Mechanical Systems I

Newton's Laws

1)First law : conservation of momentum

no external force

➔ no momentum change

linear momentum : mv

angular momentum : $J\omega$

2) Second law : $\sum F = ma = m\dfrac{dv}{dt}$

$$\sum T = J\alpha = J\frac{d\omega}{dt}$$

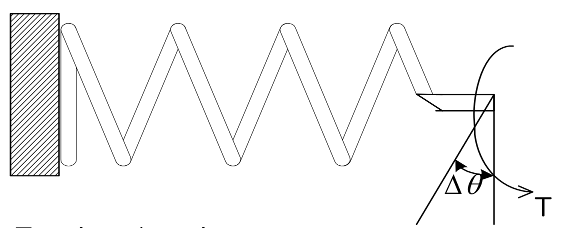
Three Basic Elements in Modeling Mechanical Systems

i) Inertial elements (kinetic energy):

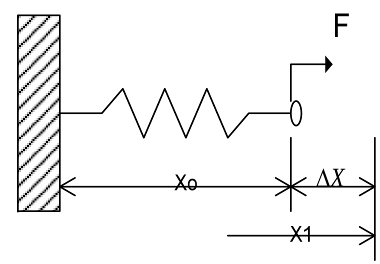
masses: M moments of inertial : J

ii) Spring elements (Potential energy)

$$T = k_\theta \cdot \Delta\theta = k(\theta_1 - \theta_0)$$



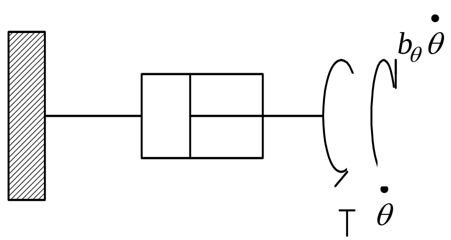
Torsional spring



F=k ΔX
=k(X1-X2)

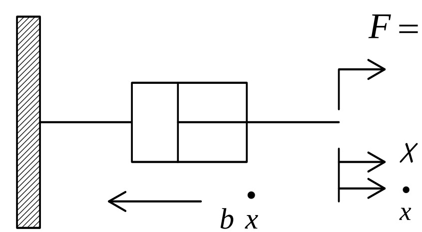
k : stiffness

iii) Damper elements (Energy dissipation)

