5. Pressure Vessels and Axial Loading Applications

5.1 Introduction

- Mechanics of materials approach (analysis)

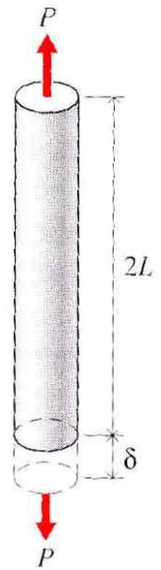
  - analyze real structural elements as idealized models subjected simplified loadings and restraints.

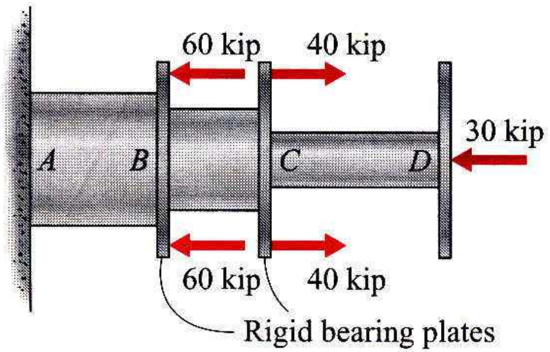
5.2 Deformation of axially loaded members

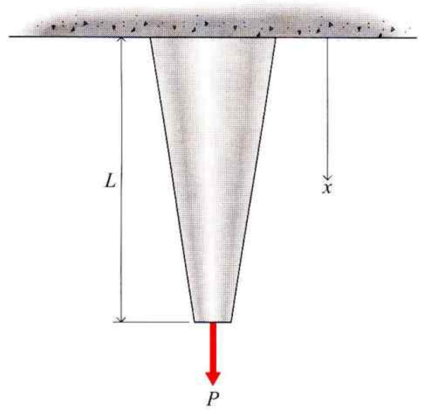
- Uniform member: $\delta = \varepsilon L = \dfrac{\sigma L}{E}, \quad \delta = \dfrac{PL}{EA}$

- Multiple loads/sizes: $\delta = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \dfrac{P_i L_i}{E_i A_i}$

- Nonuniform deformation: $\delta = \int_0^L \dfrac{P_x}{EA_x} dx$



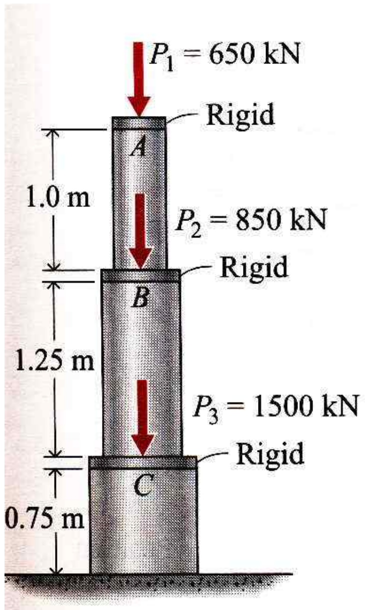




- Example Problem 5-1

$D_A = 100$ mm, $D_{B\_in} = 100$ mm, $D_{B\_out} = 150$ mm,
$D_{C\_in} = 125$ mm, $D_{C\_out} = 200$ mm, $E_A = 73$ GPa, $E_B = 100$ GPa, $E_C = 210$ GPa

- Determine the overall shortening of the member



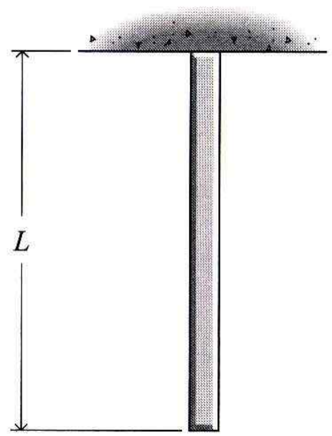
5.2 Deformation of axially loaded members

- Example Problem 5-3

A homogeneous bar of uniform cross section A hangs vertically

- Elongation of the bar due to its own weight W in terms of W, L, A, and E

- Elongation of the bar if it is also subjected to an axial tensile force P at its lower end



5.3 Deformations in a system of axially loaded bars

$$\delta_{AB} = L_f - L_i = \sqrt{(L+v_B)^2 + u_B^2} - L$$

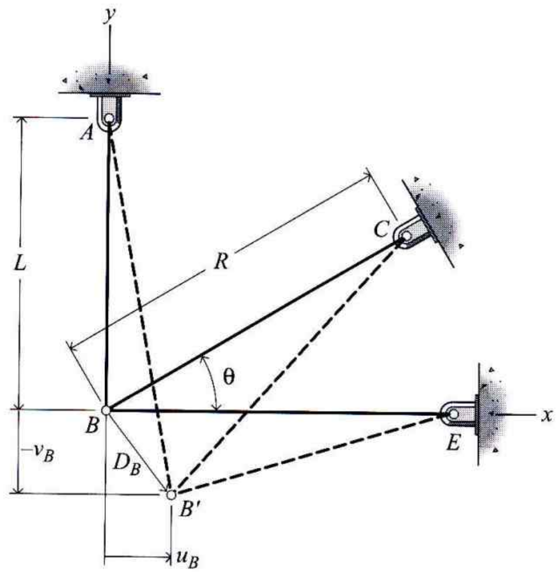
$$\delta_{AB}^2 + 2L\delta_{AB} + L^2 = L^2 + 2Lv_B + v_B^2 + u_B^2$$

$$\rightarrow \delta_{AB} \cong v_B$$

In a similar manner,

$$\delta_{BE} \cong u_B$$

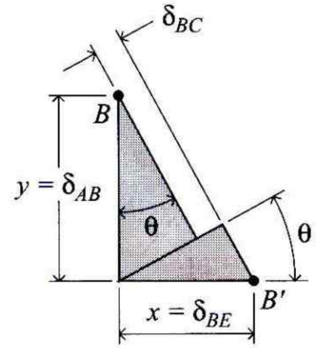
In case of $L_{BC} = R$,



$$\delta_{BC} = \sqrt{(R\cos\theta - u_B)^2 + (R\sin\theta + v_B)^2} - R$$

$$\delta_{BC}^2 + 2R\delta_{BC} + R^2 = R^2\cos^2\theta - 2Ru_B\cos\theta + u_B^2 + R^2\sin^2\theta + 2Rv_B\sin\theta + v_B^2$$

$$\rightarrow \delta_{BC} \cong v_B\sin\theta - u_B\cos\theta = \delta_{AB}\sin\theta - \delta_{BE}\cos\theta$$

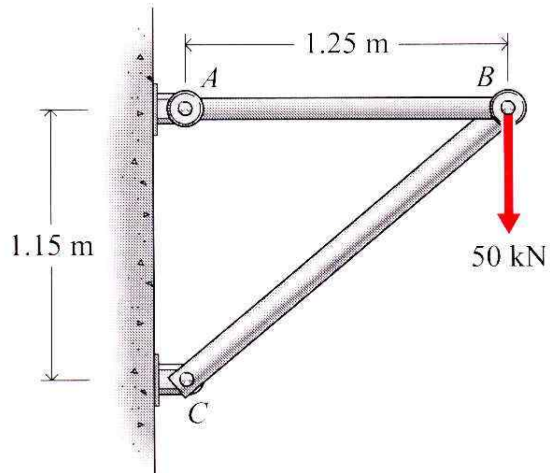


For small displacements, the axial deformation in any bar may be assumed equal to the component of the displacement of one end of the bar taken in the direction of the unstrained orientation of the bar.

