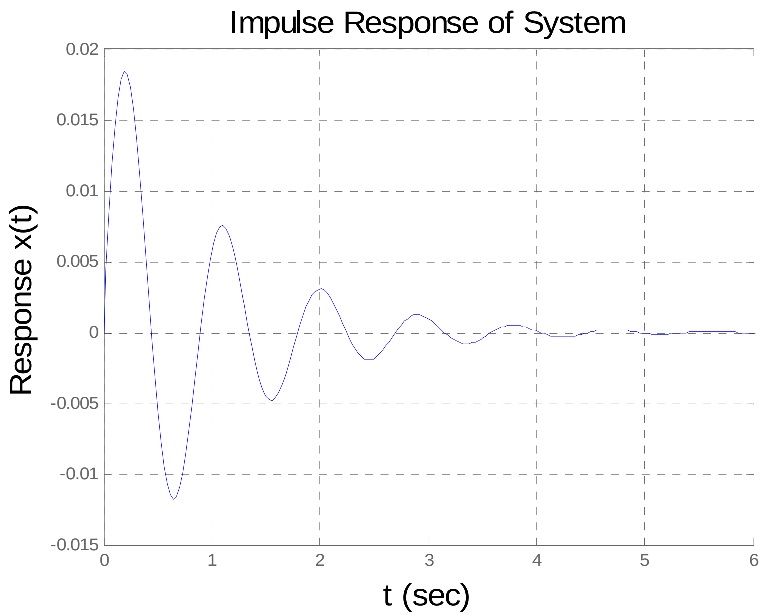
Impulse Input

ex)

$$M = 50kg, \quad m = 0.01kg, \quad b = 100Ns/m, \quad k = 2500N/m, \quad v(0-) = 800m/s$$

$$X(s) = \frac{1}{50.01s^2 + 100s + 2500} \frac{50.01 \times 0.01 \times 800}{50.02} = \frac{7.9984}{50.01s^2 + 100s + 2500}$$

```
num=[7.9984];
den=[50.01 100 2500];
sys=tf(num,den);
impulse(sys)
grid
title('Impulse Response of System','Fontsize',15')
xlabel('t(sec)','Fontsize',15')
ylabel('Response x(t)','Fontsize',15')
```



Ramp Response

$$\frac{Y(s)}{U(s)} = \frac{2s+10}{s^2+2s+10}$$

M=10kg, b=20Ns/m, k=100N/m

$$u(t): Unit\ ramp\ input, \quad u = \alpha t, \quad \alpha = 1$$

```
num=[2 10];
den=[1 2 10];
sys=tf(num,den);
t=0:0.01:4;
u=t;
lsim(sys,u,t)
grid
title('Unit-Ramp Response','Fontsize',15)
xlabel('t')
ylabel('Output y(t) and Input u(t)=t','Fontsize',15)
text(0.8,0.4,'y','Fontsize',12)
text(0.4,0.8,'u','Fontsize',12)
legend('y')
```