CHAPTER 2. VECTOR ANALYSIS

Reading assignments: Cheng Ch.2, Ulaby Ch.2, Hayt Ch.1,

1. Vector Algebra

A. Definitions

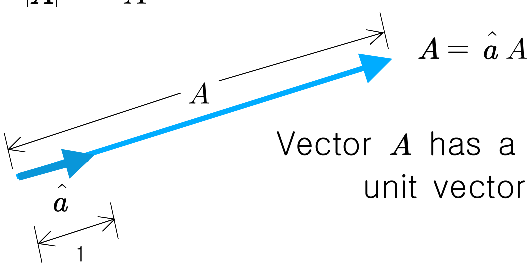
Scalar: A quantity specified by a magnitude represented by a single (positive or negative) real number (with its unit at a given position and time).

(e.g.) charge q, charge density ρ_v, mass, energy m, speed of light c, voltage V, resistance R, inductance L, ...

Vector: A quantity specified by a magnitude and a direction in space (with its unit at a given position and time).

$$\boldsymbol{A} = \hat{\boldsymbol{a}}\,|\boldsymbol{A}| = \hat{\boldsymbol{a}}\,A \quad (\text{or } = \boldsymbol{a}_A\,A) \qquad \text{(2-1), (2-2)}$$

where $\hat{\boldsymbol{a}} \equiv \dfrac{\boldsymbol{A}}{|\boldsymbol{A}|} = \dfrac{\boldsymbol{A}}{A}$: unit vector (dimensionless, unity) (2-3)



Vector $\boldsymbol{A}$ has a magnitude A and unit vector $\hat{\boldsymbol{a}}$ indicating its direction.

(e.g.) force $\boldsymbol{F}$, electric field intensity $\boldsymbol{E}$, magnetic flux density $\boldsymbol{B}$, current density $\boldsymbol{J}$, Poynting vector $\boldsymbol{S}$, ...

Notes)

i) Other vector notations: $\vec{A}$, $\bar{A}$, $\underset{\sim}{A}$, $\boldsymbol{A}$

ii) Equality of two vectors: $\boldsymbol{A} = \boldsymbol{B} \rightleftarrows$ $A = B$ and $\hat{\boldsymbol{a}} = \hat{\boldsymbol{b}}$

Equality does not necessarily mean that they are identical.

iii) Scalar and vector fields: If some quantity is defined at every point in space, a field exists and its value varies in general with both position and time.

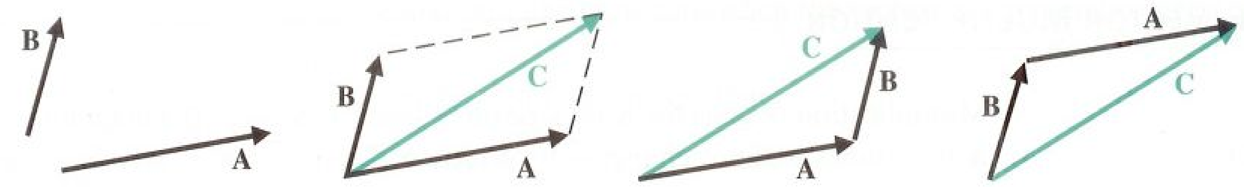
B. Vector Algebra

1) Vector Addition and Subtraction

a) Vector Addition

$C = A + B = B + A$ commutative (1)

$(A+B)+C = A+(B+C)$ associative



(a) Two vectors, **A** and **B**. (b) Parallelogram rule. (c) Head-to-tail rule, **A** + **B**. (d) Head-to-tail rule, **B** + **A**.

FIGURE 2-2 Vector addition, $\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

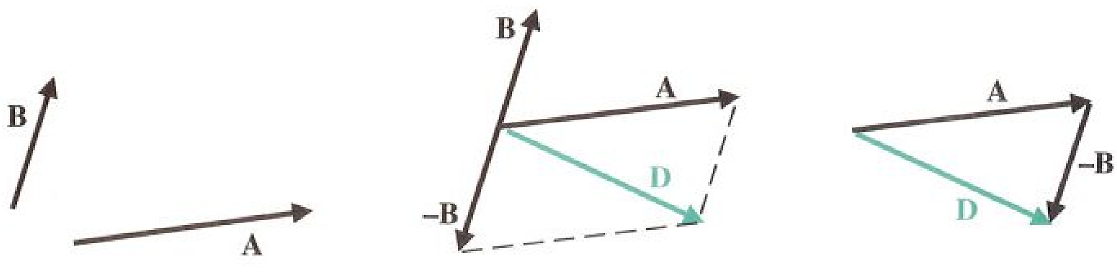
b) Vector Subtraction

For every B, vector $-B$ exists such that $B+(-B)=0$.

Then, vector subtraction can be defined in terms of vector addition:

$$D = A - B = A + (-B) \qquad (2\text{-}4)$$

FIGURE 2-3 Vector subtraction, $\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$.



(a) Two vectors, **A** and **B**. (b) Parallelogram rule. (c) Head-to-tail rule.

