457.644 Advanced Bridge Engineering

Aerodynamic Design of Bridges

Part III: Fundamental theory of Structural Dynamics

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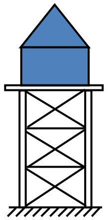
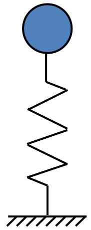
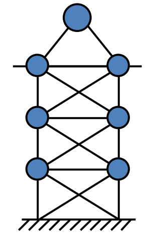
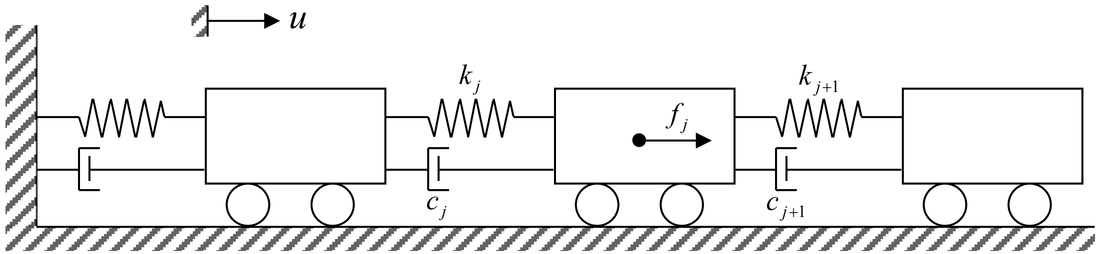
1. Equation of Multi Degree of Freedom System

Single degree of freedom vs Multi degree of freedom

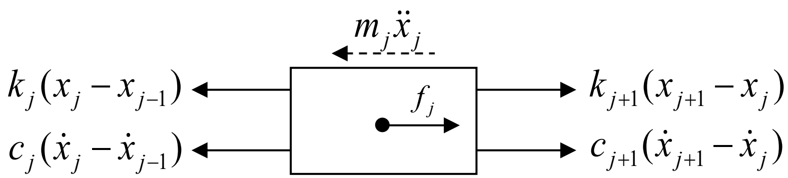
	 SDOF	 MDOF
Equation of Motion	• Single mass • One equation of motion $m\ddot{x}+c\dot{x}+kx=F(t)$	• Mass matrix • Equations of motion in matrix form • $[M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\}=\{F(t)\}$
Natural Frequency	• One Natural frequency $\omega=\sqrt{k/m}$	• Many of natural frequency & Mode shape • Eigen-value analysis
Calculation Method	• Duhamel integral • Frequency domain analysis	• Mode superposition method • Direct integration • SRSS or CQC methods

Equation of motion of Multi-degree of freedom

(a)



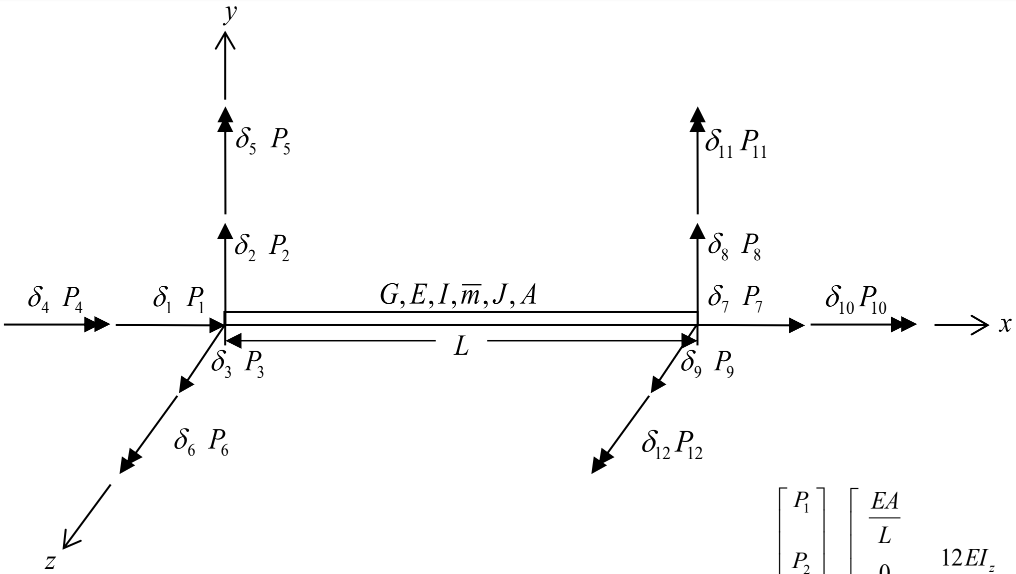
(b)



$$\begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \cdot \\ \cdot \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} c_1, & -c_1, & & & \\ -c_1, & c_1 + c_2, & -c_2, & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & c_{n-1} + c_n \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \dot{x}_n \end{Bmatrix}$$

$$+ \begin{bmatrix} k_1, & -k_1, & & & \\ -k_1, & k_1 + k_2, & -k_2, & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & k_{n-1} + k_n \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_n \end{Bmatrix} \qquad (1)$$

Stiffness matrix of frame member

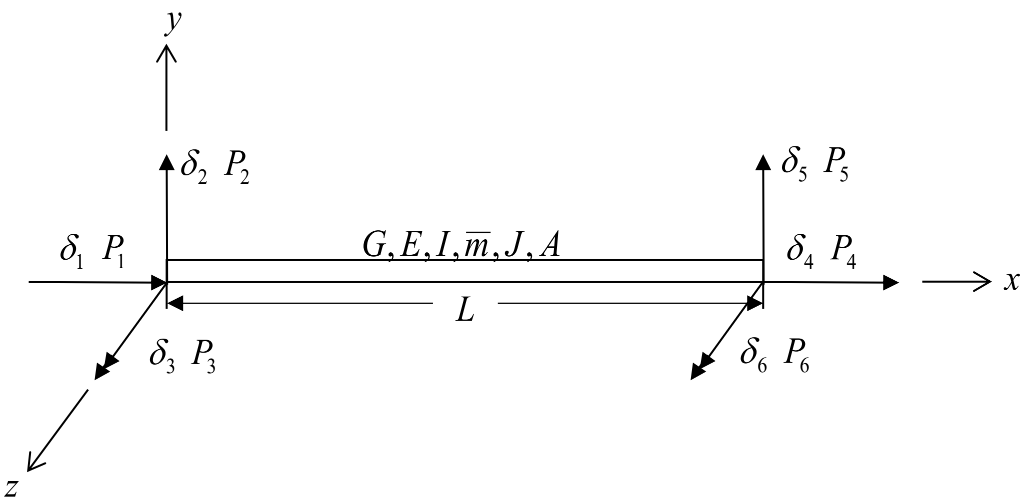


Stiffness Matrix

symmetric

$$
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{bmatrix}
=
\begin{bmatrix}
\frac{EA}{L} &&&&&&&&&&& \\
0 & \frac{12EI_z}{L^3} &&&&&&&&&& \\
0 & 0 & \frac{12EI_y}{L^3} &&&&&&&&& \\
0 & 0 & 0 & \frac{GI}{L} &&&&&&&& \\
0 & 0 & \frac{-6EI_y}{L^3} & 0 & \frac{4EI_y}{L} &&&&&&& \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_y}{L} &&&&&& \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} &&&&& \\
\frac{-EA}{L} & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} &&&& \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} &&& \\
0 & 0 & 0 & \frac{-GI}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI}{L} && \\
0 & 0 & \frac{-6EI_y}{L^3} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}
\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \\ \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{bmatrix}
$$

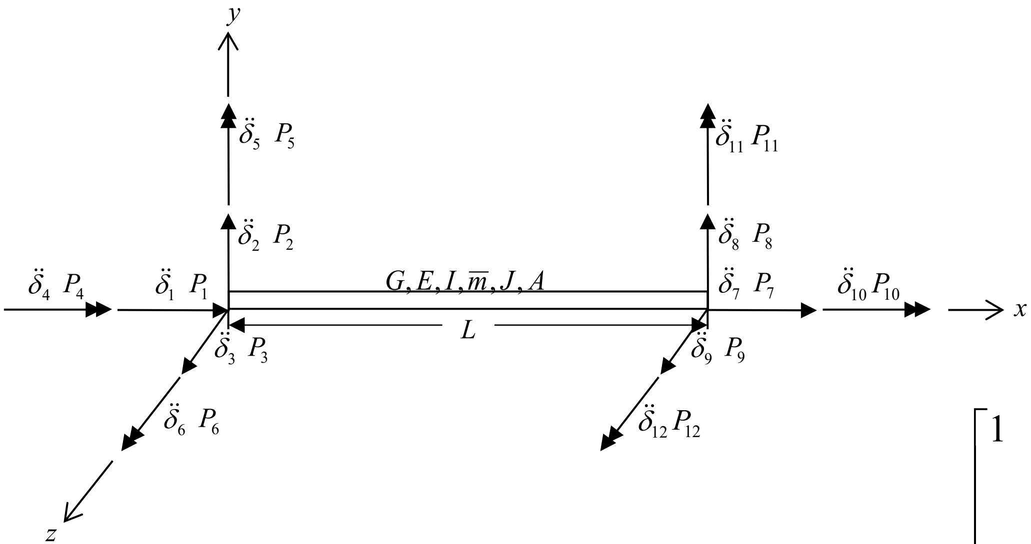
Stiffness matrix of frame member



Stiffness Matrix

$$
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}
=
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{bmatrix}
$$

Mass matrix of frame member

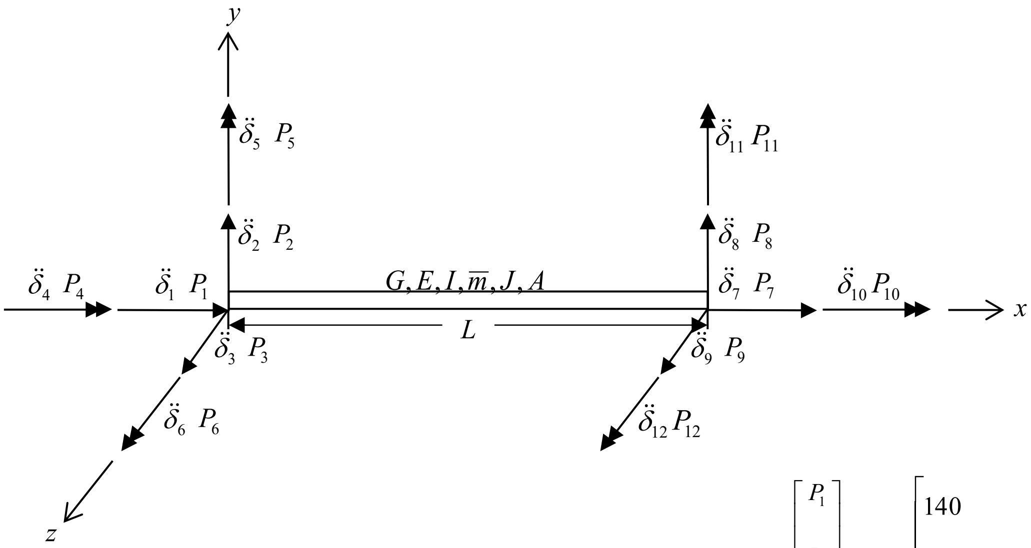


Lumped Mass Matrix

$$[M_L] = \frac{\overline{m}L}{2}\begin{bmatrix} 1 &&&&&&&&&&& \\ & 1 &&&&&&&&&& \\ && 1 &&&&&&&&& \\ &&& I_0/A &&&& \underline{0} &&&& \\ &&&& 0 &&&&&&& \\ &&&&& 0 &&&&&& \\ &&&&&& 1 &&&&& \\ &&&&&&& 1 &&&& \\ &&& \underline{0} &&&&& 1 &&& \\ &&&&&&&&& I_0/A && \\ &&&&&&&&&& 0 & \\ &&&&&&&&&&& 0 \end{bmatrix}\begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \\ \ddot{\delta}_5 \\ \ddot{\delta}_6 \\ \ddot{\delta}_7 \\ \ddot{\delta}_8 \\ \ddot{\delta}_9 \\ \ddot{\delta}_{10} \\ \ddot{\delta}_{11} \\ \ddot{\delta}_{12} \end{Bmatrix}$$