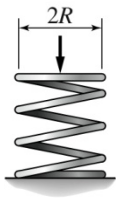


$$T = b_\theta \dot{\theta}$$

$$F = b\dot{x}$$



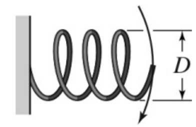
Spring Elements



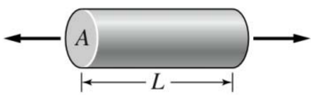
$$k = \frac{Gd^4}{64nR^3}$$

$d = wire\ diameter$

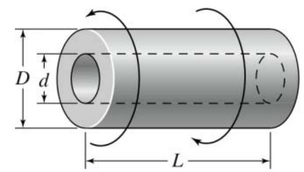
$n = number\ of\ coils$



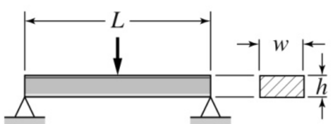
$$k_T = \frac{Ed^4}{64nD}$$



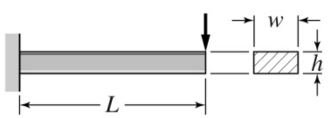
$$k = \frac{EA}{L}$$



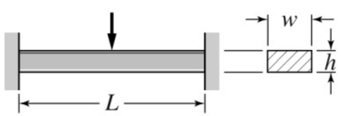
$$k_T = \frac{\pi G(D^4 - d^4)}{32L}$$



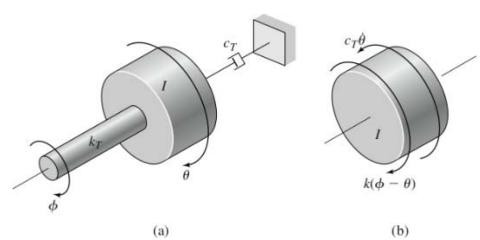
$$k = \frac{4Ewh^3}{L^3}$$



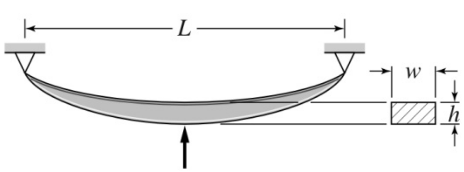
$$k = \frac{Ewh^3}{4L^3}$$



$$k = \frac{16Ewh^3}{L^3}$$

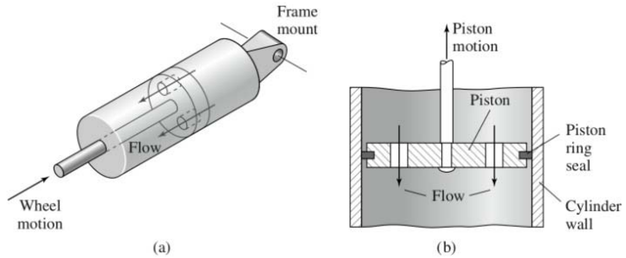


$$k_T = \frac{\pi GD^4}{32L}$$

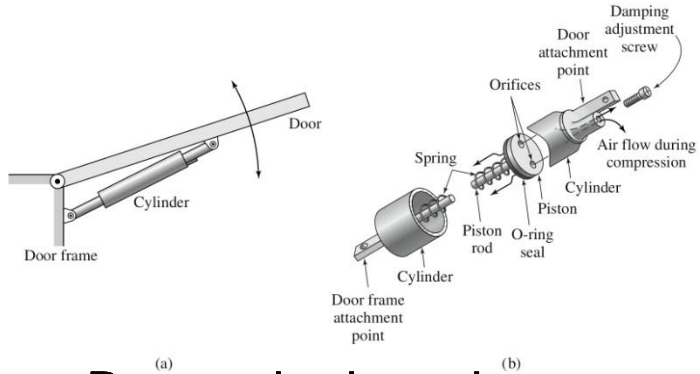


$$k = \frac{2Ewh^3}{L^3}$$

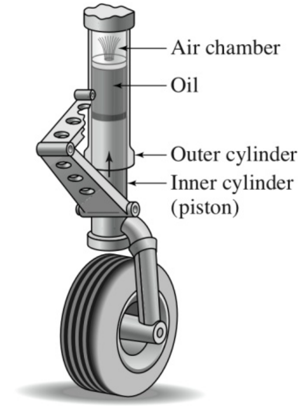
Damping Elements



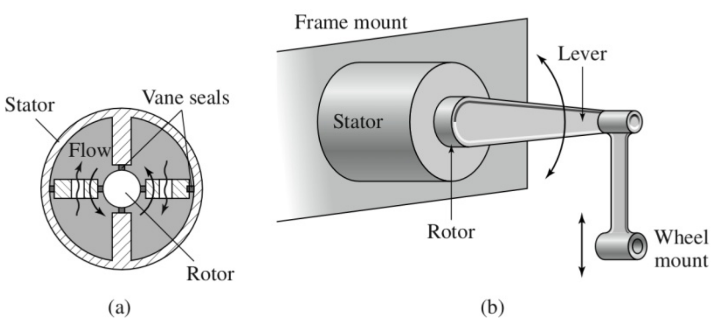
Piston damper



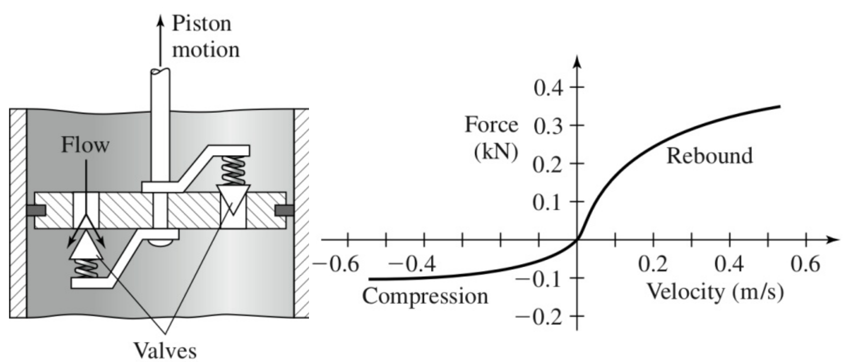
Pneumatic door closer



Oleo strut



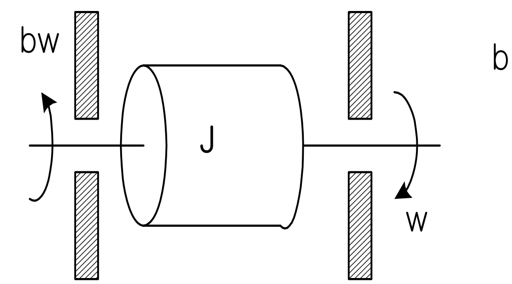
Rotary damper



Damper with spring loaded valves

Examples of Modeling Mechanical Systems

ex1) $t = 0, \quad \omega(0) = \omega_0$



equation of motion :

