Piezoelectric Constitutive Equations

❖ Constitutive Equations

$$S_{ij} = s_{ijkl}{}^{E}{}_{kl} T_{kl} + d_{mij} E_m + \alpha_{ij}{}^{E} \theta$$

$$D_n = d_{nkl} T_{kl} + \varepsilon_{nm}{}^{T} E_m + p_n{}^{T} \theta$$

Pyroelectric effect

$$\sigma = a_{ij} T_{ij} + p_m{}^{T} E_m + \left(\frac{\partial \sigma}{\partial \theta} \right)^{T,E} \theta$$

Simplify

- Pyroelectric -> 0
- Ignore entropy
- Assume constant θ, H

- Linear piezoelectric constitutive Eq.

$$S_{ij} = s_{ijkl}^{E} T_{kl} + d_{mij} E_m$$

$$D_n = d_{nkl} T_{kl} + \varepsilon_{nm}^{T} E_m$$

Piezoelectric Constitutive Equations

❖ Constitutive Equations

- Gibbs free energy

$$G = -\frac{1}{2}\varepsilon_{mn}E_m E_n - \frac{1}{3}\varepsilon_{mn0}E_m E_n E_0 - \frac{1}{4}\varepsilon_{mn0p}E_m E_n E_0 E_p - \ldots\ldots$$
$$-\frac{1}{2}s_{ijkl}T_{ij}T_{kl} - \frac{1}{3}s_{ijklmn}T_{ij}T_{kl}T_{mn} - \ldots\ldots$$
$$-u_{mijkl}E_m T_{ij}T_{kl} - r_{mnijkl}E_m E_n T_{ij}T_{kl} - \ldots\ldots$$
$$-d_{mij}E_m T_{ij} - m_{mnij}E_m E_n T_{ij} - \ldots..$$

$$D_m = -\left(\frac{\partial G}{\partial E_m}\right)^T$$

$$S_{ij} = \left(\frac{\partial G}{\partial T_{ij}}\right)^E$$

$$D_m = \varepsilon_{mn}E_n + \varepsilon_{mn0}E_m E_n + \ldots\ldots$$
$$+u_{mijkl}T_{ij}T_{kl} + 2r_{mnij}E_n T_{ij}\ldots.$$
$$+d_{mij}T_{ij} + 2m_{mnij}E_n T_{ij}\ldots$$

$$S_{ij} = s_{ijkl}T_{kl} + s_{ijklmn}T_{kl}T_{mn} + \ldots.$$
$$+2u_{mijkl}E_m T_{kl} + 2r_{mnijkl}E_m E_n T_{kl} + \ldots$$
$$+d_{mij}E_m + m_{mnij}E_m E_n + \ldots\ldots$$

Piezoelectric Constitutive Equations

❖ **Quadratic Electrostrictor Equations**

Simplifications….electrostrictors are symm.

→ odd rank permittivity → 0

$\rightarrow m_{ijmn} = m_{mnij} \quad d, q, u \rightarrow 0$

→ Drop higher order terms

$$D_m = \varepsilon^T_{mn} E_n + 2 m_{mnij} E_n T_{ij}$$

$$S_{ij} = s^E_{ijkl} T_{kl} + 2 r_{mnijkl} E_m E_n T_{kl} + m_{mnij} E_m E_n$$

$$m : \text{electrostrictive coupling}$$

$$r : \text{electrostriction}$$

$$s^*_{ijkl} = s^E_{ijkl} + 2 r_{mnijkl} E_m E_n$$: electric field varying component

Piezoelectric Constitutive Equations

❖ Piezoelectric Constitutive Behavior

$$S_{ij} = s^{E}_{ijkl} T_{kl} + d_{mij} E_m$$

$$D_n = d_{nkl} T_{kl} + \varepsilon^{T}_{mn} E_m$$

$$\{S\} = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{13} \\ 2S_{12} \end{Bmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{pmatrix},$$

$$\{D\} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}$$

: electrical displacement = charge/area

Piezoelectric Constitutive Equations

❖ **Piezoelectric Constitutive Behavior**

Electric field and stress are similar

$$\begin{Bmatrix} D \\ S \end{Bmatrix} = \begin{bmatrix} \varepsilon^T & d \\ d_t & s^E \end{bmatrix} \begin{Bmatrix} E \\ T \end{Bmatrix}$$

