Mathematical Modeling of Dynamic Systems in State Space II

System without Input Derivatives

ex1) Consider a system defined by $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

Choose state variables,

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

→ Phase variables (each subsequent state variable is defined to be the derivative of the previous state variable.)

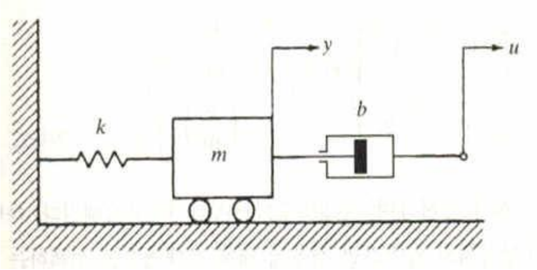
Then we obtain,

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \quad \\ \\ \end{bmatrix} u, \quad y = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

System with Input Derivates

ex2) Consider a mechanical system,

$$, \quad \ddot{y} =$$



Choose state variables,

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}\dot{u}$$

The right side includes $\dot{u}$ term. To explain the reason we should not include differentiation of u, assume $u = \delta(t)$ $(unit\ impulse\ function)$

$$x_2 = -\frac{k}{m}\int y\,dt - \frac{b}{m}y + \frac{k}{m}\delta(t)$$

x_2 includes $(k/m)\,\delta(t)$ term. It means $x_2(0) = \infty$ and cannot be accepted as a state variable.

That's why the standard form is

System with Input Derivatives

Method 1: Choose a state variable that includes *u*

To eliminate $\dot{u}$ term, $\ddot{y} = -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u} \rightarrow$

$$\frac{d}{dt}\left(\qquad \right) = -\frac{k}{m}y - \frac{b}{m}\left(\qquad \right) - \left(\frac{b}{m}\right)^2 u$$

So we choose state variables as, $x_1 = y,\ x_2 =$

$$\dot{x}_2 = \ddot{y} - \frac{b}{m}\dot{u} = \left(-\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u}\right) - \frac{b}{m}\dot{u} = -\frac{k}{m}x_1 - \frac{b}{m}\left(x_2 + \frac{b}{m}u\right)$$

$$= -\frac{k}{m}x_1 - \frac{b}{m}x_2 - \left(\frac{b}{m}\right)^2 u$$

$\longrightarrow$ $\dot{u}$ term has been eliminated.

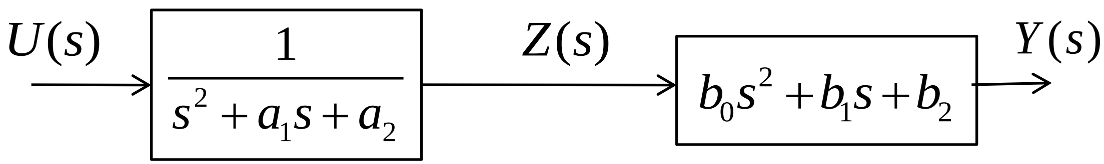
$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \qquad & \qquad \\ \qquad & \qquad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \qquad \\ \qquad \end{bmatrix} u$$

System with Input Derivatives

Method 2: Include the input derivates in the output equation

Consider a second-order system, $\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} \rightarrow \quad ,$$



$$\ddot{z} + a_1\dot{z} + a_2 z = u$$
$$y = b_0\ddot{z} + b_1\dot{z} + b_2 z$$

$let, \quad x_1 = z, \quad x_2 = \dot{z}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -$$

$$b_0\ddot{z} + b_1\dot{z} + b_2 z = \qquad\qquad = y$$

$$\therefore \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -a_2 x_1 - a_1 x_2 + u$$
$$y = (b_2 - a_2 b_0)x_1 + (b_1 - a_1 b_0)x_2 + b_0 u$$

System with Input Derivates

Method 2: Include the input derivates in the output equation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} b_2 - a_2 b_0 & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$$

N-th order differential equation, $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y$