$I_0 =$ Polar moment of inertia

Mass matrix of frame member

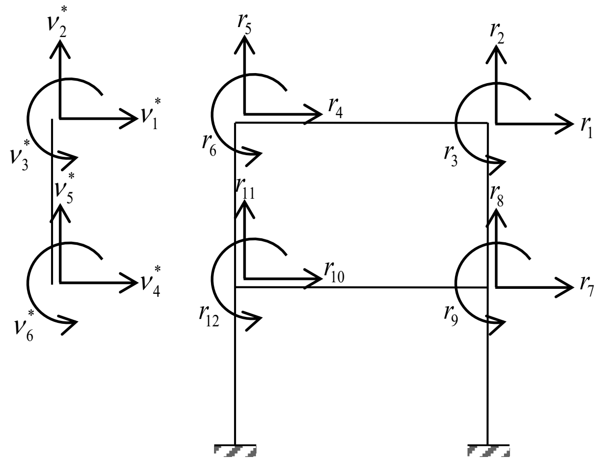


Consistent Mass Matrix

$$
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{bmatrix}
= \frac{\overline{m}L}{420}
\begin{bmatrix}
140 & & & & & & & & & & & \\
0 & 156 & & & & & & & \text{symmetric} & & & \\
0 & 0 & 156 & & & & & & & & & \\
0 & 0 & 0 & \frac{140I_0}{A} & & & & & & & & \\
0 & 0 & -22L & 0 & 4L^2 & & & & & & & \\
0 & 22L & 0 & 0 & 0 & 4L^2 & & & & & & \\
70 & 0 & 0 & 0 & 0 & 0 & 140 & & & & & \\
0 & 54 & 0 & 0 & 0 & 13L & 0 & 156 & & & & \\
0 & 0 & 54 & 0 & -13L & 0 & 0 & 0 & 156 & & & \\
0 & 0 & 0 & \frac{70I_0}{A} & 0 & 0 & 0 & 0 & 0 & \frac{140I_0}{A} & & \\
0 & 0 & 13L & 0 & -3L^2 & 0 & 0 & 0 & 22L & 0 & 4L^2 & \\
0 & -13L & 0 & 0 & 0 & -3L^2 & 0 & -22L & 0 & 0 & 0 & 4L^2
\end{bmatrix}
\begin{bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \\ \ddot{\delta}_5 \\ \ddot{\delta}_6 \\ \ddot{\delta}_7 \\ \ddot{\delta}_8 \\ \ddot{\delta}_9 \\ \ddot{\delta}_{10} \\ \ddot{\delta}_{11} \\ \ddot{\delta}_{12} \end{bmatrix}
$$

Direct assemblage

Plane Frame (ndof=12)



Element Stiffness Matrix

$$K_e = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

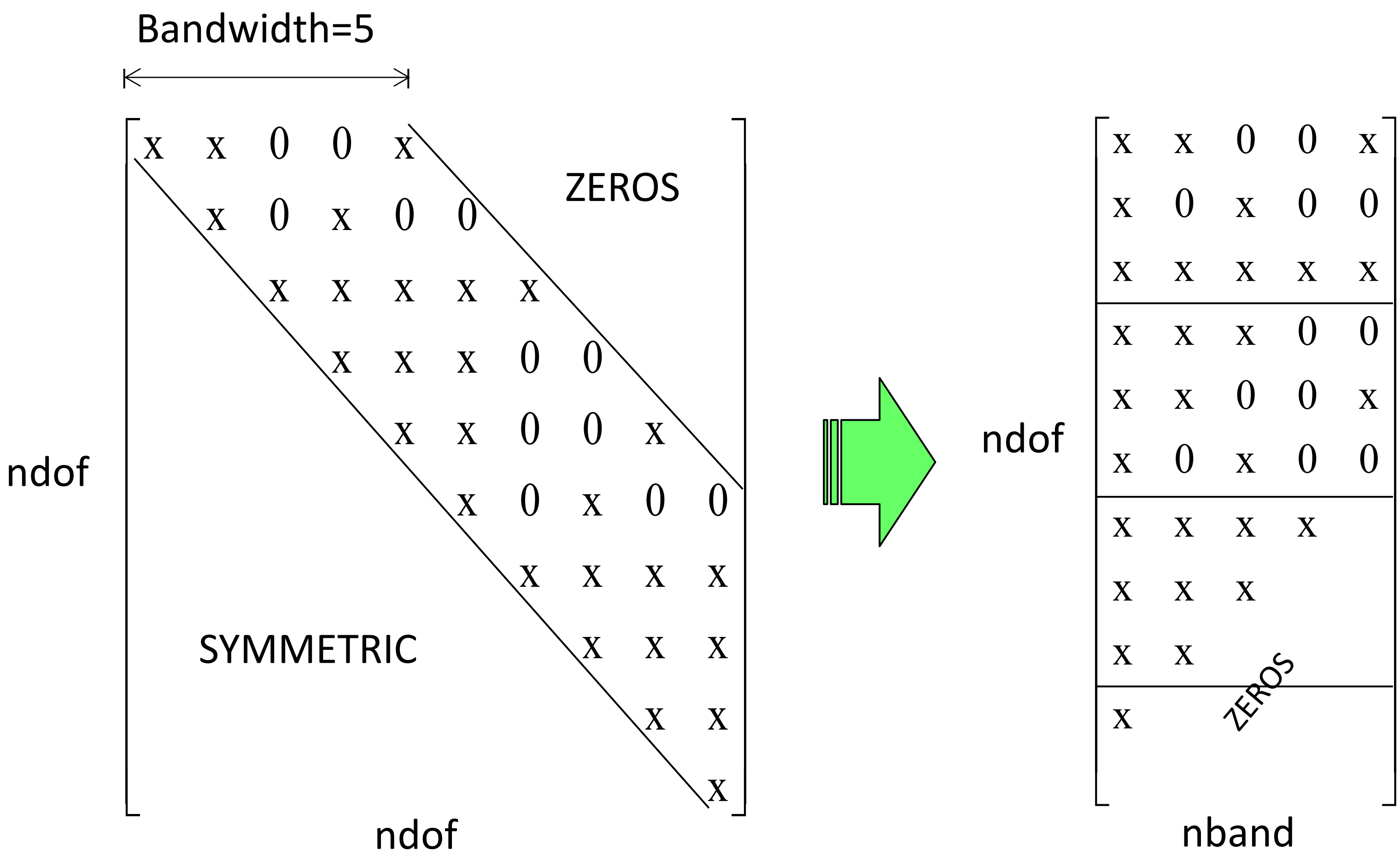
$$K_{eg} = T^T K_e T$$

Global Stiffness Matrix

$$\begin{array}{c} \begin{array}{cccccccccccc} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 & r_9 & r_{10} & r_{11} & r_{12} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{11} & K_{12} & K_{13} & 0 & 0 & 0 & K_{14} & K_{15} & K_{16} \\ 0 & 0 & 0 & K_{21} & K_{22} & K_{23} & 0 & 0 & 0 & K_{24} & K_{25} & K_{26} \\ 0 & 0 & 0 & K_{31} & K_{32} & K_{33} & 0 & 0 & 0 & K_{34} & K_{35} & K_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{41} & K_{42} & K_{43} & 0 & 0 & 0 & K_{44} & K_{45} & K_{46} \\ 0 & 0 & 0 & K_{51} & K_{52} & K_{53} & 0 & 0 & 0 & K_{54} & K_{55} & K_{56} \\ 0 & 0 & 0 & K_{61} & K_{62} & K_{63} & 0 & 0 & 0 & K_{64} & K_{65} & K_{66} \end{bmatrix} \end{array}$$

(ndof x ndof)=(12 x 12)

Direct assemblage



2. Undamped Free Vibration

Undamped Free Vibration

Equation of motion

$$\underset{(n\times n)}{\mathbf{M}}\underset{(n\times 1)}{\{\ddot{\mathbf{X}}\}}+\underset{(n\times n)}{\mathbf{K}}\underset{(n\times 1)}{\{\mathbf{X}\}}=\underset{(n\times 1)}{\{\mathbf{0}\}} \tag{1}$$

$$\sum_{j=1}^{n} m_{ij}\ddot{x}_j+\sum_{j=1}^{n} k_{ij}x_j=0 \quad , \quad i=1,2,\ldots,\mathrm{n} \tag{2}$$

Separation of variables

Spatial function

$$\underset{(n\times 1)}{\{\mathbf{X}\}}=\underset{(n\times 1)}{\{\boldsymbol{\Phi}(\mathbf{x})\}}\,q(t) \tag{3}$$

Time function

$$\therefore \sum_{j} m_{ij}\phi_j\ddot{q}(t)+\sum_{j} k_{ij}\phi_j q(t)=0 \quad , \quad i=1,2,\ldots,\mathrm{n} \tag{4}$$

$$\Rightarrow \frac{\ddot{q}(t)}{q(t)}=-\frac{\sum_{j} k_{ij}\phi_i}{\sum_{j} m_{ij}\phi_i}=-\omega^2=\text{constant} \tag{5}$$

Solution

(i) $\ddot{q}(t)+\omega^2 q(t)=0 \longrightarrow q(t)=a\cos(\omega t+\theta)$ (6)

(ii) $\sum_{j}^{n} k_{ij}\phi_j - \omega^2 \sum_{j}^{n} m_{ij}\phi_j = 0$, $i=1,2,\ldots,\text{n}$ (7)

or $\underset{(n\times n)}{\mathbf{K}}\underset{(n\times 1)}{\{\boldsymbol{\Phi}\}}+\omega^2\underset{(n\times n)}{\mathbf{M}}\underset{(n\times 1)}{\{\boldsymbol{\Phi}\}}=\mathbf{0}$ (8)

$\left(\underset{(n\times n)}{\mathbf{K}}-\omega^2\underset{(n\times n)}{\mathbf{M}}\right)\underset{(n\times 1)}{\{\boldsymbol{\Phi}\}}=\mathbf{0}$ (9)

Natural frequency

- To get solution which is not $\underset{(n\times 1)}{\{\boldsymbol{\Phi}\}} = \underset{(n\times 1)}{\mathbf{0}}$

$$\left| \underset{(n\times n)}{\mathbf{K}} - \omega^2 \underset{(n\times n)}{\mathbf{M}} \right| = 0 \tag{10}$$

- Get n number of ω^2 by eigen-value analysis

$$\omega_1 < \omega_2 < \omega_3 \ldots < \omega_n \tag{11}$$

Fundamental natural frequency

Natural mode shape

- k^{th} natural frequency ω_k

$$\left(\mathbf{K}-\omega_k{}^2\mathbf{M}\right)\{\mathbf{\Phi}\}^{(k)}=\mathbf{0} \tag{12}$$

- Natural mode matrix

$$\underset{(n\times n)}{[\mathbf{\Phi}]}=\left[\underset{(n\times 1)}{\{\mathbf{\Phi}\}^{(1)}}\ \underset{(n\times 1)}{\{\mathbf{\Phi}\}^{(2)}},\ldots,\underset{(n\times 1)}{\{\mathbf{\Phi}\}^{(n)}}\right]=\begin{bmatrix}\phi_1^{(1)} & \phi_1^{(2)} & \cdots & \phi_1^{(n)}\\ \phi_2^{(1)} & \phi_2^{(2)} & \cdots & \phi_2^{(n)}\\ \vdots & \vdots & \cdots & \vdots\\ \phi_n^{(1)} & \phi_n^{(2)} & \cdots & \phi_n^{(n)}\end{bmatrix} \tag{13}$$

Undamped Free Vibration

Responses

$$\underset{(n\times 1)}{\{\mathbf{X}\}} = \sum_{i=1}^{n} \underset{(n\times 1)}{\{\mathbf{\Phi}\}^{(i)}} q_i(t) \tag{14}$$

$$= \sum_{i=1}^{n} a_i \underset{(n\times 1)}{\{\mathbf{\Phi}\}^{(i)}} \cos(\omega_i t + \theta_i) \tag{15}$$

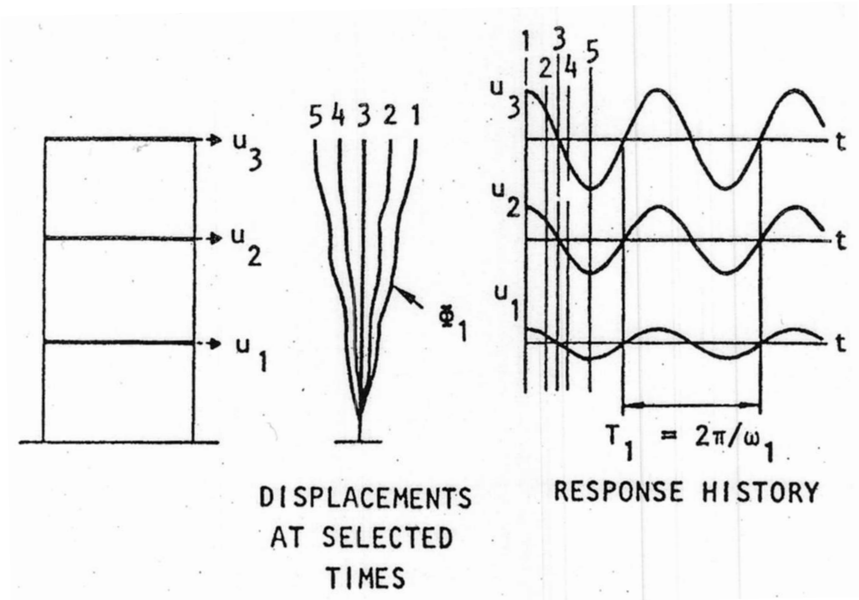
$$= \underset{(n\times n)}{\mathbf{\Phi}} \underset{(n\times 1)}{\{\mathbf{q(t)}\}} \tag{16}$$

