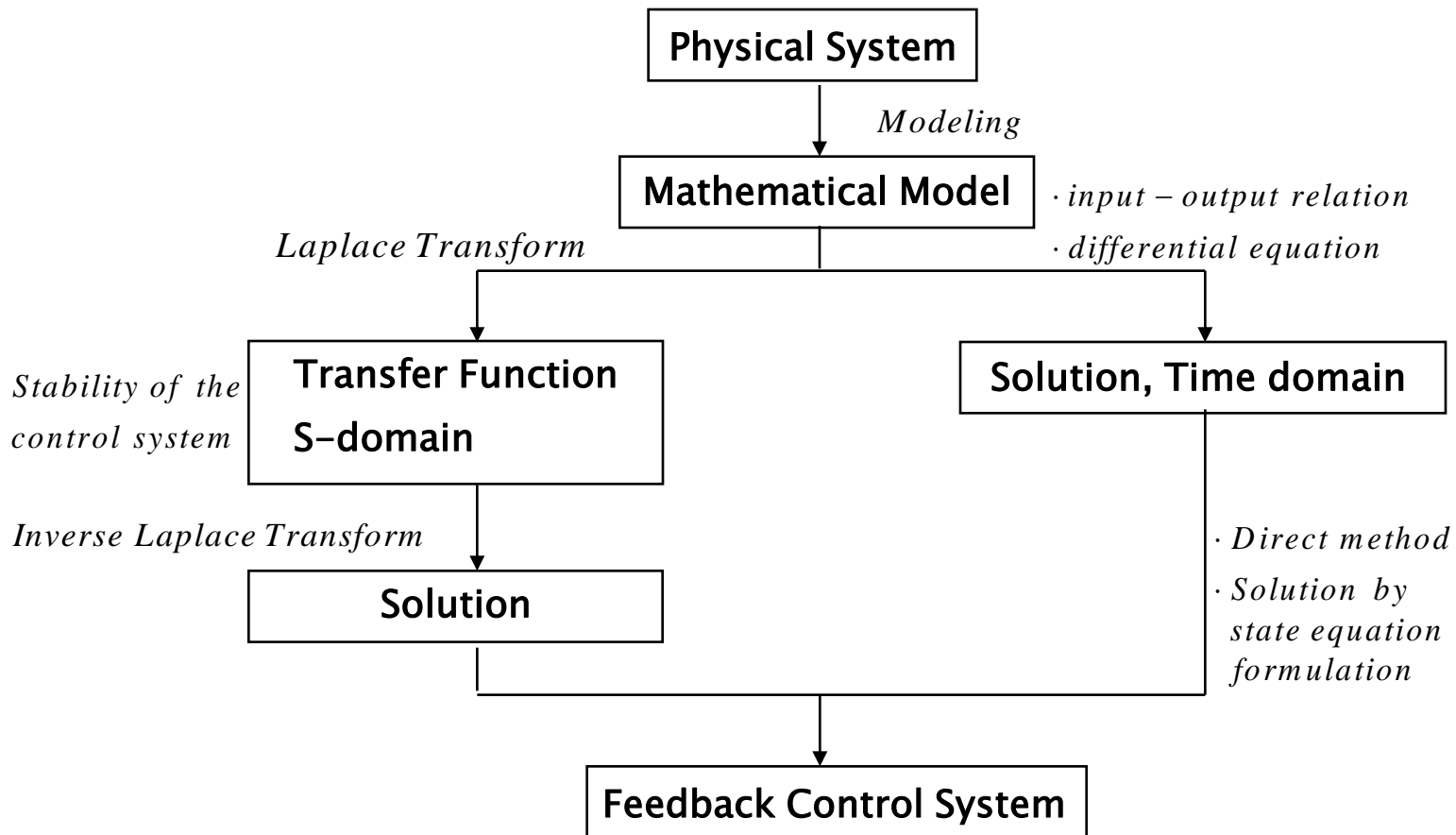
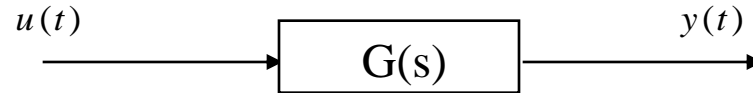