→ transpose

$$\varepsilon^T = \begin{bmatrix} \varepsilon_{11}^T & 0 & 0 \\ 0 & \varepsilon_{22}^T & 0 \\ 0 & 0 & \varepsilon_{33}^T \end{bmatrix}$$

$$s^E = \begin{bmatrix} s_{11}^E & s_{12}^E & s_{13}^E & & & \\ s_{12}^E & s_{11}^E & s_{13}^E & & 0 & \\ s_{13}^E & s_{13}^E & s_{33}^E & & & \\ & & & s_{55}^E & 0 & 0 \\ & 0 & & 0 & s_{55}^E & 0 \\ & & & 0 & 0 & s_{66}^E \end{bmatrix}$$

$$d = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix}$$

Piezoelectric Constitutive Equations

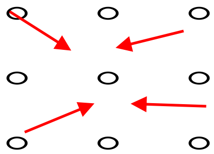
❖ **Electric field and stress are similar**

IEEE Standard STD-176-1978

ferroelectric : - able to be poled by electric field

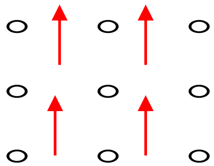
- transversely isotropic in 1-2 directions,
- poling in 3-direction

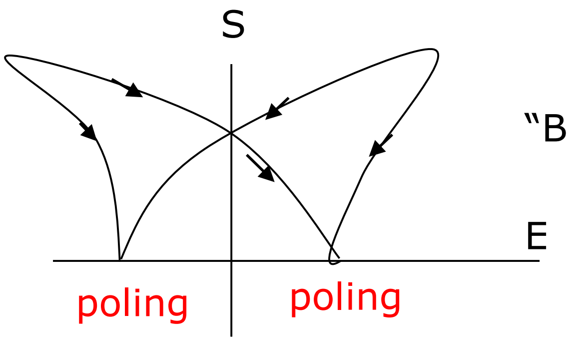
Poling



In the beginning

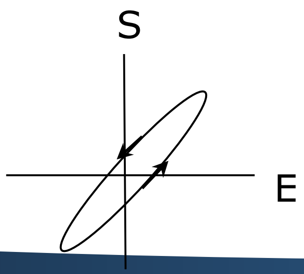
Apply large electric field





"Butterfly Curve"

Used



Piezoelectric Constitutive Equations

❖ Relation of Coupling terms from Form 2

$$T = (s^E)^{-1} S - (s^E)^{-1} d_t E$$

1st eqn. of Form 2

$$D = d(s^E)^{-1} S - d(s^E)^{-1} d_t E + \varepsilon^T E$$

Compare to Form #1

$$c^E = (s^E)^{-1}$$
$$e = dc^E$$
$$\varepsilon^S = \varepsilon^T - dc^E d_t$$

Clamped dielectric < free dielectric

- Example 2

from Form 2 $$E = \left(\varepsilon^T\right)^{-1} D - \left(\varepsilon^T\right)^{-1} dT$$

Substitute into equation for S

$$S = s^E T + d_t (\varepsilon^T)^{-1} D - d_t (\varepsilon^T)^{-1} dT$$

compare to Form 3,

$$s^D = s^E - d_t \left(\varepsilon^T\right)^{-1} d$$

Piezoelectric Constitutive Equations

❖ Material Constants

Electrical $\varepsilon_{mn} = \kappa \varepsilon_0$

where $\varepsilon_0 = 8.85 \times 10^{-12}\ F / \mathrm{m}$

Material	κ $= \varepsilon_{33} / \varepsilon_0$
Air(vacuum)	1
Rubber	6
Epoxy	3-6
Water	80
PZT	3,400