Initial condition

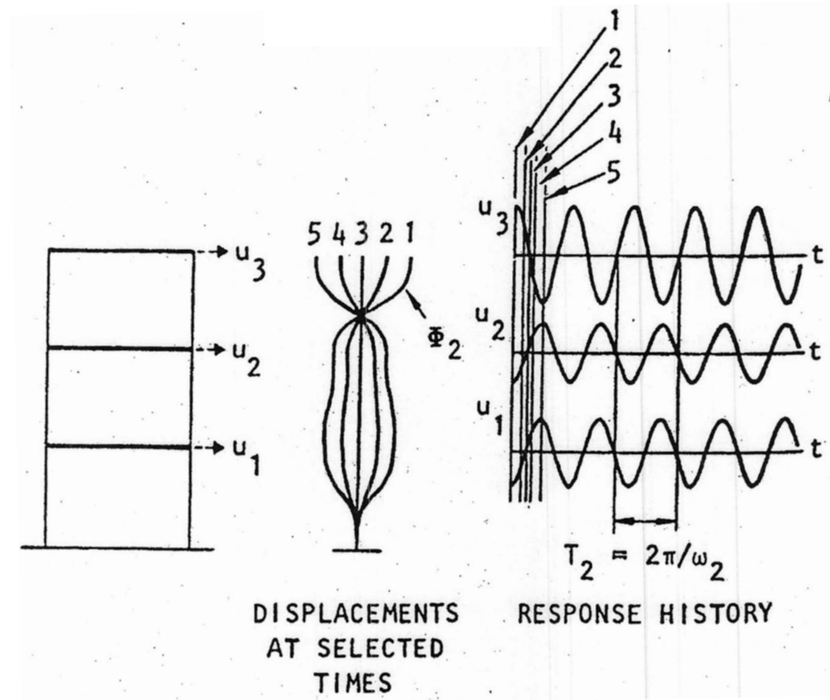
$$x_j(0) \text{ and } \dot{x}_j(0) \quad , \quad i = 1, 2, \ldots, \mathrm{n} \tag{17}$$

$$\Rightarrow a_i \text{ and } \theta_i \quad , \quad i = 1, 2, \ldots, \mathrm{n} \tag{18}$$

Mode superposition

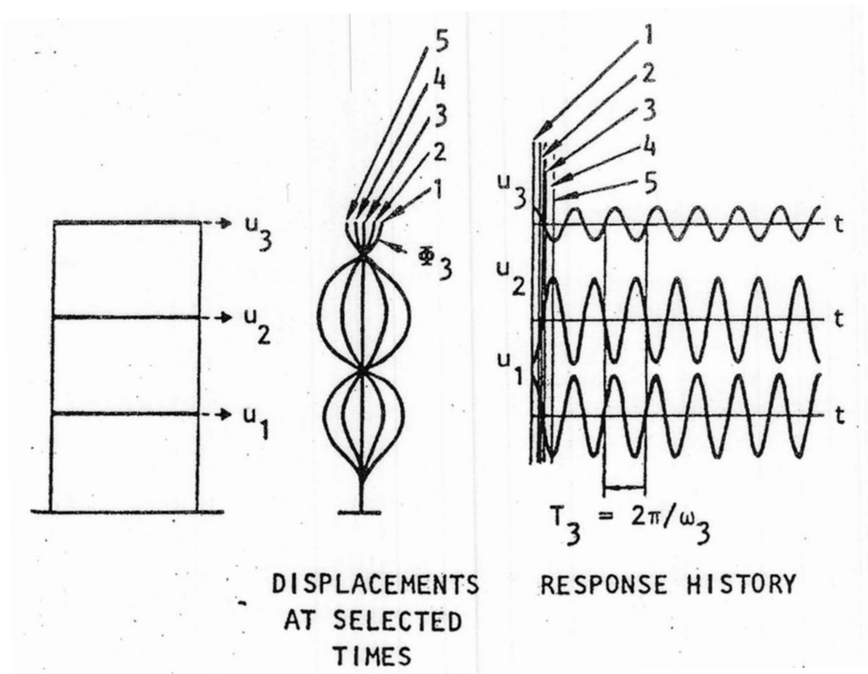


< 1st mode >

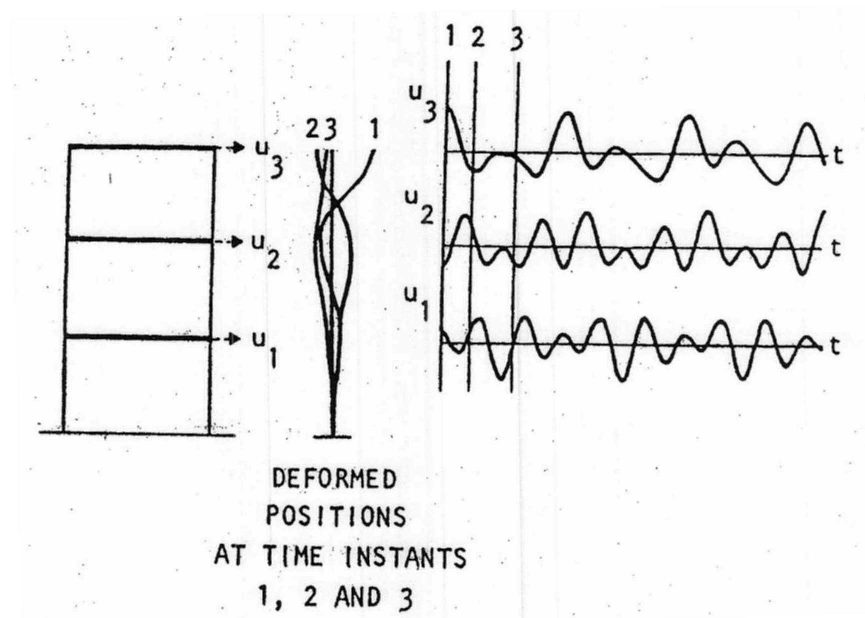


< 2nd mode >

Mode superposition



< 3rd mode >



< Mode Superposition >

Mode orthogonality

- $$\underset{(n\times n)(n\times n)(n\times n)}{\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}} = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & \underline{0} \\ & & \cdot & \\ & \underline{0} & & \cdot & \\ & & & & \mu_n \end{bmatrix}$$

or $$\underset{(1\times n)}{\{\mathbf{\Phi}\}_i^{\mathrm{T}}}\underset{(n\times n)}{\mathbf{M}}\underset{(n\times 1)}{\{\mathbf{\Phi}\}_j} = \begin{cases} 0 & for\ i \neq j \\ \mu_i & for\ i = j \end{cases}$$

μ_i = Generalized mass

- $$\underset{(n\times n)(n\times n)(n\times n)}{\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi}} = \begin{bmatrix} \mu_1\omega_1^2 & & & \\ & \mu_2\omega_2^2 & & \underline{0} \\ & & \cdot & \\ & \underline{0} & & \cdot & \\ & & & & \mu_n\omega_n^2 \end{bmatrix}$$

or $$\underset{(1\times n)}{\{\mathbf{\Phi}\}_i^{\mathrm{T}}}\underset{(n\times n)}{\mathbf{K}}\underset{(n\times 1)}{\{\mathbf{\Phi}\}_j} = \begin{cases} 0 & for\ i \neq j \\ k_i = \mu_i\omega_i^2 & for\ i = j \end{cases}$$

k_i = Generalized stiffness

3. Mode Superposition Method

Generalized coordinates

Displacements by superposition of generalized modal coordinates

$$\underset{(n\times 1)}{\{\mathbf{X(t)}\}} = \underset{(n\times 1)}{\{\boldsymbol{\Phi}\}}^{(1)} q_1(t) + \underset{(n\times 1)}{\{\boldsymbol{\Phi}\}}^{(2)} q_2(t) + \ldots + \underset{(n\times 1)}{\{\boldsymbol{\Phi}\}}^{(n)} q_n(t)$$

$$= \underset{(n\times n)}{\boldsymbol{\Phi}}\,\underset{(n\times 1)}{\{\mathbf{q}\}} \qquad (1)$$

Generalized modal coordinates

Equation of motion

$$\underset{(n\times n)}{\mathbf{M}}\underset{(n\times 1)}{\{\ddot{\mathbf{X}}\}} + \underset{(n\times n)}{\mathbf{C}}\underset{(n\times 1)}{\{\dot{\mathbf{X}}\}} + \underset{(n\times n)}{\mathbf{K}}\underset{(n\times 1)}{\{\mathbf{X}\}} = \underset{(n\times 1)}{\{\mathbf{P(t)}\}} \qquad (2)$$

$$\underset{(n\times n)}{\mathbf{M}}\underset{(n\times n)}{\boldsymbol{\Phi}}\underset{(n\times 1)}{\{\ddot{\mathbf{q}}\}} + \underset{(n\times n)}{\mathbf{C}}\underset{(n\times n)}{\boldsymbol{\Phi}}\underset{(n\times 1)}{\{\dot{\mathbf{q}}\}} + \underset{(n\times n)}{\mathbf{K}}\underset{(n\times n)}{\boldsymbol{\Phi}}\underset{(n\times 1)}{\{\mathbf{q}\}} = \underset{(n\times 1)}{\{\mathbf{P(t)}\}} \qquad (3)$$

$$\underset{(n\times n)}{\boldsymbol{\Phi}^{\mathrm{T}}}\underset{(n\times n)}{\mathbf{M}}\underset{(n\times n)}{\boldsymbol{\Phi}}\underset{(n\times 1)}{\{\ddot{\mathbf{q}}\}} + \underset{(n\times n)}{\boldsymbol{\Phi}^{\mathrm{T}}}\underset{(n\times n)}{\mathbf{C}}\underset{(n\times n)}{\boldsymbol{\Phi}}\underset{(n\times 1)}{\{\dot{\mathbf{q}}\}} + \underset{(n\times n)}{\boldsymbol{\Phi}^{\mathrm{T}}}\underset{(n\times n)}{\mathbf{K}}\underset{(n\times n)}{\boldsymbol{\Phi}}\underset{(n\times 1)}{\{\mathbf{q}\}} = \underset{(n\times n)}{\boldsymbol{\Phi}^{\mathrm{T}}}\underset{(n\times 1)}{\{\mathbf{P(t)}\}} \qquad (4)$$

$$\begin{bmatrix}\ddots & & \\ & \mu & \\ & & \ddots\end{bmatrix} \qquad \qquad \begin{bmatrix}\ddots & & \\ & k & \\ & & \ddots\end{bmatrix}$$

Suppose Diagonal Matrix

generalized mass | generalized damping | generalized stiffness | generalized force

Generalized coordinates

Equation of motion

$$\underset{(n\times n)}{\begin{bmatrix} \ddots & & \\ & \mu & \\ & & \ddots \end{bmatrix}}\underset{(n\times 1)}{\{\ddot{\mathbf{q}}\}}+\underset{(n\times n)}{\begin{bmatrix} \ddots & & \\ & c & \\ & & \ddots \end{bmatrix}}\underset{(n\times 1)}{\{\dot{\mathbf{q}}\}}+\underset{(n\times n)}{\begin{bmatrix} \ddots & & \\ & k & \\ & & \ddots \end{bmatrix}}\underset{(n\times 1)}{\{\mathbf{q}\}}=\underset{(n\times 1)}{\{\mathbf{Q(t)}\}} \qquad (5)$$

or

$$\mu_i\ddot{q}_i + c_i\dot{q}_i + k_i q_i = Q_i(t) \qquad i = 1, 2, \ldots, n \qquad (6)$$

N-number of uncoupled equation of motion $\Longrightarrow$ N-number of SDOF equation of motion

Equation of motion

$$c_i = 2\omega_i \xi_i \mu_i \quad (7)$$

$$\ddot{q}_i + 2\xi_i \omega_i \dot{q}_i + \omega_i^2 \dot{q}_i = \frac{Q_i(t)}{\mu_i} \quad (8)$$

$$q_i(t) = e^{-\xi_i \omega_i t}\left[q_i(0)\cos\omega_{D_i} t + \frac{\xi_i \omega_i q_i(0) + \dot{q}_i(0)}{\omega_{D_i}} \sin\omega_{D_i} t \right] \quad (9)$$

$$+ \frac{1}{\mu_i \omega_{D_i}} \int_0^t \mathrm{Q}_\mathrm{i}(\tau) e^{-\xi_i \omega_i (t-\tau)} \sin\omega_{D_i}(t-\tau)\, d\tau \quad (10)$$

Solve $q_i(0)$ & $\dot{q}_i(0)$ with initial condition

- In real problem, getting the solution by superposing l number of the low order modes

$$\{\mathbf{X(t)}\} = \mathbf{\Phi}_\mathbf{l} \{\mathbf{q(t)}\} \qquad (l << n) \quad (11)$$