Frequency Response I





Concept of Frequency Response



$$u(t) = P \sin \omega t$$

$$G(s) = \frac{K (s + z_1)(s + z_2) \cdots (s + z_n)}{(s + s_1)(s + s_2) \cdots (s + s_n)}$$

$$Y(s) = G(s) \cdot u(s), \quad u(s) = P \cdot \frac{\omega}{s^2 + \omega^2}$$

$$Y(s) = G(s) \cdot \frac{P\omega}{s^2 + \omega^2}$$
$$= \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} + \frac{b_1}{s + s_1} + \frac{b_2}{s + s_2} + \cdots + \frac{b_n}{s + s_n}$$

$$y(t) = \text{ , }$$



Frequency Response

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$$

$$a = \quad = -$$

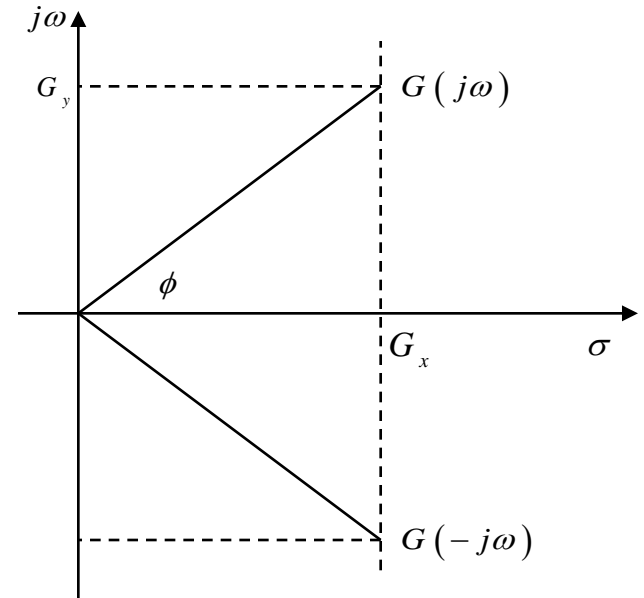
$$\bar{a} = G(s) \cdot \frac{P\omega}{s^2 + \omega^2} (s - j\omega) \Big|_{s=j\omega} = \frac{P}{2j} G(j\omega)$$

$$\begin{aligned} G(j\omega) &= G_x + jG_y \\ &= |G(j\omega)| \cos \phi + j |G(j\omega)| \sin \phi \\ &= |G(j\omega)| (\cos \phi + j \sin \phi) = \end{aligned}$$

Similarly, $G(-j\omega) = |G(-j\omega)| e^{-j\phi}$

$$\Rightarrow a = -\frac{P}{2j} G(-j\omega) = -\frac{P}{2j} |G(j\omega)| e^{-j\phi}$$

$$\bar{a} = \frac{P}{2j} G(j\omega) = \frac{P}{2j} |G(j\omega)| e^{j\phi}$$



Frequency Response

$$\begin{aligned}\therefore y(t) &= ae^{-j\omega t} + \bar{a}e^{j\omega t} \\ &= -\frac{P}{2j} |G(j\omega)| e^{-j\phi} e^{-j\omega t} + \frac{P}{2j} |G(j\omega)| e^{j\phi} e^{j\omega t} \\ &= |G(j\omega)| \frac{P}{2j} (e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}) \\ &= \end{aligned}$$