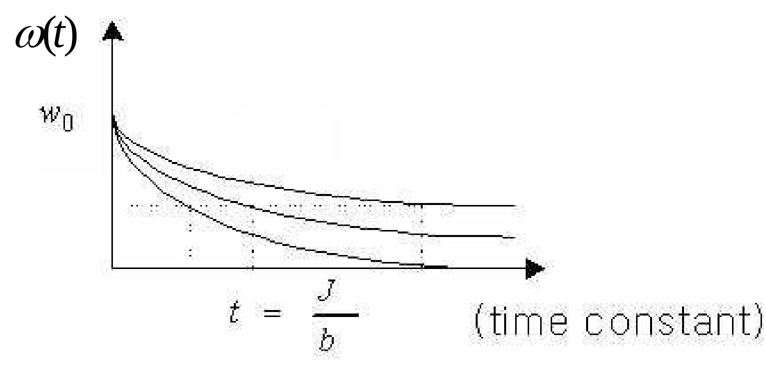
$$J\frac{d\omega}{dt} = -b\omega$$

$$J\frac{d\omega}{dt} + b\omega = 0 \rightarrow \frac{d\omega}{dt} + \frac{b}{J}\omega = 0$$

$$let,\ \omega(t) = ce^{\lambda t} \rightarrow \lambda e^{\lambda t} + \frac{b}{J}e^{\lambda t} = 0$$

$$\lambda = -\frac{b}{J}$$

$\omega_0 e^{-1} = 0.368\omega_0$



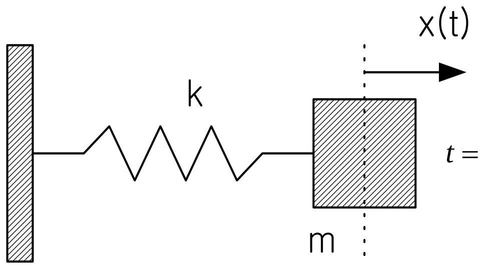
$$\omega(t) = ce^{-\frac{b}{J}t},\ t = 0,\ \omega(0) = \omega_0 = C$$

$$\therefore \omega(t) = \omega_o e^{-\frac{b}{J}t}$$

Examples of Modeling Mechanical Systems

Spring-Mass

ex2)



$t=0, x(0)=x_{0,}\ \dot{x}(0)=0$

$$m\frac{d^2x}{dt^2}=\sum F=-kx$$

$$m\ddot{x}+kx=0 \quad \rightarrow \quad \ddot{x}+\frac{k}{m}x=0$$

$$x(t)=A\cos\sqrt{\frac{k}{m}}t+B\sin\sqrt{\frac{k}{m}}t$$

$$x(0)=A=x_o$$

$$\dot{x}(0)=-A\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t+B\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t \quad =B\sqrt{\frac{k}{m}}=0$$

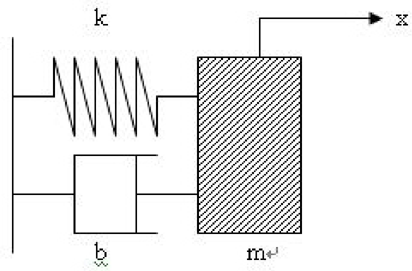
$$x(t)=x_0\cos\sqrt{\frac{k}{m}}t=x_0\cos\omega_n t$$

- **Period** $T=\dfrac{2\pi}{\sqrt{k/m}}[\sec]$
- **Frequency** $f=\dfrac{1}{T}=\dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}} \quad [Hz]$
- **Natural frequency**

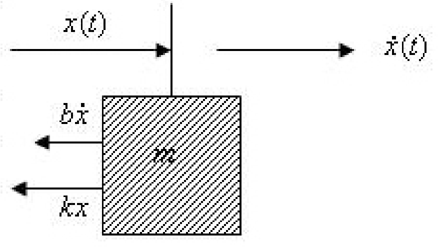
$$\omega_n=2\pi f=\sqrt{\frac{k}{m}} \quad [rad/\sec]$$