5.3 Deformations in a system of axially loaded bars

• Example Problem 5-5

Cross-sectional areas of tie rod *AB* and pipe strut *BC* are $650\ mm^2$ and $925\ mm^2$, respectively. $E = 200\ GPa$



- Normal stress in *AB* and *BC*

- Lengthening or shortening of *AB* and *BC*

- Horizontal and vertical components of the displacement of *B*

- Angles through which members *AB* and *BC* rotate

5.4 Statistically indeterminate axially loaded members

- The number of unknowns should be the same as the number of equations.

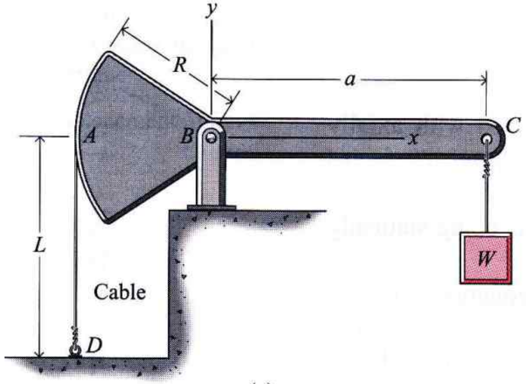
- For many mechanical systems, the equations of equilibrium are not sufficient for the determination of axial forces in the members and reactions at the supports.

- Additional equations involving the geometry of the deformations of the system can be helpful to solve the problems.

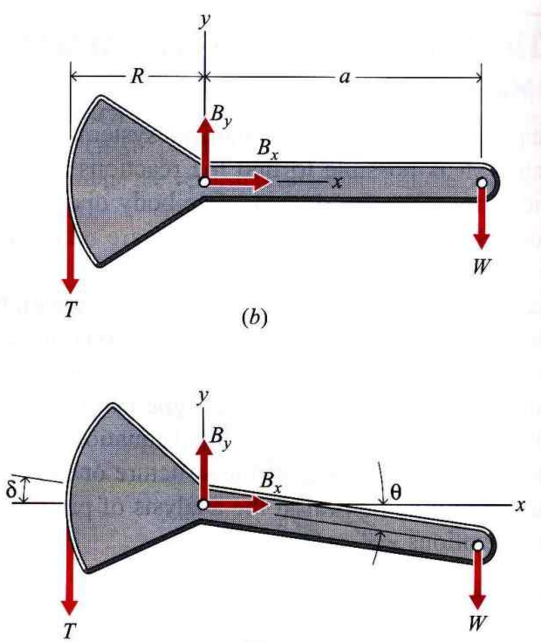
- Hooke's law and the definition of stress and strain can be used to relate deformations and forces when all stresses are under the proportional limit of the materials.

5.4 Statistically indeterminate axially loaded members

- Calculation of the tension of the cable



(a)



(b)

(c)

1) Rigid cable

$$\sum M_B = 0: \quad TR - Wa = 0 \quad \rightarrow \quad T = \frac{Wa}{R}$$

2) Deformable cable

$$\sum M_B = 0: \quad TR - W\left(a\cos\theta\right) = 0 \quad \rightarrow \quad T = \frac{Wa}{R}\cos\theta$$

$$\delta = \frac{TL}{AE} \quad \rightarrow \quad \frac{\delta AE}{L} = \frac{Wa}{R}\cos\theta$$

$$\delta = R\theta \quad \rightarrow \quad R^2 EA\theta = WaL\cos\theta$$

5.4 Statistically indeterminate axially loaded members

- Comparison of tensions with various elastic moduli of the cable

Example 1: The cable is rigid.
If $W = 100$ lb, $a = 30$ in., and $R = 15$ in.
Equation (a) yields: $T = 200$ lb

Example 2: The cable is a 3/32-in.-diameter steel ($E = 29{,}000$ ksi) wire.
If $W = 100$ lb, $a = 30$ in., $R = 15$ in., and $L = 45$ in.
Equation (f) yields: $\theta = 0.002997 \text{ rad} = 0.1717°$
Equation (b) yields: $T = 199.999$ lb
The percent difference in T in the two examples is

$$\%D = \frac{200 - 199.999}{199.999}(100) = 0.0005\%$$

Example 3: The cable is a 3/32-in.-diameter aluminum ($E = 10{,}600$ ksi) wire.
If $W = 100$ lb, $a = 30$ in., $R = 15$ in., and $L = 45$ in.
Equation (f) yields: $\theta = 0.00820 \text{ rad} = 0.4698°$
Equation (b) yields: $T = 199.993$ lb
The percent difference in T for Examples 1 and 3 is
$\% D = 0.0035\ \%$.

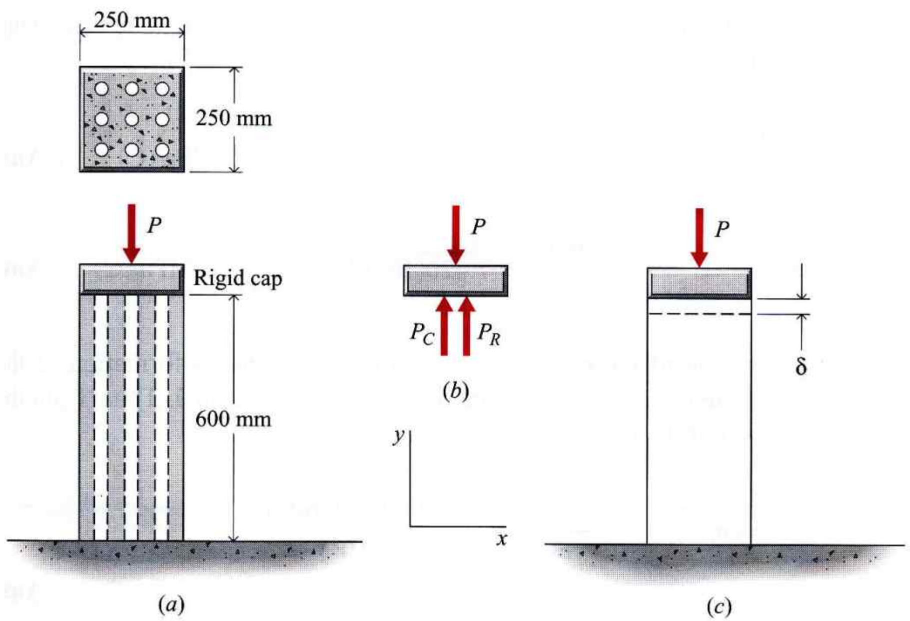
5.4 Statistically indeterminate axially loaded members

• Example Problem 5-7

A pier has nine 25-mm-diameter steel reinforcing bars ($E = 200\ GPa$) in the concrete ($E = 30\ GPa$). $P = 650\ kN$

- Stresses in the concrete and the steel bars

- Shortening of the pier

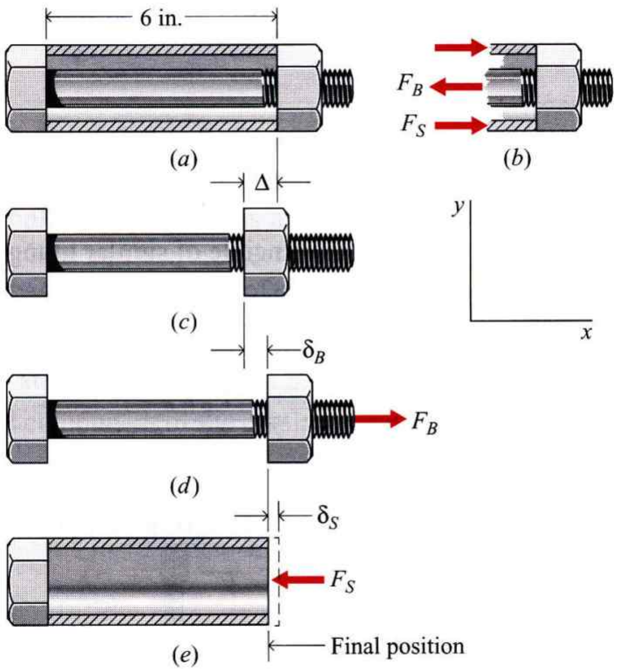


5.4 Statistically indeterminate axially loaded members

• Example Problem 5-10

A 0.5 in.-diameter bolt (*E = 30,000 ksi*) & a sleeve (*E = 15,000 ksi*) of 0.375 in^2 cross-sectional area are deformed by a nut (*0.02 in.*).