Mechanical

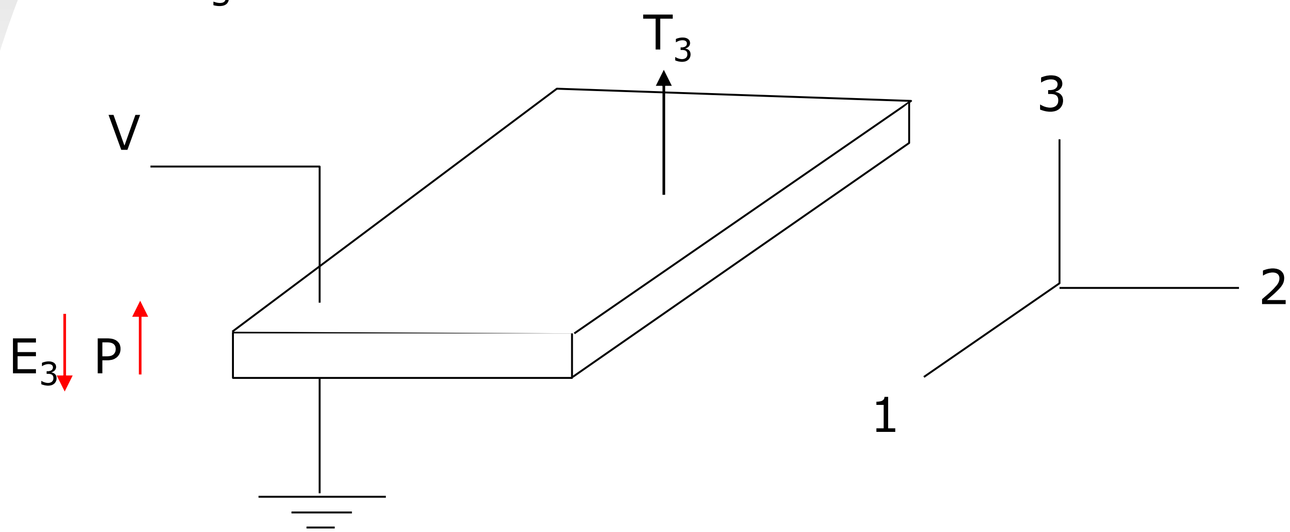
Material	E(MPa) $= 1 / s_{11}$
epoxy	~2,800
steel	200,000
Al	70,000
PZT-5H	60,600

Piezoelectric Constitutive Equations

- **Modes of Operation**

  -Uni-axial stress cases

  - Longitudinal mode



Characterized

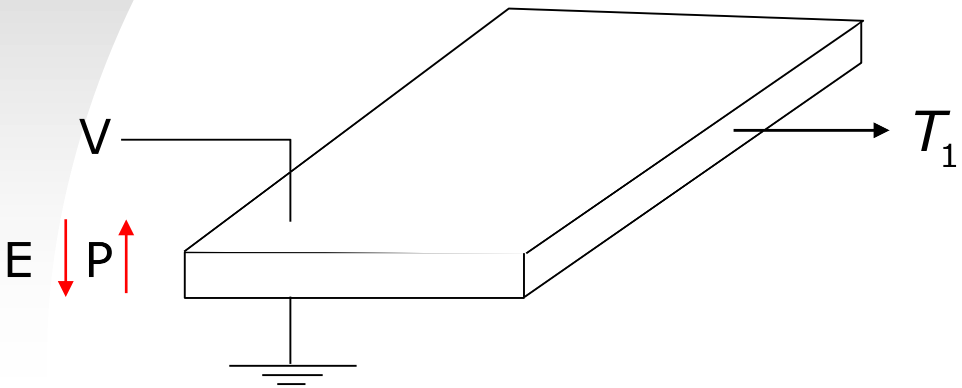
E_3, T_3

All the other $T \rightarrow 0$

$$\begin{Bmatrix} S_3 \\ D_3 \end{Bmatrix} = \begin{bmatrix} s_{33}^E & d_{33} \\ d_{33} & \varepsilon_{33}^T \end{bmatrix} \begin{Bmatrix} T_3 \\ E_3 \end{Bmatrix}$$