Frequency Response of First Order Systems



Consider, $\frac{Y(s)}{R(s)} = G(s) = \frac{1}{Ts + 1}$

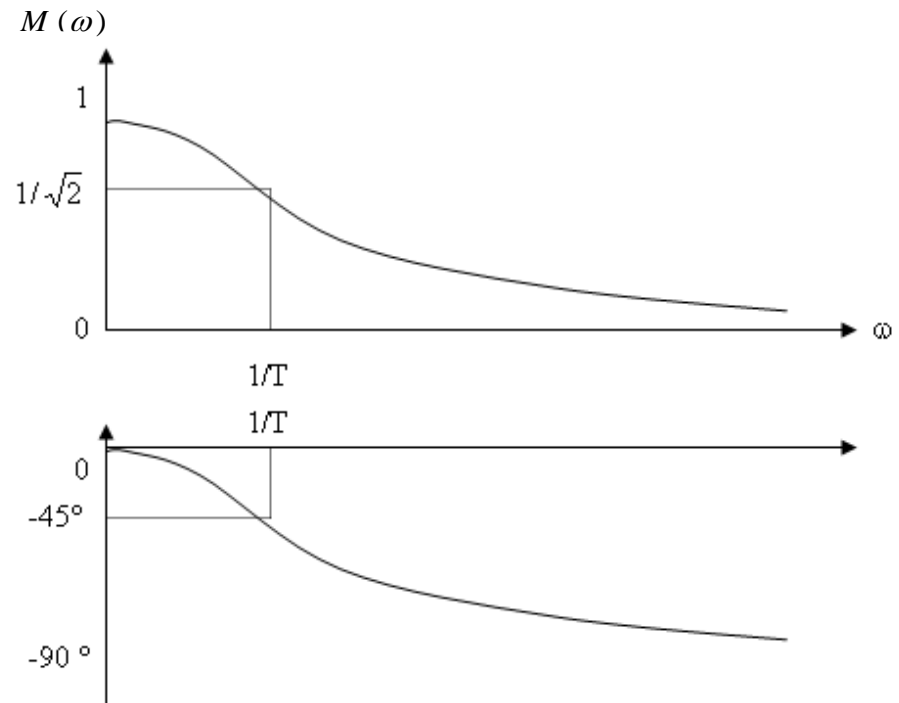
$$G(j\omega) = \frac{1}{Tj\omega + 1}$$

$$M(j\omega) = |G(j\omega)| =$$

$$\phi(j\omega) = \angle G(j\omega) = -\angle(Tj\omega + 1)$$

=

$$y(t) = \frac{1}{\sqrt{T^2\omega^2 + 1}} \sin(\omega t - \tan^{-1} T\omega)$$



Frequency Response of Second Order Systems



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad 0 < \zeta < 1$$

$$R(s) = \frac{\omega}{s^2 + \omega^2}, \quad r(t) = \sin \omega t$$

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{\omega}{s^2 + \omega^2} = \frac{as + b}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{cs + d}{s^2 + \omega^2}$$

$$y(t) = ae^{-\zeta\omega_n t} \cos \omega_d t + \frac{b}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t + c \cos \omega_d t + d \sin \omega_d t$$

=



Steady State Frequency Response

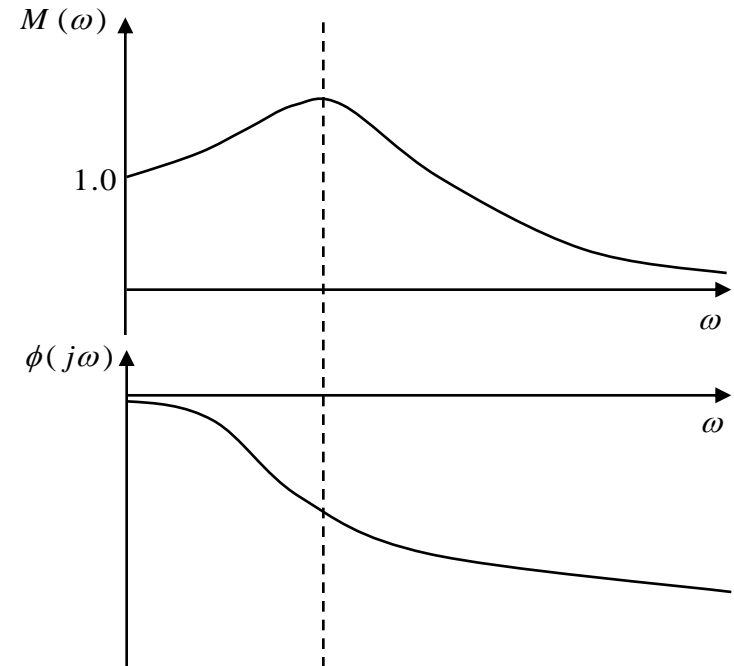
$$y(t) = A(j\omega) \sin(\omega t + \phi)$$