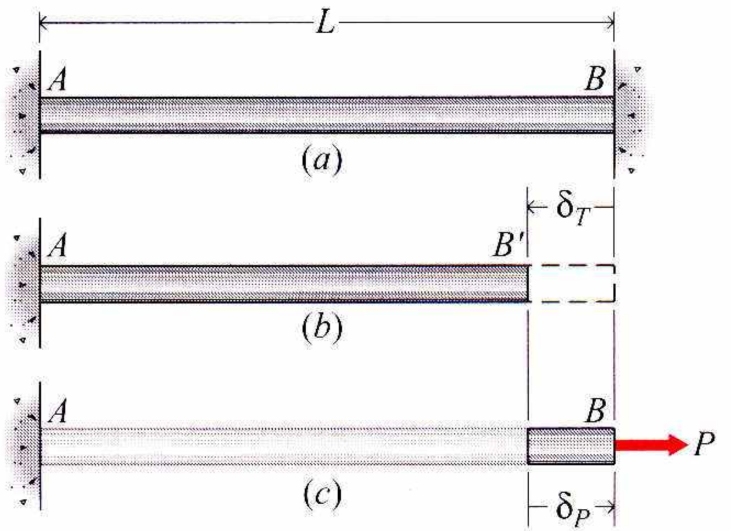
- Stresses in the bolt and the sleeve



5.5 Thermal Effects

- When the deformation of a bar is prevented the thermal strain is offset by the mechanical strain in opposite direction.

$$\delta_{total} = \delta_T + \delta_\sigma = \varepsilon_T L + \varepsilon_\sigma L = \alpha \Delta T L + \frac{\sigma}{E} L = 0$$



• Example Problem 5-11

A 10-m section steel rail ($E = 200$ GPa, $\alpha = 11.9(10^{-6})/^{\circ}C$) has a cross-sectional area of 7,500 mm^2. Deformations in all directions are restricted. For an increase in temperature of 50 $^{\circ}C$, determine

- Normal stress in the rail

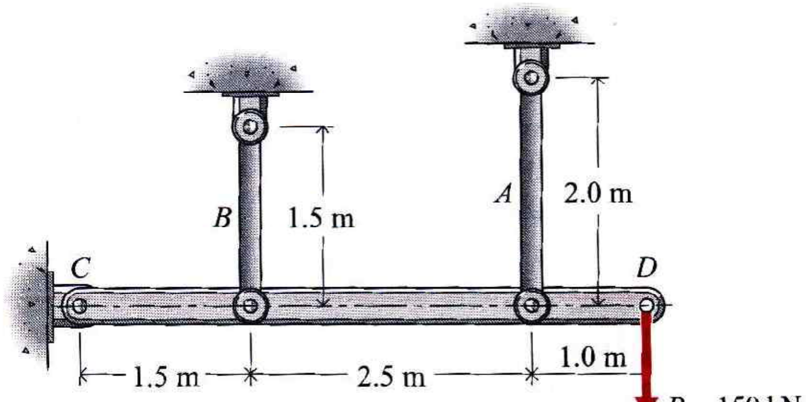
- Internal force on a cross section of the rail

- Example Problem 5-12

The temperature increases 100 ˚C. The thermal coefficients of expansion and the modulus of elasticity are $22(10^{-6})/^{\circ}C$ and 75 GPa for the rod A, and $12(10^{-6})/^{\circ}C$ and 200 GPa for the rod B. The cross sectional area of A and B are 1000 mm^2 and 500 mm^2, respectively. The rod CD is rigid.

- Normal stress in bars A and B

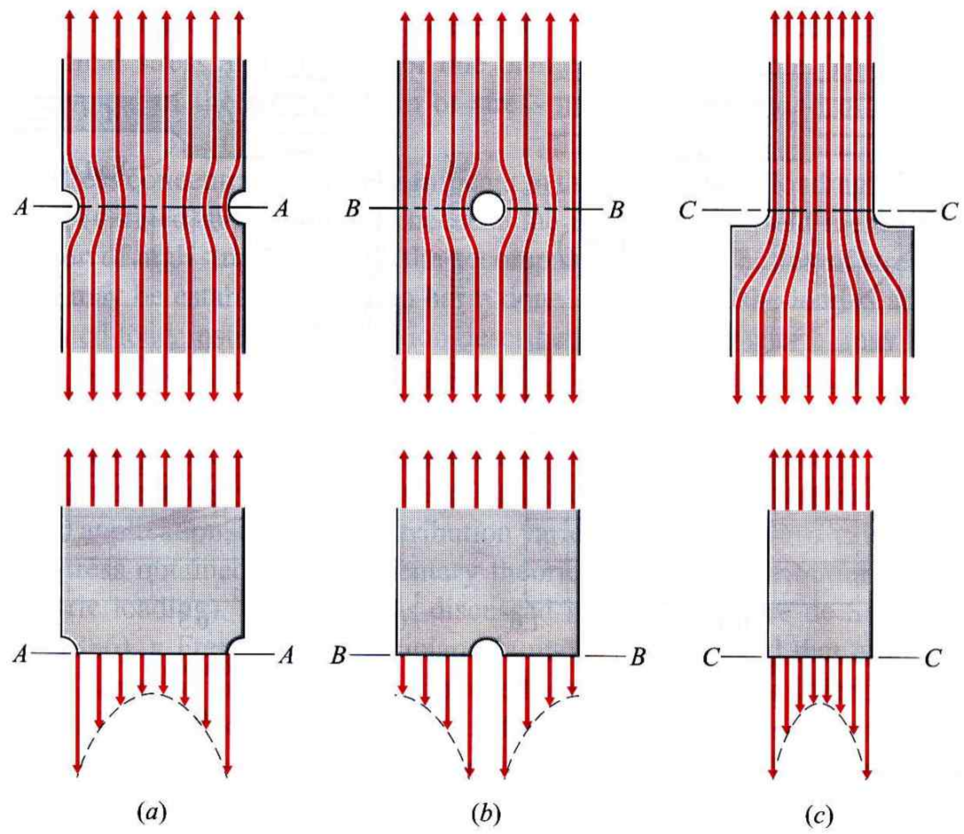
- Vertical component of the displacement of point D



(a)