Piezoelectric Constitutive Equations

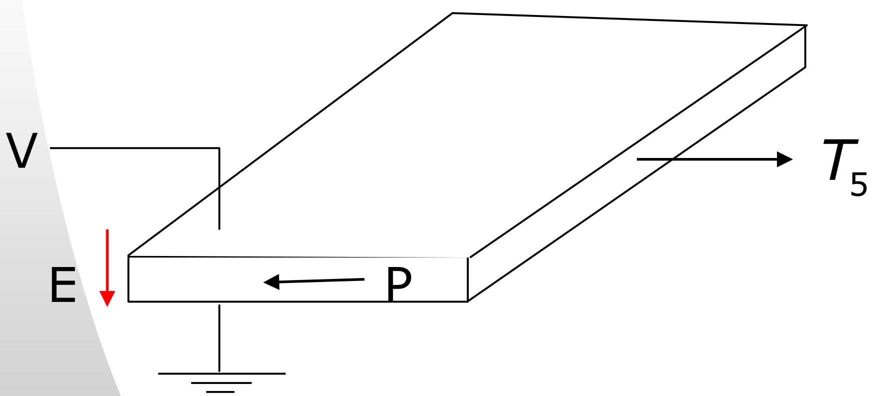
❖ Transverse mode



Characterized by E_3, T_1

$$\begin{Bmatrix} S_1 \\ D_3 \end{Bmatrix} = \begin{bmatrix} s_{11}^E & d_{31} \\ d_{31} & \varepsilon_{33}^T \end{bmatrix} \begin{Bmatrix} T_1 \\ E_3 \end{Bmatrix}$$

❖ Shear mode



Characterized by $E_1 (\perp P),\ T_5$

$$\begin{Bmatrix} S_5 \\ D_1 \end{Bmatrix} = \begin{bmatrix} s_{55}^E & d_{15} \\ d_{15} & \varepsilon_{11}^T \end{bmatrix} \begin{Bmatrix} T_5 \\ E_1 \end{Bmatrix}$$

Piezoelectric Constitutive Equations

❖ **Form Relationships**

$$\varepsilon^S = \varepsilon^T - dc^E d_t$$

$$\varepsilon^S = \varepsilon^T \left(1 - \frac{dc^E d_t}{\varepsilon^T}\right)$$

$$= \varepsilon^T \left(1 - \left(\frac{d\ d_t}{s^E \varepsilon^T}\right)\right)$$

Coupling Coefficient

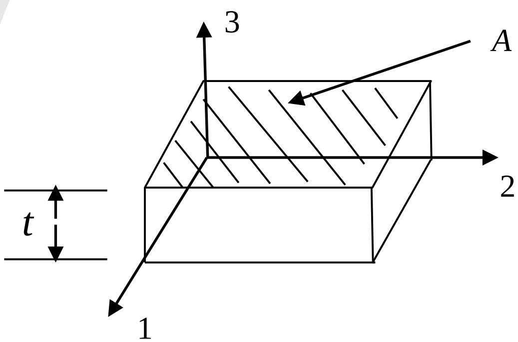
$$s^D = s^E - d_t\ \varepsilon^{T-1}\ d$$