Examples of Modeling Mechanical Systems

Spring-mass-damper



$$x(0) = x_0$$
$$\dot{x}(0) = \dot{x}_0$$



$$m\frac{d^2x}{dt^2} = -kx - b\dot{x}$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

$$let, \quad \frac{k}{m} = \omega_n^2, \qquad \frac{b}{m} = 2\zeta\omega_n, \qquad \frac{b}{2\sqrt{mk}} = \zeta$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \qquad x(t) = c \cdot e^{\lambda t} \qquad \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 x = 0$$

$$\therefore \lambda = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

Examples of Modeling Mechanical Systems

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \qquad \begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= \dot{x}_0 = 0 \end{aligned}$$

$$let, \quad \frac{k}{m} = \omega_n^2, \qquad \frac{b}{m} = 2\zeta\omega_n, \qquad \frac{b}{2\sqrt{mk}} = \zeta$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Laplace Transform

$$(s^2 X(s) - s \cdot x_0 - \dot{x}_0) + 2\zeta\omega_n(sX(s) - x_0) + \omega_n^2 X(s) = 0$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)X(s) - s \cdot x_0 - 2\zeta\omega_n x_0 = 0$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)X(s) = s \cdot x_0 + 2\zeta\omega_n x_0$$

$$X(s) = \frac{s \cdot x_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Examples of Modeling Mechanical Systems

$$X(s)=\frac{s\cdot x_0+2\zeta\omega_n x_0}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{s\cdot x_0}{s^2+2\zeta\omega_n s+\omega_n^2}+\frac{2\zeta\omega_n x_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$s^2+2\zeta\omega_n s+\omega_n^2$: charateristic polynomial

$s^2+2\zeta\omega_n s+\omega_n^2=0$: characteistic equation

$s=-\zeta\omega_n\pm\sqrt{(\zeta\omega_n)^2-\omega_n^2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$: charateristic roots

1) $\zeta<1$ underdamped

$$s=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}$$

$$X(s)=\frac{(s+\zeta\omega_n)\cdot x_0}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}+\frac{\zeta\omega_n x_0}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

$$x(t)=e^{-\zeta\omega_n t}\left(x_0\cos\sqrt{1-\zeta^2}\,\omega_n t+\frac{\zeta}{\sqrt{1-\zeta^2}}x_0\sin\sqrt{-\zeta^2}\,\omega_n t\right)$$

Examples of Modeling Mechanical Systems

2) $\zeta > 1$ overdamped

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$X(s) = \frac{(s + 2\zeta\omega_n)\cdot x_0}{\left(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1}\right)}$$

$$x(t) = k_1 e^{\left(-\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n\right)t} + k_2 e^{\left(-\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n\right)t}$$

3) $\zeta = 1$ critically damped

$$s = -\zeta\omega_n$$

$$X(s) = \frac{(s + 2\zeta\omega_n)\cdot x_0}{\left(s + \zeta\omega_n\right)^2}$$

$$x(t) = k_1 e^{(-\zeta\omega_n)t} + k_2 t e^{(-\zeta\omega_n)t}$$

Examples of Modeling Mechanical Systems

1) $\zeta < 1$ Underdamped

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta}{\sqrt{1-\zeta^2}} x_0 \sin\sqrt{-\zeta^2}\,\omega_n t \right)$$

$$\begin{cases} \omega_n : \text{natural frequency} \\ \zeta : \text{damping ratio} \end{cases} \qquad \omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

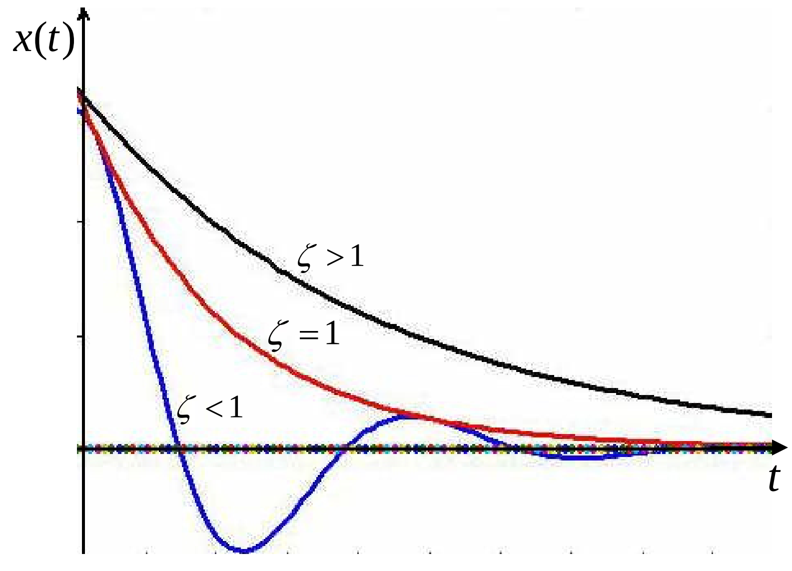
2) $\zeta > 1$ Overdamped

$$x(t) = k_1 e^{\left(-\zeta\omega_n + \sqrt{\zeta^2-1}\,\omega_n\right)t} + k_2 e^{\left(-\zeta\omega_n - \sqrt{\zeta^2-1}\,\omega_n\right)t}$$