5.6 Stress concentrations

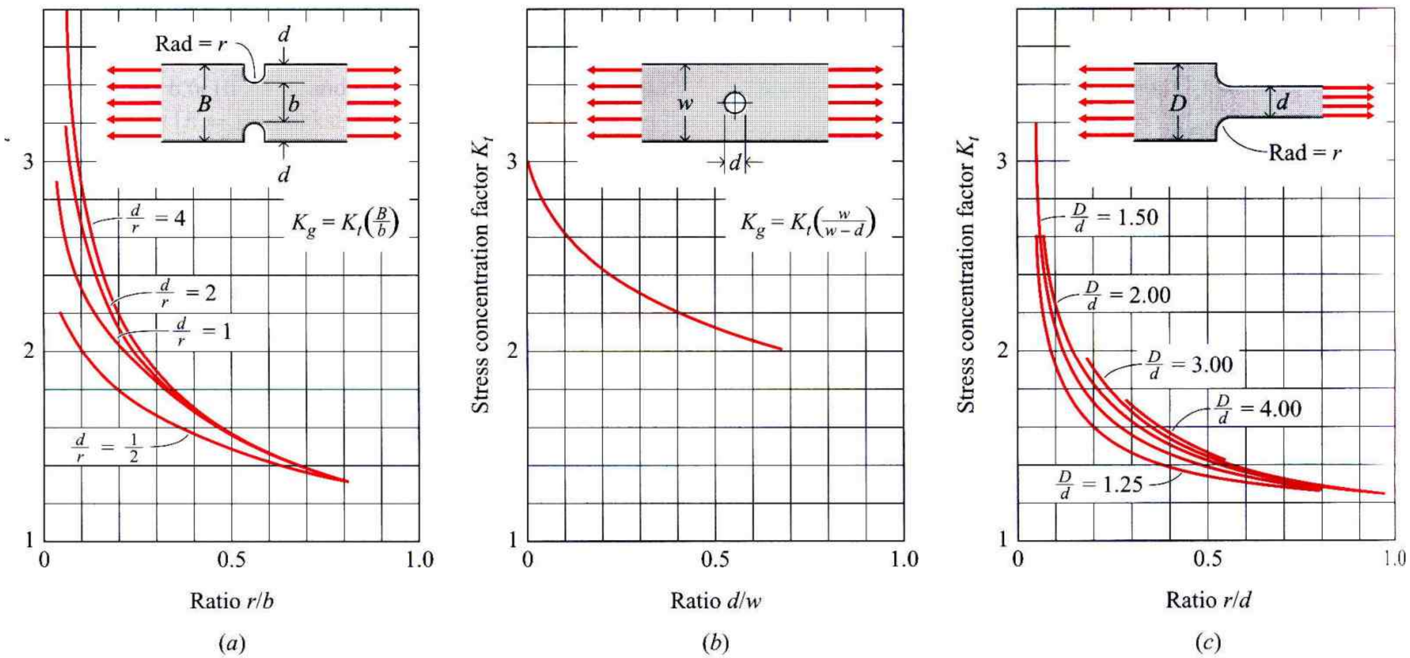
- The stress is concentrated around discontinuities that interrupt the stress path.



5.6 Stress concentrations

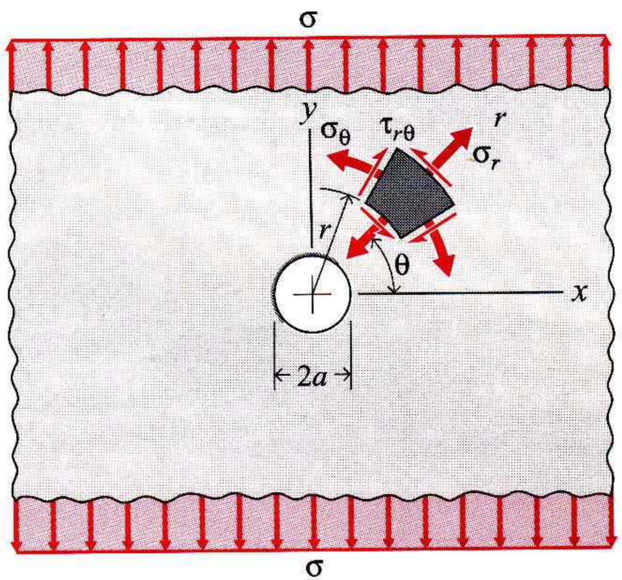
- Stress concentration factor, K, is defined based on an area at the reduced section (net area) or on the gross area:

$$\sigma = K\frac{P}{A}$$



5.6 Stress concentrations

- Kirsch's solution: stress distribution around a small circular hole in a wide plate under uniform unidirectional tension.



$$\sigma_r = \frac{\sigma}{2}\left(1-\frac{a^2}{r^2}\right)-\frac{\sigma}{2}\left(1-\frac{4a^2}{r^2}+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{\sigma}{2}\left(1+\frac{a^2}{r^2}\right)+\frac{\sigma}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = \frac{\sigma}{2}\left(1+\frac{2a^2}{r^2}+\frac{3a^4}{r^4}\right)\sin 2\theta$$

$$\sigma_r = 0;\quad \sigma_\theta = \sigma\left(1+2\cos 2\theta\right);\quad \tau_{r\theta} = 0 \quad at\ \ r = a$$

$$\sigma_{\theta=0^\circ} = \frac{\sigma}{2}\left(2+\frac{a^2}{r^2}+\frac{3a^4}{r^4}\right) \qquad \sigma_{\theta=0^\circ,r=3a} = 1.074\sigma$$

5.6 Stress concentrations

- Saint Venant's Principle:

Localized stress concentration disappears at some distance.

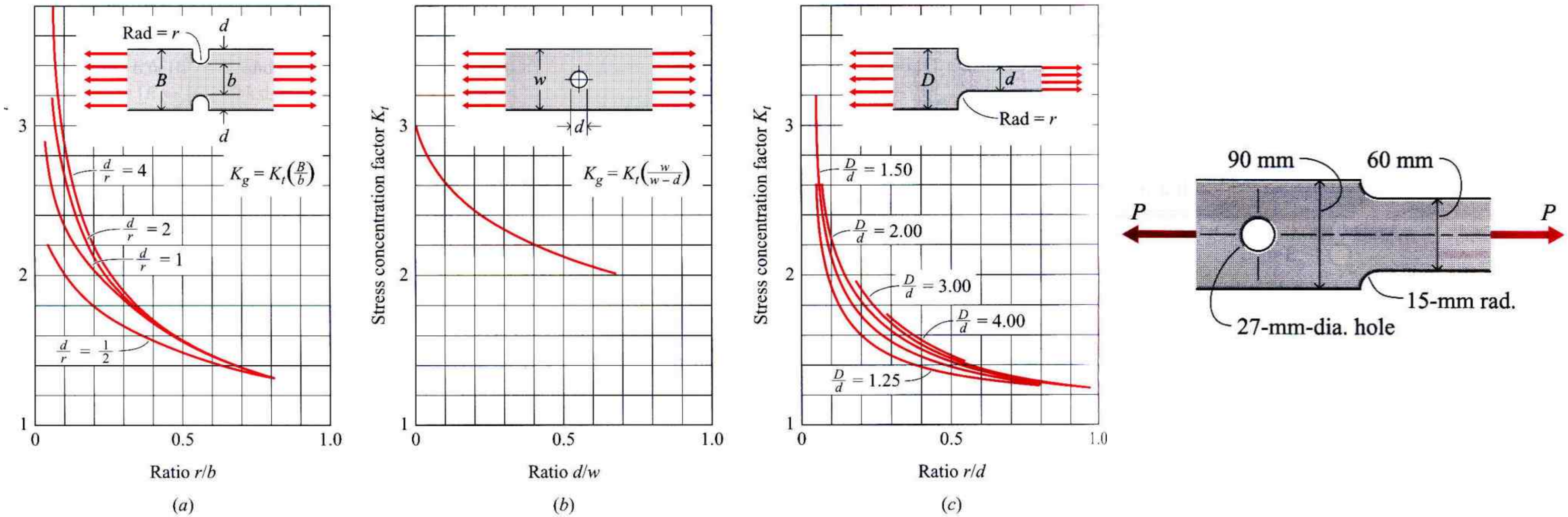
→ The difference between the stresses caused by statically equivalent load systems is insignificant at distances greater than the largest dimension of the area over which the loads are acting.