Procedure of Mode Superposition

(1) Equations of Motion :

$$\underset{(n\times n)}{\mathbf{M}}\ddot{\mathbf{X}}+\underset{(n\times n)}{\mathbf{C}}\dot{\mathbf{X}}+\mathbf{KX}=\underset{(n\times 1)}{\mathbf{P(t)}} \tag{1}$$

(2) Mode shapes and Natural Frequencies :

$$\left[\mathbf{K}-\omega^2\mathbf{M}\right]\mathbf{X}=\mathbf{0} \tag{2}$$

and obtain $\underset{(n\times 1)}{\{\boldsymbol{\omega}\}}$ and $\underset{(n\times l)}{\boldsymbol{\Phi}_\mathbf{l}}$, $\underset{(n\times 1)}{\mathbf{X}}=\underset{(n\times l)(l\times 1)}{\boldsymbol{\Phi}_\mathbf{l}\mathbf{q}}$ (3)

(3) Generalized Mass & Force :

$$\boldsymbol{\mu}_\mathbf{i}=\underset{(1\times n)(n\times n)\,(n\times 1)}{\boldsymbol{\Phi}_\mathbf{i}^{\mathbf{T}}\mathbf{M}\boldsymbol{\Phi}_\mathbf{i}}\ ,\quad Q_i(t)=\underset{(1\times n)\,(n\times 1)}{\boldsymbol{\Phi}_\mathbf{i}^{\mathbf{T}}\mathbf{P(t)}}\ ,\quad i=1,2,\ldots,l \tag{4}$$

(4) Uncoupled Equation of Motion :

$$\ddot{q}_i+2\xi_i\omega_i\dot{q}_i+\omega_i^2\dot{q}_i=\frac{Q_i(t)}{\mu_i}\quad ,\qquad i=1,2,\ldots,l \tag{5}$$

Procedure of Mode Superposition

(5) Modal Response to Loading :

$$q_i = e^{-\xi_i \omega_i t}\left\{\cdots\cdots \qquad \right\} + \int_0^t \frac{Q_i(\tau)}{\mu_i} h(t-\tau) \tag{6}$$

$$\underset{(l\times 1)}{\{\mathbf{q}\}} = \underset{(l\times l)(l\times n)(n\times n)(n\times 1)}{\boldsymbol{\mu}^{-1}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\,\mathbf{X}} \quad \begin{cases} \mathbf{q}_{\mathrm{i}}(0) = \boldsymbol{\mu}_{\mathrm{i}}^{-1}\boldsymbol{\Phi}_{\mathrm{i}}^{\mathrm{T}}\mathbf{M}\mathbf{X}(0) \\ \dot{\mathbf{q}}_{\mathrm{i}}(0) = \boldsymbol{\mu}_{\mathrm{i}}^{-1}\boldsymbol{\Phi}_{\mathrm{i}}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{X}}(0) \end{cases} \tag{7}$$

(6) Displacement in Geometric Coordinates :

$$\underset{(n\times 1)}{\mathbf{X}(t)} = \underset{(n\times l)}{\boldsymbol{\Phi}_{\mathrm{l}}}\,\underset{(l\times 1)}{\mathbf{q}(t)} \tag{8}$$

(7) Elastic Force Response :

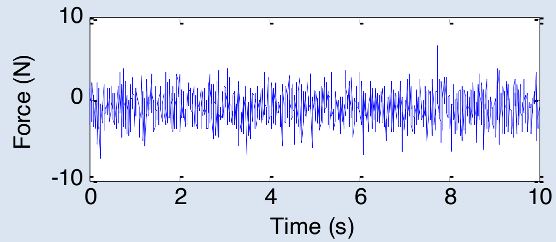
$$\underset{(12\times 1)}{\mathbf{f}_{\mathrm{s}}(t)} = \underset{(12\times 12)(12\times 1)}{\mathbf{K}_{\mathrm{e}}\mathbf{X}_{\mathrm{e}}(t)} = \underset{(12\times 12)(12\times l)(l\times 1)}{\mathbf{K}_{\mathrm{e}}\boldsymbol{\Phi}_{\mathrm{e}}\mathbf{q}(t)} \overset{\text{or}}{=\!=} \underset{(12\times 12)(12\times l)(l\times l)}{\mathbf{M}_{\mathrm{e}}\,\boldsymbol{\Phi}\left[\diagdown \omega^2 \diagdown\right]}\underset{(l\times 1)}{\mathbf{q}(t)} \tag{9}$$

4. Basics of Random Process

Deterministic analysis vs. Random analysis

<table>
<tr><th></th><th>Deterministic analysis</th><th>Random analysis</th></tr>
<tr><td>Input force</td><td>

• Harmonic wave



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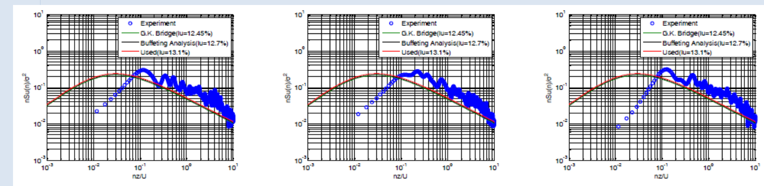
• Spectrum: $S_n(\omega)$



</td></tr>
<tr><td>Response</td><td>

• Response time history: X(t)

$$x(t) = \int_0^t h(t-\tau)f(\tau)d\tau$$

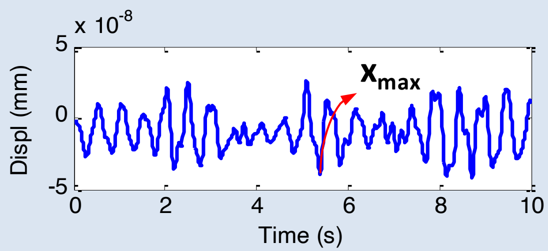
</td><td>

• Response spectrum: $S_x(\omega)$

$$S_x(\omega) = |H(\omega)|^2 S_f(\omega)$$

</td></tr>
<tr><td>Maximum response</td><td>

• Pick the maximum value from the time history



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• Calculate the standard deviation from the response spectrum:

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_x(\omega)d\omega$$

• In general, obtain the maximum response by 3 times of the standard deviation

$$|X(t)|_{max} \simeq 3\sigma_x$$

</td></tr>
<tr><td>Notes</td><td>

• For evaluation of maximum response

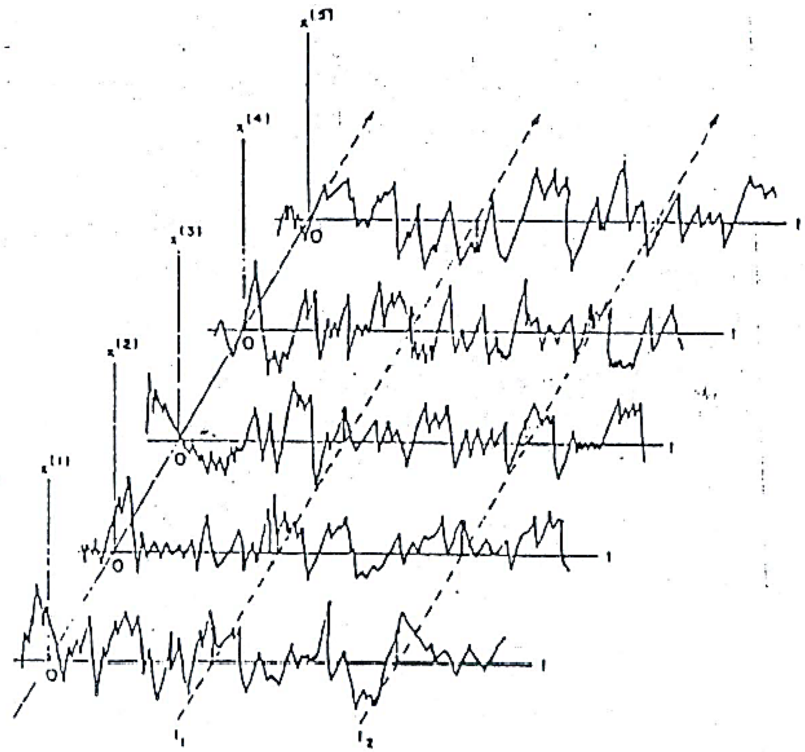
</td><td>

• For fatigue analysis of structure

</td></tr>
</table>

Random variable and random process

- **A random variable $X(\omega)$ is a function defined over a sample space Ω in such a way that a specific value x is assigned to each and every outcome ω of a random phenomenon.**
- **A random process (stochastic process) is an ensemble of parametered random variables with the parameter (or parameters) belong to an indexing set (sets): $X(t), X(s), X(t,s), \ldots,$**

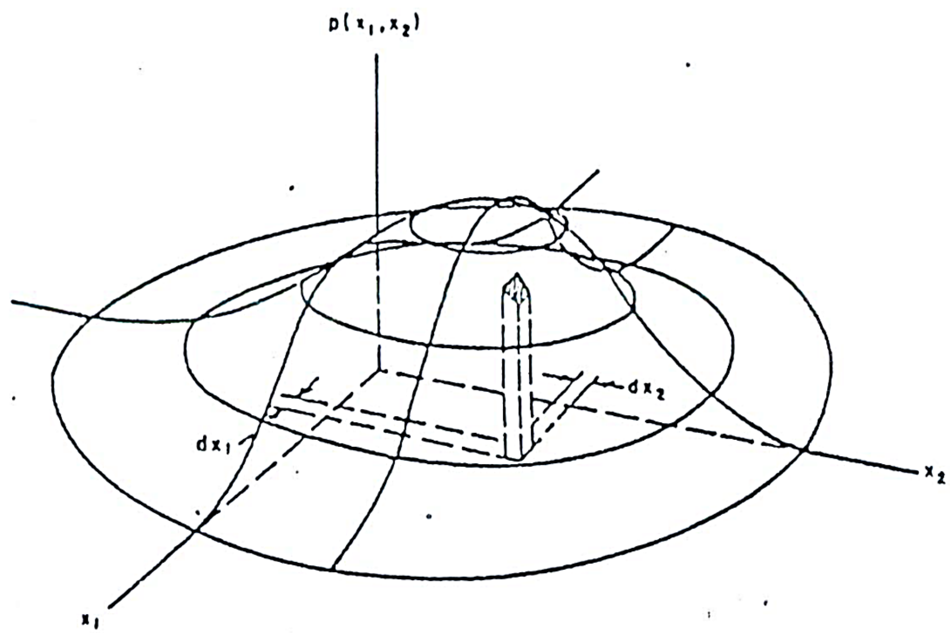


Schematic representation of a random process x(t). Each $x_n(t)$ is a sample function of the ensemble

Joint probability density function

- $f_{X_1X_2X_3,\cdots,X_n}(x_1, x_2, x_3, \cdots, x_n)dx_1dx_2\cdots dx_n$
$$= \text{Prob}[(x_1 < X_1 < x_1 + dx_1) \cap (x_2 < X_2 < x_2 + dx_2) \cap \cdots \cap (x_n < X_n < x_n + dx_n)]$$

- **Since a random process x(t) is an ensemble of random variables, its probabilistic properties can be described by the joint probability density functions:** $f_{X(t_1)X(t_2)X(t_3),\cdots,X(t_n)}(x_1, x_2, x_3, \cdots, x_n)$



Mathematical Expectation

- **Mean value function (Ensemble average)**

$$E[X(t_1)] = \int_{-\infty}^{\infty} x_1 f_{X(t_1)}(x_1)dx_1 = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} {}^{k}x(t_1) = \mu_X(t_1)$$

- **Mean square function**

$$E[X^2(t_1)] = \int_{-\infty}^{\infty} x_1^2 f_{X(t_1)}(x_1)dx_1$$

- **Variance function**

$$E\left[\left(X(t_1) - \mu_X(t_1)\right)^2\right] = \int_{-\infty}^{\infty} \left(x_1 - \mu_X(t_1)\right)^2 f_{X(t_1)}(x_1)dx_1 = \sigma_X^2(t_1)$$

- **Auto correlation function**

$$E[X(t_1)X(t_2)] = \iint_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1,x_2)dx_1 dx_2 = R_{XX}(t_1,t_2)$$

- **Covariance function**

$$E[\{X(t_1) - \mu_{X1}(t_1)\}\{X(t_2) - \mu_{X2}(t_2)\}] = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$$

- **Cross correlation function**

$$E[X(t_1)\ Y(t_2)] = R_{XY}(t_1,t_2)$$

Stationary random process

Strongly stationary (Stationary in the strict sense)

- The probabilistic characteristics is completely independent of a shift of the time origin

$$f_{X(t_1)X(t_2)\cdots X(t_n)}(x_1, x_2, \cdots, x_n) = f_{X(t_1+t_o)X(t_2+t_0)\cdots X(t_n+t_0)}(x_1, x_2, \cdots, x_n)$$

Weakly stationary (Stationary in the wide sense)

- If a random process is stationary up to the 2nd order

1) $f_{X(t_1)}(x) = f_{X(t_1+t_0)}(x) \xRightarrow[(\text{if } t_0=-t_1)]{} f_{X(0)}(x) \rightarrow \text{not a function of t}$