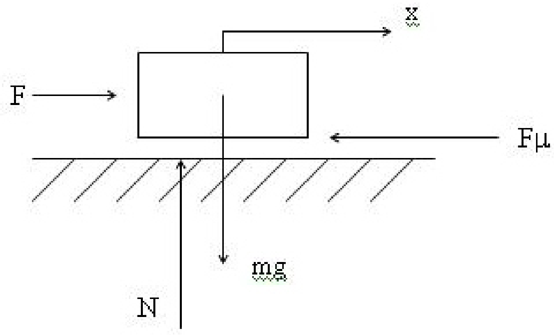
$$\lambda = -\zeta\omega_n \pm \sqrt{\zeta^2-1}\,\omega_n$$

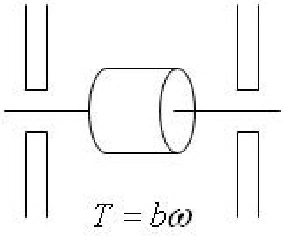
3) $\zeta = 1$ $\lambda = -\zeta\omega_n$

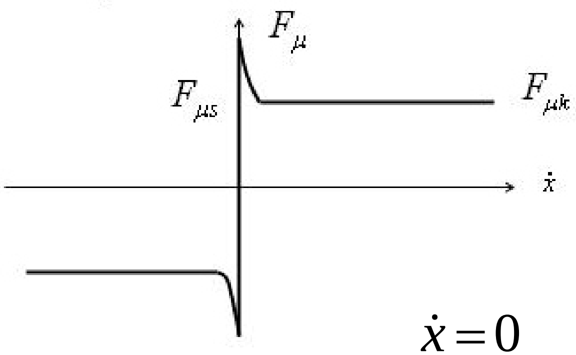
$$x(t) = k_1 e^{-\zeta\omega_n t} + k_2 t e^{-\zeta\omega_n t}$$



Dry Friction (no lubricant)







- $F\mu s$ = Static Friction Force

  $F\mu k$ = Kinetic Friction Force

- $F\mu s = \mu s \cdot N$ μs : Static Friction Coefficient

  $F\mu k = \mu k \cdot N$ μk : Kinetic Friction Coefficient

$\dot{x} = 0$

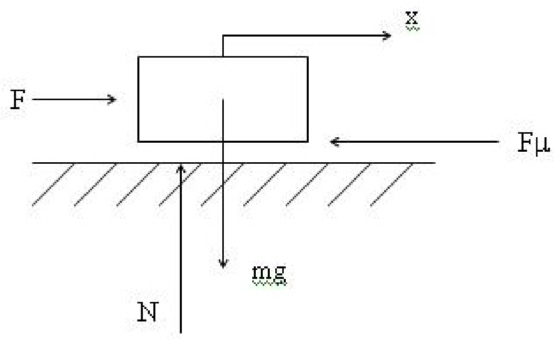
$$F_\mu = \begin{cases} F & \text{if } F \le F_{\mu s} \\ F_{\mu s}\,\mathrm{sgn}(F) & \text{if } F > F_{\mu s} \end{cases}$$

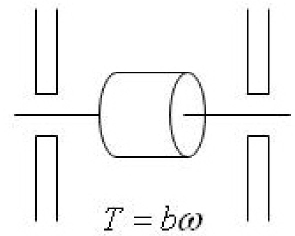
$\dot{x} \neq 0$

$$F_\mu = F_{\mu k}\,\mathrm{sgn}(\dot{x})$$

ε

Friction (with lubricant)





$$F_\mu = \begin{cases} b\dot{x} + G \cdot N \operatorname{sgn}(\dot{x}) & \forall \dot{x} > \varepsilon \\ F & \text{if } |\dot{x}| < \varepsilon \text{ and } |\mathrm{F}| \le (F_s + G \cdot N) \\ (F_s + G \cdot N)\operatorname{sgn}(F) & \textit{otherwise} \quad \text{i.e., if } |\dot{x}| < \varepsilon \text{ and } |\mathrm{F}| > (F_s + G \cdot N) \end{cases}$$

b: viscous friction coefficient

G: load-dependent factor

N: normal force

Fs: the maximum static friction

ε **: a small bound for zero velocity detection**