5.6 Stress concentrations

- Example Problem 5-14

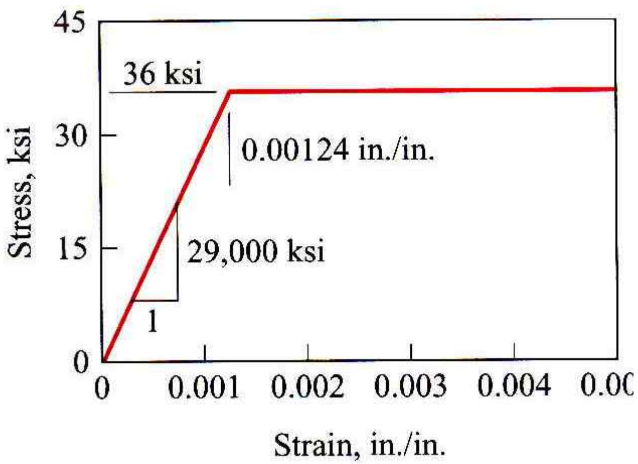
The machine part is 20 mm thick and the maximum allowable stress is 144 MPa.

- The maximum value of *P*

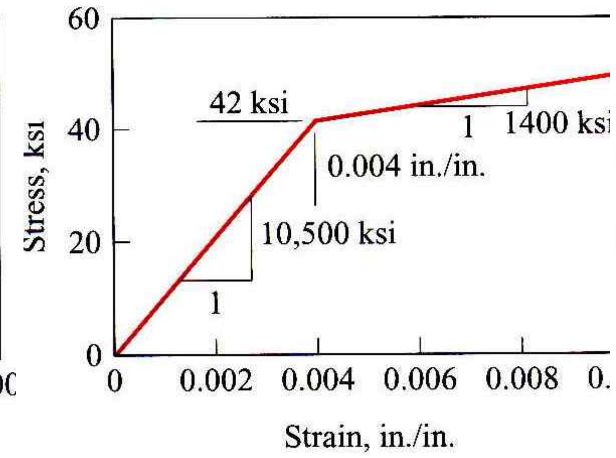


5.7 Inelastic behavior of axially loaded members

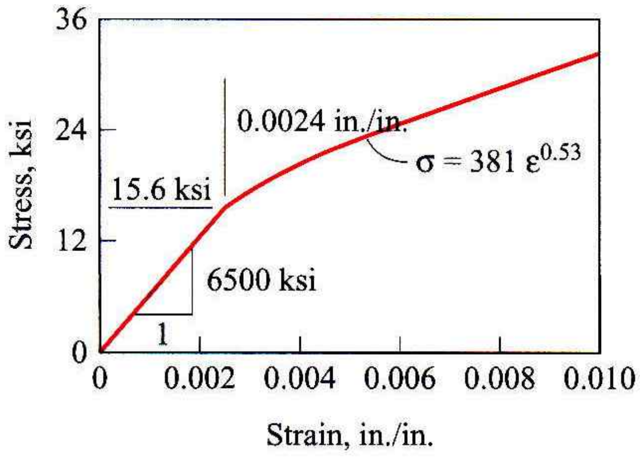
- When the stresses in some members extend into the inelastic range, stress-strain diagrams must be used to relate the loads and the deflections and solve the problem.



Steel (elastoplastic)



Aluminum alloy
(strain hardening)



Magnesium alloy
(strain hardening)

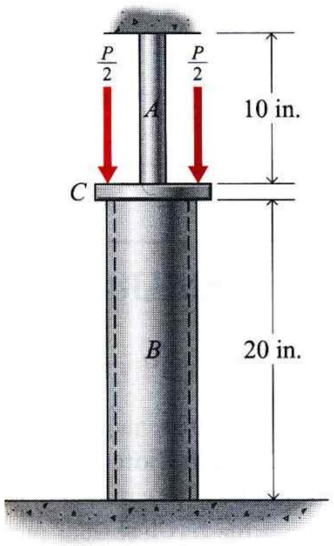
5.7 Inelastic behavior of axially loaded members

• Example Problem 5-15

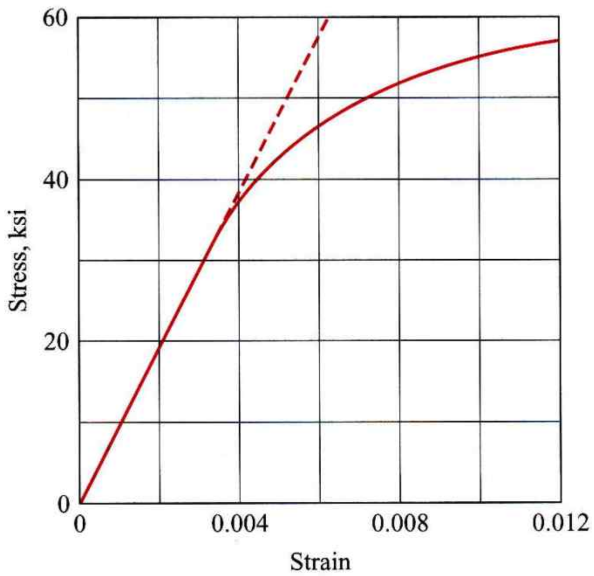
A is a 0.5-in.-diameter steel rod (elastoplastic) which has a proportional limit of 40 ksi and a modulus of elasticity of 30,000 ksi. Pipe *B* has a cross-sectional area of 2 in.^{2} and shows strain-hardening as below. Load *P* is 30 kip.

- Normal stress in *A* and *B*

- Displacement of plate *C*



(*a*)



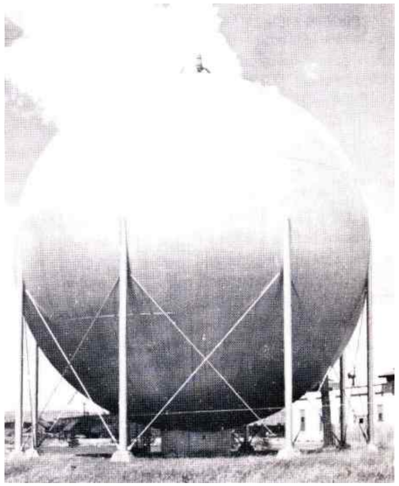
(*b*)

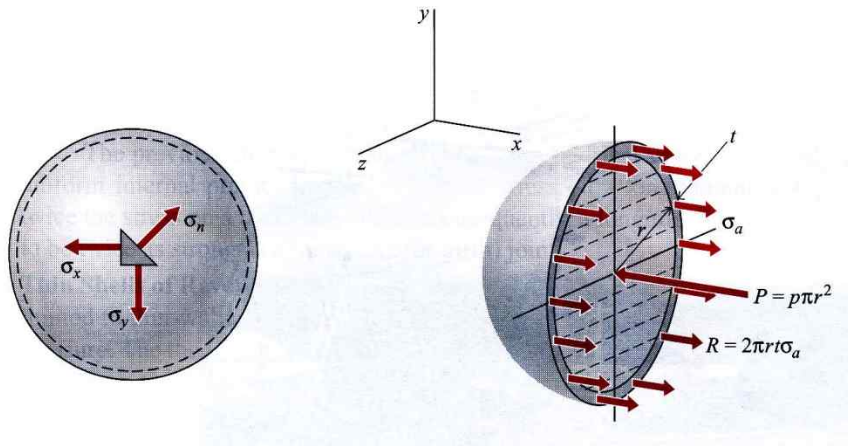
5.8 Thin-walled pressure vessels

- A pressure vessel is a thin walled container whose wall thickness is so small that the normal (axial, meridional) stress on a plane perpendicular to the surface is uniform throughout the thickness.