Magnitude ratio

$$M(\omega) = \left| \frac{y(t)}{r(t)} \right| = |G(j\omega)|$$

Phase $\phi(j\omega) = \angle G(j\omega)$

⇒ Frequency response



Steady State Frequency Response

$$\frac{Y(s)}{R(s)} = G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

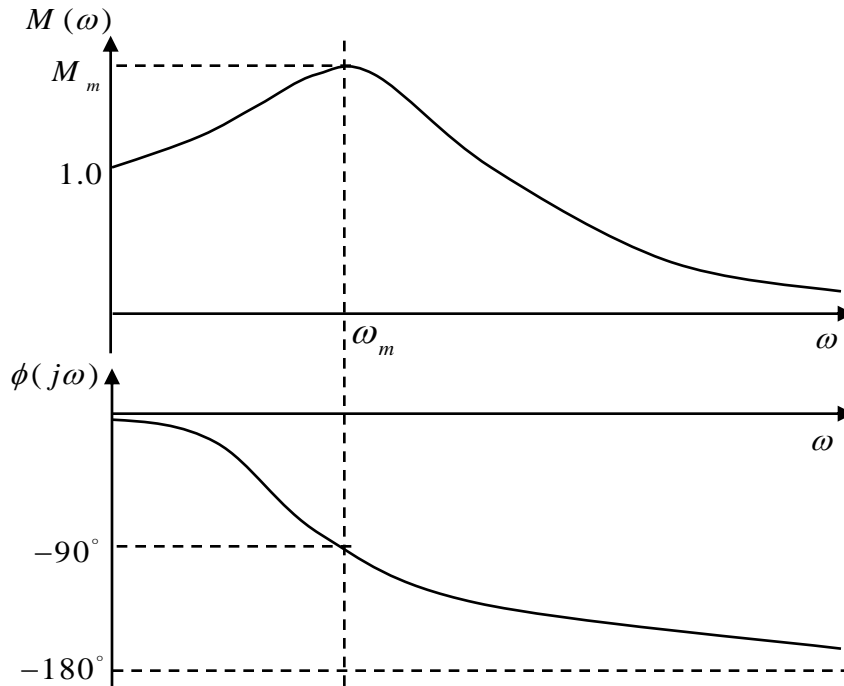
$$\frac{Y(j\omega)}{R(j\omega)} = G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n \omega j + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega_n \omega j}$$

$$\begin{aligned} M(\omega) &= \left| \frac{Y(j\omega)}{R(j\omega)} \right| = \left| \frac{\omega_n^2}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2} \cdot (\omega_n^2 - \omega^2 - 2\zeta\omega_n \omega j) \right| \\ &= \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2}} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \cdot \frac{\omega^2}{\omega_n^2}}} \end{aligned}$$