2) Vector Multiplication

a) Simple Product: Multiplication by a scalar

$$k\,A = \hat{a}\,(kA) \qquad (2\text{-}5)$$

$k(A+B) = kA + kB$ distribution law

$k(sA) = (ks)A$ associative law

$(k+s)A = kA + sA$ distribution law

Notes)

i) Null vector $\mathbf{0}$:

$\exists$ a null vector $\mathbf{0}$ such that, for all $\boldsymbol{A}$, $\boldsymbol{A}+\mathbf{0}=\boldsymbol{A}$

ii) Linearly independent vectors $\boldsymbol{A}, \boldsymbol{B}, \ldots, \boldsymbol{V}$:

$a\boldsymbol{A}+b\boldsymbol{B}+\cdots+v\boldsymbol{V}=\mathbf{0}$ holds only for the trivial one with $a=b=\cdots=v=0$

iii) An n-dimensional(n-D) vector space:

There exist n linearly independent vectors, but no set of n+1 linearly independent one.

iv) Base vectors and a coordinate system

Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ be a set of n linearly independent vectors in the n-D vector space. If $\boldsymbol{A}$ is an arbitrary vector in this space,

$\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n, \boldsymbol{A}$ $\neq$ linearly independent (n+1 vectors)

$\therefore\ \alpha\boldsymbol{a}_1+\beta\boldsymbol{a}_2+\cdots+\tau\boldsymbol{A}=\mathbf{0}$

where $\alpha, \beta, \cdots, \tau$ are not all zero, especially $\tau\neq 0$.

$$\boldsymbol{A}=\left(-\frac{\alpha}{\tau}\right)\boldsymbol{a}_1+\left(-\frac{\beta}{\tau}\right)\boldsymbol{a}_2+\cdots+\left(-\frac{\nu}{\tau}\right)\boldsymbol{a}_n$$

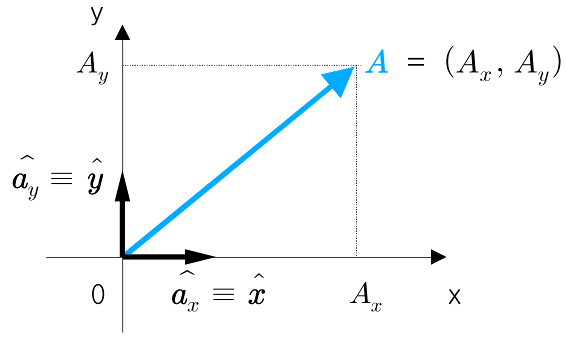
$$\equiv\ A_1\ \boldsymbol{a}_1+\ A_2\ \boldsymbol{a}_2+\cdots+\ A_n\ \boldsymbol{a}_n=\sum_{i=1}^{n}\boldsymbol{a}_i A_i \qquad (2)$$

The vectors $\boldsymbol{a}_i$ $(i=1, 2, \cdots, n)$ are said to form a **basis** or a **coordinate system**.

$A_1, A_2, \cdots, A_n$ are called **components** of vector $\boldsymbol{A}$, and $\boldsymbol{a}_i$ $(i=1, 2, \cdots, n)$ are called **base vectors**.

(e.g.) In a 2-D Cartesian coordinate system,

$$\boldsymbol{A}=\boldsymbol{a}_x A_x+\boldsymbol{a}_y A_y=\hat{\boldsymbol{x}}A_x+\hat{\boldsymbol{y}}A_y=(A_x, A_y) \qquad (2)^*$$



b) Scalar (or Dot) Product

Definition:

$$\boxed{\mathbf{A}\cdot\mathbf{B} \triangleq AB\cos\theta_{AB}.} \tag{2-6}$$

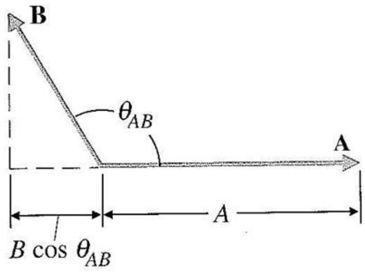
where

$$\theta_{AB} \equiv \measuredangle(A, B)$$

$$= \cos^{-1}\left(\frac{A\cdot B}{AB}\right)$$

$$0 \le \theta_{AB} \le \pi$$

FIGURE 2-4 Illustrating the dot product of **A** and **B**.