1) Spherical pressure vessels: no shear stress (strain)

$$P\left(p\pi r^2\right) = R\left(2\pi rt\sigma_a\right) \rightarrow \sigma_a = \frac{pr}{2t}$$





(a) (b)

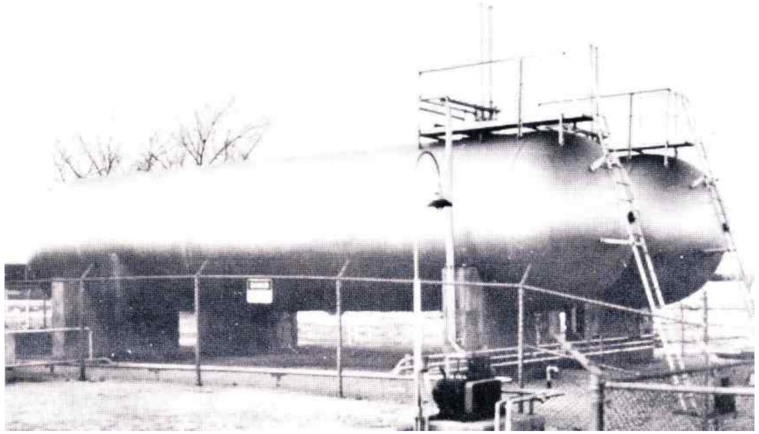
5.8 Thin-walled pressure vessels

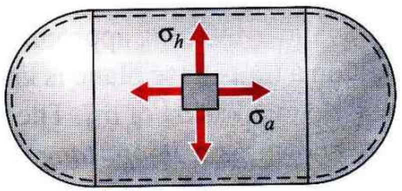
2) Cylindrical pressure vessels:

Hoop (tangential, circumferential) stress vs. axial (meridional) stress

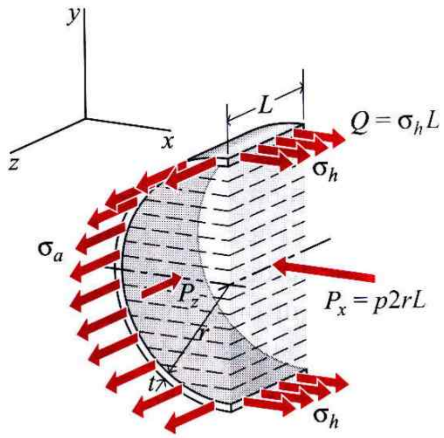
$$2Q\left(\sigma_h L t\right) = P_x\left(p2rL\right) \rightarrow \sigma_h = \frac{pr}{t}$$

$$\sigma_a = \frac{pr}{2t} \qquad \therefore \sigma_h = 2\sigma_a$$





(a)



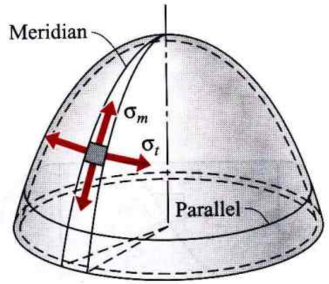
(b)

5.8 Thin-walled pressure vessels

3) Thin shells of revolution:

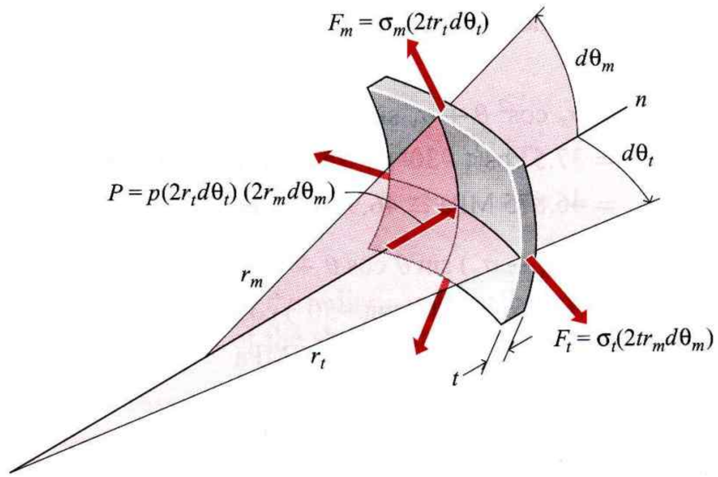
Shapes made by rotating plane curves (meridian) about an axis – sphere, hemisphere, torus (doughnut), cylinder, cone, and ellipsoid.

$$P - 2F_m \sin d\theta_m - 2F_t \sin d\theta_t = 0$$



$$p\left(2r_t d\theta_t\right)\left(2r_m d\theta_m\right) = 2\sigma_m\left(2tr_t d\theta_t\right)\sin d\theta_m + 2\sigma_t\left(2tr_m d\theta_m\right)\sin d\theta_t$$

$$d\theta_t \approx d\theta_m \rightarrow \quad \frac{\sigma_m}{r_m} + \frac{\sigma_t}{r_t} = \frac{p}{t}$$

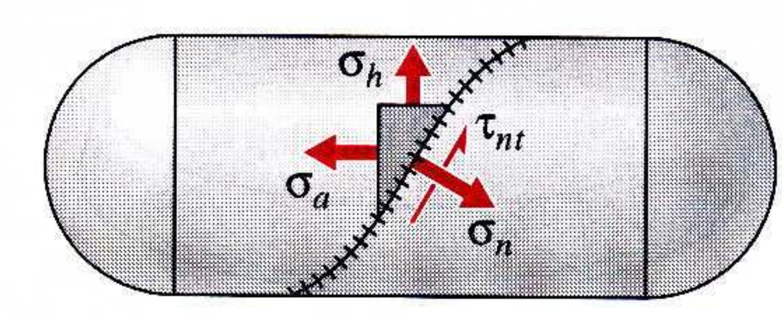


5.8 Thin-walled pressure vessels

• Example Problem 5-16

A cylinder whose diameter is 1.5 m is constructed by 15-mm-thick steel plate, and is under 1500 kPa of inner pressure.

- Normal and shear pressure at welded edge forming 30˚ from hoop direction

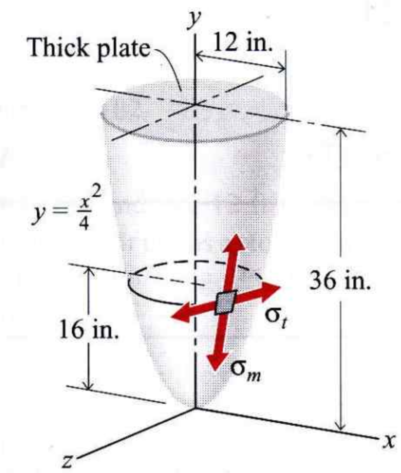


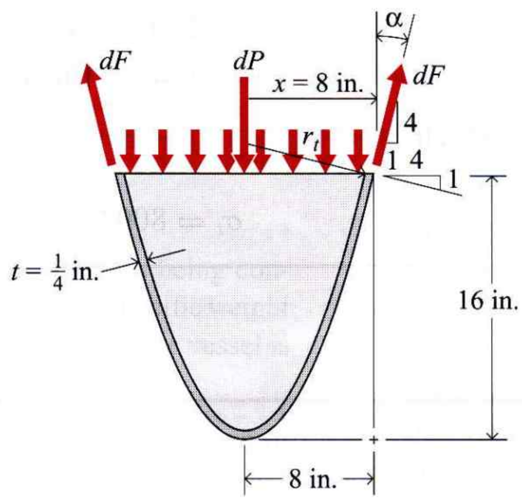
5.8 Thin-walled pressure vessels

- Example Problem 5-17

A pressure vessel of ¼ in. steel plate is closed by a flat plate. The meridian line follows a parabola of $y = x^2/4$. $P = 250$ psi, $r_m = (1+(dy/dx)^2)^{1.5}/(d^2y/dx^2)$