2) $f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(t_1+t_0)X(t_2+t_0)}(x_1, x_2)$
$\xRightarrow[(\text{if } t_0=-t_1)]{} f_{X(0)X(t_2-t_1)}(x_1, x_2) \rightarrow \text{a function of } (\mathrm{t}_2 - \mathrm{t}_1)$

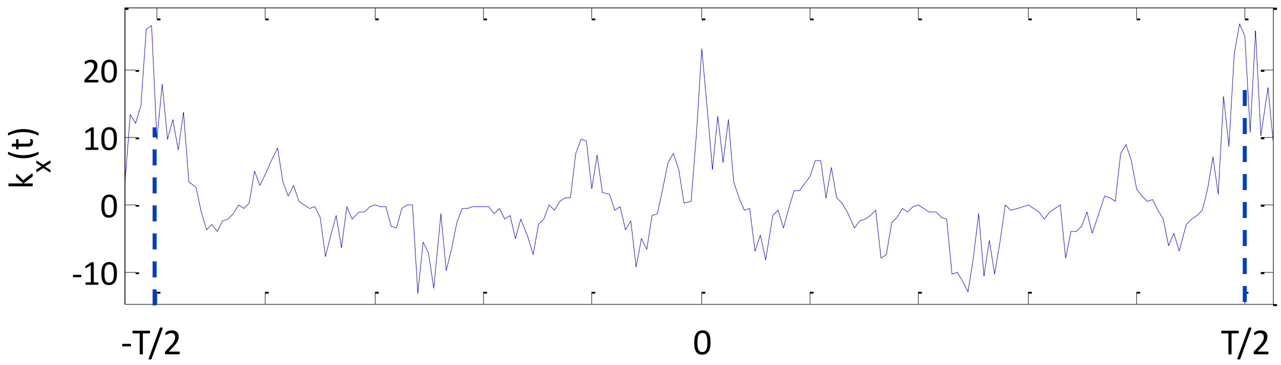
Autocorrelation function of a stationary process

- $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \iint_{-\infty}^{\infty} x_1 x_2 f_{X_1X_2}(x_1, x_2) dx_1 dx_2$
  $= R_{XX}(t_2 - t_1) \xRightarrow[(\tau=t_2-t_1)]{} R_{XX}(\tau)$
- $R_{XX}(t_1, t_2) = R(t_2, t_1) \quad \leftrightarrow \quad R_{XX}(\tau) = R_{XX}(-\tau)$
- $R_{XX}(0) = E[X^2(t)]$

Temporal average and ergodic process

Temporal average

- Statistical average obtained by averaging with respect to time along the ensemble



- $< {}^k x(t) >= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} {}^k x(t) dt$

Temporal mean square

- $< {}^k x(t)^2 >= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} {}^k x(t)^2 dt$

Temporal auto-correlation

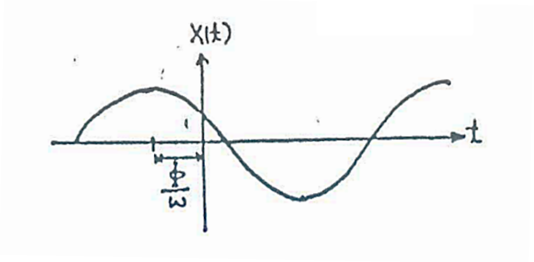
- $< {}^k x(t)\, {}^k x(t+\tau) >= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} {}^k x(t)\, {}^k x(t+\tau) dt$

Temporal average and ergodic process

Ergodic process

- A special case of a stationary process in which all the temporal averages are equal to the corresponding ensemble averages.

(e.g.) Consider a random variable X(t)



$$X(t) = A\cos(\omega t + \phi)$$
$$A, \omega = \text{positive constants}$$
$$\phi = \text{a random variable distributed uniformly between } \pi \ and \ -\pi$$

$$E[X(t)] = E[A\cos(\omega t + \phi)] = 0$$
$$E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} A\cos(\omega t_1 + \varphi)A\cos(\omega t_2 + \varphi)f_\phi(\varphi)d\varphi$$
$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\{\cos[\omega(t_1 + t_2) + 2\varphi)] + \cos[\omega(t_1 - t_2)]\}d\varphi$$
$$= \frac{A^2}{2}\cos[\omega(t_1 - t_2)] = \frac{A^2}{2}\cos(\omega\tau)$$

Temporal average and ergodic process

$$< {}^k x(t) > = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A\cos(\omega t + {}^k\varphi) dt = 0$$

$$< {}^k x(t)\, {}^k x(t+\tau) >$$

$$= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \cos(\omega t + {}^k\varphi) \cos(\omega(t+\tau) + {}^k\varphi)\, dt$$

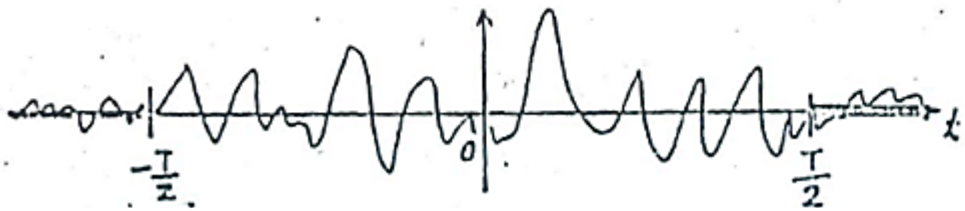
$$= \lim_{T\to\infty} \frac{A^2}{2T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [\cos(2\omega t + \omega\tau + 2\, {}^k\varphi) + \cos(\omega\tau)] dt$$

$$= \frac{A^2}{2} \cos(\omega\tau)$$

- X(t) is ergodic up to the 2^{nd} moment

Temporal average and ergodic process

- **Consider the kth sample function taken from a stationary random process X(t)**



- ${}^{k}x_T(t)$**can be represented by using Fourier series:**

$$ {}^{k}x_T(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos(m\Delta\omega t) + b_m \sin(m + \Delta\omega t)) $$

where $\Delta\omega = 2\pi/T$

$$ \mathrm{a_m} = \frac{2}{\mathrm{T}} \int_{-T/2}^{T/2} {}^{k}x_T(t) \cos(m\Delta\omega t)\, dt \;\; b_\mathrm{m} = \frac{2}{\mathrm{T}} \int_{-T/2}^{T/2} {}^{k}x_T(t) \sin(m\Delta\omega t)\, dt $$

- **Or by using complex Fourier series:**

$$ {}^{k}x_T(t) = \sum_{-\infty}^{\infty} c_m e^{im\Delta\omega t} $$

where $\Delta\omega = 2\pi/T,\ \mathrm{c_m} = (a_m - ib_m)/2, c_{-\mathrm{m}} = (\mathrm{a_m} + \mathrm{ib_m})/2 = \mathrm{c_m^*}$

Power spectral density (PSD) of a stationary process

- **Temporal mean square of ${}^{k}X_T(t)$ over the interval –T/2 < t < T/2**

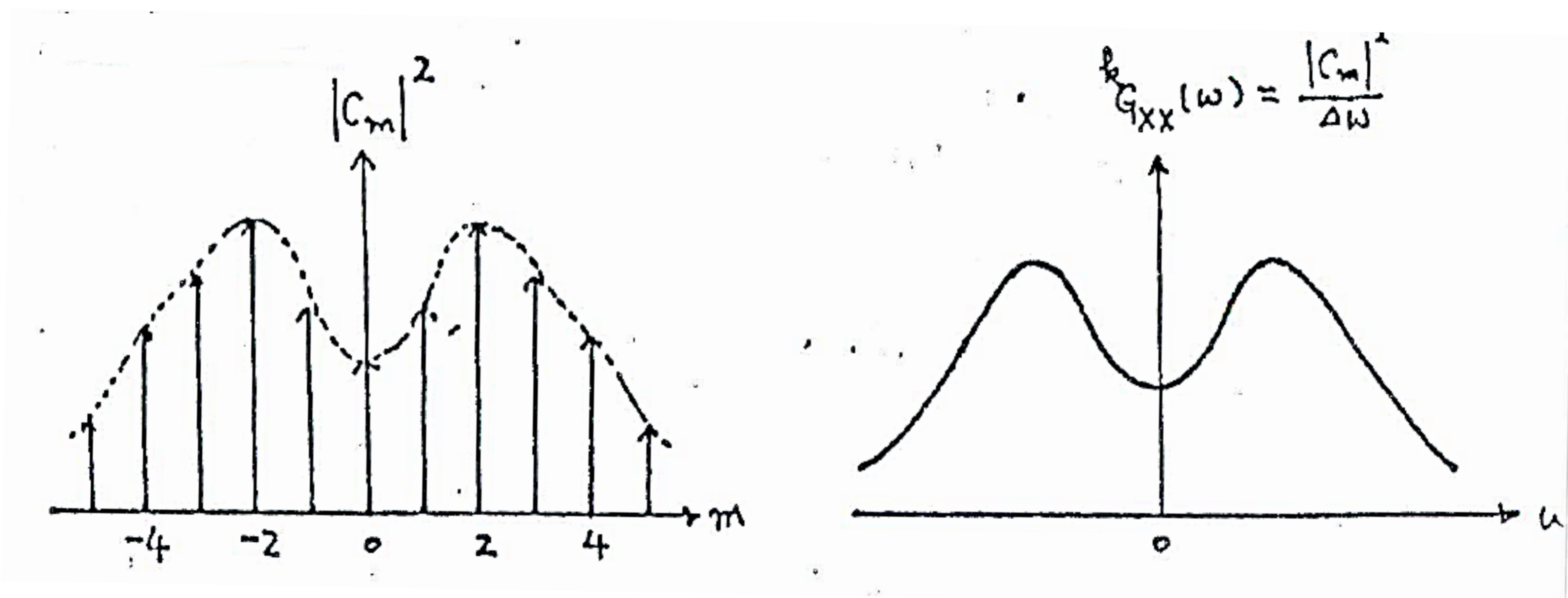
$$
\begin{aligned}
< {}^{k}x_T(t)^2 > &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} {}^{k}x_T(t)^2 dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_m c_n e^{i(m+n)\Delta\omega t}\, dt \\
&= \lim_{T\to\infty} \sum_{m=-\infty}^{\infty} |c_m|^2 \qquad (\because \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(m+n)\Delta\omega t} dt \ = T\cdot\delta((m+n)\Delta\omega) \\
&= \lim_{T\to\infty} \sum_{m=-\infty}^{\infty} \frac{|c_m|^2}{\Delta\omega}\Delta\omega \\
&= \int_{-\infty}^{\infty} {}^{k}G_{XX}(\omega) d\omega
\end{aligned}
$$

where ${}^{k}G_{XX}(m\Delta\omega) = |c_m|^2/\Delta\omega \leftarrow$ <u>Power Spectral Density (PSD)</u> of ${}^{k}x_T(t)$

Power spectral density (PSD) of a stationary process

- ${}^{k}G_{XX}(\omega)d\omega$ **represents the amount of the <u>mean square (average power) value</u> contained in the frequency band lying between ω-dω/2 and ω+dω/2**

- ${}^{k}G_{XX}(\omega)$ **represents <u>the distribution of the average power along the frequency</u>**



Relationship between PSD and Autocorrelation Function

$$^{k}G_{XX}(\omega) = \lim_{\Delta\omega\to\infty} \frac{1}{\Delta\omega}|C_m|^2 = \lim_{\Delta\omega\to\infty} \frac{1}{2\pi/T}|C_m|^2$$

$$^{k}G_{XX}(\omega) = \frac{1}{2\pi T}\left|\int_{-\frac{T}{2}}^{\frac{T}{2}} {}^{k}x_T(t)e^{-i\omega t}dt\right|^2$$

$$= \frac{1}{2\pi T}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} {}^{k}x_T(t)e^{-i\omega t}dt\right]\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} {}^{k}x_T(t')e^{-i\omega t'}dt'\right]$$

$$= \frac{1}{2\pi T}\iint_D {}^{k}x_T(t)\,{}^{k}x_T(t')e^{-i\omega(t-t')}dtdt' \qquad (\text{Let } t - t' = \tau)$$

$$= \frac{1}{2\pi T}\iint_{D'} {}^{k}x_T(t)\,{}^{k}x_T(t-\tau)e^{-i\omega\tau}dtd\tau$$

$$= \frac{1}{2\pi}\int_{-T}^{T} {}^{k}\emptyset(\tau)e^{-i\omega\tau}d\tau \qquad (\text{as } T \to \infty)$$

$$\left.\begin{aligned} {}^{k}G_{XX}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} {}^{k}\emptyset(\tau)e^{-i\omega\tau}d\tau \\ {}^{k}\emptyset(\tau) &= \int_{-\infty}^{\infty} {}^{k}G_{XX}(\omega)e^{i\omega\tau}d\tau \end{aligned}\right\} \text{Wiener} - \text{Khintchine Transform}$$