$$= s^E \left(1 - \frac{d_t\ d}{s^E \varepsilon^T}\right)$$

Piezoelectric Constitutive Equations

❖ **Coupling Coefficient**

Definition : Ratio of Electrical/Mechanical

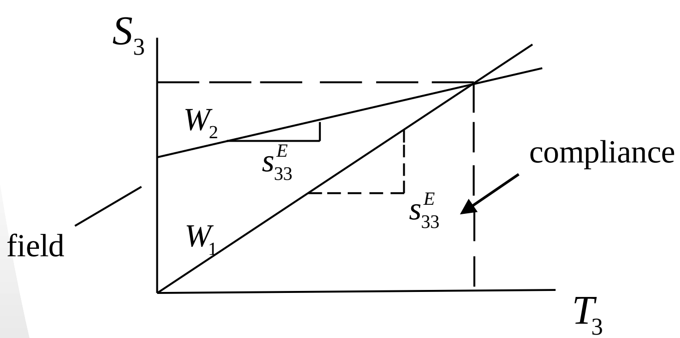


i) Load up mechanically, $T_3, \quad E_3 = 0$

$$U = \int T_3 \, \delta S_3 = \frac{1}{2} s_{33}{}^{E} \; T_3{}^{2}$$

ii) Field

$$E_3 = -\frac{T_3 d_{33}}{\varepsilon_{33}{}^{T}}$$



Coupling Coefficient

$$k^2 = \frac{W_1}{W_1 + W_2} = \frac{W^E}{W^m}$$

$$W^E = \frac{1}{2} \varepsilon_{33}{}^{T} E_3{}^{2} = \frac{1}{2} \frac{T_3 d_{33}{}^{2}}{\varepsilon_{33}^{T}}$$

$$W^m = \frac{1}{2} T_3{}^{2} s_{33}{}^{E}$$

$$\frac{\square}{\square} = \frac{d_{33}{}^{2}}{\varepsilon_{33}{}^{T} s_{33}{}^{E}} = k_{33}{}^{2}$$

Piezoelectric Constitutive Equations

Transverse made

$$k_{33}^{\;2} = \frac{d_{31}^{\;2}}{\varepsilon_{33}^{\;T} s_{11}^{\;E}}$$

Shear made

$$k_{15}^{\;2} = \frac{d_{15}^{\;2}}{\varepsilon_{33}^{\;T} s_{15}^{\;E}}$$

- **Change of coordinates**

  Longitudinal

$$\begin{bmatrix} S_3 \\ D_3 \end{bmatrix} = \begin{bmatrix} s_{33}^{\;E} & d_{33} \\ d_{33} & \varepsilon_{33}^{\;T} \end{bmatrix} \begin{bmatrix} T_3 \\ E_3 \end{bmatrix}$$

Piezoelectric Constitutive Equations

Define a transformation

$$\begin{bmatrix} \tilde{S}_3 \\ \tilde{D}_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{s_{33}{}^E} & 0 \\ 0 & 1/\sqrt{\varepsilon_{33}{}^T} \end{bmatrix} \begin{bmatrix} S_3 \\ D_3 \end{bmatrix} = [T] \begin{bmatrix} S_3 \\ D_3 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{T}_3 \\ \tilde{E}_3 \end{bmatrix} = [T^{-1}] \begin{bmatrix} T_3 \\ E_3 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{S}_3 \\ \tilde{D}_3 \end{bmatrix} = \underbrace{\left\{ T \begin{bmatrix} & \\ & \end{bmatrix} T \right\}}_{\begin{bmatrix} 1/\sqrt{s_{33}{}^E} & 0 \\ 0 & 1/\sqrt{\varepsilon_{33}{}^T} \end{bmatrix} \begin{bmatrix} s_{33}{}^E & d_{33} \\ d_{33} & \varepsilon_{33}{}^T \end{bmatrix} \begin{bmatrix} 1/\sqrt{s_{33}{}^E} & 0 \\ 0 & 1/\sqrt{\varepsilon_{33}{}^T} \end{bmatrix}} \begin{bmatrix} \tilde{T}_3 \\ \tilde{E}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & k_{33} \\ k_{33} & 1 \end{bmatrix} \qquad k_{33} = \sqrt{\frac{d_{33}{}^2}{\varepsilon_{33}{}^T s_{33}{}^E}}$$