- Meridional and tangential stress σ_m and σ_t at a point 16 in. above the bottom

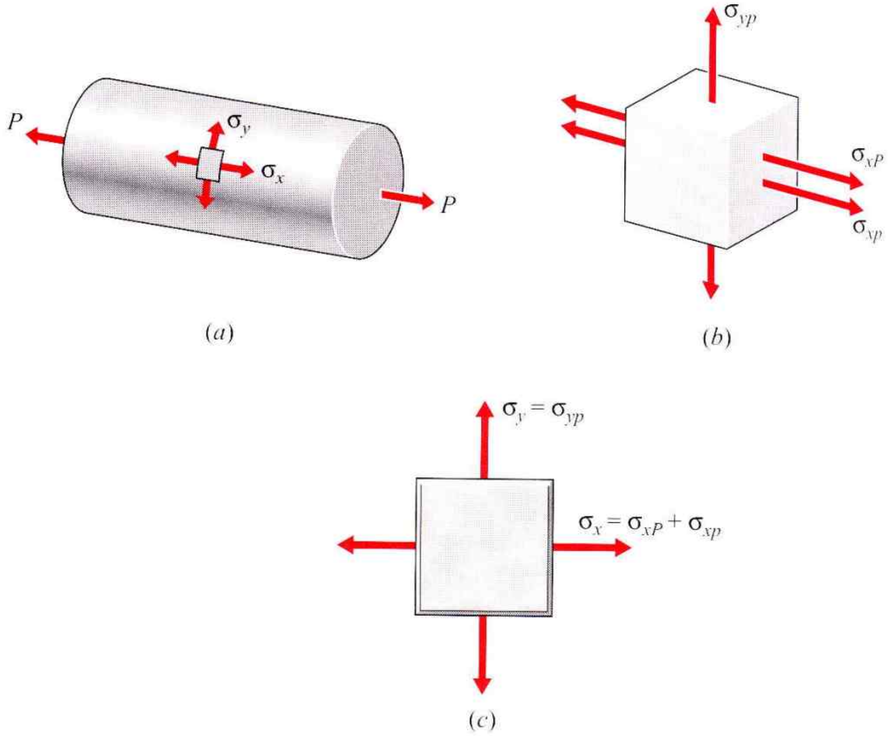




5.9 Combined effects – axial and pressure loads

- A case of an axial load (P) combindedly added to an axial pressure (p)

$$\sigma_x = \sigma_{xP} + \sigma_{xp}$$

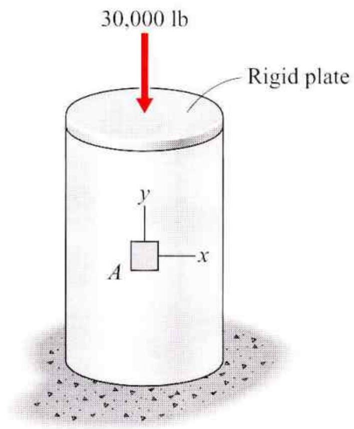


• Example Problem 5-18

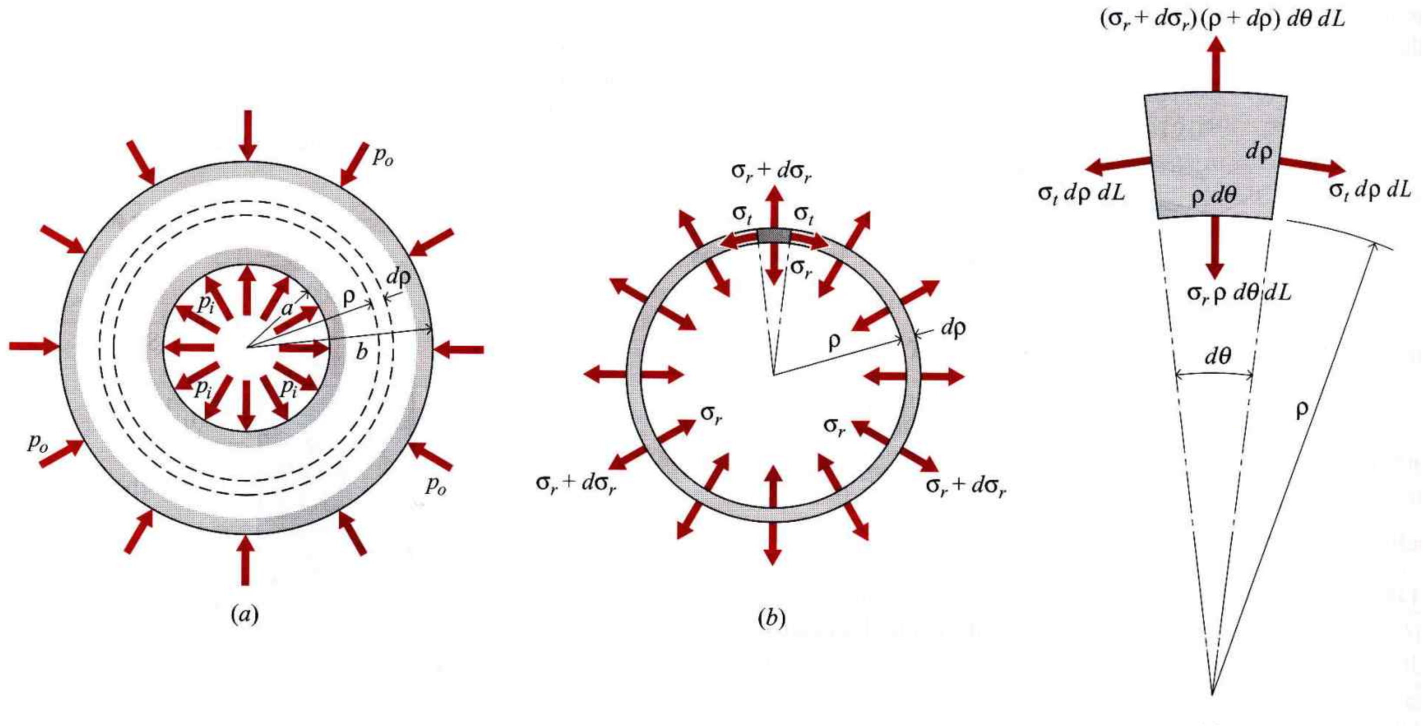
A cylindrical pressure tank whose diameter and wall thickness are 4 ft and ¾ in., respectively is under 400 psi of pressure and 30,000 lb of axial load.