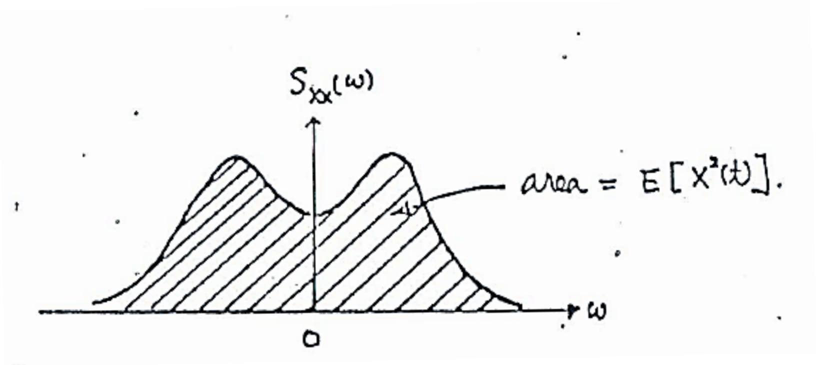
Spectral Density Function of Ergodic Process

$$S_{XX}(\omega) = E[G_{XX}(\omega)] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} {}^{k}G_{XX}(\omega)$$

▶ **If X(t) is an ergodic process**

- $$\left.\begin{aligned} S_{XX}(\omega) &= {}^{k}G_{XX}(\omega) \\ R_{XX}(\tau) &= {}^{k}\emptyset_{XX}(\tau) \end{aligned}\right\rangle \textit{for each } k$$

  and



- $S_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$
- $R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega$
- $R_{XX}(0) = E[X^2(t)] = \int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = 2\int_{0}^{\infty} S_{XX}(\omega)d\omega$

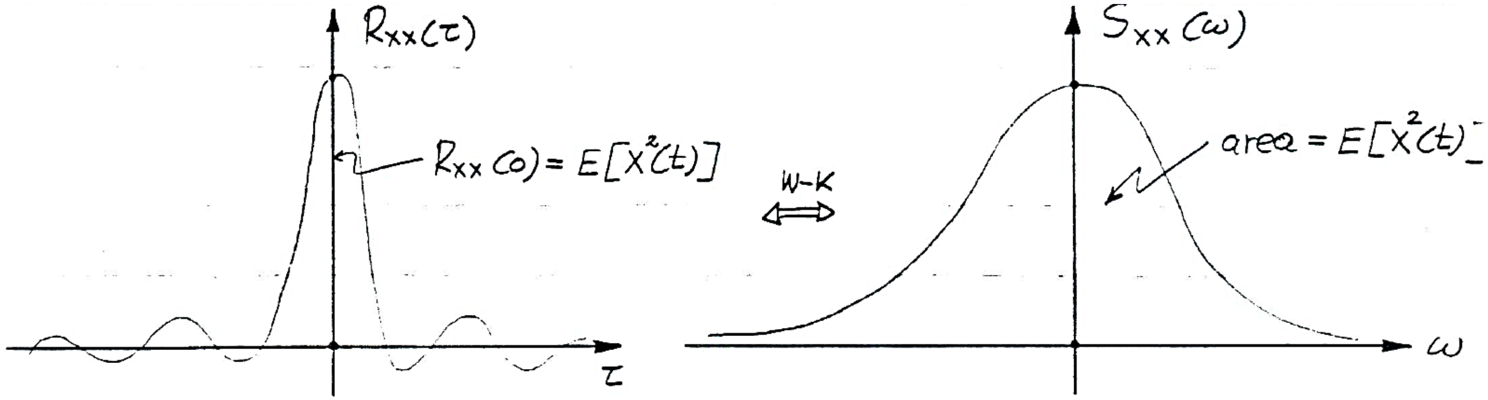
Spectral Density Function of Ergodic Process

Properties of $S_{XX}(\omega)$& $R_{XX}(\tau)$

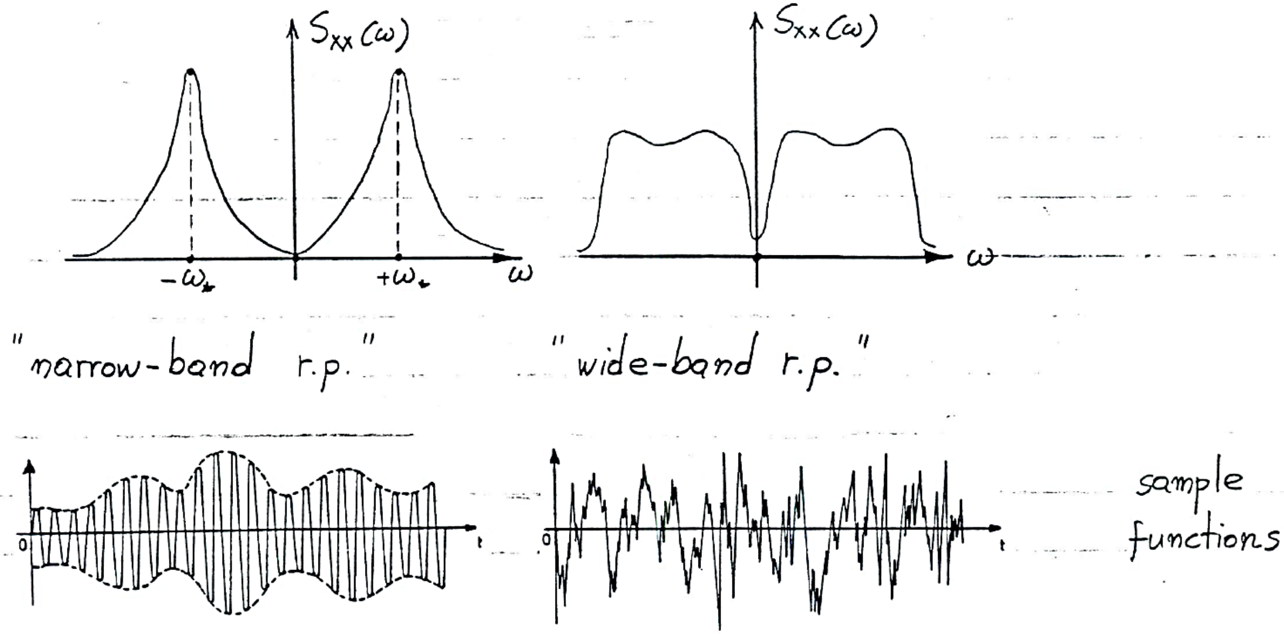
- $E[X^2] = R_{XX}(o) = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega \xRightarrow[\text{if assuming zero mean}]{} \sigma_x{}^2$
- If X(t) is real, $\begin{cases} S(\omega) = S(-\omega) \\ S(\omega) \geq 0 \qquad for\ all\ \omega \end{cases}$
- If X(t) is not periodic, $\begin{cases} R(\tau) \to 0, as\ \tau \to \infty \\ S(\omega) \text{is finite} \end{cases}$
- If X(t) has a periodic component with frequency ω_m,

  $\begin{cases} R(\tau) \text{ is a also periodic with frequency } \omega_m \\ S(\omega) \text{ has infinitely high peak of zero width(spike)} at\ \omega = \omega_m \end{cases}$
- If X(t) has a nonzero mean, S(ω)has a spike at ω=0

Spectral Density Function of Ergodic Process



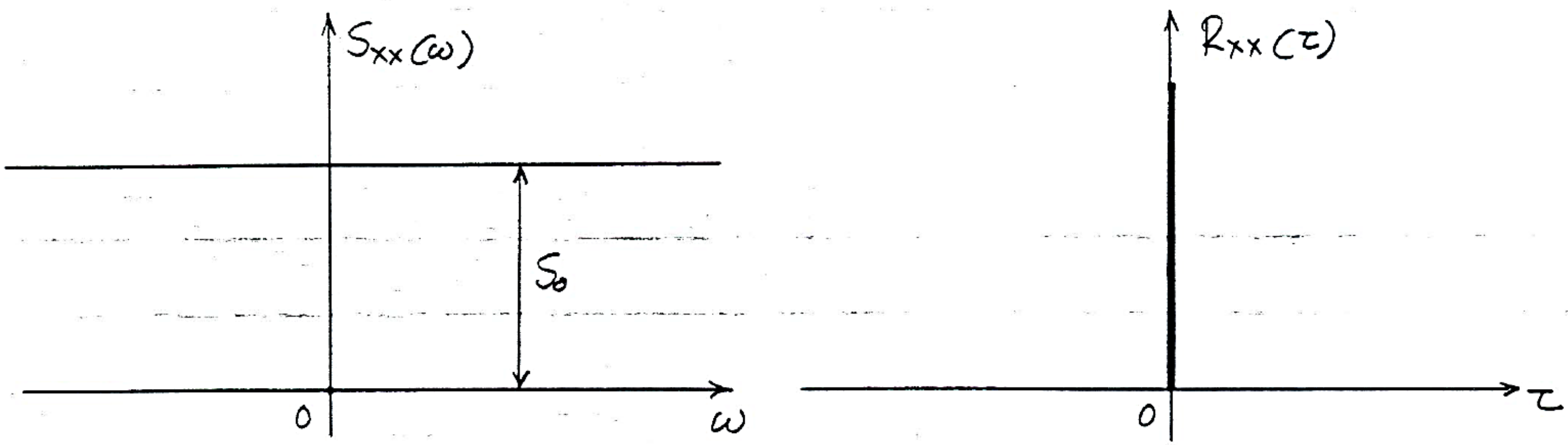
▶ Two Types of Stationary Random Process



Some commonly Used Stationary Random Process

Idealized White-Noise

$$\begin{cases} S_{XX}(\omega) = S_0 \quad \text{for} \quad -\infty < \omega < +\infty \\ R_{XX}(\tau) = \displaystyle\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\omega\tau}\, d\omega = 2\pi S_0 \cdot \delta(\tau) \end{cases}$$



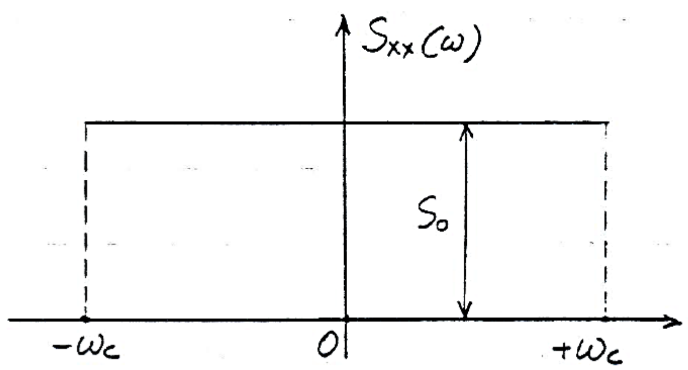
- Since $R_{XX}(\tau) = 0$ for $\tau \neq 0$, $X(t)$ and $X(t+\tau)$ are uncorrelated for any non-zero value of τ.
- Idealized white-noise is physically unrealizable, since its mean square is infinite: $E[X^2(t)] = R_{XX}(0) \to \infty$

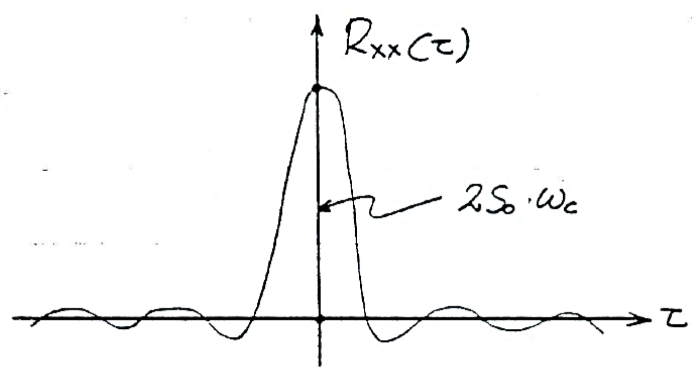
Band limited white-noise

$$S_{xx}(\omega) = \begin{cases} S_0 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases} \qquad \text{where } \omega_c = \text{cutoff frequency}$$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(\omega) \cdot e^{i\omega\tau} d\omega = \int_{-\omega_c}^{\omega_c} S_0 \cdot e^{i\omega\tau} d\omega = S_0 \int_{-\omega_c}^{\omega_c} \cos\omega\tau \, d\omega \Rightarrow$$

$$\Rightarrow R_{xx}(\tau) = 2 \cdot S_0 \cdot \frac{\sin\omega_c \cdot \tau}{\tau}$$





- Mean Square : $E[x^2(t)] = R_{xx}(0) = \int_{-\omega_c}^{\omega_c} S_0 \, d\omega = 2S_0 \cdot \omega_c$

PSD and Autocorrelation Function for Derivatives

- $\mathrm{E}\left[\frac{\mathrm{d^n X(t)}}{dt^n}\right] = \frac{\mathrm{d^n}}{\mathrm{dt^n}}\mathrm{E[X(t)]}$