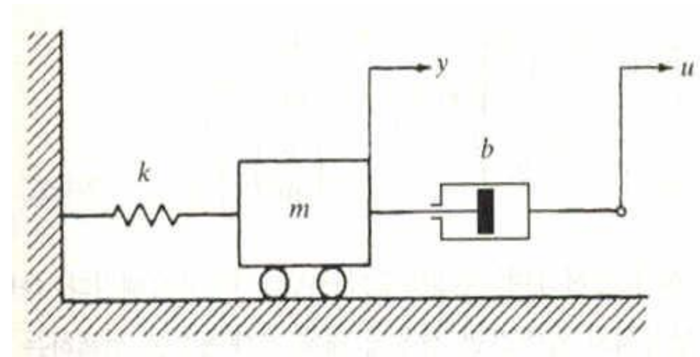
$$= b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} b_n - a_n b_0 & \vdots & b_{n-1} - a_{n-1} b_0 & \vdots & \cdots & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

System with Input Derivatives

ex2) Consider this mechanical system again,



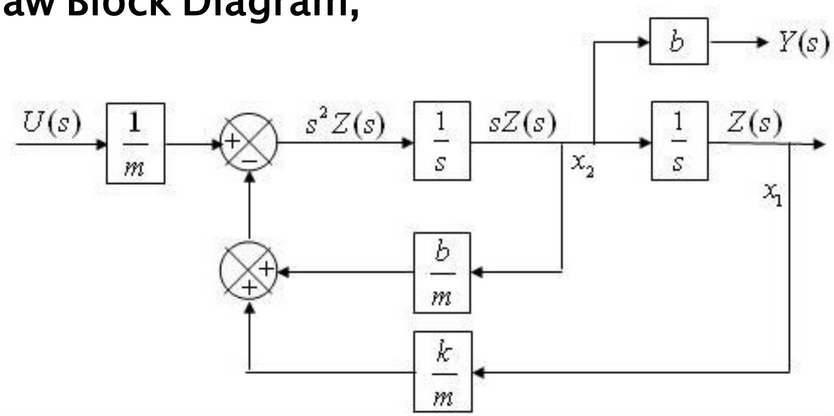
$$m\ddot{y} = -ky - b(\dot{y} - \dot{u}), \quad m\ddot{y} + b\dot{y} + ky = b\dot{u}$$

$$\frac{Y(s)}{U(s)} = \frac{bs}{ms^2 + bs + k}, \quad \frac{Z(s)}{U(s)} = \qquad\qquad, \quad \frac{Y(s)}{Z(s)} =$$

$$(ms^2 + bs + k)Z(s) = U(s), \quad bs\,Z(s) = Y(s)$$

$$s^2 Z(s) = \frac{1}{m}U(s) - \frac{b}{m}s\,Z(s) - \frac{k}{m}Z(s)$$

Draw Block Diagram,



State variables,

$x_1 = \qquad x_2 =$

$\dot{x}_1 = x_2$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

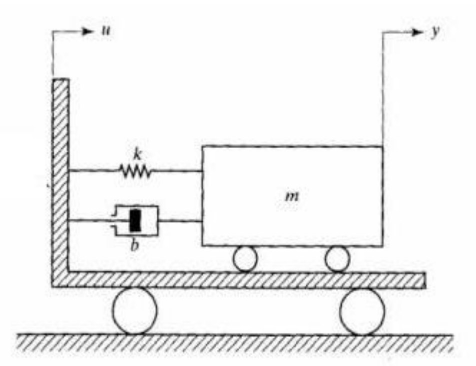
$y =$

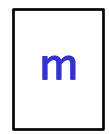
System Matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix}$$

System with Input Derivatives

ex) Consider a spring-mass-damper system





$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

$$or \quad m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

$$Transfer\ Function = \frac{Y(s)}{U(s)} = \frac{bs+k}{ms^2+bs+k}$$

State Eq:

Output Eq:

State variables:

Rewrite equations:

$$b_0 = 0, \quad b_1 = b, \ b_2 = k \qquad b_2 - a_2 b_0 = \frac{k}{m} - \frac{k}{m} + 0 = \frac{k}{m}$$

$$a_1 = \frac{b}{m}, \quad a_2 = \frac{k}{m} \qquad b_1 - a_1 b_0 = \frac{b}{m} - \frac{b}{m} + 0 = \frac{b}{m}$$

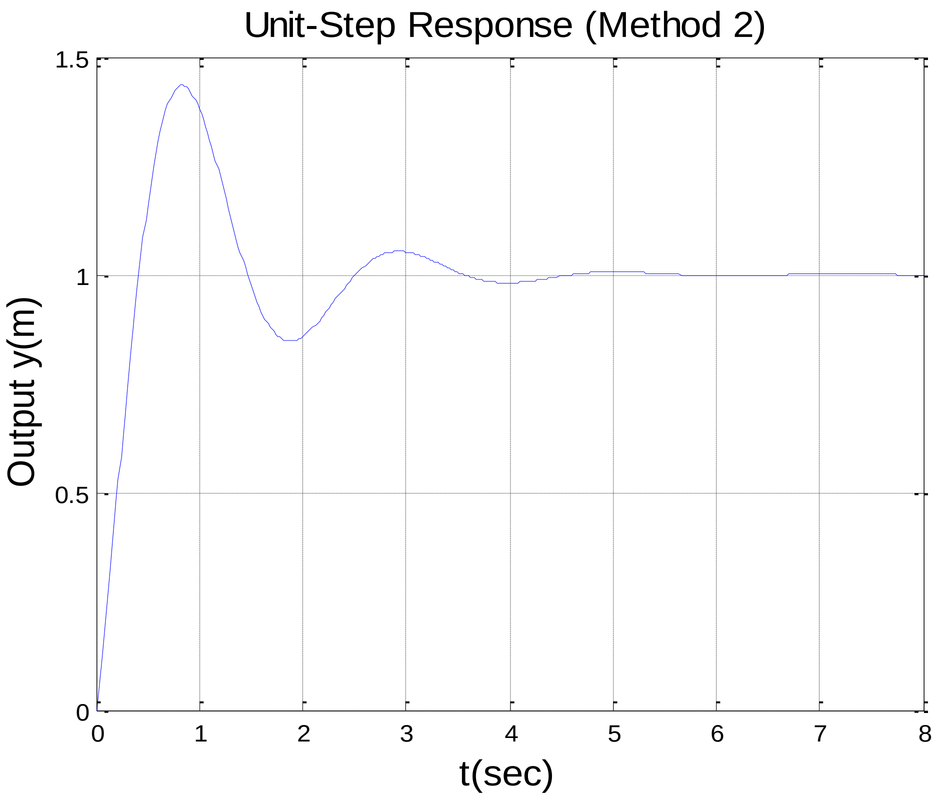
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}\mathrm{u}, \quad y = \begin{bmatrix}\frac{k}{m} & \frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Matlab Example

If, m=10kg, b=20N-s/m, k=100N/m

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{u}, \quad y = \begin{bmatrix} 10 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

```
t=0:0.02:8;
A=[0 1;-10 -2];
B=[0;1];
C=[10 2];
D=[0];
sys=ss(A,B,C,D);
[y,t]=step(sys,t);
plot(t,y)
grid
title('Unit-Step Response (Method 2)','Fontsize',15)
xlabel('t(sec)','Fontsize',15)
ylabel('Output y(m)','Fontsize',15)
```



Transformation of Mathematical Models with MATLAB

$$\frac{Y(s)}{U(s)} = \frac{numerator\ \ polynomial\ in\ s}{denominator\ \ polynomial\ in\ s} = \frac{num}{den}$$

MATLAB command, [A, B, C, D] = tf2ss(num,den) **gives a state space representation.**

ex) Consider, $$\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160}$$

One of many possible state-space representations is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & 160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

```
>> num=[0 0 1 0];
>> den=[1 14 56 160];
>> [A,B,C,D]=tf2ss(num,den)
A =
   -14   -56  -160
     1     0     0
     0     1     0
B =
     1
     0
     0
C =
     0     1     0
D =
     0
```