Notes)

i) $A\cdot B$ = scalar $\le AB$

ii) $A\cdot B = A(B\cos\theta_{AB}) = B(A\cos\theta_{AB})$

= |one vector| x projection of the other vector on the first one

iii) $A\cdot B > 0$ if $\theta < \frac{\pi}{2}$ and $A\cdot B < 0$ if $\theta > \frac{\pi}{2}$

iv) $A\cdot B = 0 \leftrightarrows A\perp B$: perpendicular to each other (3)

v) $A\cdot B = B\cdot A$ commutative (2-7)

$A\cdot(B+C) = A\cdot B + A\cdot C$ distributive

vi) $A\cdot A = A^2 \Rightarrow A \equiv |A| = \sqrt{A\cdot A}$ magnitude of A (2-8), (2-9)

c) Vector (or Cross) Product

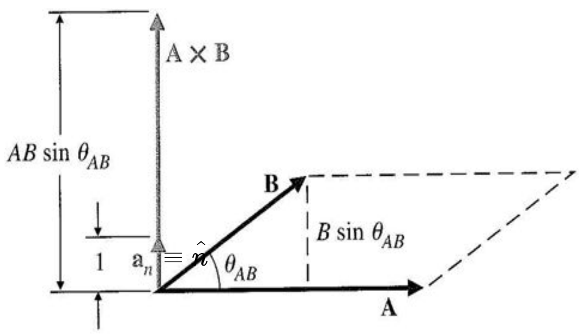
Definition:

$$\boxed{\mathbf{A}\times\mathbf{B} \triangleq \mathbf{a}_n AB\sin\theta_{AB},} = \hat{n}AB\sin\theta_{AB} \tag{2-12}$$

where $\theta_{AB} \equiv \measuredangle(A, B)$, $0 \le \theta_{AB} \le \pi$

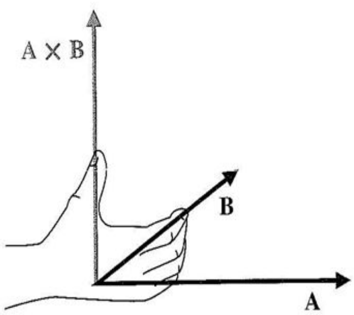
$a_n \equiv \hat{n}$ = unit vector normal to a plane containing A & B, directing in accordance with the right-hand rule.

FIGURE 2-6 Cross product of **A** and **B**, **A** × **B**.



(a) $\mathbf{A}\times\mathbf{B} = \mathbf{a}_n AB\sin\theta_{AB}$

FIGURE 2-6 Cross product of **A** and **B**, **A** × **B**.



(b) The right-hand rule.

Notes)

i) $A \times B$ = vector whose direction $\hat{n}$ obtained by the right-hand rule

ii) $|A \times B| = A(B\sin\theta_{AB})$ = area of parallelogram by A & $B \geq 0$

iii) $A \times B = 0 \rightleftarrows A \swarrow\swarrow B$ (parallel) or $A \swarrow\nearrow B$ (anti-parallel) (4)

iv) $A \times B = -B \times A$ anti-commutative (2-13)

$A \times (B + C) = A \times B + A \times C$ distributive (5)

c) Triple Products

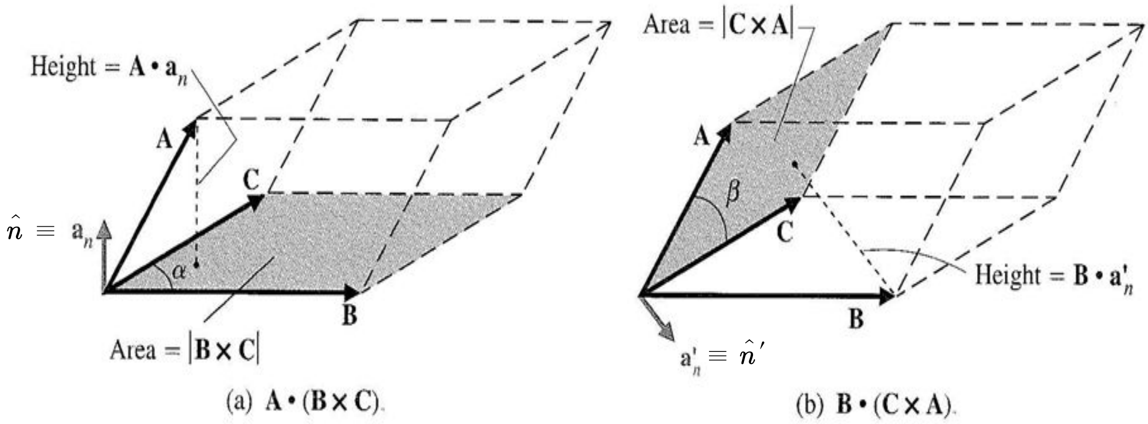
(1) Scalar Triple Product

$A \cdot (B \times C) = (A \cdot \hat{n})(BC\sin\alpha)$ (2-14)

= (height)x(area) of the parallelepiped formed by A, B, C

= (volume) of the parallelepiped formed by A, B, C

FIGURE 2-7 Illustrating scalar triple products.



Notes)

$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$ *in cyclic permutation* (2-15)

(2) Vector Triple Product

$A \times (B \times C) = B(C \cdot A) - C(A \cdot B)$ (6), (2-113)

Notes)

$A \times (B \times C) \neq (A \times B) \times C$ not associative

Note) Dyadic: Direct product of two vectors $\Rightarrow$ Tensor

$AB = \overleftrightarrow{A}$