- $\mathrm{R}_{\dot{X}\dot{X}}(t-s) = E\left[\dot{X}(s)\dot{X}(t)\right] = \frac{\partial^2}{\partial s \partial t}E[X(s)X(t)]$

$$= \frac{\partial^2}{\partial s \partial t}R_{XX}(t-s) = -\frac{\partial^2}{\partial^2 \tau}R_{XX}(\tau)$$

$$\therefore \mathrm{R}_{\dot{X}\dot{X}}(\tau) = -\frac{\partial^2}{\partial^2 \tau}R_{XX}(\tau)$$

- $\mathrm{R}_{X\dot{X}}(t-s) = E\left[X(s)\dot{X}(t)\right] = \frac{\partial}{\partial t}E[X(s)X(t)]$

$$= \frac{\partial}{\partial t}R_{XX}(t-s) = \frac{\partial}{\partial \tau}R_{XX}(\tau)$$

$$\therefore \mathrm{R}_{X\dot{X}}(\tau) = \frac{\partial}{\partial \tau}R_{XX}(\tau)$$

PSD and Autocorrelation Function for Derivatives

- $$\mathrm{R}_{\dot{X}\dot{X}}(\tau) = -\frac{\partial^2}{\partial^2\tau} R_{XX}(\tau) = -\frac{\partial^2}{\partial^2\tau}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega$$

  $$= \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)e^{i\omega\tau}d\omega = \int_{-\infty}^{\infty} S_{\dot{X}\dot{X}}(\omega)e^{i\omega\tau}d\omega$$

  $$\therefore \mathrm{S}_{\dot{X}\dot{X}}(\omega) = \omega^2 \mathrm{S}_{XX}(\omega)$$

  Similarly,

  $$\therefore \mathrm{S}_{\ddot{X}\ddot{X}}(\omega) = \omega^4 \mathrm{S}_{XX}(\omega)$$

- $$\mathrm{R}_{X\dot{X}}(\tau) = \frac{\partial}{\partial\tau}\mathrm{R}_{XX} = \int_{-\infty}^{\infty} i\omega S_{XX}(\omega)e^{i\omega\tau}d\omega \underset{\tau=0}{\Longrightarrow} 0$$

  $\therefore$ X & $\dot{\mathrm{X}}$ are uncorrelated.

Cross-correlation Function and Cross-spectral Density Function

- $R_{XY}(\tau) = E[X(t)Y(t+\tau)] = E[X(t-\tau)Y(t)]$
  $= E[Y(t)X(t-\tau)] = R_{YX}(-\tau)$

  $\therefore R_{XY}(\tau) = R_{YX}(-\tau)$

- $S_{XY}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau)e^{-i\omega\tau}d\tau$
- $R_{XY}(\omega) = \int_{-\infty}^{\infty} S_{XY}(\omega)e^{i\omega\tau}d\omega$
- $S_{XY}(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XY}(\tau)e^{i\omega\tau}d\tau = S_{XY}(\omega)^* \rightarrow complex\ conjugate$
- $E[X(t)Y(t)] = R_{XY}(0) = \int_{-\infty}^{\infty} S_{XY}(\omega)d\omega = 2\int_{0}^{\infty} Re[S_{XY}(\omega)]d\omega$

5. Random Vibration of Linear Structures

- $m\ddot{X} + C\dot{X} + KX = f(t)$

$$X(t) = \int_{-\infty}^{t} h(t-\tau)f(\tau)d\tau = \int_{0}^{\infty} h(\tau)f(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)f(t-\tau)d\tau$$

where $\mathrm{h(t)} =$ Impulse response function

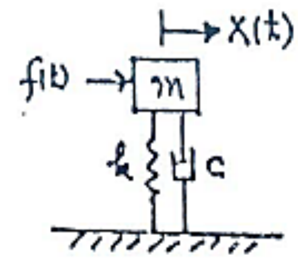
$=$ Response due to $\mathrm{f(t)} = \delta(\mathrm{t})$

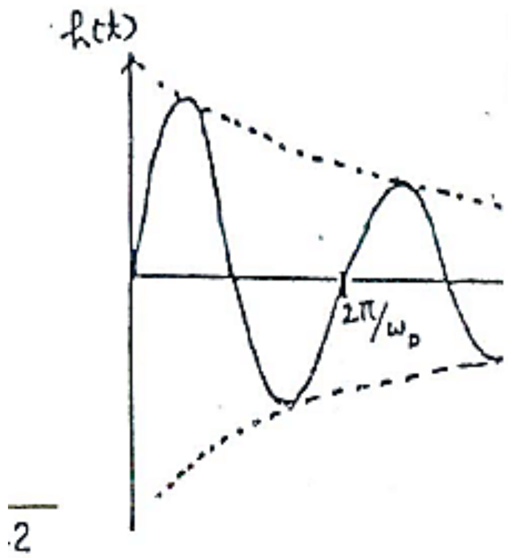
$$= \frac{1}{\mathrm{m}\omega_{\mathrm{D}}} e^{-\xi\omega t} \sin \omega_D t$$

$$\omega = \text{natural frequency} = \sqrt{\frac{\mathrm{k}}{\mathrm{m}}}$$

$$\xi = \text{damping ratio} = \frac{\mathrm{c}}{2\mathrm{m}\omega}$$

$$\omega_{\mathrm{D}} = \text{damped natural frequency} = \omega\sqrt{1-\xi^2}$$





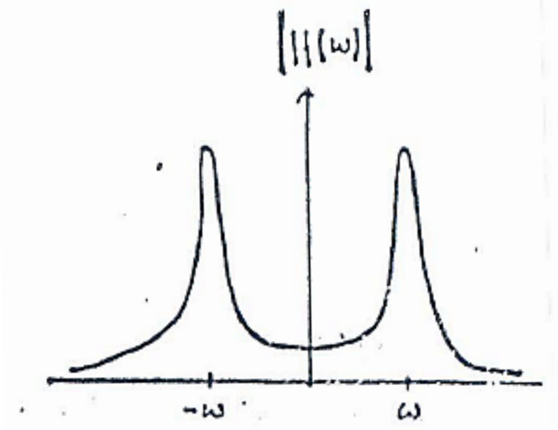
- **Apply the Fourier transform to the M-C-K equation:**

$$(-m\bar{\omega}^2 + i\bar{\omega}C + K)\bar{X}(\bar{\omega}) = \bar{F}(\bar{\omega})$$

$$\bar{X}(\bar{\omega}) = \frac{1}{-m\bar{\omega}^2 + i\bar{\omega}C + K}\bar{F}(\bar{\omega}) = H(\bar{\omega})\bar{F}(\bar{\omega})$$

where $\bar{X}(\bar{\omega}) = \mathcal{F}\{X(t)\} = \int_{-\infty}^{\infty} X(t)e^{-i\bar{\omega}t}dt$

$$\bar{F}(\bar{\omega}) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-i\bar{\omega}t}dt$$



$$\bar{H}(\bar{\omega}) = \frac{1}{-m\bar{\omega}^2 + i\bar{\omega}C + K}$$

$$= \text{Complex Frequency Response Function}$$

$$= \mathcal{F}\{h(t)\}$$

Mean value of the response

$$\mathrm{E[X(t)]} = E\left[\int_0^t h(\tau)f(h-\tau)d\tau\right] = E\left[\int_{-\infty}^{\infty} h(\tau)f(t-\tau)d\tau\right]$$

$$= E[f(t)]\int_{-\infty}^{\infty} h(\tau)d\tau = E[f(t)]H(0)$$

$$= \frac{1}{k}E[f(t)]$$

$\leftarrow$ mean value of the response can be obtained statically

c.f)

$$\overline{\mathrm{H}}(\overline{\omega}) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-i\overline{\omega}t}dt$$

$$\therefore\ \overline{\mathrm{H}}(0) = \int_{-\infty}^{\infty} h(t)dt = \left.\frac{1}{-\mathrm{m}\omega^2 + i\omega C + K}\right|_{\omega=0} = \frac{1}{k}$$

Auto-correlation of the response

$$
\begin{aligned}
\mathrm{R_{XX}}(\tau) &= E[X(t)X(t+\tau)] \\
&= E\left[\int_0^\infty h(\tau_1)f(t-\tau_1)d\tau_1 \int_0^\infty h(\tau_2)f(t+\tau-\tau_2)d\tau_2\right] \\
&= \int_0^\infty \int_0^\infty E[f(t-\tau_1)f(t+\tau-\tau_2)]h(\tau_1)h(\tau_2)d\tau_1 d\tau_2 \\
&= \int_0^\infty \int_0^\infty R_{ff}(\tau+\tau_1-\tau_2)h(\tau_1)h(\tau_2)d\tau_1 d\tau_2
\end{aligned}
$$

Power spectral density of the response

$$S_{XX}(\bar{\omega}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\bar{\omega}\tau}d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^{\infty}\int_0^{\infty} R_{ff}(\tau+\tau_1-\tau_2)h(\tau_1)h(\tau_2)d\tau_1 d\tau_2\right)e^{-i\bar{\omega}\tau}d\tau$$

$$= \left(\int_0^{\infty} h(\tau_1)e^{i\bar{\omega}\tau_1}d\tau_1\right)\left(\int_0^{\infty} h(\tau_2)e^{-i\bar{\omega}\tau_2}d\tau_2\right)$$

$$\times\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\tau+\tau_1-\tau_2)e^{-i\bar{\omega}(\tau+\tau_1-\tau_2)}d\tau\right)$$

$$= \mathrm{H}(\bar{\omega})^{*}H(\bar{\omega})S_{ff}(\bar{\omega}) = |H(\bar{\omega})|^2 S_{ff}(\bar{\omega})$$

$$\therefore S_{XX}(\bar{\omega}) = |H(\bar{\omega})|^2 S_{ff}(\bar{\omega})$$

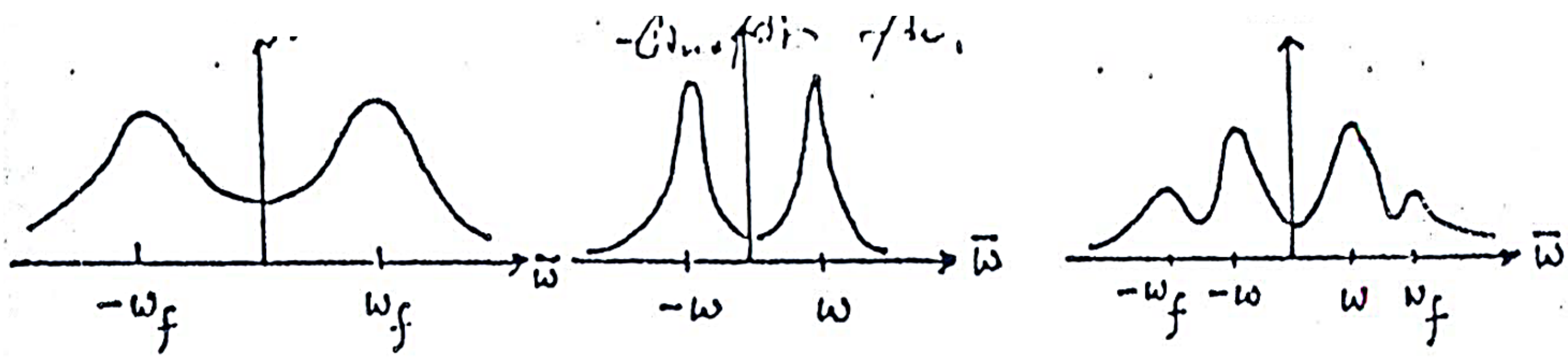
$$\therefore S_{\dot{X}\dot{X}}(\bar{\omega}) = \bar{\omega}^2 S_{XX}(\bar{\omega}) = \bar{\omega}^2|H(\bar{\omega})|^2 S_{ff}(\bar{\omega})$$

Mean square response

$$E[X(t)^2] = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\bar{\omega})d\bar{\omega} = \int_{-\infty}^{\infty} |H(\bar{\omega})|^2 S_{ff}(\bar{\omega})d\bar{\omega}$$

$f(t) \longrightarrow$ Structure $\longrightarrow X(t)$

$$S_{ff}(\bar{\omega}) \quad \times \quad |H(\bar{\omega})|^2 \quad = \quad S_{XX}(\bar{\omega})$$



Response of Multi-Degree-of-Freedom Structure

$$M\{\ddot{X}\} + C\{\dot{X}\} + K\{X\} = \{f\}$$

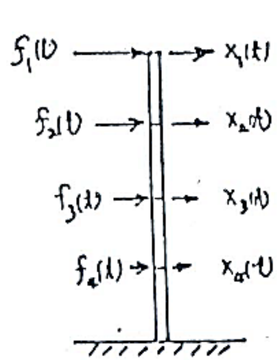
- If $f_j(t) = \delta(t)$ & $f_l(t) = 0 \quad (l \neq j)$