- $\sigma_x, \sigma_y,$ and τ_{xy}

- Normal and shearing stresses on an inclined plane oriented at +30˚ from the *x*-axis



5.10 Thick-walled cylindrical pressure vessels



- Force equilibrium in radial direction

$$\left(\sigma_r + d\sigma_r\right)\left(\rho + d\rho\right)d\theta dL - \sigma_r \rho d\theta dL - 2\sigma_t d\rho dL \sin\frac{d\theta}{2} = 0$$

$$\rightarrow \rho\frac{d\sigma_r}{d\rho} + \sigma_r - \sigma_t = 0 \quad \text{(neglecting higher-order terms)}$$

5.10 Thick-walled cylindrical pressure vessels

- Obtaining σ_r and σ_t

$$\varepsilon_a = \frac{1}{E}\left(\sigma_a - \nu\left(\sigma_r + \sigma_t\right)\right) \quad \rightarrow \quad \sigma_r + \sigma_t = \frac{1}{\nu}\left(\sigma_a - E\varepsilon_a\right) = 2C_1 \quad \text{from Hooke's law}$$

$$\rho\frac{d\sigma_r}{d\rho} + \sigma_r - \sigma_t = 0 \quad \rightarrow \quad \rho\frac{d\sigma_r}{d\rho} + 2\sigma_r = 2C_1$$

$$\frac{d\left(\rho^2\sigma_r\right)}{d\rho} = \rho^2\frac{d\sigma_r}{d\rho} + 2\rho\sigma_r = 2C_1\rho$$

$$\text{Integrating } \frac{d\left(\rho^2\sigma_r\right)}{d\rho} = 2C_1\rho \quad \rightarrow \quad \rho^2\sigma_r = C_1\rho^2 + C_2$$

$$\therefore \sigma_r = C_1 + \frac{C_2}{\rho^2}, \quad \sigma_t = C_1 - \frac{C_2}{\rho^2} \ \left(\because \sigma_r + \sigma_t = 2C_1\right)$$

5.10 Thick-walled cylindrical pressure vessels

- Applying boundary conditions

$$\sigma_r = -p_i \quad \text{at} \quad \rho = a$$

$$\sigma_r = -p_o \quad \text{at} \quad \rho = b$$

$$\rightarrow C_1 = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} \qquad C_2 = -\frac{a^2 b^2 \left(p_i - p_o\right)}{b^2 - a^2}$$

$$\therefore \sigma_r = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} - \frac{a^2 b^2 \left(p_i - p_o\right)}{\left(b^2 - a^2\right)\rho^2}$$

$$\sigma_t = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} + \frac{a^2 b^2 \left(p_i - p_o\right)}{\left(b^2 - a^2\right)\rho^2}$$

5.10 Thick-walled cylindrical pressure vessels

- Displacements

$$\delta_t = 2\pi\delta_r = \varepsilon_t\left(2\pi\rho\right)$$

$$\delta_r = \varepsilon_t\rho$$

When $\sigma_a = 0$

$$\varepsilon_t = \frac{1}{E}\left(\sigma_t - \nu\sigma_r\right)$$

$$\rightarrow \delta_r = \frac{\rho}{E}\left(\sigma_t - \nu\sigma_r\right)$$

5.10 Thick-walled cylindrical pressure vessels

- Case 1: Internal pressure only ($p_o = 0$)

$$\sigma_r = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} - \frac{a^2 b^2 (p_i - p_o)}{(b^2 - a^2)\rho^2} \rightarrow \sigma_r = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{a^2}\right)$$

$$\sigma_t = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} + \frac{a^2 b^2 (p_i - p_o)}{(b^2 - a^2)\rho^2} \rightarrow \sigma_t = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{a^2}\right)$$

$$\delta_r = \frac{\rho}{E}(\sigma_t - \nu\sigma_r) \rightarrow \delta_r = \frac{a^2 p_i}{(b^2 - a^2)E\rho}\left[(1-\nu)\rho^2 + (1+\nu)b^2\right]$$

- Case 2: External pressure only ($p_i = 0$)

$$\sigma_r = -\frac{b^2 p_o}{b^2 - a^2}\left(1 - \frac{a^2}{\rho^2}\right)$$

$$\sigma_t = -\frac{b^2 p_o}{b^2 - a^2}\left(1 + \frac{a^2}{\rho^2}\right)$$

$$\delta_r = -\frac{b^2 p_o}{(b^2 - a^2)E\rho}\left[(1-\nu)\rho^2 + (1+\nu)b^2\right]$$

-Case 3: External pressure on a solid circular cylinder (a = 0)

$$\sigma_r = -p_o$$

$$\sigma_t = -p_o$$

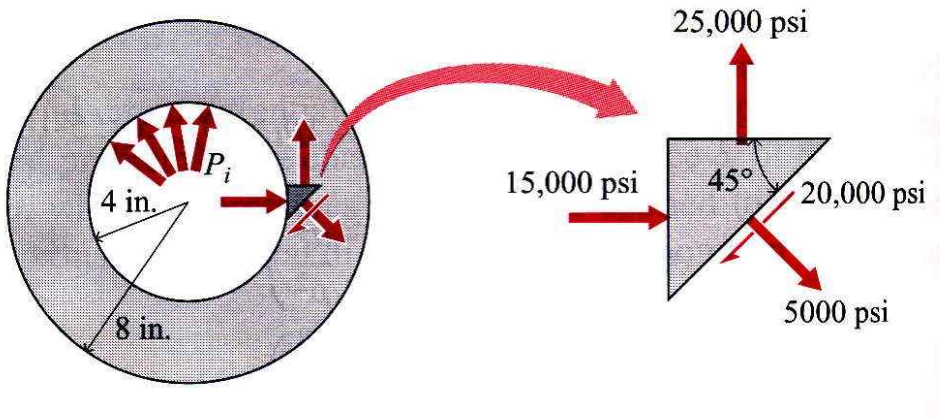
$$\delta_r = -\frac{1-\nu}{E} p_o \rho$$

5.10 Thick-walled cylindrical pressure vessels

• Example Problem 5-19

A steel cylinder (E = 30,000 ksi and ν = 0.30) whose inside and outside diameters are 8 in. and 16 in., respectively is subjected to an internal pressure of 15,000 psi. The axial load is zero.

- The maximum tensile stress
- The maximum shearing stress
- The increase of the inside diameter
- The increase of the outside diameter



5.11 Design

- Failure is defined as the state in which a member or structure no longer functions as intended.

- Failure modes are limited to elastic failure (yielding) here.

- Allowable stress design: Strength $\geq$ Stress

- Factor of safety: Strength $\geq$ Factor of safety $\times$ Stress

5.11 Design

- Example Problem 5-20

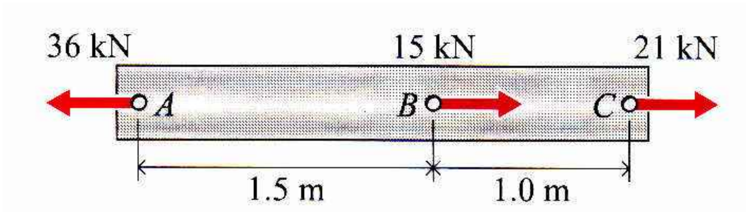
An axially loaded circular bar is subjected to a load of 6500 lb with the factor of safety of 1.5. The yield strength of the bar is 36 ksi.

- Minimum diameter of the bar

- Example Problem 5-21

An axially loaded circular bar has a yielding stress of 250 MPa. The factor of safety is to be 1.8.

- Minimum diameter of the bar

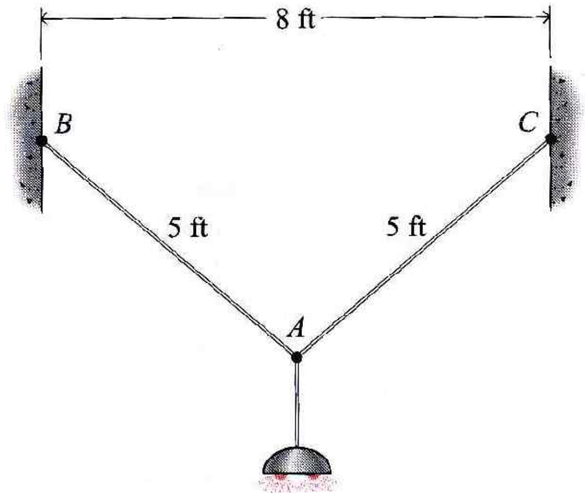


5.11 Design

- Example Problem 5-22

A 40-lb light is supported by a rigid wire whose yield stress is 62 ksi. The factor of safety should be 3.

- Minimum diameter of the wire



Problems: 5-1, 13, 28, 42, 59, 71, 89, 105, 124, 136 by : May 14