$$\frac{Y(j\omega)}{R(j\omega)} = M(j\omega) \angle G(j\omega)$$

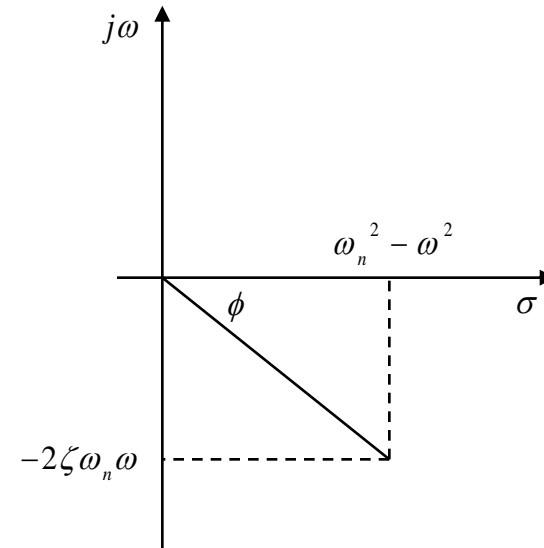


Steady State Frequency Response



$$\phi(j\omega) = -\tan^{-1} \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}$$

$$\lim_{\omega \rightarrow \infty} \phi(j\omega) = -\tan^{-1} 0 = -180^\circ$$



Steady State Frequency Response

$$M^2(\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \cdot \frac{\omega^2}{\omega_n^2}}$$

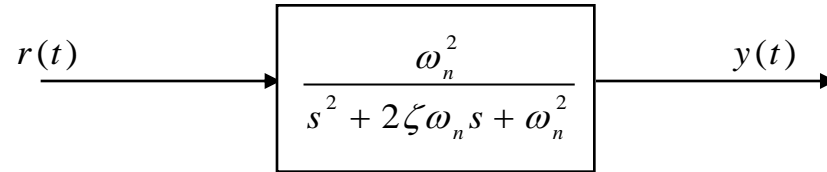
$$\frac{dM^2(\omega)}{d\omega} = \frac{-\left[2\left(1 - \frac{\omega^2}{\omega_n^2}\right)\left(-\frac{2\omega}{\omega_n^2}\right) + 8\zeta^2 \cdot \frac{\omega}{\omega_n^2}\right]}{\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \cdot \frac{\omega^2}{\omega_n^2}\right]^2} = 0$$

$$\Rightarrow 2\left(1 - \frac{\omega^2}{\omega_n^2}\right)\left(-\frac{2\omega}{\omega_n^2}\right) + 8\zeta^2 \cdot \frac{\omega}{\omega_n^2} = 0, \quad -\left(1 - \frac{\omega^2}{\omega_n^2}\right) + 2\zeta^2 = 0$$

$$\therefore \omega = \omega_n \sqrt{1 - 2\zeta^2} = \omega_m, \quad M_m =$$



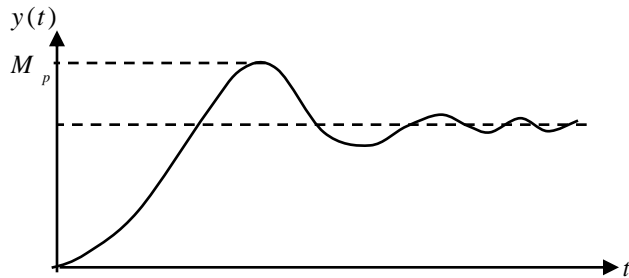
Unit Step Response VS Frequency Response



$$r(t) = u_{-1}(t)$$

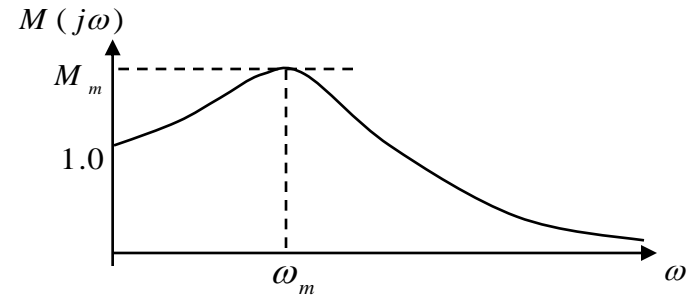
$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t + \cos^{-1} \zeta\right)$$

Unit step response



$$M_p = 1 + \exp\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

Frequency response



$$\omega_m = \omega_n \sqrt{1-2\zeta^2}, \quad M_m = \frac{1}{2\zeta \sqrt{1-\zeta^2}}$$