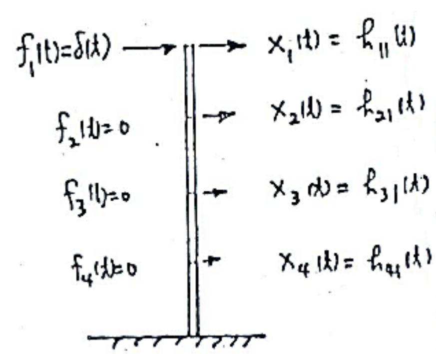
  $X_k(t) = h_{kj}(t)$ → Impulse response function at node k due to excitation δ(t) at node j

- For a set of general loading $f_1(t), f_2(t), \ldots, f_n(t)$

$$X_k(t) = \sum_{j=1}^{m} \int_{-\infty}^{\infty} h_{kj}(t-\tau) f_j(\tau) d\tau \quad \{x(t)\} = \int_{-\infty}^{\infty} [h(t-\tau)]\{f(\tau)\} d\tau$$

where $[h(t-\tau)] = \begin{bmatrix} h_{11}(t-\tau) & \cdots & h_{1n}(t-\tau) \\ \vdots & \ddots & \vdots \\ h_{n1}(t-\tau) & \cdots & h_{nn}(t-\tau) \end{bmatrix}$ → matrix of impulse response function





Response of Multi-Degree-of-Freedom Structure

- If $f_j(t) = e^{i\bar{\omega}t}$, $f_l(t) = 0 \;\; (l \neq j)$

  $X_k(t) = H_{kj}(\bar{\omega})e^{i\bar{\omega}t}$

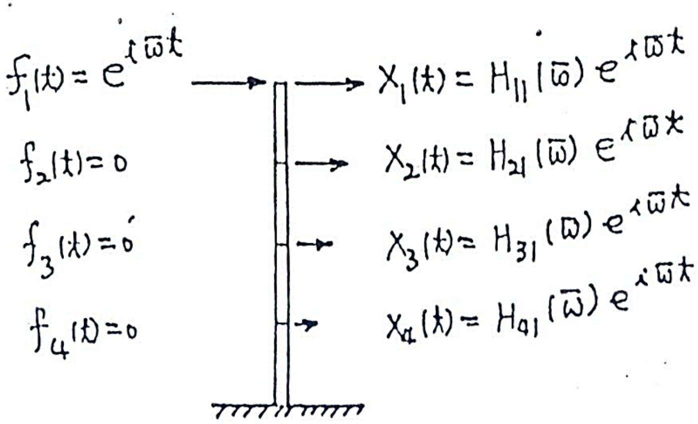
  where $H_{kj}(\bar{\omega}) = \mathcal{F}\{h_{kj}(t)\} = \int_{-\infty}^{\infty} h_{kj}(t)e^{-i\bar{\omega}t}dt$

- For a set of general loadings $\{f(\bar{\omega})\} = \mathcal{F}\{f(t)\}$

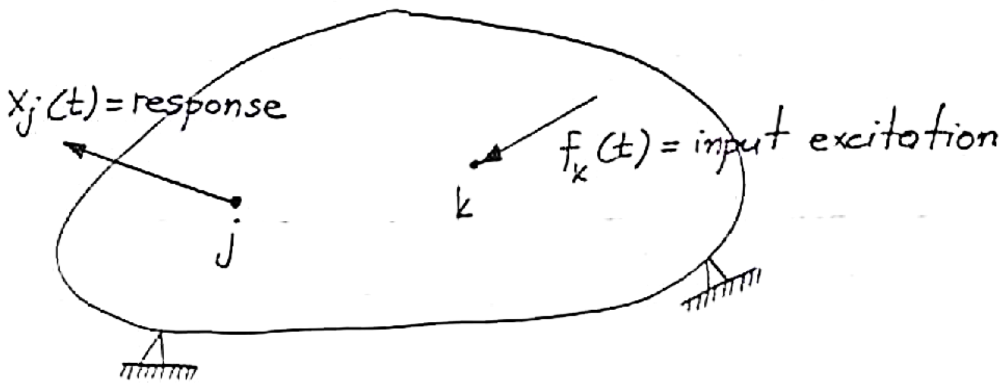
  $\{X(\bar{\omega})\} = [H(\bar{\omega})]\{f(\bar{\omega})\}$

  where $[H(\bar{\omega})] = \begin{bmatrix} H_{11}(\bar{\omega}) & \cdots & H_{1n}(\bar{\omega}) \\ \vdots & \ddots & \vdots \\ H_{n1}(\bar{\omega}) & \cdots & H_{nn}(\bar{\omega}) \end{bmatrix}$ → Matrix of frequency response function

  $\{X(\bar{\omega})\} = \mathcal{F}\{X(t)\}$



Physical Meaning of $H_{jk}(\omega)$



- **Neglecting the effect of initial conditions, we assume that the response has reached steady-state.**
- **Excitation at point k :** $f_k(t) = A \cdot e^{i\omega t}$;   A=constant
- **Response at point j :** $X_j(t) = A \cdot H_{jk}(\omega) \cdot e^{i\omega t}$

$$H_{jk}(\omega) = \frac{[X_j(t)]_{steady-state}}{[f_k(t)]_{sinusoidal}}$$

Response of Multi-Degree-of-Freedom Structure

- Using modal superposition, $\{X\} = \mathbf{\Phi}\{q\}$

  the equation of motion can be rewritten as

$$\mathbf{M}\{\ddot{X}\} + \mathbf{C}\{\dot{X}\} + \mathbf{K}\{X\} = \{f\}$$

$$\boldsymbol{\phi}^T\mathbf{M}\boldsymbol{\phi}\{\ddot{q}\} + \boldsymbol{\phi}^T\mathbf{C}\boldsymbol{\phi}\{\dot{q}\} + \boldsymbol{\phi}^T\mathbf{K}\boldsymbol{\phi}\{q\} = \boldsymbol{\phi}^T\{f\}$$

$$[\ddots\ \mu\ \ddots]\{\ddot{q}\} + [\ddots\ c\ \ddots]\{\dot{q}\} + [\ddots\ k\ \ddots]\{q\} = \{Q\}$$

$$\mu_j\ddot{q}_j + 2\xi_j\omega_j\mu_j\dot{q}_j + \omega_j^2\mu_j q_j = Q_j$$

where, $[\ddots\ \mu\ \ddots]$, $[\ddots\ k\ \ddots]$; generalized mass, stiffness matrices (diagonal)

$[\ddots\ c\ \ddots]$; generalized damping matrix (assumed to be diagonal)

$\{Q\}$; $\phi^T\{f\}$ = generalized force

ξ_j ; modal damping ratio for j^th mode

ω_j ; natural frequency for the j^th mode

$$\mathbf{M}\{\ddot{X}\} + \mathbf{C}\{\dot{X}\} + \mathbf{K}\{X\} = \{f\}$$

$$\begin{aligned}
S_{X_k X_j}(\bar{\omega}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{X_k X_j}(\tau)e^{-i\bar{\omega}\tau}d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} E\left[X_k(t)X_j(t+\tau)\right]e^{-i\bar{\omega}\tau}d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} E\left[\left(\int_{-\infty}^{\infty}\sum_{l=1}^{n} h_{kl}(\tau_1)f_l(t-\tau_1)d\tau_1\right)\left(\int_{-\infty}^{\infty}\sum_{m=1}^{n} h_{jm}(\tau_2)f_m(t+\tau-\tau_2)d\tau_2\right)\right]e^{-i\bar{\omega}\tau}d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{l=1}^{n}\sum_{m=1}^{n}\left(\int_{-\infty}^{\infty} h_{kl}(\tau_1)e^{i\bar{\omega}\tau_1}d\tau_1\right)\left(\int_{-\infty}^{\infty} h_{jm}(\tau_2)e^{-i\bar{\omega}\tau_2}d\tau_2\right) \\
&\qquad \times\left(E[f_l(t-\tau_1)f_m(t+\tau-\tau_2)]\right)e^{-i\bar{\omega}(\tau+\tau_1-\tau_2)}d\tau \\
&= \sum_{l=1}^{n}\sum_{m=1}^{n} H_{kl}^{*}(\bar{\omega})H_{jm}(\bar{\omega})\left\{\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} R_{f_l f_m}(\tau+\tau_1-\tau_2)e^{-i\bar{\omega}(\tau+\tau_1-\tau_2)}d\tau}_{S_{f_l f_m}(\bar{\omega})}\right\} \\
&= \sum_{l=1}^{n}\sum_{m=1}^{n} H_{kl}^{*}(\bar{\omega})S_{f_l f_m}(\bar{\omega})H_{jm}(\bar{\omega}) \\
&= [\mathbf{H}^{*}(\bar{\omega})\mathbf{S_{ff}}(\bar{\omega})\mathbf{H}(\bar{\omega})^{T}]_{kj}
\end{aligned}$$

In matrix form

$$[S_{XX}(\bar{\omega})] = \mathbf{H}^{*}(\bar{\omega})\mathbf{S_{ff}}(\bar{\omega})\mathbf{H}(\bar{\omega})^{T}$$

Response of MDOF Structure to Random Loading

- Then we can obtain PSD for the generalized coordinate q_j, due to the random force Q_j:

$$S_{q_j q_j}(\bar{\omega}) = \left|H_j(\bar{\omega})\right|^2 S_{Q_j Q_j}(\bar{\omega})$$

where $H_j(\bar{\omega}) = \dfrac{1}{\mu_j\left(-\bar{\omega}^2 + 2\xi_j\omega_j(i\bar{\omega}) + \omega_j^2\right)}$

= Frequency response function for the j^{th} mode

- In matrix form,

$$\left[S_{qq}(\bar{\omega})\right] = [\ddots\ H(\bar{\omega})\ \ddots]^* \left[S_{QQ}(\bar{\omega})\right] [\ddots\ H(\bar{\omega})\ \ddots]$$

where $\left[S_{QQ}(\bar{\omega})\right] = \boldsymbol{\phi}^T [S_{ff}(\bar{\omega})] \boldsymbol{\phi}$

$$[S_{ff}(\bar{\omega})] = \begin{bmatrix} S_{f_1 f_1}(\bar{\omega}) & \cdots & S_{f_1 f_n}(\bar{\omega}) \\ \vdots & \ddots & \vdots \\ S_{f_n f_1}(\bar{\omega}) & \cdots & S_{f_n f_n}(\bar{\omega}) \end{bmatrix}$$

- Therefore,

$$
\begin{aligned}
[S_{XX}(\overline{\omega})] &= \boldsymbol{\phi}\mathbf{S_{qq}}(\overline{\omega})\boldsymbol{\phi}^T \\
&= \boldsymbol{\phi}[\ddots\ H(\overline{\omega})\ \ddots]^*\left[S_{QQ}(\overline{\omega})\right][\ddots\ H(\overline{\omega})\ \ddots]\boldsymbol{\phi}^T \\
&= \boldsymbol{\phi}[\ddots\ H(\overline{\omega})\ \ddots]^*\boldsymbol{\phi}^T[S_{ff}(\overline{\omega})]\boldsymbol{\phi}[\ddots\ H(\overline{\omega})\ \ddots]\boldsymbol{\phi}^T
\end{aligned}
$$

- Comparing the above result from the previous one, we can obtain

$$[H(\overline{\omega})] = \boldsymbol{\phi}[\ddots\ H(\overline{\omega})\ \ddots]\boldsymbol{\phi}^T$$

Response of MDOF Structure to Random Loading

- **Procedures to compute $S_{XX}(\bar{\omega})$ and $\sigma^2_{X_k}$**

1) Obtain mode shape and natural frequencies; $\{\phi_j\},\ \omega_j;\ \ j = 1,2,\dots,l,$
2) Obtain PSD for the generalized force

$$\underset{(l\times l)}{[S_{QQ}(\bar{\omega})]} = \boldsymbol{\phi}^T[S_{ff}(\bar{\omega})]\boldsymbol{\phi}$$

3) Obtain PSD for the generalized modal coordinates

$$[S_{qq}(\bar{\omega})] = [\ddots\ H(\bar{\omega})\ \ddots]^*[S_{QQ}(\bar{\omega})][\ddots\ H(\bar{\omega})\ \ddots]$$

4) Obtain PSD for the structural displacement

$$[S_{XX}(\bar{\omega})] = \boldsymbol{\phi}\mathbf{S_{qq}}(\bar{\omega})\boldsymbol{\phi}^T$$

5) Obtain PSD for member force; $\{S(t)\} = [D]\{X(t)\}$

$$[S_{SS}(\bar{\omega})] = [D]\mathbf{S}_{\boldsymbol{XX}}(\bar{\omega})[D]^T$$

6) Compute $\sigma^2_{X_k}$ & $\sigma^2_{S_k}$

$$\sigma^2_{X_k} = \int_{-\infty}^{\infty} S_{X_kX_k}(\bar{\omega})d\bar{\omega}$$

$$\sigma^2_{S_k} = \int_{-\infty}^{\infty} S_{S_kS_k}(\bar{\omega})d\bar{\omega}$$

THANK YOU

for your attention!